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The new approach to the construction of parametric membership functions for fuzzy sets with unequal supports

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Abstract

The current research is devoted to developing methods of a novel mathematical interpretation of term-sets of linguistic variables. To the term-sets of the linguistic variables fuzzy sets are assigned. We intend to adopt the *s*-functions and the π-functions to derive formulas of membership functions of these sets. The fuzzy sets are divided in three families in the case of an odd number of the term-sets. To each family, we assign only one parametric formula, which depends on two parameters: the width of a non-fuzzy set, which contains all supports of the fuzzy sets being representatives of the term-sets, and a number of the term-sets. Provided that the supports of fuzzy sets will be unequal, the membership function of the set, belonging to one of the families, is computed by means of a functional modifier, inserted in the common equation typical of this family. Medical examples explain how to use cumulated membership functions practically. The procedure can be easily computerized.

Keywords: s-function; π-function; term-set of linguistic variable; variable intensity level; set-parametric boundaries; functional modifier.

1. Introduction

Membership functions of fuzzy sets and fuzzy numbers have a powerful significance in applications, involving the designs of these mappings. Shapes of functions and their boundaries have a decisive character in such models as approximate reasoning, fuzzy control and others.

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Very often, in order to divide measurable or abstract features into intensity levels, we state a list of these levels, provided as verbal expressions. The list of expressions, named "term-sets of a linguistic variable", was proposed by L. Zadeh in[1]. Each term, originating from a natural language, was replaced by a fuzzy set to operate with numbers instead of words in practical calculations. The grammar and axioms of linguistic variables were developed, e.g., in[1,2,3] to explain a creation of the linguistic variables and their contents. In order to furnish fuzzy sets with appropriate membership functions, different techniques were tested after appointing so called atoms and hedges in the lists of term-sets.

In the first trials of generating membership functions of linguistic terms, the operations of concentration, dilution and complement were added to some selected membership functions, which resulted in the narrowing, the widening and the inverse effects in membership functions of all terms[1,2,3,4]. The operations made changes in membership function patterns, but they still preserved the same supports of fuzzy sets, even if these sets represented different terms. It was expected to find some modifiers, which should fulfil the double role, namely, these should alter supports of fuzzy sets and change forms of their membership functions[5,6,7]. The insertion of modifiers like scalar product, normalization, Bouchon-Meunier modifiers, perturbation and (weakening and reinforcement) power, into the membership functions of atoms[8], led to substantial progress, due to expectations mentioned above.

A model for the parametric representation of linguistic hedges in Zadeh's fuzzy logic gave rise to the induction of a linguistic truth variable in[4]. Interesting attempts of generating the term-set lists' contents are discussed as: order relation among terms[9], weighted linguistic labels[10], applications of the OWA and FLIOWA operators[11], and a measure of the goodness of a linguistic summary[12]. The modification of term-sets by the application of linguistic quantifiers, involving descriptions of probability values or Bayesian approach, was discussed in[13,14].

The rapid development of computational intelligence brought such algorithms as, e.g., the c-means classification of objects into clusters and neural networks. It is remarkable to see their influence on the formation of term-sets of linguistic variables in such research works as[15] and[16,17], respectively. At last, the terms of the linguistic variable are affected by several distance measures, when defining complex numbers verbally[18].

A linguistic variable can be stated for intensity levels of some parameters, like levels of clinical markers. The names of levels create a list of term-sets, replaced by fuzzy sets.

During many personal contacts with practitioners and users of fuzzy set theory, we have heard the opinion that a pattern of membership functions of verbally defined intensity levels should be based on two values. The users have noticed that the width of a common non-fuzzy reference set, including all supports of fuzzy sets, and a number of terms should be the most desirable parameters, easily determined for the purpose of practical and computerized actions.

To meet the users' suggestions, we propose a new system of determining membership functions of fuzzy sets without guessing their boundary values ad hoc. We thus divide all term-sets of the linguistic variable into three families, when the number of terms is odd. As membership functions, the s-class and π-class functions[19,20] will be adopted. Each family is determined by only one formula depending on width E of the crisp "*reference set*" containing the supports of all fuzzy sets assisting term-sets, number m of all term-sets and number t, which a term set receives in its family. Further, in order to find formulas of membership functions of fuzzy sets, common for each family of sets, the parameters of the classical s- and π-functions will be constructed by operations on units E and m. By means of common formulas of membership functions, the user is provided with automatically computed boundaries of the supports of fuzzy sets, which prevents a technician, not familiar with fuzzy sets, from predetermining boundary values of fuzzy sets in an intuitive or a random manner. The term number t, marking the position of the fuzzy set in its family, will be involved in the value of a modifier, included in the family's formula. The method of parametrization of three general formulas, providing us with all membership functions of term-sets due to the definition of the linguistic variable, was already discussed in[21,22,23].

