Conditional Heteroskedasticity in Long Memory Model 'FIMACH' for Return Volatilities in Equity Markets

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Abstract:

This paper incorporates conditional heteroscedasticity properties in the long memory model and applies the model on squared returns of BRICS (Brazil, Russia, India, China, and South Africa), UK and USA equity markets to capture the volatility of stock return. The conditional first- and second-order moments are provided. The CLS, FGLS and QML are discussed and 2SQML estimator is proposed. The simulation study suggests that the proposed 2SQML estimator performs better than the other three estimators. Both in simulation and empirical studies, we find that the proposed model FIMACH outperforms FIGARCH in terms of eliminating serial correlations.

Key Words: Long Memory Conditional Heteroskedastic Model, Return Volatility.

JEL Classification: C13, C22, C25, C51, G01, G12, G14, G17.

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1. Introduction

The volatility of stock returns reflects the response to macroeconomic news and rumors. Engle and Patton (2001), and Poon and Granger (2003) stress that volatility surface has empirically been proved to have persistence for a long time against market shocks. The long-memory phenomenon in time series is first considered by Hurst (1951, 1956). In these studies, he explains the long-term storage requirements of the Nile River. He shows that the cumulated water flows in a year depend not only on the water flows in recent years, but also on water flows in years much earlier prior to the present year. Mandelbrot and Van Ness (1968) explain and advance Hurst's studies by employing fractional Brownian motion. In analogy with Mandelbrot and Van Ness (1968), Granger (1980), Granger and Joyeux (1980) and Hosking (1981) develop Autoregressive Fractionally Integrated Moving Average (ARFIMA) models to account for the long memory in time series data. However, an empirical study regarding the usefulness of ARFIMA model is conducted by Bhardwaj and Swanson (2006), who find strong evidence in favor of ARFIMA in absolute, squared and log-squared stock index returns. In this regard, Ding and Granger (1996) point out that a number of other processes can also have the long-memory property. Further, a fractionally integrated generalized autoregressive conditional heteroskedasticity (FIGARCH) is primarily developed by Baillie, Bollerslev and Mikkelsen (1996), but later modified by Chung (1999). Nevertheless, Quoreshi (2014) develops an Integer-valued ARFIMA (INARFIMA) model to account for the long-memory property in a high frequency count data framework.

This paper incorporates conditional heteroscedasticity properties in the long memory model and applies the model on squared returns of BRICS (Brazil, Russia, India, China, and South Africa), UK and USA. The new model is called Fractionally Integrated Moving Average Conditional Heteroskedasticity (FIMACH). This model is designed, in a similar fashion to Quoreshi (2014), for non-integer data. The main difference between the introduced model and model in ARFIMA class is that this model class can study the heteroskedasticity property on the level series, while the ARFIMA-FIGARCH class studies the same on the fractionally differenced series through Fourier transformation. One obvious advantage of the FIMACH model over the ARFIMA-FIGARCH class is that the model can easily be extended to multivariate settings for the level series. The model may additionally be used to measure the reaction times for macro-economic news or rumors, and captures information spread through the system. The model is specified in terms of first and second order moments conditioned on historical observations. We perform a Monte-Carlo simulation, where we find that ARFIMA or FIGARCH is not suitable for data that are generated according to the FIMACH model. Empirically, we find evidence of long memory for squared stock return of UK, USA and BRICS countries. It is also found that the FIMACH model outperforms both FIGARCH and ARFIMA models in terms of eliminating serial correlations.

The paper is organized as follows. The ARFIMA-FIGARCH model class is discussed, and the FIMACH model is introduced in section 2. The estimation procedure of FIMACH is discussed in section 3. Section 4 presents a brief Monte Carlo experiment. The description of the empirical data is presented in section 5. The empirical results on the stock return volatilities are presented in section 6, and the concluding comments are included in section 7.

2. Model

We assume that $r_t = p_t - p_{t-1}$ is a stock index return time series, where p_t is price for the index at time t. Let σ_t^2 be the degree of index return volatility, proxied by the squared return r_t^2, which has a slow decaying autocorrelation function. The moving average representation of ARFIMA (0,d,0) of the series y_t is

$$\sigma_t^2 = u_t + d_1 u_{t-1} + d_2 u_{t-2} + d_3 u_{t-3} \dots$$

or

$$\sigma_t^2 = (1+L)^{-d} u_t. \qquad (1)$$

where $y_t = \sigma_t^2$, t = 1, . . . , T time intervals and σ_t^2 has long memory properties. Note that σ_t^2 has long memory in a sense that the variable has a slow decaying autocorrelation function and the parameters $d_i = \Gamma(i+d)/[\Gamma(i+1)\Gamma(d)]$, $i = 0, 1, 2, \dots$ where $d_0 = 1$. The u_t is i.i.d. sequence of random variables with unconditional mean $E(u) = \lambda$ and variance $V(\alpha u)$=$\alpha^2\emptyset^2$ where $V(u) = E(u)^2$-λ^2=$\emptyset^2$. Conditionally, it holds that $E(u|u) = u$ and $V(\alpha u|u)$=$\alpha^2 V(u|u)$ where $V(u|u) = u^2 - 2\lambda u + \lambda^2$. The conditional mean and variance for the moving average representation of ARFIMA (0,d,0) are

$$E(\sigma_t^2\,|Y_{t-1}) = E_{t-1} = \lambda + \sum_{i=1}^{\infty} d_i\, u_{t-i} \qquad (2a)$$

$$V(\sigma_t^2\,|Y_{t-1}) = V_{t-1} = \emptyset^2 + \sum_{i=1}^{\infty} d_i^2\big(u_{t-i}^2 - 2\lambda u_{t-i} + \lambda^2\big). \qquad (2b)$$

where,Y_{t-1} is the information set available at time *t-1*. The conditional mean and variance vary with u_{t-i}. Since the conditional variance varies with u_{t-i}, there is a conditional heteroskedasticity property of moving average type that Brännäs and Hall (2001) called MACH(q). As λ and $\emptyset^2$ are not functions of time and $|\sum_{i=1}^{\infty} d_i| \leq \left|\sum_{i=1}^{\infty} d_i^2\right|$ for $d \in [-1, 1]$, it is sufficient that $\sum_{i=1}^{\infty} d_i < \infty$ for $\{\sigma_t^2\}$ to be a stationary sequence. We call the model Fractionally Integrated Moving Average Conditional Heteroskedasticity FIMACH (*d*) where *d* represents the long memory parameter. The main difference between this model and the model in the ARFIMA class is that this model can study the heteroskedasticity property on the level series, while, e.g., FIGARCH of ARFIMA class studies the same on the fractionally differenced series through Fourier transformation. The autocorrelation functions of the ARFIMA model class are assumed to be a hyperbolic function, while the general mathematical expression of the autocorrelation function for FIMACH is considerably complicated to derive, although possible. Assuming $E(u_t u_t|Y_{t-1}) = u_t^2$ and $E(u_t u_{t-i}|Y_{t-1}) = 0$ where $i = 1, 2, \dots, \infty$, we can provide a simple form of conditional auto-correlation function at lag *k* for FIMACH as

