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Extending Characters of Fixed Point Algebras

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Abstract: A dynamical system is a triple (A, G, α) consisting of a unital locally convex algebra A, a topological group G, and a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$ that induces a continuous action of G on A. Furthermore, a unital locally convex algebra A is called a continuous inverse algebra, or CIA for short, if its group of units $A^\times$ is open in A and the inversion map $\iota : A^\times \to A^\times, a \mapsto a^{-1}$ is continuous at 1_A. Given a dynamical system (A, G, α) with a complete commutative CIA A and a compact group G, we show that each character of the corresponding fixed point algebra can be extended to a character of A.

Keywords: dynamical system; continuous inverse algebra; character; maximal ideal; fixed point algebra; extension

MSC: 46H05; 46H10 (primary); 37B05 (secondary)

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1. Introduction

Let $\sigma : P \times G \to P$ be a smooth action of a Lie group G on a manifold P. It is well-known (see e.g., [1], Proposition 2.1) that σ induces a smooth action of G on the unital Fréchet algebra $C^\infty(P)$ of smooth functions on P defined by $\alpha_\sigma : G \times C^\infty(P) \to C^\infty(P)$, $(g, f) \mapsto f \circ \sigma_g$. The corresponding fixed point algebra is given by

$$C^\infty(P)^G := \{f \in C^\infty(P) : (\forall g \in G) \;\; \alpha_\sigma(g, f) = f\}.$$

The origin of this short article is, in a manner of speaking, "commutative geometry", namely the question whether *each character* $\chi : C^\infty(P)^G \to \mathbb{C}$ *extends to a character* $\tilde{\chi} : C^\infty(P) \to \mathbb{C}$ (cf. [2,3]).

One possible way to approach this problem is to classify the characters under consideration. Indeed, it follows from ([1], Lemma A.1) that each character $\chi : C^\infty(P) \to \mathbb{C}$ is an evaluation in some point $p \in P$, that is, of the form $\delta_p : C^\infty(P) \to \mathbb{C}$, $f \mapsto f(p)$. If the action σ is additionally free and proper, then the orbit space P/G has a unique manifold structure such that the canonical quotient map $q : P \to P/G$, $p \mapsto [p]$ is a submersion. Moreover, in this situation, the map

$$\Phi : C^\infty(P)^G \to C^\infty(P/G), \quad f \mapsto ([p] \mapsto f(p))$$

is an isomorphism of unital Fréchet algebras showing that each character $C^\infty(P)^G \to \mathbb{C}$ is of the form $\delta_{[p]} \circ \Phi$ for some $p \in P$ which may simply be extended by δ_p.

In this note, however, we approach the above problem in a more systematic way. In fact, given a dynamical system (A, G, α) with a complete commutative continuous inverse algebra (CIA) A and a compact group G, we show that each character of the corresponding fixed point algebra

$$A^G := \{a \in A : (\forall g \in G) \;\; \alpha(g)(a) = a\}$$

extends to a character of A (Theorem 2). Our approach is motivated by the following three facts:

(i) Our initial question is, after all, of purely topological nature.
(ii) If P is compact, then $C^\infty(P)$ is the prototype of a complete commutative CIA.
(iii) CIA's provide a class of algebras for which characters are automatically continuous (cf. [4], Lemma 2.3).

We would also like to mention that CIAs are naturally encountered in K-theory and noncommutative geometry, usually as dense unital subalgebras of C^*-algebras. Finally, we point out that a classical result for actions of finite groups can be found in ([5], Chapter 5, §2.1, Corollary 4).