We now concentrate on a special case of the model, namely, we wish the middle set from the list of the linguistic variable to be widest. Consequently, the left and right neighbors of the widest set will partially diminish their widths.

The theorem, formulated and proved in the paper, is an original proposal of modeling the membership functions. Some uncomplicated parts of larger medical queries are added to test the reliability of formulas.

The arrangement of the paper is organized in five sections. Section 2 samples the basic information about the nature of fuzzy sets. In Section 3, we derive formulas of parametric membership functions assigned to term-sets of a linguistic variable. The case study is tested in Section 4. In Section 5, we summarize the results of the current research.

2. Introduction to fuzzy sets

Before discussing mathematical patterns, including *s*- and π-functions[19,20], let us recall the definition of a fuzzy set. If X is a set of non-negative real numbers, denoted generically by x, then a fuzzy set A in X is a set of ordered pairs $A = \{(x, \mu_A(x)) : x \in X\}$, where $\mu_A(x) \in [0,1]$[24]. Each element x gets a membership degree $\mu_A(x)$, which expresses the strength of the relationship between x and A. Membership degrees equal to 1 inform about the total relation between the element and the set. The function $\mu_A: X \to [0,1]$ is called "*the membership function*" of A.

We will also use the following definitions.

The support of a fuzzy set A, supp(A), is a non-fuzzy (crisp) set of all $x \in X$ such that $\mu_A(x) > 0$[24].

The α-cut of a fuzzy set A, A_α, is a non-fuzzy set of all $x \in X$ such that $\mu_A(x) \geq \alpha$.

The Euclidean distance $d(P_1, P_2)$ between points $P_1 = (x_1,\mu(x_1))$ and $P_2 = (x_2,\mu(x_2))$, in the two dimensional system with x- and $\mu(x)$-axes, is estimated as $d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (\mu(x_2) - \mu(x_1))^2}$. If we refer to the concept of the distance between two points in the text below, then we will involve the formula of $d(P_1, P_2)$ in practical computations.

Apart from linear shapes, mostly employed as membership functions of fuzzy sets, we prefer using the *s*-class function, formed as a specially designed polynomial of the second degree. Set A has the membership function $\mu_A(x)$ given by

$$\mu_A(x) = s(x, \alpha_A, \beta_A, \gamma_A) = \begin{cases} 0 & \text{for} \quad x \leq \alpha_A, \\ 2\left(\frac{x-\alpha_A}{\gamma_A-\alpha_A}\right)^2 & \text{for} \quad \alpha_A \leq x \leq \beta_A, \\ 1-2\left(\frac{x-\gamma_A}{\gamma_A-\alpha_A}\right)^2 & \text{for} \quad \beta_A \leq x \leq \gamma_A, \\ 1 & \text{for} \quad x \geq \gamma_A, \end{cases} \tag{1}$$

where α_A, β_A, γ_A are parameters of the *s*-function, $\beta_A = (\alpha_A + \gamma_A)/2$, and x-values belong to the domain of the *s*-function. Since the *s*-function is adopted as a membership function of fuzzy set A, then x will be in practice an element of space X, in which set A is defined. The parameters α_A and γ_A are interpreted as elements of A, for which the membership function of A takes values between 0 and 1.