$$\rho_{k|t-1} = \frac{\sum_{i=0}^{\infty} d_i d_{k+i} u_{t-i-k}^2}{V(\sigma_t^2\,|Y_{t-1})} \qquad (2c)$$

where $k = -j, j$ and represent lag, and $d_0 = 1$. Note that this autocorrelation function varies with u_{t-i} which captures the heteroscedasticity property in autocorrelation function. The heteroscedasticity in autocorrelation function for absolute return of stock is illustrated by Ding et al. (1993), although the authors assume a smooth function for explaining the autocorrelation. The model can be extended with random parameters as

$$\sigma_t^2 = u_t + d_1 u_{t-1} + d_2 u_{t-2} + d_3 u_{t-3} \ldots + \sum_{i=1}^{p} \theta_i u_{t-i} \tag{3}$$

where d_i capture the long memory properties and have the same definitions as in equation 1. The $\theta_i, i = 1,2,\ldots$, comprise the random parameters and are independent of each other. These parameters capture the short term deviation from the long memory trend. We name this model FIMACH(*d, p*) model. The conditional mean and variance of the random coefficients of FIMACH(*d, p*) representation can be written as

$$E(\sigma_t^2 \,|Y_{t-1}) = E_{t-1} = \lambda + \sum_{i=1}^{\infty} d_i u_{t-i} + \sum_{i=1}^{p} \theta_i u_{t-i} \tag{4a}$$

$$V(\sigma_t^2 \,|Y_{t-1}) = V_{t-1} = \emptyset^2 + \sum_{i=1}^{\infty} d_i^2 \left(u_{t-i}^2 - 2\lambda u_{t-i} + \lambda^2\right) + \sum_{i=1}^{p} \theta_i \left(u_{t-i}^2 - 2\lambda u_{t-i} + \lambda^2\right). \tag{4b}$$

Note that the moments are conditioned only on the previous observations, Y_{t-1}. The same stationary condition is applicable as for equation (2). The model can be used to measure mean and median reaction time to macroeconomic news and rumors[3]. The model can easily be extended to a multivariate setting. Hence, the covariance and Granger-Causality between two or several series can easily be studied in the same fashion as the VARMA model. These possibilities are limited in the FIGARCH of ARFIMA class, at least on the level series.

[3]This is the reaction to macroeconomic news/rumours in the (u_{jt}) sequence, we use the mean lag $\sum_{i=0}^{q_j} i\,\alpha_{ji}/w$, where $w = \sum_{i=0}^{q_j} \alpha_{ji}$ and $\alpha_{j0} = 1$ (see Quoreshi, 2012).

3. Estimation

If we do not assume a full density function, we may estimate the Quasi Maximum Likelihood (QML) Estimator as discussed by Weiss (1986) and Bollerslev and Wooldridge (1992) instead of Maximum Likelihood (ML) Estimator. Conditional Least Square (CLS), Feasible Generalized Least Square (FGLS), Generalized Methods of Moments (GMM) and possibly others, e.g. Two Stage Least Square (2SLS), are candidates for estimation. In the previous studies, it turns out that FGLS is the best estimator among the three in terms of eliminating serial correlation (Quoreshi, 2014). The CLS comes in the second position, which is almost as good as FGLS. Here, we only consider CLS, FGLS and ML class for estimation.

The Conditional least square (CLS) estimator for FIMACH(*d, p*) representation model have the following residual

$$e_t = y_t - E_{t-1} = \sigma_t^2 - \lambda - \sum_{i=1}^{\infty} d_i u_{t-i} - \sum_{i=1}^{p} \theta_i u_{t-i} \tag{5}$$

and the criterion function $S_{CLS} = \sum_{i=m+1}^{T} e_t^2$is minimized with respect to unknown parameters, i.e. $\psi = (\lambda, \theta'$and $d')$ where θ' and d' are vector of parameters with elements θ_i respective d_i. Using a finite maximum lag *m* in (5) instead of infinite lags may cause biasing effects. Due to omitted variables, i.e. $u_{t-m-1}, \dots, u_{t-\infty}$, we may expect a positive bias on the parameters λ, θ_iandd_i (Brännäs and Quoreshi, 2010). These moment conditions correspond to the normal equations of the CLS estimator that focuses on the unknown parameters of the conditional mean function. Alternatively and equivalently, the properties E(e_t) = 0 and E($e_t e_{t-j}$) = 0, $j \geq 1$ could be used. Note that the moment conditions for FIMACH(*d, 0*) can be obtained by setting $\theta_i = 0$.

The FGLS estimator minimizes

$$S_{FGLS} = \sum_{t=m+1}^{T} e_t \hat{V}^{-1} \tag{6}$$

with $\hat{V}^{-1}$ as given. The variance of error from CLS estimates may be used for approximation of $\hat{V}^{-1}$ in equation (6). Alternatively, $\hat{V}^{-1}$ can be estimated as specified in (4b) by employing estimates from CLS. The covariance matrix estimators for CLS and FGLS are

$$Cov(\hat{\psi}_{CLS}) = \left(\sum_{t=m+1}^{T} \frac{\partial e_t}{\partial \psi} \frac{\partial e_t}{\partial \psi'} \right)^{-1}$$

$$Cov(\hat{\psi}_{FGLS}) = \left(\sum_{t=m+1}^{T} \hat{V}^{-1} \frac{\partial e_t}{\partial \psi} \frac{\partial e_t}{\partial \psi'} \right)^{-1} .$$

The ML or QML estimator for FIMACH(*d, p*) representation model have the same residual as in equation (5), and maximize the following criterion function

$$f(\sigma_1^2, \sigma_2^2 \ldots, \sigma_T^2 \,|Y_{t-1}, \lambda, \theta_i \text{and } d_i) = \prod_{t=1}^{T} f(\sigma_t^2 \,|Y_{t-1}, \psi_i) \; = \left(\frac{1}{2\pi V_{t-1}} \right)^{T/2} exp\left(-\frac{\sum_{i=m+1}^{T} e_t^2}{2V_{t-1}} \right) (6)$$

