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MATHEMATICAL PHYSICS

Equation Admitting Linearization and Describing Waves in Dissipative Media with Modular, Quadratic, and Quadratically Cubic Nonlinearities

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Abstract—A second-order partial differential equation admitting exact linearization is discussed. It contains terms with nonlinearities of three types—modular, quadratic, and quadratically cubic—which can be present jointly or separately. The model describes nonlinear phenomena, some of which have been studied, while others call for further consideration. As an example, individual manifestations of modular nonlinearity are discussed. They lead to the formation of singularities of two types, namely, discontinuities in a function and discontinuities in its derivative, which are eliminated by dissipative smoothing. The dynamics of shock fronts is studied. The collision of two single pulses of different polarity is described. The process reveals new properties other than those of elastic collisions of conservative solitons and inelastic collisions of dissipative shock waves.

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Second-order nonlinear partial differential equations admitting exact linearization are of interest in mathematical physics and for the understanding of the dynamics of nonlinear phenomena. Especially useful are equations adequate for actual systems.

In this work, we consider the equation

$$\frac{\partial V}{\partial Z} = \frac{\partial}{\partial \theta}\left(\alpha |V| + \frac{\beta}{2}V^2 + \frac{\gamma}{2}V|V|\right) + \Gamma\frac{\partial^2 V}{\partial \theta^2}. \qquad (1)$$

The coefficients α, β, and γ are involved in the terms with nonlinearities, which are called modular, quadratic, and quadratically cubic, respectively. If there is only a quadratic nonlinearity ($\alpha = \gamma = 0$, $\beta \neq 0$), Eq. (1) turns into the well-studied Burgers equation (see, e.g., [1, 2]). This equation, first, can be linearized by a simple change of variable and, second, has an important physical interpretation. Specifically, for acoustic waves, it correctly describes the formation of compression shock waves, the transformation of a harmonic signal into a sawtooth wave with triangular teeth, harmonic generation, the transformation of broad (including noise) spectra, and a variety of other observed phenomena [3–5].

The quadratically cubic (QC) equation ($\alpha = \beta = 0$, $\gamma \neq 0$) has been recently proposed and studied [6–9]. It can also be linearized and has an important physical interpretation. The QC equation describes compression and rarefaction shock waves, which are stable only for certain shock parameter values, the transformation of a harmonic wave into a sawtooth waveform with trapezoidal teeth, self-action, nonlinear damping, and other effects.

Recently, studies were initiated of waves in media with a modular nonlinearity ($\beta = \gamma = 0$, $\alpha \neq 0$) [10–12], which exhibit different elasticity properties under tensile and compressive strains. Examples of such materials are reinforced polymers and concretes (see [13, Chapter 1, Section 10]). This equation is even simpler than the above two linearizable ones. It is linear for the function preserving its sign, i.e., for $V > 0$ or $V < 0$. Nonlinear effects are observed only in sign-changing solutions.

Is easy to see that the substitution

$$V = \frac{2\Gamma}{\beta \pm \gamma}\frac{\partial}{\partial \theta}\ln U \qquad (2)$$

made in (1) reduces it to a linear equation for the auxiliary function U:

$$\frac{\partial U}{\partial Z} = \pm\alpha\frac{\partial U}{\partial \theta} + \Gamma\frac{\partial^2 U}{\partial \theta^2}. \qquad (3)$$

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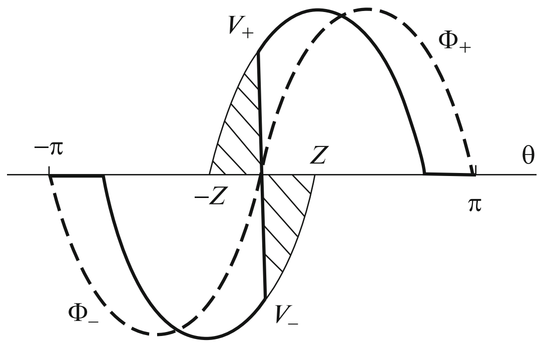

Fig. 1. Profile of a single period of the initial harmonic wave (dashed curve) and its distorted shape at a distance of Z (thick solid curve).

In formulas (2) and (3), the upper and lower signs correspond to $V > 0$ and $V < 0$, respectively.