If we desire to maintain the shape of the *s*-function, but we want to set $\delta\alpha_A$ instead of α_A and $\delta\gamma_A$ instead of γ_A in (1), then

$$\mu_A(x) = s(x, \delta\alpha_A, \delta\beta_A, \delta\gamma_A) = \begin{cases} 0 & \text{for} \quad x \leq \delta\alpha_A, \\ 2\left(\frac{x-\delta\alpha_A}{\delta\gamma_A-\delta\alpha_A}\right)^2 = 2\left(\frac{x-\delta\alpha_A}{(\gamma_A-\alpha_A)\delta}\right)^2 & \text{for} \quad \delta\alpha_A \leq x \leq \delta\beta_A, \\ 1-2\left(\frac{x-\delta\gamma_A}{\delta\gamma_A-\delta\alpha_A}\right)^2 = 1-2\left(\frac{x-\delta\gamma_A}{(\gamma_A-\alpha_A)\delta}\right)^2 & \text{for} \quad \delta\beta_A \leq x \leq \delta\gamma_A, \\ 1 & \text{for} \quad x \geq \delta\gamma_A. \end{cases} \tag{2}$$

Parameter δ initiates a scale of values $0 < \delta < 1$ for narrowed domains of functions described by Eq. (2) and, apparently, for narrowed supports of fuzzy sets, utilizing Eq. (2) as formulas of membership functions. Values $\delta > 1$ have an effect of enlarging domains in Eq. (2). Value $\delta = 1$ allows returning to Eq. (1)[7].

In many applications of fuzzified mathematical models, e.g., fuzzy control systems, we need to introduce a linguistic variable specified by list of verbal expressions (term-sets) to facilitate the communication with a professional adviser.

Our purpose is to put together fuzzy sets with the same properties in common families to derive one formula for membership functions characteristic of sets, placed in the same family. We propose a mathematical formalization of the membership functions of fuzzy sets, belonging to a common family, by adopting the *s*-function $s(x, \alpha, \beta, \gamma)$.

3. The construction of parametric membership functions for families of fuzzy sets with unequal supports

In some technical and medical applications, demanding the initialization of term sets, researchers wish to assign the widest support to the medium fuzzy set, whereas other sets, allocated before and after the widest set will narrow their supports in conformity with a certain scale.

Example 1
Let us differ the clinical marker "*systolic blood pressure*" in 7 levels: L_1 = "*critically low*", L_2 = "*very low*", L_3 = "*low*", L_4 = "*rather normal*", L_5 = "*high*", L_6 = "*very high*" and L_7 = "*critically high*". Due to the physician experience, we state the reference set $L = [x_{\min}, x_{\max}] = [40Hg/mm, 260Hg/mm]$. Rather normal values can approximately be found in interval (110, 180), which overlaps supports of "*low*" and "*high*". Extremely low blood pressure values among 40 and 55 can appear in the support of "*critically low*", where a patient is exposed to the voluminous bleeding. The support of "*critically high*" may be estimated as interval (245, 260). We wish to implement functions sketched in Fig. 1.

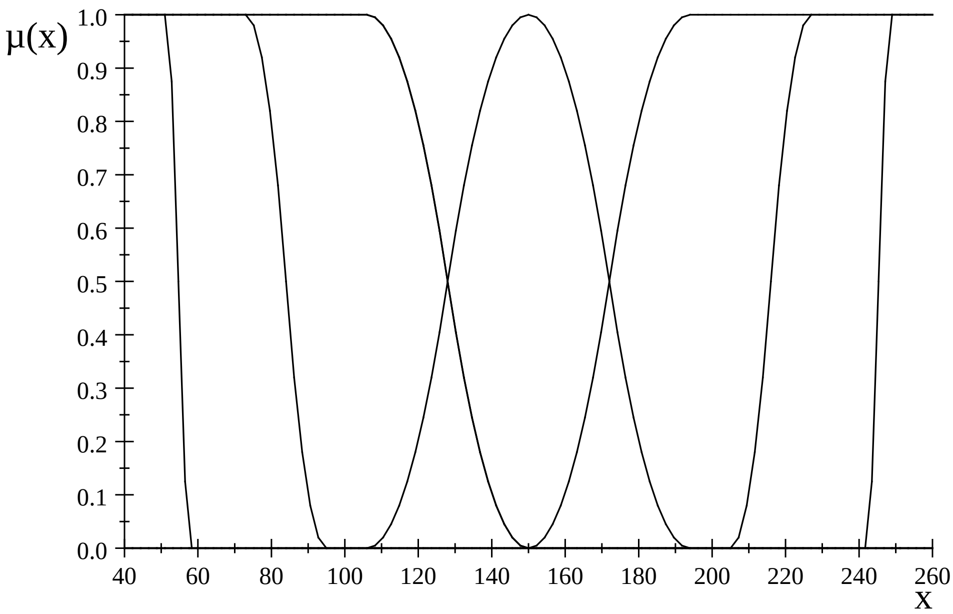

Fig. 1. The expected membership functions for levels of "systolic blood pressure"