Where $\psi_i = (\lambda, \theta_i$ and $d_i)$ and V_{t-1} is as in equation$(4b)$. Taking logarithm of equation (6), we may simply use the criterion function and minimize the function as

$$Lnf(\sigma_1^2, \sigma_2^2 \ldots, \sigma_T^2 \,|Y_{t-1}, \lambda, \theta_i, d_i \text{and} \hat{V}_{t-1}) = -\frac{T}{2} \ln(\hat{V}_{t-1}) - \ln(2\pi) - \left(\frac{\sum_{t=m+1}^{T} e_t^2}{2\hat{V}_{t-1}} \right) \;\; (7)$$

where $\hat{V}_{t-1}$is an estimate for V_{t-1} that is to be estimated. Since T, 2 and π are constants, we can equivalently minimize the following criterion function

$$Lnf(\sigma_1^2, \sigma_2^2 \ldots, \sigma_T^2 \,|Y_{t-1}, \lambda, \theta_i, d_i \text{and} \hat{V}_{t-1}) = -\ln(\hat{V}_{t-1}) - \left(\frac{\sum_{t=m+1}^{T} e_t^2}{\hat{V}_{t-1}} \right). \qquad (8)$$

Note that the $\hat{V}_{t-1}$ is to be estimated at the same time as the other parameters. If the estimation is sensitive to the start value of $\hat{V}_{t-1}$, we can obviously estimate CLS at the first stage and calculate the $\hat{V}_{t-1}$ which can be used as the start value for QML. We call this estimation procedure Two Stage Quasi Maximum Likelihood (2SQML) Estimation. The covariance matrix estimators for QML and 2SQML are

$$Cov(\hat{\psi}_{QML}) = \left(\sum_{t=m+1}^{T} \hat{V}^{-1} \frac{\partial e_t}{\partial \psi} \frac{\partial e_t}{\partial \psi'} \right)^{-1}$$

$$Cov(\hat{\psi}_{2SQML}) = \left(\sum_{t=m+1}^{T} \hat{V}^{-1} \frac{\partial e_t}{\partial \psi} \frac{\partial e_t}{\partial \psi'} \right)^{-1}.$$

4. Monte Carlo experiment

Smith et al. (1996) and Quoreshi (2014) have studied the bias and misspecification in ARFIMA respective INARFIMA models. Drost et al. (2009) investigated finite sample behavior of semiparametric integer-valued AR(p) models, while Brännäs and Quoreshi (2010) studied finite lag misspecification when the data is generated according to an infinite-lag INMA model. In this brief Monte Carlo experiment we study the bias, MSE, Ljung–Box statistics, AIC and SBIC properties of the ML estimators for finite-lag specifications, when data is generated according to FIMACH (*d*,*0*). The data generating process is as in (1), with $d = 0.1, 0.25 \text{ and } 0.4$ and lag length $m = 70$. The u_t sequence is generated from i.i.d. normal distribution, with mean 25 and standard deviation 10. Six time series with length $T = 4500 \text{ and } T = 900$ are generated. The first 500 observations are discarded to avoid the start-up effect. The results for the Monte Carlo experiment are given in Table 1. We also generate four other series in a similar fashion, with mean 25 and standard deviation 4 and 100 to study the performance of CLS, FGLS, 2SQML and QML estimators. We also evaluate the FIMACH(*d, 0*), ARFIMA (*0, d, 0*), FIGARCH (1,1) and GARCH (1,1) models in terms of eliminating serial correlations when the data are generated in accordance with FIMACH(*d, 0*). These results are presented in Table 2.

We set $\hat{\lambda}$ equal to mean values for the generated series instead of $\lambda = 25$. By doing so, we eliminate biased effect of λ in the generated series. Hence, we can study the bias of the estimated parameter d due to misspecification of lag length m more appropriately. The Monte Carlo study shows that as m increases towards $m = 70$, the bias in d decreases (see Table 1). For m less than 70 we find positive bias in d, while the bias is negative for $m = 90$. Biases are smaller for $T = 4500$ when m is less than 70 than those for $T = 9000$. MSE decreases as sample size increases or m increases. Like Brännäs and Quoreshi (2010) and Quoreshi (2014), we conclude that we may expect a positive biasing effect on the parameters due to omitting variables, i.e. $u_{t-m-1}, \dots, u_{t-\infty}$. The statistics for AIC and SBIC decrease as lag length increases, and are noted lowest at $m = 90$. Hence, the standard AIC and SBIC need to be corrected in order to choose optimal lag lengths. As expected, the LB statistics are, with some exceptions, lowest at $m = 70$. The exceptions may arise due to the conditional heteroskedasticity nature of the data set. Hence, it is appropriate to evaluate at more than one time point or to use an average value of LB statistics for a number of time points.