Equation (1) is derived by applying the standard scheme [1, 2]. Considering, for example, a viscous medium, we reduce the Navier–Stokes and continuity equations for small pressure p' and density ρ' perturbations to the relation

$$\Delta p' = \frac{\partial^2 \rho'}{\partial t^2} - \frac{b}{\rho_0}\frac{\partial}{\partial t}\Delta\rho'. \tag{4}$$

Here, b is the dissipation coefficient and ρ_0 is the equilibrium density. The equation of state is specified as

$$c_0^2\rho' = p' - g|p'| - \frac{\varepsilon}{c_0^2\rho_0}p'^2 - \frac{q}{c_0^2\rho_0}p'|p'|. \tag{5}$$

We eliminate the density increment and pass to the coordinate system $z, \tau = t - \frac{z}{c_0}$ moving along the z axis at the speed of sound c_0. Assume that the wave is plane and the distortion is slow in z, which is ensured by small nonlinear and dissipative terms. Neglecting the terms of orders higher than the first, we obtain the evolution equation

$$\frac{\partial p'}{\partial z} = \frac{\partial}{\partial \tau}\left(\frac{g}{2c_0}|p'| + \frac{\varepsilon}{2c_0^3\rho_0}p'^2 + \frac{q}{2c_0^3\rho_0}|p'|p'\right) + \frac{b}{2c_0^3\rho_0}\frac{\partial^2 p'}{\partial \tau^2}. \tag{6}$$

To reduce Eq. (6) to dimensionless form (1), we use the notation

$$Z = \frac{z}{z_*}, \quad \theta = \omega\tau, \quad V = \frac{p'}{p_0'}, \quad \Gamma = \frac{z_*}{z_{\text{DISS}}} = \frac{b\omega^2 z_*}{2\rho_0 c_0^3}. \tag{7}$$

The constants are interpreted as follows: ω and p_0' are the characteristic frequency and amplitude of the initial signal, and z_* and z_{DISS} are the characteristic nonlinearity and dissipation distances; Γ is a similarity criterion: dissipation dominates for $\Gamma \gg 1$, while nonlinearity dominates for $\Gamma \ll 1$. With notation (7), Eq. (6) takes the form of (1), where

$$\frac{\alpha}{z_*} = \frac{g\omega}{2c_0}, \quad \frac{\beta}{z_*} = \frac{\varepsilon\omega p_0'}{2c_0^3\rho_0}, \quad \frac{\gamma}{z_*} = \frac{q\omega p_0'}{2c_0^3\rho_0}. \tag{8}$$

Here, we consider only the effects of the modular nonlinearity, setting $\alpha = \beta = 0$ and $\gamma \neq 0$ in Eq. (1). We can set $\gamma = 1$ by choosing a suitable z_*.

First, we neglect dissipation, setting $\Gamma = 0$ in the modular equation. The solution of the degenerate problem for the positive V_+ and negative V_- branches has the form

$$V_\pm = \Phi_\pm(\theta \pm Z). \tag{9}$$

Here, $\Phi_\pm(\theta)$ is the function $V_\pm(\theta)$ (i.e., the initial waveform) at $Z = 0$.

Figure 1 illustrates the behavior of solution (9). The dashed curve depicts one period of the original $(Z = 0)$ harmonic wave. As the distance Z grows, the positive and negative half-periods get overlapped near the point $\theta = 0$. The ambiguity of the function is eliminated by constructing a discontinuity that "cuts off" equal areas (see below). In Fig. 1, these areas are shaded.

It can be seen that the nonlinearity gives rise to singularities of two types: a discontinuity of the function $V(\theta)$ (at $\theta = 0$) and a derivative discontinuity (at $\theta = \pi - Z$, $\theta = -\pi + Z$). There is nonlinear damping, so the wave completely disappears at a finite distance of $Z = \pi$.

Now, we find a solution eliminating the first singularity (the discontinuity of the function) under finite dissipation, $\Gamma \neq 0$. The jump is assumed to be asymmetric with respect to the level $V = 0$. The desired solution must have the form a shock wave, i.e., a drop between the negative value of the variable $V \to -|V_-|$ as $\theta \to -\infty$ and its positive value $V \to |V_+|$ as $\theta \to +\infty$. Such a wave is stationary. Its shape does not change with increasing Z. The solution is sought in the form $V = \Phi(T = \theta + \delta Z)$. Substituting this expression into the equation and integrating the result yields

$$\Gamma\frac{dV}{dT} = \delta V - |V| + D. \tag{10}$$

The constants are determined using the boundary conditions at infinity:

$$\delta = \frac{|V_+| - |V_-|}{|V_+| + |V_-|}, \quad D = \frac{2|V_+||V_-|}{|V_+| + |V_-|}. \tag{11}$$