In general, the term-sets are assigned to fuzzy sets $L_1,\dots,L_m$, where m is an odd positive integer. We assume that supports of L_l, $l = 1,\dots,m$, being continuous sets, will cover parts of the common continuous crisp reference set $L = [x_{\min}, x_{\max}]$. Due to the definition of the α-cut set of a fuzzy set, we denote by $L_{l,\alpha}$ a set of $x \in L$, for which $\mu_{L_l}(x) \geq \alpha, l = 1, \dots, m$.

Example 2
For, e.g., $\alpha = 0.5$ in L_1, we approximate $L_{1,0.5} = (40,54.667)$. If $\alpha = 1$, then, $L_{3,1} - L_{2,1} = (40,106) - (40,73) = (73,106)$. We explain how to compute the borders of intervals considered after deriving membership functions of sets $L_1, \dots, L_7$. Example 2 helps us to understand the assumptions of the following theorem.

Theorem 1
Let us suppose that term-sets L_l, $l = 1,\dots,m$, have supports included in the common non-fuzzy reference set $L = [x_{\min}, x_{\max}]$, where $x_{\min} = \min(L)$ and $x_{\max} = \max(L)$, for $x \in L \subset X$. The width E of L is computed as $E = x_{\max} - x_{\min}$. A family of "*left*" sets $L_1,\dots, L_{(m-1)/2}$ contains L_t sets, where $t = 1,\dots,(m-1)/2$. The set $L_{(m+1)/2}$ is called "*in the middle*". A

family of "*right*" sets $L_{(m+3)/2},\ldots,L_m$ consists of $L_{(m+3)/2+t-1}$ sets for $t = 1,\ldots,(m-1)/2$, where t is a function number in the respective family. Assume that the widths of sets $L_{1,1}$, $L_{t,1}$–$L_{t-1,1}$, $t = 2,...,(m-1)/2$, created by α-cuts of $L_1,\ldots,L_{(m-1)/2}$ for $\alpha = 1$, satisfy the following conditions: the width of $L_{1,1}$ is equal to r, the width of $L_{2,1}$–$L_{1,1}$ is equal to $2r,\ldots$,the width of $L_{(m-1)/2,1}$–$L_{(m-3)/2}$ is equal to $((m-1)/2)r$, and suppose that the Euclidean distance between points $(\alpha_{L_{(m-1)/2}},1)$ and $(E/2),1)$ is equal to $((m+1)/2)r$, where r is a solution of equation

$$1\cdot r + 2\cdot r + \ldots + (\tfrac{m-1}{2})\cdot r + (\tfrac{m+1}{2})\cdot r = \tfrac{E}{2} \tag{3}$$

Next, we assume that the membership functions of $L_{(m-1)/2}$ and $L_{(m+3)/2}$ have the intersection points with the membership function of $L_{(m+1)/2}$ on membership level 0.5.

As a result, we determine the common equation for membership functions of "*left*" fuzzy sets $L_1,\ldots,L_{(m-1)/2}$, as

$$\mu_{L_t}(x,t,m,r,E) = \begin{cases} 1 & \text{for } x_{\min} \le x \le x_{\min} + \frac{E-(m+1)r}{2}\delta(t), \\ 1-2\left(\frac{x-(x_{\min}+\frac{E-(m+1)r}{2}\delta(t))}{\frac{(m+1)r}{2}\delta(t)}\right)^2 & \text{for } x_{\min} + \frac{E-(m+1)r}{2}\delta(t) \le x \le x_{\min} + \frac{2E-(m+1)r}{4}\delta(t), \\ 2\left(\frac{x-(x_{\min}+\frac{E}{2}\delta(t))}{\frac{(m+1)r}{2}\delta(t)}\right)^2 & \text{for } x_{\min} + \frac{2E-(m+1)r}{4}\delta(t) \le x \le x_{\min} + \frac{E}{2}\delta(t), \\ 0 & \text{for } x_{\min} + \frac{E}{2}\delta(t) \le x \le x_{\max}, \end{cases} \tag{4}$$

for $\delta(t) = (\sum_{k=1}^{t} k)/(\sum_{k=1}^{(m-1)/2} k)$, $t = 1,\ldots,(m-1)/2$.