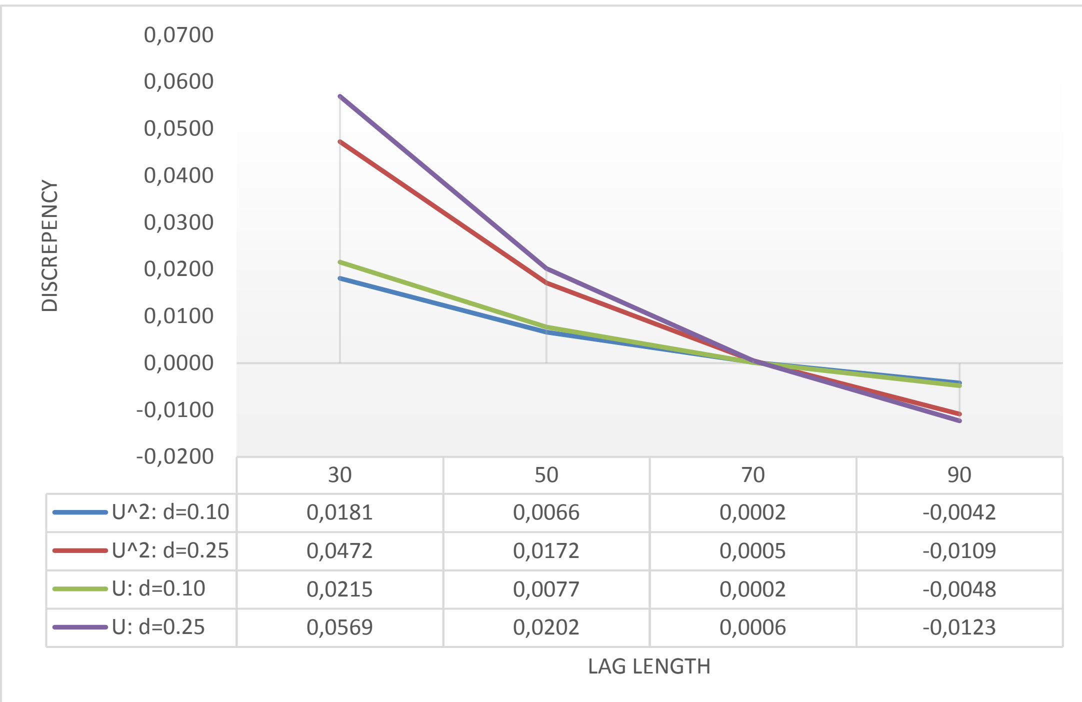
Insert Table 1 about here

When the data is generated according to FIMACH ($d, 0$), it is not appropriate to use ARFIMA, FIGARCH or GARCH model (see Table 2). The ARFIMA model reduces the serial correlation successfully, but it does not perform as well as FIMACH. This shows that there is need of using FIMACH instead of ARFIMA when we need to take account of the heteroskedasticity property in the long memory. FIGARCH and GARCH take account of heteroskedasticity in the short memory. Hence, these models did not perform well. FGLS and CLS perform consistently well, and somewhat better than QML. It turns out that QML is sensitive to start values. Estimating CLS at the first stage and using the CLS estimates as start value for QML estimator, we estimate 2SQML which performs best out of these estimators.

Insert Table 2 about here

We have also conducted similar Monte-Carlo studies with innovation U^2. The results are quite similar. Average discrepancy ($\hat{d}$-d) with number of simulation equals 100 for FIMACH time series, with innovation U^2 respective U; when the data are generated with lag length 70 and d=0.1 respective d=0.25 are presented in Figure 1 below.

Figure 1: Average discrepancy ($\hat{d} - d$) with number of simulation equals 100 for FIMACH time-series with innovation U^2 respective U when the data are generated with lag length 70 and $d = 0.1$ respective $d = 0.25$.



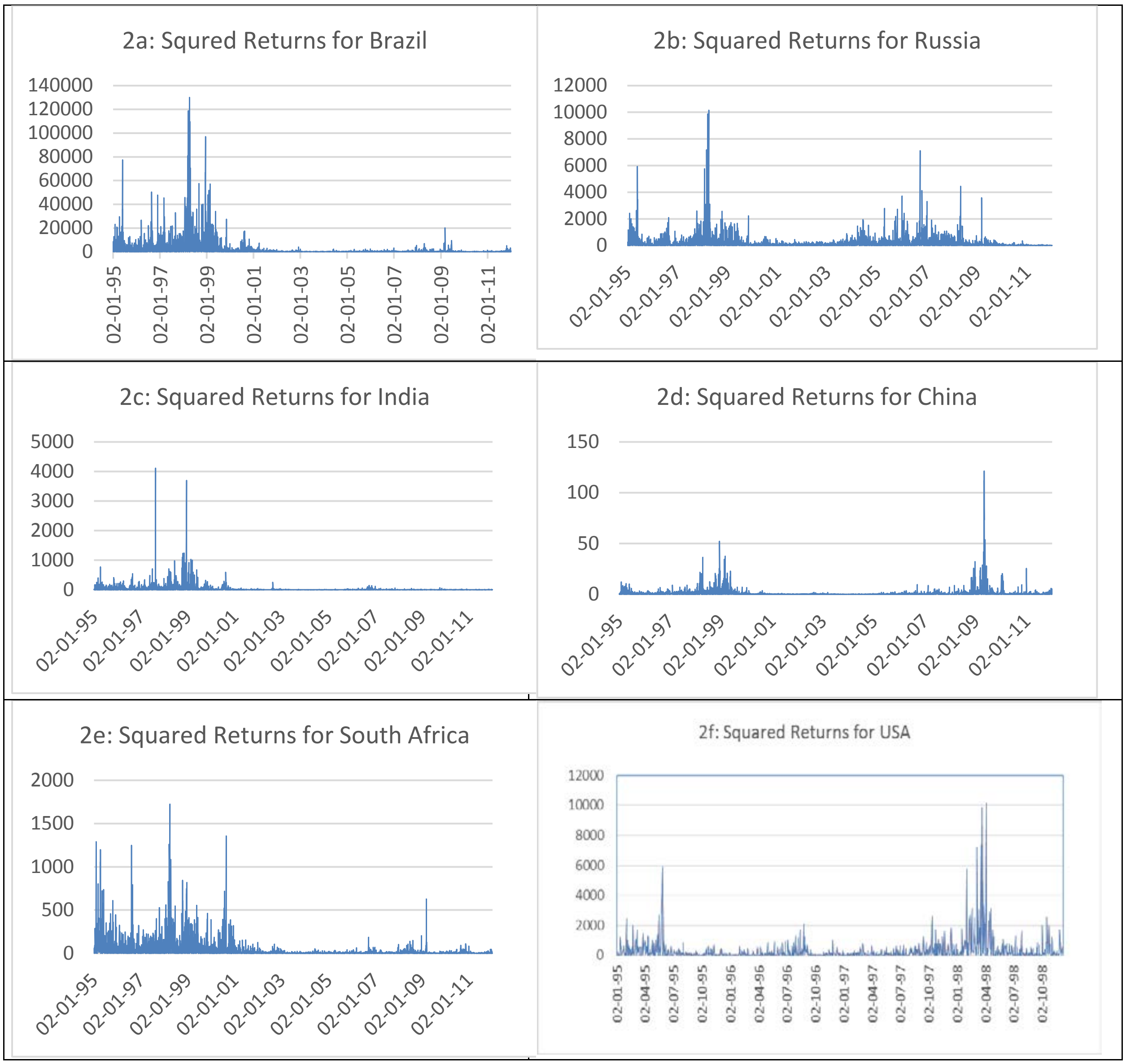
	30	50	70	90
U^2: d=0.10	0,0181	0,0066	0,0002	-0,0042
U^2: d=0.25	0,0472	0,0172	0,0005	-0,0109
U: d=0.10	0,0215	0,0077	0,0002	-0,0048
U: d=0.25	0,0569	0,0202	0,0006	-0,0123

5. Empirical Data

The daily squared stock MSCI index return series over the period of 17 years from 1995 to 2011 for the UK, USA and BRICS are applied in this paper (see Figure 2 below). Each series comprises 4435 observations. The descriptive statistics of the data set are given in Table 3. The mean squared stock index return for Brazil is about 1800, which is the largest among the BRICS counties. The corresponding number for China is about 1, which is the smallest among the BRICS countries. The corresponding mean for USA and UK are 172 and 195 respectively. The skewness, kurtosis and Jarque-Bera statistics indicate that the data are not from normal distribution. The autocorrelation functions are presented in Figure 3 below. The autocorrelation functions for all of the series decay very slowly, which indicates a long memory behavior in the squared return series.