2. Preliminaries and Notations

All algebras are assumed to be complex. The spectrum of an algebra A is the set $\Gamma_A := \mathrm{Hom}_{\mathrm{alg}}(A, \mathbb{C})\backslash\{0\}$ (endowed with the topology of pointwise convergence on A) and its elements are called characters. Moreover, given a compact group G, we denote by $\hat{G}$ the (countable) set of equivalence classes of finite-dimensional irreducible representations of G. For $\pi \in \hat{G}$ we write χ_π for the function defined by $G \mapsto \mathbb{C}$, $g \mapsto \mathrm{tr}(\pi(g))$ and we put $d_\pi := \chi_\pi(1_G)$ for the corresponding dimension. We also need the following well-known structure theorem for dynamical systems:

Lemma 1. ([6], [Lemma 3.2 and Theorem 4.22]). *Let (A, G, α) be a dynamical system with a complete unital locally convex algebra A and a compact group G. Furthermore, given $\pi \in \hat{G}$ and $a \in A$, let*

$$P_\pi(a) := d_\pi \int_G \overline{\chi_\pi}(g)\, (\alpha(g)(a))\; dg,$$

where dg denotes the normalized Haar measure on G. Then the following assertions hold:

(a) *For each $\pi \in \hat{G}$ the map $P_\pi : A \to A$ is a continuous G-equivariant projection onto the G-invariant subspace $A_\pi := P_\pi(A)$. In particular, A_π is algebraically and topologically a direct summand of A.*
(b) *The module direct sum $A_{\mathrm{fin}} := \bigoplus_{\pi\in\hat{G}} A_\pi$ is a dense subalgebra of A.*

3. Extension Results

In this section our main results are stated and proved. We begin with some general statements on the extendability of ideals.

Lemma 2. *Let (A, G, α) be a dynamical system with a complete unital locally convex algebra A and a compact group G. Then the following assertions hold:*

(a) *If I is a proper left ideal in A^G, then $A_{\mathrm{fin}} \cdot I = \bigoplus_{\pi\in\hat{G}} A_\pi \cdot I$ defines a proper left ideal in A_{fin} that contains I.*
(b) *If I is a proper closed left ideal in A^G and J is the closure of $A_{\mathrm{fin}} \cdot I$ in A_{fin}, then J is a proper closed left ideal in A_{fin} that contains I.*

Proof. (a) We first observe that A^G coincides with A_1 (where 1 stands for the equivalence class of the trivial representation). Hence $I \subseteq A^G$ is contained in A_{fin} and thus $A_{\mathrm{fin}} \cdot I$ is the left ideal of A_{fin} generated by I. Using the integral formula for P_π from Lemma 1, we see that $A_\pi \cdot I \subseteq A_\pi$, entailing that the sum in part (a) is direct. To see that $A_{\mathrm{fin}} \cdot I$ is proper, we assume the contrary, that is,

$$1_A \in A_{\mathrm{fin}} \cdot I = \bigoplus_{\pi\in\hat{G}} A_\pi \cdot I.$$

Then $1_A \in A^G$ implies that $1_A \in A^G \cdot I = I$, which contradicts the fact that I is a proper left ideal of A^G. We conclude that $A_{\mathrm{fin}} \cdot I$ is a proper left ideal in A_{fin} that contains I.

(b) Part (a) and the definition of J imply that J is a closed left ideal in A_{fin} that contains I. To see that J is proper, we again assume the contrary, that is, $1_A \in J$. Then there exists a net $(a_\gamma)_{\gamma\in\Gamma}$ in $A_{\mathrm{fin}} \cdot I$

such that $\lim_\gamma a_\gamma = 1_A$ and the continuity of the projection map $P_1 : A \to A$ onto the fixed point algebra A^G implies that

$$1_A = P_1(1_A) = P_1(\lim_\gamma a_\gamma) = \lim_\gamma P_1(a_\gamma).$$

Since I is closed in A^G and $P_1(a_\gamma) \in A^G \cdot I = I$ for all $\gamma \in \Gamma$, we conclude that $1_A \in I$. This contradicts the fact that I is a proper ideal of A^G and therefore J is a proper closed left ideal in A_{fin} that contains I. □

Lemma 3. *Let A be a topological algebra and B a dense subalgebra of A. If I is a proper closed left ideal in B, then $\overline{I}$ is a proper closed left ideal in $\overline{B} = A$.*

Proof. It is easily seen that $\overline{I}$ is a closed left ideal in $\overline{B} = A$. Moreover, we have $I = \overline{I} \cap B$. Indeed, the inclusion " $\subseteq$ " is obvious and for the other inclusion we use the fact that I is closed in B. Consequently, if $\overline{I}$ is not proper, that is, $\overline{I} = A$, then $I = B$, which yields a contradiction. Hence, $\overline{I}$ is a proper closed left ideal in A. □

We are now ready to state and prove our main extension results.