The function of the set "*in the middle*" is exhibited by

$$\mu_{L_{\frac{m+1}{2}}}(x,m,r,E) = \begin{cases} 0 & \text{for } x_{\min} \le x \le x_{\min} + \frac{E-(m+1)r}{2}, \\ 2\left(\frac{x-(x_{\min}+\frac{E-(m+1)r}{2}}{\frac{(m+1)r}{2}}\right)^2 & \text{for } x_{\min} + \frac{E-(m+1)r}{2} \le x \le x_{\min} + \frac{2E-(m+1)r}{4}, \\ 1-2\left(\frac{x-(x_{\min}+\frac{E}{2})}{\frac{(m+1)r}{2}}\right)^2 & \text{for } x_{\min} + \frac{2E-(m+1)r}{4} \le x \le x_{\min} + \frac{E}{2}, \\ 1-2\left(\frac{x-(x_{\min}+\frac{E}{2})}{\frac{(m+1)r}{2}}\right)^2 & \text{for } x_{\min} + \frac{E}{2} \le x \le x_{\min} + \frac{2E+(m+1)r}{4}, \\ 2\left(\frac{x-(x_{\min}+\frac{E+(m+1)r}{2})}{\frac{(m+1)r}{2}}\right)^2 & \text{for } x_{\min} + \frac{2E+(m+1)r}{4} \le x \le x_{\min} + \frac{E+(m+1)r}{2}, \\ 0 & \text{for } x_{\min} + \frac{E+(m+1)r}{2} \le x \le x_{\max} \end{cases} \tag{5}$$

and, lastly, the "*right*" sets are provided by the membership function

$$\mu_{L_{\frac{m+3}{2}+t-1}}(x,t,m,r,E) =$$

$$\begin{cases} 0 & \text{for } x_{\min} \le x \le x_{\min} + E - \frac{E}{2}\varepsilon(t), \\ 2\left(\frac{x-\left(x_{\min}+E-\frac{E}{2}\varepsilon(t)\right)}{\frac{(m+1)r}{2}\varepsilon(t)}\right)^2 & \text{for } x_{\min} + E - \frac{E}{2}\varepsilon(t) \le x \le x_{\min} + E - \frac{2E-(m+1)r}{4}\varepsilon(t), \\ 1-2\left(\frac{x-\left(x_{\min}+E-\frac{E-(m+1)r}{2}\varepsilon(t)\right)}{\frac{(m+1)}{2}\varepsilon(t)}\right)^2 & \text{for } x_{\min} + E - \frac{2E-(m+1)r}{4}\varepsilon(t) \le x \le x_{\min} + E - \frac{E-(m+1)r}{2}\varepsilon(t), \\ 1 & \text{for } x_{\min} + E - \frac{E-(m+1)r}{2}\varepsilon(t) \le x \le x_{\max}, \end{cases} \tag{6}$$

Proof:
1). Let first $x_{\min} = 0$.

In the current model, the widths of sets $L_{1,1}$, $L_{t,1}$–$L_{t-1,1}$, $t = 2,...,(m-1)/2$ are not equal.

Let us state the formula of the membership function characteristic of set "*in the middle*"= $L_{(m+1)/2}$. The function is constructed as a π-function

$$\mu_{L_{\frac{m+1}{2}}}(x) = \pi(x, \alpha_{1,L_{\frac{m+1}{2}}}, \beta_{1,L_{\frac{m+1}{2}}}, \gamma_{1,L_{\frac{m+1}{2}}} = \alpha_{2,L_{\frac{m+1}{2}}}, \alpha_{2,L_{\frac{m+1}{2}}} = \gamma_{1,L_{\frac{m+1}{2}}}, \beta_{2,L_{\frac{m+1}{2}}}, \gamma_{2,L_{\frac{m+1}{2}}}) = \begin{cases} s(x, \alpha_{1,L_{\frac{m+1}{2}}}, \beta_{1,L_{\frac{m+1}{2}}}, \gamma_{1,L_{\frac{m+1}{2}}}) & \text{for } x \le \gamma_{1,L_{\frac{m+1}{2}}}, \\ 1 - s(x, \alpha_{2,L_{\frac{m+1}{2}}}, \beta_{2,L_{\frac{m+1}{2}}}, \gamma_{2,L_{\frac{m+1}{2}}}) & \text{for } x \ge \alpha_{2,L_{\frac{m+1}{2}}}, \end{cases} \tag{7}$$