Insert Table 3 about here

Figure 2: The daily squared stock index return series over the period of 17 years from 1995 to 2011 for the BRICS countries, USA and UK.



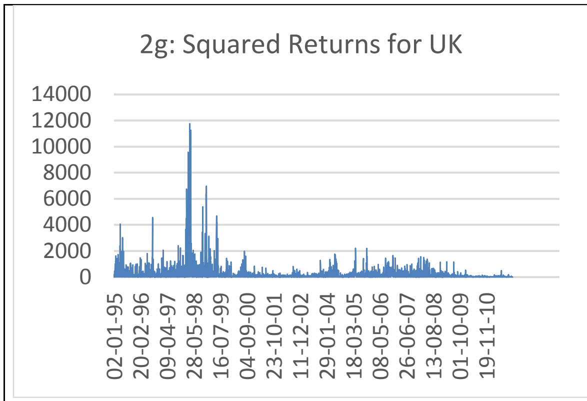
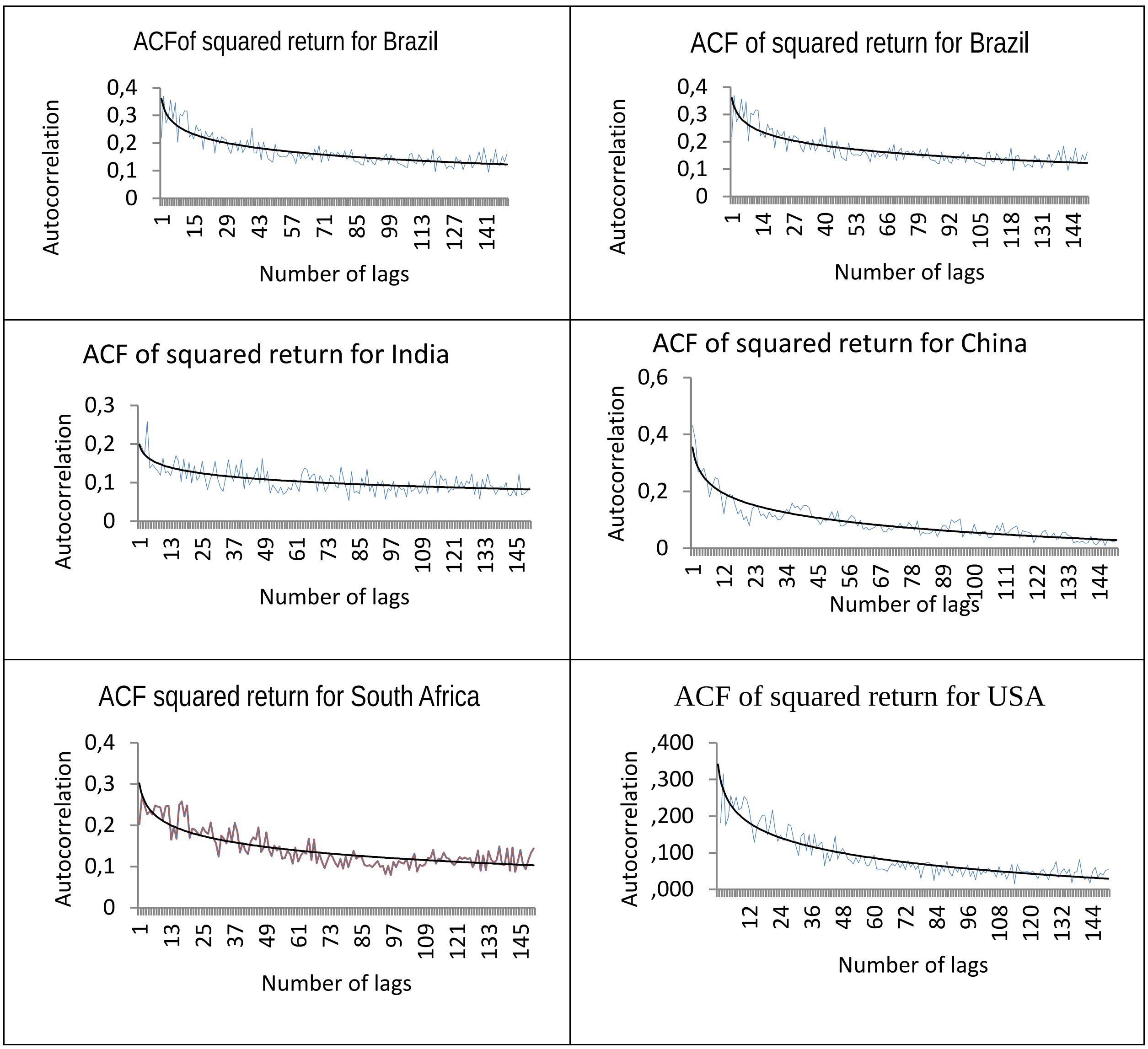
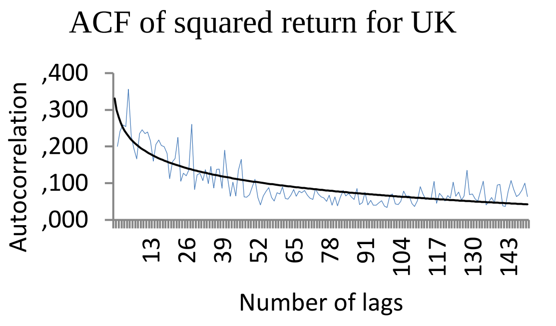


Figure 3: Autocorrelation Function for BRICS countries, USA and UK.





6. Empirical Results

The empirical results of the squared stock index returns for USA, UK and BRICS countries are presented in Table 4a-g. CLS, FGLS and QML estimators have been employed to evaluate the performance of the estimators. FIMACH (d,0), FIGARCH (0,d,0), FIGARCH (1,d,1) and GARCH (1,1) are estimated to find out the most suitable model for the volatility return. It turns out that CLS, FGLS and QML estimators have performed equally well for the time series of USA, Brazil, Russia and China in terms of reducing serial correlation. However, the QML estimator outperforms CLS and FGLS for the time series of India and South Africa, while CLS and FGLS perform better than QML estimator for the time series of UK. Hence, we conclude that QML is somewhat better estimator among these. It is to be noted that the performance the QML estimator is highly sensitive to the start values, and may turn out worse than that of FGLS or CLS if the start values are not selected carefully. We suggest that the value for the autocorrelation at lag one may be chosen as the start values for the fractional integration parameter. The variance of residuals from the CLS estimator may be used as the start value for $\hat{V}_{t-1}$ of the QML estimator which we call 2SQML. We also recommend to use both QML and CLS estimators in order to determine the best estimates in terms of reducing serial correlation.