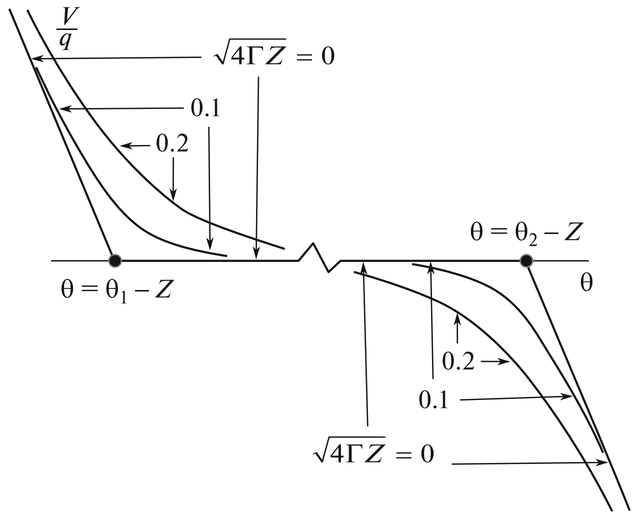

Fig. 2. Removing the singularity (derivative discontinuity) arising for $\sqrt{4\Gamma Z} = 0$ at two points marked by solid circles. The smoothed curves correspond to $\sqrt{4\Gamma Z} = 0.1$ and 0.2.

Substituting (11) into (10), we find two branches of the solution: for $V > 0$ and for $V < 0$. Requiring that the solution and its first derivative be continuous at the point $T = 0$ gives

$$V = |V_+|\left[1 - \exp\left(-\frac{2|V_-|}{|V_+| + |V_-|}\frac{T}{\Gamma}\right)\right], \quad T > 0,$$
$$V = -|V_-|\left[1 - \exp\left(\frac{2|V_+|}{|V_+| + |V_-|}\frac{T}{\Gamma}\right)\right], \quad T < 0. \tag{12}$$

For a symmetric jump (see Fig. 1), $|V_+| = |V_-|$. Moreover, $\delta = 0$ and $T = \theta$. A similar form of a symmetric shock wave for the usual Burgers equation is given by $V = \tanh\left(\frac{\theta}{2\Gamma}\right)$ [1, 2].

Figure 1 shows that a discontinuity in the function arises when a domain of negative values of V is followed by a domain of its positive values. A singularity of the second type (derivative discontinuity) arises in the opposite case, i.e., when a domain with $V \geq 0$ is followed by a domain with $V \leq 0$. Moreover, the two domains do not get overlapped; on the contrary, they move away from each other. The structure shown in Fig. 2 is formed. The locations of the singularities for $\Gamma Z = 0$ are shown by solid circles. Dissipation leads to the smoothing of the corner. The smoothing functions are given by the expressions

$$V = \frac{q_1}{2}(\theta - \theta_1 + Z)\left[\operatorname{erf}\left(\frac{\theta - \theta_1 + Z}{\sqrt{4\Gamma Z}}\right) - 1\right],$$
$$V = -\frac{q_2}{2}(\theta - \theta_2 - Z)\left[\operatorname{erf}\left(\frac{\theta - \theta_2 - Z}{\sqrt{4\Gamma Z}}\right) + 1\right]. \tag{13}$$

Here, $\operatorname{erf}(\theta)$ is the error integral. The constants q_1 and q_2 determine the slopes of the left and right rays, while θ_1 and θ_2 are the initial locations of the singularities. As Γ and Z grow, smoothing is enhanced (see the curves for $\sqrt{4\Gamma Z} = 0.1$ and 0.2).

Note that the described smoothing of singularities agrees with the numerical results obtained by Nazarov's research team [10–12]. Smoothed solutions can also be constructed by applying the standard method of matched asymptotic expansions (see, e.g., [14]). In that case, the locations of the derivative discontinuities are obvious, while the coordinates $\theta_{SH}(Z)$ of the shock fronts are to be calculated.