in which functions $s(x, \alpha_{1,L_{\frac{m+1}{2}}}, \beta_{1,L_{\frac{m+1}{2}}}, \gamma_{1,L_{\frac{m+1}{2}}})$ and $s(x, \alpha_{2,L_{\frac{m+1}{2}}}, \beta_{2,L_{\frac{m+1}{2}}}, \gamma_{2,L_{\frac{m+1}{2}}})$ are provided by Eq. (1) on condition that $A = L_{(m+1)/2}$. Set $L_{(m+1)/2}$ is planned to have the widest support. Let us adopt Eq. (7) to find parameters $\alpha_{1,L_{(m+1)/2}}, \beta_{1,L_{(m+1)/2}}, \gamma_{1,L_{(m+1)/2}} = \alpha_{2,L_{(m+1)/2}}, \alpha_{2,L_{(m+1)/2}} = \gamma_{1,L_{(m+1)/2}}, \beta_{2,L_{(m+1)/2}}, \gamma_{2,L_{(m+1)/2}}$ affected by E, m and r.

In set $L_{(m+1)/2}$, element $\gamma_1 = \alpha_2 = E/2$ reaches the largest membership degree equal to 1.

As the membership function of $L_{(m-1)/2}$ has the intersection point with the membership function of $L_{(m+1)/2}$ on membership level 0.5 and the distance between points $(\alpha_{L_{(m-1)/2}},1)$ and $(E/2),1)$ is equal to $((m+1)/2)r$, then the distance between points $(\beta_{1,L_{(m+1)/2}},0.5)$ and $(E/2,0.5)$ is estimated as $((m+1)/4)r$.

Hence, $\beta_{1,L_{(m+1)/2}} = E/2 - ((m+1)/4)r = (2E-(m+1)r)/4$ and $\alpha_{1,L_{(m+1)/2}} = E/2 - ((m+1)/2)r = (E-(m+1)r)/2$.

Further, $\beta_{2,L_{(m+1)/2}} = E/2 + ((m+1)/4)r = (2E+(m+1)r)/4$ and $\alpha_{2,L_{(m+1)/2}} = E/2 + ((m+1)/2)r = (E+(m+1)r)/2$, which proves Eq. (5) for $x_{\min} = 0$.

The membership function of the last "*left*" set $L_{(m-1)/2}$ has obtained $\alpha_{L_{(m-1)/2}} = \alpha_{1,L_{(m+1)/2}} = (E-(m+1)r)/2$, $\beta_{L_{(m-1)/2}} = \beta_{1,L_{(m+1)/2}} = (2E-(m+1)r)/4$ and $\gamma_{L_{(m-1)/2}} = E/2$ in order to ensure the existence of the intersection point of $L_{(m-1)/2}$ with $L_{(m+1)/2}$ on membership level 0.5. Thus

$$\mu_{L_{\frac{m-1}{2}}}(x) = 1 - s\left(x, \tfrac{E-(m+1)r}{2}, \tfrac{2E-(m+1)r}{4}, \tfrac{E}{2}\right) \tag{8}$$

We insert function $\delta(t) = (\sum_{k=1}^{t} k)/(\sum_{k=1}^{(m-1)/2} k)$, $t = 1,\ldots,(m-1)/2$, in Eq. (8), expanded due to Eq. (2), to obtain Eq. (4). In the numerator of $\delta(t)$, we compute a number of units r between the points ($x_{\min}$, 1) and ($\max(L_{t,1})$,1), where t is a number of fuzzy set L_t in the "*left*" family of fuzzy sets, for which the membership function is constructed.

The value of the numerator is divided by a number of all units r between the points ($x_{\min}$, 1) and ($\max(L_{(m-1)/2,1})$, 1). The value of $\delta((m-1)/2) = (\sum_{k=1}^{(m-1)/2} k)/(\sum_{k=1}^{(m-1)/2} k) = 1$, confirms no influence of function δ on the pattern of the membership function of fuzzy set $L_{(m-1)/2}$, when $x_{\min} = 0$.