We find that the GARCH is not an appropriate model for the time series, as the estimated parameters deviate substantially from the expected values. The FIMACH (0,*d*) turns out to be the best in terms of eliminating serial correlations. This model performs much better than that of FIGARCH (0,d,0) and FIGARCH (1,d,1) for all the five time series. For Brazil, the Ljung-Box (LB) statistics for residuals for FIMACH is about 1430, while the corresponding number for FIGARCH is about 16,882 (see Table 4a). The corresponding statistics for USA are 698 for FIMACH and 7147 for FIGARCH (see Table 4f). The Ljung-Box statistics for standardized residuals for FIGARCH are smaller than that of residuals, but these are much larger than the corresponding statistics for FIMACH. Note that the Ljung-Box statistics for standardized residuals, and residuals are the same for all FIMACH estimations. This may imply that the FIMACH model captures heteroskedasticity properly.

Employing the proposed FIMACH ($d, 0$) model, we find that the squared stock return index for each of the USA, UK and BRICS countries have long memory properties. The persistence in terms of days varies among the countries. The effects of macroeconomics news and rumors on stock market volatility for India persist up to 35 days, while the corresponding number of days for South Africa is 70. The stock market in China reacts initially more than any other BRICS countries, since the fractional integration parameter (*d*) for China is 0.283, which is the largest among the BRICS countries. The stock market in Russia reacts initially least ($d = 0.167$) among those countries. The volatility intensity increases for all the seven stock markets when the macro-economic news or rumors break out, and the impact remains between 35 and 70 days, and fades away very slowly with time.

Insert Tables 4a-g about here

7. Conclusion

This paper introduces a new class of long memory model for volatility of stock returns. The model introduced is capable of taking account of heteroskedasticity in long memory. The conditional first- and second-order moments are provided. The CLS, FGLS and QML are discussed and 2SQML estimator is proposed. In Monte Carlo experiments we find that it is not appropriate to use ARFIMA, FIGARCH or GARCH model if the data is generated according to FIMACH ($d, 0$). The ARFIMA model reduces the serial correlation successfully, but it does not perform as well as FIMACH. From the empirical results, we establish that the squared returns of stock index for the BRICS countries, UK and USA have long memory properties. However, the effects of macroeconomics news and rumors on stock return volatility vary among the countries. We also find that the volatility intensity increases for all the seven stock markets when the macro-economic news or rumors break out, and that the impact remains between 35 and 70 days and fades away very slowly with time. CLS and FGLS estimators perform equally well in terms of residual properties, while the QML estimator performs in somewhat better among the three estimators. The results of the simulation study indicate that 2SQML performs best among the five estimators and hence 2SQML is suggested to be used when QML does not perform well. Both in simulation and empirical studies, we find that the proposed model FIMACH outperforms FIGARCH in terms of eliminating serial correlations.

Acknowledgement:

We acknowledge financial support from NASDAQ OMX Nordic Foundation.

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Table 1: Bias, MSE, Ljung–Box statistics, AIC and SBIC properties of the ML estimators for finite-lag specifications, when data is generated according to FIMACH (0,d,0) Model with $d = 0.1, 0.25$ and 0.4 and $m = 70$ and σ^2=100.

		T=4500			T=9000		
Lag	Parameters	**0.10**	**0.25**	**0.40**	**0.10**	**0.25**	**0.40**
	δ	0.168***	0.432***	0.698***	0.167***	0.431***	0.696***
	(s.e.)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
	MSE	99.115	104.898	130.848	97.579	103.611	129.173
	LB100	130.669	297.170	1047.228	166.566	525.401	2045.455
	LB200	236.154	417.668	1214.240	285.267	651.793	2200.088
	AIC	20659.704	20907.183	21811.986	41201.319	41723.602	43516.296
M10	SBIC	20730.210	20977.689	21882.492	41279.461	41801.744	43594.438
	δ	0.122***	0.308***	0.496***	0.122***	0.307***	0.495***
	(s.e.)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
	MSE	98.180	99.305	103.427	96.901	97.936	101.736
	LB100	101.855	116.564	167.503	108.976	151.998	284.131
	LB200	202.485	219.121	268.172	225.197	268.950	405.750
	AIC	20566.007	20616.690	20797.669	41048.905	41143.489	41482.381
M30	SBIC	20596.463	20815.250	20996.228	41127.023	41221.607	41560.499
	δ	0.108***	0.271***	0.434***	0.108***	0.270***	0.433***
	(s.e.)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
M50	MSE	98.177	100.234	104.320	97.025	97.903	100.766

	LB100	101.107	111.853	150.678	105.672	116.088	180.584
	LB200	200.968	217.921	264.321	224.083	234.140	299.255
	AIC	20514.152	20606.353	20784.158	41049.001	41129.531	41387.421
	SBIC	20840.585	20932.787	21110.592	41411.071	41491.601	41749.491
	δ	0.101***	0.251***	0.402***	0.100***	0.250***	0.401***
	(s.e.)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.001)
	MSE	98.027	99.939	105.527	96.821	97.604	100.086
	LB100	103.897	115.392	160.502	103.300	108.986	134.357
	LB200	200.740	212.267	264.265	222.201	226.818	249.751
	AIC	20455.626	20541.172	20782.114	40989.067	41060.157	41281.722
M70	SBIC	20909.753	20995.299	21236.241	41492.967	41564.057	41785.621
	δ	0.096***	0.238***	0.380***	0.095***	0.237***	0.379***
	(s.e.)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
	MSE	98.110	100.157	106.060	96.917	97.794	100.522
	LB100	110.666	162.701	332.128	108.179	143.667	292.239
	LB200	210.420	267.350	445.421	228.784	266.039	422.550
	AIC	20407.668	20498.686	20750.943	40936.006	41016.225	41261.143
M90	SBIC	20989.306	21080.324	21332.581	41581.644	41661.864	41906.782

Table 2: Comparing between FIMACH, FIGARCH and GARCH and between QMLE, 2SQMLE, FGLS and CLS estimators, when data is generated according to FIMACH $(0, d, 0)$ models with $d = 0.25$ and $m = 70$.