Theorem 1. (*Extending ideals*). *Let (A, G, α) be a dynamical system with a complete unital locally convex algebra A and a compact group G. Then each proper closed left ideal in A^G is contained in a proper closed left ideal in A.*

Proof. Let I be a proper closed left ideal in A^G. Then Lemma 2 (b) implies that I is contained in a proper closed left ideal in A_{fin}. Since A_{fin} is a dense subalgebra of A by Lemma 1 (b), the claim is a consequence of Lemma 3. □

Theorem 2. (*Extending characters*). *Let (A, G, α) be a dynamical system with a complete commutative* CIA *A and a compact group G. Then each character $\chi : A^G \to \mathbb{C}$ is continuous and extends to a continuous character $\tilde{\chi} : A \to \mathbb{C}$.*

Proof. Let $\chi : A^G \to \mathbb{C}$ be a character. Since A^G carries the structure of a CIA in its own right, it follows from ([4], Lemma 2.3) that χ is continuous which shows that $I := \ker \chi$ is a proper closed ideal in A^G. Hence, Theorem 1 implies that I is contained in a proper closed ideal in A. In particular, it is contained in a proper maximal ideal J of A which, according to ([7], Lemma 2.2.2) and ([4], Lemma 2.3), is the kernel of some continuous character $\tilde{\chi} : A \to \mathbb{C}$. Since I is a maximal ideal in the unital algebra A^G and

$$I = I \cap A^G \subseteq J \cap A^G \subsetneq A^G,$$

we conclude that $I = J \cap A^G$. Therefore, the decomposition $A^G = I \oplus \mathbb{C} = (J \cap A^G) \oplus \mathbb{C}$ finally proves that $\tilde{\chi}$ extends χ. □

Remark 1. *It is not clear how to extend Theorem 2 beyond the class of CIAs. For instance, given a non-compact manifold P, the set $C_c^\infty(P)$ of compactly supported smooth functions on P is a proper ideal in $C^\infty(P)$. As such it is contained in a proper maximal ideal in $C^\infty(P)$ that cannot be closed since $C_c^\infty(P)$ is dense in $C^\infty(p)$. However, in the more general situation of a complete commutative unital locally convex algebra A, a similar argument as in the proof of Theorem 2 shows that each continuous character $\chi : A^G \to \mathbb{C}$ can be extended to a character $\tilde{\chi} : A \to \mathbb{C}$.*

We conclude with the following two immediate corollaries.

Corollary 1. *Suppose we are in the situation of Theorem 2. Then the natural map on the level of spectra* $\Gamma_A \to \Gamma_{A^G}, \chi \mapsto \chi_{|A^G}$ *is surjective.*

Corollary 2. *Let* $(C^\infty(P), G, \alpha)$ *be a dynamical system with a compact manifold* P *and a compact group* G*. Then each character* $\chi : C^\infty(P)^G \to \mathbb{C}$ *extends to a character* $\tilde{\chi} : C^\infty(P) \to \mathbb{C}$*.*

Remark 2. *Given a dynamical system* $(C^\infty(P), G, \alpha)$ *with a compact manifold* P *and a compact group* G*, we would like to describe* $\Gamma_{C^\infty(P)^G}$ *as a set of points associated to* P *and* G*. As already explained in the introduction, it is not hard to see that* $\Gamma_{C^\infty(P)^G}$ *is homeomorphic to* P/G *if* G *is a Lie group and* α *is induced by a free and smooth action of* G *on* P*. However, even if we do not have any additional information, it is still possible to show that the map*

$$P/G \to \Gamma_{C^\infty(P)^G}, \quad q(p) \mapsto \delta_p$$

is a homeomorphism (see e.g., [2]*, Proposition 8.7) and Corollary 2 may be used to verify its surjectivity.*

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