The initialization of the membership function of the first "*right*" fuzzy set $L_{(m+3)/2}$ is based on the assumption that $\mu_{L_{(m+3)/2}}(x)$ has an inverse shape of $\mu_{L_{(m-1)/2}}(x)$ to secure the existence of the intersection point between functions of $L_{(m+1)/2}$ and $L_{(m+3)/2}$ on membership 0.5. The membership function of $L_{(m+3)/2}$ will be found as

$$\mu_{L_{\frac{m+3}{2}}}(x) = s\left(x, E - \tfrac{E}{2}, E - \tfrac{2E-(m+1)r}{4}, E - \tfrac{E-(m+1)r}{2}\right). \tag{9}$$

The "*right*" family of sets $L_{(m+3)/2},\dots,L_m$ demands an insertion of a new function $\varepsilon(t)=(\sum_{k=1}^{(m+1)/2-t}k)/(\sum_{k=1}^{(m-1)/2}k)$, t = 1,..., $(m-1)/2$, in Eq. (9), expressed in the pattern of Eq. (2).

In the numerator of $\varepsilon(t)$, we estimate a number of units r between the points (min($L_{(t+3)/2+t-1,1}$),1) and ($x_{\max}$, 1), t = 1,…,(m–1)/2, where t is the number of a fuzzy set $L_{(t+3)/2+t-1}$ in the "*right*" family of fuzzy sets, for which the membership function is derived.

The value of the numerator is divided by a number of all units r between the points (min$(L_{(m+3)/2,1})$, 1) and ($x_{\max}$, 1).

The value of t = 1 is followed by $\varepsilon(1)=(\sum_{k=1}^{(m+1)/2-1}k)/(\sum_{k=1}^{(m-1)/2}k)=1$, which agrees with maintaining the formula of Eq. (6).

2). $x_{\min} \neq 0$. In order to move $x_{\min} = 0$ to another beginning $x_{\min}$ along the positive part of the x-axis, we add the value of $x_{\min}$ to borders of the intervals. We also need to transform structures in the numerators of Eqs (4), (5) and (6), containing $x–b$, to $x–(x_{\min}+b)$, where b is a symbolic parameter composed of E, r and m.

4. Case study

In this section, we aim to build membership functions in compliance with formulas, proved in Theorem 1.

Example 3

We refer to Example 1 to expand membership functions of $L_1,\dots,L_7$, when testing the formulas found in Theorem 1. For r = 110/(1+2+3+4) = 11, we adopt Eq. (4) to extract membership functions of "*left*" sets as structures

$$\mu_{L_t}(x,t,7,11,220)=\begin{cases}1 & \text{for} \quad 40\le x\le 40+66\delta(t),\\ 1-2\left(\frac{x-(40+66\delta(t))}{44\delta(t)}\right)^2 & \text{for} \quad 40+66\delta(t)\le x\le 40+88\delta(t),\\ 2\left(\frac{x-(40+110\delta(t))}{44\delta(t)}\right)^2 & \text{for} \quad 40+88\delta(t)\le x\le 40+110\delta(t),\\ 0 & \text{for} \quad 40+110\delta(t)\le x\le 220,\end{cases}. \tag{10}$$

in which $\delta(1)=1/(1+2+3)=1/6$ for "*critically low*", $\delta(2)=(1+2)/(1+2+3)=1/2$ for "*very low*" and $\delta(3)=(1+2+3)/(1+2+3)=1$ for "*low*".

The widest set "*rather normal*" has, by Eq. (5), the membership function

$$\mu_{L_{\frac{m+1}{2}}}(x,7,11,220)=\begin{cases}0 & \text{for } 40\le x\le 106,\\ 2\left(\frac{x-106}{44}\right)^2 & \text{for } 106\le x\le 128,\\ 1-2\left(\frac{x-150}{44}\right)^2 & \text{for } 128\le x\le 150,\\ 1-2\left(\frac{x-150}{44}\right)^2 & \text{for } 150\le x\le 172,\\ 2\left(\frac{x-194}{44}\right)^2 & \text{for } 172\le x\le 194,\\ 0 & \text{for } 194\le x\le 260.\end{cases}. \tag{11}$$