		T=4500				T=9000			
Lag	Ljung-Box	FIMACH	ARFIMA	FIGARCH	GARCH	FIMACH	ARFIMA	FIGARCH	GARCH
σ=4	LB_{100}	90.521	138.064	2704.712	2730.476	90.521	138.064	1455.198	1578.196
	LB_{200}	198.883	247.337	2781.968	2805.809	198.883	247.337	1564.312	1687.964
σ=10	LB_{100}	90.521	138.064	2704.712	2730.476	90.521	138.064	1455.198	1578.196
	LB_{200}	198.883	247.337	2781.968	2805.809	198.883	247.337	1564.312	1687.964
σ=100	LB_{100}	90.521	138.064	1455.198	1578.196	90.521	138.064	1455.198	1578.196
	LB_{200}	198.883	247.337	1564.312	1687.964	198.883	247.337	1564.312	1687.964
		QMLE	2SQMLE	FGLS	CLS	QMLE	2SQMLE	FGLS	CLS
σ=4	LB_{100}	679.924	90.521	100.409	100.409	291.590	90.521	291.586	291.592
	LB_{200}	843.340	198.883	200.232	200.232	406.221	198.883	406.217	406.224
σ=10	LB_{100}	109.710	90.521	100.229	100.229	108.557	90.521	108.557	108.557
	LB_{200}	206.859	198.883	199.617	199.617	226.373	198.883	226.373	226.373
σ=100	LB_{100}	99.964	90.521	100.139	100.139	102.549	90.521	102.549	102.549
	LB_{200}	198.430	198.883	199.260	199.260	221.728	198.883	221.728	221.728
		σ=4	σ=10	σ=100		σ=4	σ=10	σ=100	
Generated series	LB_{100}	1885.1	1885.1	1885.1		1885.1	1885.1	1885.1	
	LB_{200}	1991.9	1991.9	1991.9		1991.9	1991.9	1991.9	

Table 3: Descriptive Statistics

This table reports descriptive statistics of squared returns for BRICS (Brazil, Russia, India, China, and South Africa) countries, UK and US for the period 1995-2011.

Country	Mean	Median	Std. Dev.	Minimum	Maximum	Skewness	Kurtosis	Jarque-Bera	Observations
Brazil	1799.718	158.722	6476.492	0.000	130113.100	9.301	125.839	2851677.000***	4435
Russia	216.787	23.981	949.744	0.000	35765.620	19.805	594.868	65009317.000***	4435
India	26.928	2.151	119.300	0.000	4115.351	18.851	542.786	54093083.000***	4435
China	1.032	0.123	4.089	0.000	121.396	14.887	341.988	21393884.000***	4435
South Africa	33.414	4.351	95.426	0.000	1725.239	7.215	78.675	1096490.000***	4435
UK	195.152	46.909	548.819	0.000	11764.870	10.107	152.599	4210143.000***	4435
US	171.966	34.018	472.666	0.000	10158.220	9.380	141.160	3591553.000***	4435

Table 4a: Empirical results for squared stock return index for Brazil.

Coefficients	FIMACH (*d,0*)			FIGARCH (*0,d,0*)	FIGARCH (1,d,1)	GARCH (1,1)
	QMLE Estimates (s.e.)	FGLS Estimates (s.e.)	CLS Estimates (s.e.)	QML Estimates (s.e.)	QML Estimates (s.e.)	QML Estimates (s.e.)
Constant (Mean)	737.541*** (72.54)	737.541*** (73.65)	737.541*** (72.83)	129.996*** (8.71)	126.228*** (6.28)	101.091*** (8.41)
Constant (Variance)					-46632.882*** (1915.54)	634.873 (487.658)
d	0.208*** (0.029)	0.208*** (0.029)	0.208*** (0.028)	0.661*** (0.004)	0.751*** (0.003)	
α						1.081*** (0.03)
β					0.513*** (0.00)	0.879*** (0.02)
VAR	34453328*** (5577746)	34468875.171	34468875.171	46284279	41936202	41927397
Q(100)	1430.501***	1430.501***	1430.501***	16882.451***	16882.451***	16881.811***
Q(200)	2345.913***	2345.913***	2345.913***	25714.538***	25714.538***	25713.570***
LB_{SD} (100)	1430.501***	1430.501***	1430.501***	5142.817***	4164.267***	1712.273***
LB_{SD} (200)	2345.913***	2345.913***	2345.913***	8786.100***	6739.816***	2668.345***
Lag length	45	45	45			

Table 4b: Empirical results for squared stock return index for Russia.

Coefficients	FIMACH(d,0)			FIGARCH (*0,d,0*)	FIGARCH(1,d,1)	GARCH (1,1)
	QMLE Estimates (s.e.)	FGLS Estimates (s.e.)	CLS Estimates (s.e.)	QML Estimates (s.e.)	QML Estimates (s.e.)	QML Estimates (s.e.)
Constant (Mean)	107.259*** (12.47)	107.259*** (14.48)	107.258*** (14.22)	24.075*** (1.86)	10.409*** (0.276)	2.901 (1.84)
Constant (Variance)					-1860.043*** (61.58)	5.340 (7.66)
d	0.167*** (0.04)	0.167*** (0.04)	0.167*** (0.04)	0.615*** (0.00)	0.728*** (0.00)	
α						1.302*** (0.10)
β					0.537*** (0.00)	0.808*** (0.03)
VAR	823353*** (309982.45)	823727	823727		901821	901627
LB (100)	997.362***	997.362***	997.362***	4487.831***	4487.831***	4488.051***
LB (200)	1374.943***	1374.943***	1374.943***	5959.304***	5959.304***	5959.476***
LB_{SD} (100)	997.362***	997.362***	997.362***	3363.808***	1348.205***	112.467***
LB_{SD} (200)	1374.943***	1374.943***	1374.943***	5564.297***	2043.733***	232.909***
Lag length	41	41	41			

Table 4c: Empirical results for squared stock return index for India

	FIMACH			FIGARCH (*0,d,0*)	FIGARCH (1,d,1)	GARCH (1,1)
	QML Estimates (s.e.)	FGLS Estimates (s.e.)	CLS Estimates (s.e.)	QML Estimates (s.e.)	QML Estimates (s.e.)	QML Estimates (s.e.)
Constant (Mean)	12.709*** (2.57)	17.614*** (1.59)	17.614*** (1.59)	1.499*** (0.11)	1.579*** (0.11)	1.469*** (0.22)
Constant (Variance)					0.311* (0.18)	0.035 (0.23)
d	0.185*** (0.05)	-0.169*** (0.04)	-0.169*** (0.04)	0.549*** (0.11)	0.999*** (0.00)	
α			0.319 (0.18)			1.204*** (0.10)
β					0.934*** (0.02)	0.854*** (0.04)
VAR	12829.171*** (4439.03)	13166.002	13166.002	14229.523	14229.523	14226.460
LB (100)	717.844***	1572.995***	1572.995***	6214.240***	6214.240***	6214.247***
LB (200)	1150.830***	2582.103***	2582.103***	9987.077***	9987.077***	9987.540
LB_{SD} (100)	717.844***			1319.669***	1052.742***	1057.400***
LB_{SD} (200)	1150.830***			2479.399***	1840.481***	1791.520***
Lag length	35	35	35			

Table 4d: Empirical results for squared stock return index for China.