This can be done as follows. In the case of weak dissipation ($\Gamma \ll 1$), the front position θ_{SH} is determined by the momentum conservation law for the wave; its geometric interpretation is that the cut off areas are equal [1, 2] (shown by shaded areas in Fig. 1):

$$\frac{d}{dZ}\int_{-|V_-|}^{|V_+|}[\theta(V) - \theta_{SH}(Z)]dV = 0,$$
$$\frac{d\theta_{SH}}{dZ} = -\frac{|V_+(Z)| - |V_-(Z)|}{|V_+(Z)| + |V_-(Z)|}. \tag{14}$$

Equation (14) involves three unknown functions of Z: θ_{SH}, $|V_+|$, and $|V_-|$. Therefore, (14) has to be supplemented with two equations following from the fact that $|V_+|$ and $|V_-|$ both lie on the shock front and the smooth segments of profile (9):

$$\theta_{SH} = \Phi_+^{-1}(|V_+|) - Z, \quad \theta_{SH} = \Phi_-^{-1}(-|V_-|) + Z. \tag{15}$$

Here, $\Phi_\pm^{-1}$ denotes the inverses of the functions $\Phi_\pm$. System (14), (15) is complete, and it can be used to calculate the shape of the wave profile for a given initial shape with allowance for the displacement of the arising discontinuity along the θ axis.

For the harmonic initial wave shown in Fig. 1, the problem is trivial. The front is not displaced, i.e., $\theta_{SH}(Z) \equiv 0$, and the discontinuity parameters vary according to the law $|V_+| = |V_-| = \sin Z$. The period-averaged intensity of this wave decreases due to nonlinear damping:

$$\overline{V^2} = \frac{1}{2} + \frac{1}{4\pi}(\sin 2Z - 2Z). \tag{16}$$

For initial profiles of complex geometry, the solution of nonlinear system (14), (15) has a cumbersome form. Consider a simple case. Suppose that the initial wave consists of two pulses of different polarity with flat segments in the form of straight line ones and with a common discontinuity (curve *2* in Fig. 3). The positive parameters θ_1, θ_2, V_1, and V_2, which determine the durations and amplitudes of the pulses, are shown

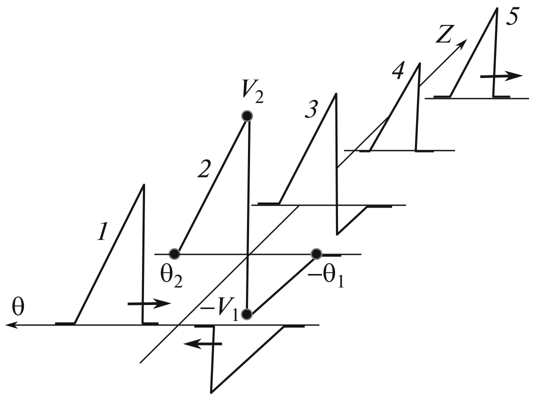

Fig. 3. Collision of two pulses of different polarity.

by solid circles in Fig. 3. For this profile, Eq. (14) becomes

$$\frac{d\theta_{SH}}{dZ} = -\frac{(V_2 - V_1) - \left(\frac{V_2}{\theta_2} - \frac{V_1}{\theta_1}\right)Z - \left(\frac{V_2}{\theta_2} + \frac{V_1}{\theta_1}\right)\theta_{SH}}{(V_2 + V_1) - \left(\frac{V_2}{\theta_2} + \frac{V_1}{\theta_1}\right)Z - \left(\frac{V_2}{\theta_2} - \frac{V_1}{\theta_1}\right)\theta_{SH}}. \quad (17)$$

Equation (17) is reduced to a homogeneous one and is easy to solve. After determining the discontinuity coordinate $\theta_{SH}(Z)$, we calculate the shock parameters $V_+(Z), V_-(Z)$.

The results are shown in Fig. 4 for $\theta_1 = \theta_2$ and $\frac{V_1}{V_2} =$ 0 (dashed curves *0*), 0.11 (curves *1*), 0.33 (curves *2*), and 0.82 (curves *3*). When there is no negative pulse (i.e., $V_1 = 0$), the positive pulse propagates with a constant velocity and amplitude. If $\frac{V_1}{V_2} \neq 0$, the pulses interact. As a result, the negative pulse with a smaller amplitude is absorbed by the positive pulse (i.e., $|V_-|$ vanishes at some distance $Z = Z_*\left(\frac{V_1}{V_2}\right)$), and $|V_+|$ decreases up to some positive value. In the course of the interaction, the positive pulse "slows down," i.e., its "velocity" $\frac{d\theta_{SH}}{dZ}$ decreases. When the negative pulse disappears at distances $Z > Z_*$, the nonlinear interaction ceases and the positive signal speeds up to the initial value $\frac{d\theta_{SH}}{dZ} = 1$.