Membership functions of the fuzzy sets, included in the "*right*" family, are provided, by Eq. (6), as functions

$$\mu_{L_{\frac{m+3}{2}+t-1}}(x,t,7,11,220)=\begin{cases}0 & \text{for } 40\le x\le 260-110\varepsilon(t),\\ 2\left(\frac{x-(260-110\varepsilon(t))}{44\varepsilon(t)}\right)^2 & \text{for } 260-110\varepsilon(t)\le x\le 260-88\varepsilon(t),\\ 1-2\left(\frac{x-(260-66\varepsilon(t))}{44\varepsilon(t)}\right)^2 & \text{for } 260-88\varepsilon(t)\le x\le 260-66\varepsilon(t),\\ 1 & \text{for } 260-66\varepsilon(t)\le x\le 260,\end{cases}. \tag{12}$$

for $\varepsilon(1)=(1+2+3)/(1+2+3)=1$ in "*high*", $\varepsilon(2)=(1+2)/(1+2+3)=1/2$ in "*very high*" and $\varepsilon(3)=1/(1+2+3)=1/6$ in "*critically high*".

The computation of a membership degree for a selected *x*-value is a much uncomplicated process. For instance, "*systolic blood pressure*" = x = 178 is classified in L_4 = "*rather normal*" in interval $172 \le x \le 194$, which is tied to function $2\left(\frac{x-194}{44}\right)^2$. Hence, $\mu_{L_4}(178) = 2\left(\frac{178-194}{44}\right)^2 = 0.264$. Value x = 178 is also an element of L_5 = "*high*". After inserting $\varepsilon(1) = 1$, for $t = 1$, in intervals of Eq. (12) we also place $x = 178$ in $172 \le x \le 194$, where the corresponding membership function is $1-2\left(\frac{x-(260-66\varepsilon(t))}{44\varepsilon(t)}\right)^2$. Then, $\mu_{L_5}(178) = 1-2\left(\frac{178-(260-66\cdot 1)}{44\cdot 1}\right)^2 = 0.735$.

The membership functions, mathematically formalized, can find useful applications in fuzzy inference rules of the type IF…THEN…, where membership degrees of predetermined variable values are crucial components of solutions, e.g., in fuzzy control[22,23].

5. Conclusions

When meeting some wishes of fuzzy set theory users, we have tried to bring on different mathematical techniques, which help to sample membership functions of fuzzy sets in few formulas. The sets, as representatives of the term-sets of a linguistic variable, often express intensity levels of a certain marker-variable. The sets' supports are scattered along a non-fuzzy reference set, which is common for them. The width of the reference set is a binding factor for all membership functions of fuzzy sets. The number of term-sets constitutes another essential factor, included in the formulas of membership functions of fuzzy sets replacing term-sets.

Among the term-sets, the meaning of a right expression usually is the reverse of a left expression, like e.g., "high-low"; therefore it is rational to select terms in two families as "*left*" and "*right*". The set "*in the middle*" has a moderate character.

The *s*-class functions are regarded by us to be most applicable as samplings in one combined formula for deriving membership functions of all fuzzy sets, situated in the "*left*" family", the "*right*" family or in the "*in the middle*" family, respectively. In the formulas of the classical *s*-class functions there exist parameters, which mark beginnings and ends of the domains of functions (at the same time they mark the beginnings and ends of supports of fuzzy sets possessing s-functions as membership functions). In the membership functions of fuzzy sets, collected within one family, these parameters are stated as combinations of the width of a common reference set, including all supports of fuzzy sets, and the number of sets. The *s*-functions are easily invertible, that is why they act as membership functions for opposed terms.

Particularly designed modifiers of the integer variable, being the number of a function in its family, are inserted in the common formula of membership functions of fuzzy sets in respective families in order to narrow the supports of these sets.

By doing this research, we have considered a case concerning the design of fuzzy sets, characterized by unequal supports.

The simple medical application is added in order to explain how the collected membership functions work in practice. We should emphasize that the only numerical information about the term-set list, required from a user, is the number of terms and the width of the common set, including all supports of fuzzy sets. The boundaries of membership functions are computationally induced. This prevents the user from an uncertain decision concerning a placement of the boundary values.

All processes can be easily converted to computer programs, since $\delta(t)$ and $\varepsilon(t)$ are functions of the integer variable.

We hope that our mathematical formalizations of term-sets will constitute the supplements to methods of granulation already existing.

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