Coefficients	FIMACH			FIGARCH (0,d,0)	FIGARCH (1,d,1)	GARCH (1,1)
	QML Estimates (s.e.)	FGLS Estimates (s.e.)	CLS Estimates (s.e.)	QML Estimates (s.e.)	QML Estimates (s.e.)	QML Estimates (s.e.)
Constant (Mean)	0.283*** (0.11)	0.283*** (0.10)	0.283*** (0.10)	0.220*** (0.02)	0.077*** (0.01)	0.055*** (0.01)
Constant (Variance)					-0.021*** (0.00)	-0.000 (0.00)
d	0.312*** (0.11)	0.312*** (0.11)	0.312*** (0.11)	0.490*** (0.00)	0.714*** (0.00)	
α						1.078*** (0.04)
β					0.684 (0.00)	(0.916) (0.02)
VAR	12.567*** (3.08)	12.569	12.569	16.716	16.716	16.713
LB (100)	437.879***	437.879***	437.879***	8062.879***	8062.879***	8063.962***
LB (200)	638.522***	638.522***	638.522***	9148.716***	9148.716***	9150.167***
LB_{SD} (100)	437.879***	437.879***	437.879***	4259.401***	1871.683***	820.221***
LB_{SD} (200)	638.522***	638.522***	638.522***	7478.183***	3316.763***	1354.586***
Lag length	42	42	42			

Table 4e: Empirical results for squared stock return index for South Africa.

	FIMACH			FIGARCH (0,d,0)	FIGARCH (1,d,1)	GARCH (1,1)
	QML Estimates (s.e.)	FGLS Estimates (s.e.)	CLS Estimates (s.e.)	QML Estimates (s.e.)	QML Estimates (s.e.)	QML Estimates (s.e.)
Constant (Mean)	12.782*** (1.37)	20.314*** (1.28)	20.314*** (1.27)	4.181*** (0.298)	3.274*** (0.15)	3.600*** (0.28)
Constant (Variance)					-22.094*** (1.12)	0.587 (0.43)
d	0.194*** (0.02)	-0.166*** (0.02)	-0.166*** (0.02)	0.533 (0.00)	0.689*** (0.00)	
α						1.115*** (0.05)
β					0.572*** (0.00)	0.867*** (0.03)
VAR	7036.353*** (996.96)	7381.231	7381.231	9104.279	9104.279	9102.426
LB (100)	772.396***	2684.763***	2684.764***	11702.479***	11702.479***	11693.869***
LB (200)	1266.134***	4513.014***	4513.015***	18138.175***	18138.175***	18126.195***
LB_{SD} (100)	772.396***	2684.763***	2684.764***	3285.210***	1606.489 ***	1393.617***
LB_{SD} (200)	1266.134***	4513.014***	4513.015***	5571.753***	2285.839***	1831.270***
Lag length	70	70	70			

Table 4f: Empirical results for squared stock return index for USA.

	FIMACH			FIGARCH (0,d,0)	FIGARCH (1,d,1)	GARCH (1,1)
	QML Estimates (s.e.)	FGLS Estimates (s.e.)	CLS Estimates (s.e.)	QML Estimates (s.e.)	QML Estimates (s.e.)	QML Estimates (s.e.)
Constant (Mean)	70.071*** (8.539)	70.071*** (8.412)	70.071*** (8.403)	50.926*** (2.918)	18.044*** (1.616)	13.701*** (3.349)
Constant (Variance)					-1106.968*** (47.472)	-0.662 (2.993)
d	0.187*** (0.034)	0.187*** (0.035)	0.187*** (0.035)	0.472 (0.016)	0.669*** (0.004)	
α						1.045*** (0.018)
β					0.622*** (0.005)	0.915*** (0.015)
VAR	192930.645*** (32020.920)	192974.940	192974.940	223369.828	223369.828	223325.106
LB (100)	697.590***	697.590***	697.590***	7147.323***	7147.323***	7145.920***
LB (200)	868.459***	868.459***	868.459***	8028.339***	8028.339***	8026.328***
LB_{SD} (100)	693.986***	697.590***	693.986***	4432.055***	2445.256***	1061.702***
LB_{SD} (200)	864.174***	868.459***	864.174***	6799.300***	3694.440***	1744.355***
Lag length	70	70	70			

Table 4g: Empirical results for squared stock return index for UK.

	FIMACH			FIGARCH (0,d,0)	FIGARCH (1,d,1)	GARCH (1,1)
	QML	FGLS	CLS	QML	QML	QML
	Estimates (s.e.)	Estimates (s.e.)	Estimates (s.e.)	Estimates (s.e.)	Estimates (s.e.)	Estimates (s.e.)
Constant (Mean)	123.736***	78.056***	78.056***	62.144***	36.174***	35.517***
	(7.816)	(10.074)	(10.176)	(4.688)	(1.988)	(0.360)
Constant (Variance)					-655.158***	45.497***
					(77.010)	(0.004)
d	-0.162***	0.192***	0.192***	0.444	0.682***	
	(0.027)	(0.035)	(0.035)	(0.018)	(0.007)	
α						1.039***
						(0.014)
β					0.660***	0.918***
					(0.007)	(0.013)
VAR	270893.898***	258868.144	258868.144	301142.683	301142.683	301080.523
	(52010.797)					
LB (100)	2086.949***	1012.821***	1012.821***	7671.396***	7671.396***	7670.816***
LB (200)	2984.945***	1499.144***	1499.144***	10420.796***	10420.796***	10420.151***
LB_{SD} (100)	2086.902***	1012.821***	1012.317***	3809.926***	2201.357***	1600.740***
LB_{SD} (200)	2980.607***	1499.144***	1498.329***	6070.656***	3661.447***	2832.232***
Lag length	70	70	70			