The stages of the process are shown in Fig. 3. First, the pulses are separated and begin to approach each other (curve *1*). There appears a coupled state with a common shock front (curve *2*). At this time, nonlinear damping is "switched on" (compare curves *2* and *3*), which persists until the negative pulse disappears (curve *4*). The arising positive pulse has a phase lag and a smaller amplitude as compared with the initial one. Subsequently, it propagates with its shape unchanged (cf. curves *4* and *5*).

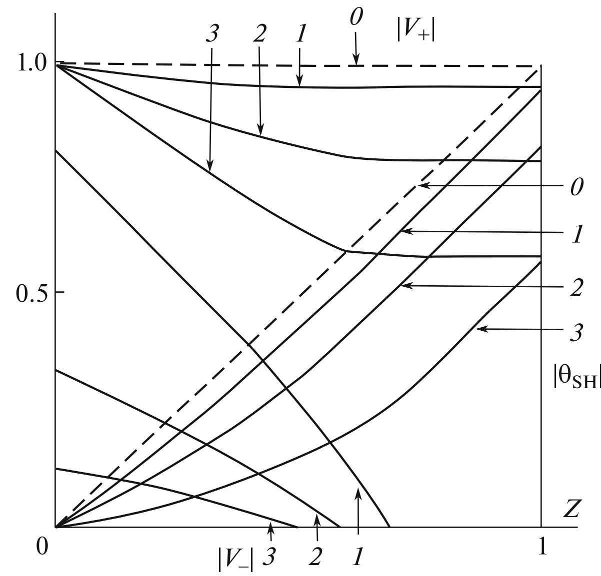

Fig. 4. Discontinuity coordinate $|\theta_{SH}|$ and the shock parameters $|V_+|$ and $|V_-|$ as functions of the distance Z.

Thus, the interaction of solitary waves reveals properties other than those observed in elastic collisions of solitons and inelastic coalescence of shock waves. There is an analogy with the interaction of blobs of chemicals, for example, inflammable and oxidizing materials. As a result of the reaction, the smaller component disappears, while the mass of the larger one decreases.

In future works, it would be of interest to study phenomena originating from the simultaneous effect of any two or even all three types of nonlinearity present in the linearizable equation (1).

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REFERENCES

1. O. V. Rudenko and S. I. Soluyan, *Theoretical Foundations of Nonlinear Acoustics* (Plenum, New York, 1977).
2. S. N. Gurbatov, O. V. Rudenko, and A. I. Saichev, *Waves and Structures in Nonlinear Nondispersive Media* (Springer, Berlin, 2011).
3. O. V. Rudenko, Phys.-Usp. **38**, 965–989 (1995).
4. O. V. Rudenko, Phys.-Usp. **29**, 620–641 (1986).
5. B. K. Novikov, O. V. Rudenko, and V. I. Timoshenko, *Nonlinear Underwater Acoustics* (Am. Inst. Physics, New York, 1987).

6. O. V. Rudenko, Phys.-Usp. **183** (7), 719–726 (2013).
7. O. V. Rudenko and C. M. Hedberg, Dokl. Math. **91** (2), 232–235 (2015).
8. O. V. Rudenko and C. M. Hedberg, Nonlin. Dyn. **85** (2), 767–776 (2016).
9. V. A. Gusev and O. V. Rudenko, Dokl. Math. **93** (1), 94–98 (2016).
10. V. E. Nazarov, S. B. Kiyashko, and A. V. Radostin, Radiophys. Quantum Electron. **59** (3), 246–256 (2016).
11. A. V. Radostin, V. E. Nazarov, and S. B. Kiyashko, Wave Motion **50** (2), 191–196 (2013).
12. V. E. Nazarov, S. B. Kiyashko, and A. V. Radostin, Nelin. Din. **11** (2), 209–218 (2015) [in Russian].
13. S. A. Ambartsumyan, *Elasticity Theory of Different Moduli* (Nauka, Moscow, 1982; China Rail. Publ. House, Beijing, 1986).
14. A. H. Nayfeh, *Introduction to Perturbation Techniques* (Wiley, New York, 1981; Mir, Moscow, 1984).

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