THE STRUCTURE OF EPSILON-STRONGLY GROUP GRADED RINGS

Daniel Lännström

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Abstract. The development of a general theory of strongly group graded rings was initiated by Dade, Năstăsescu and van Oystaeyen in the 1980s, and since then numerous structural results have been established. In this thesis we develop a general theory of so-called (nearly) epsilon-strongly group graded rings which were recently introduced by Nystedt, Öinert and Pinedo and which generalize strongly group graded rings. Moreover, we obtain applications to Leavitt path algebras, partial crossed products and algebraic Cuntz-Pimsner rings.

This thesis is based on five scientific papers (A, B, C, D, E).

Papers A and B are concerned with structural properties of epsilon-strongly graded rings. In Paper A, we consider an important construction called the induced quotient group grading. In Paper B, using results from Paper A, we obtain a Hilbert Basis Theorem for epsilon-strongly graded rings. In Paper C, we study the graded structure of algebraic Cuntz-Pimsner rings. In particular, we obtain a partial characterization of unital strongly graded, epsilon-strongly graded and nearly epsilon-strongly graded algebraic Cuntz-Pimsner rings up to graded isomorphism.

In Paper D, we give a complete characterization of group graded rings that are graded von Neumann regular.

Finally, in Paper E, written in collaboration with Lundström, Öinert and Wagner, we consider prime nearly epsilon-strongly graded rings. Generalizing Passman's work from the 1980s, we give necessary and sufficient conditions for a nearly epsilon-strongly graded ring to be prime.

Keywords. group graded ring, Leavitt path algebra, partial crossed product, Cuntz-Pimsner ring, von Neumann regular ring, non-unital ring

Contents

Preface

This thesis consists of two parts: Part I and Part II. In Part I, we introduce the subject and summarize the results of the scientific papers. Part I has been reworked and expanded from the author's licentiate's thesis [1].

Part II consists of five scientific papers (A, B, C, D, E):

A. D. Lännström. *Induced quotient group gradings of epsilon-strongly graded rings.* J. Algebra Appl. vol. 19 (2020), no. 9, 2050162, 26 pp.
`https://doi.org/10.1142/S0219498820501625`

B. D. Lännström. *Chain conditions for epsilon-strongly graded rings with applications to Leavitt path algebras.* Algebr. Represent. Theory vol. 23 (2020), no. 4, 1707–1726.
`https://doi.org/10.1007/s10468-019-09909-0`

C. D. Lännström. *The graded structure of algebraic Cuntz-Pimsner rings.* J. Pure Appl. Algebra vol. 224 (2020), no. 9, 106369, 26 pp.
`https://doi.org/10.1016/j.jpaa.2020.106369`

D. D. Lännström. *A characterization of graded von Neumann regular rings with applications to Leavitt path algebras.* J. Algebra vol. 567 (2021), 91–113.
`https://doi.org/10.1016/j.jalgebra.2020.09.022`

E. D. Lännström, P. Lundström, J. Öinert, S. Wagner. *Prime group graded rings with applications to partial crossed products and Leavitt path algebras.* Preprint.
`https://arxiv.org/abs/2105.09224`

Reprints were made with permission from the publishers. Papers A-C were also included in the author's licentiate's thesis [1].

References

[1] D. Lännström. *The structure of epsilon-strongly graded rings with applications to Leavitt path algebras and Cuntz-Pimsner rings*, number 2019:07. Licentiate thesis, Blekinge Tekniska Högskola, 2019.

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Daniel Lännström
Karlskrona, May 2021

Part I

Introduction and summary

CHAPTER 1

Introduction

The theory of C^*-algebras was originally developed to formalize quantum mechanics but later emerged as a central tool in many diverse branches of mathematics, for instance: combinatorics, geometry, group theory, logic and stochastics. The prototype example of a C^*-algebra is the algebra of bounded linear operators on a complex Hilbert space. However, there are also C^*-algebras which are defined in terms of a discrete, combinatorial structure such as a graph. An important ongoing trend is to define and study algebraic analogues of C^*-algebras. By studying these algebraic analogues, the hope is that one may learn about the core structure of the model C^*-algebras. This basic approach can be traced all the way back to the works of von Neumann and Kaplansky and their students Berberian and Rickart (see [7, 31]).

The Leavitt algebras were introduced and studied by Leavitt in a series of papers [34, 35, 36] on ring theory. More than a decade later, Cuntz [19] constructed his famous Cuntz C^*-algebras giving an explicit construction of an important class of C^*-algebras. It was realized later (see e.g. [1, p. 6]) that these two constructions are related. Shortly after Cuntz' important paper, Cuntz and Krieger [20] constructed the Cuntz-Krieger C^*-algebras. Moreover, to a row-finite graph E, Kumjian, Pask and Raeburn [33] associated a Cuntz-Krieger algebra $C^*(E)$. This construction is known as the graph C^*-algebra associated to E. The *Leavitt path algebra* $L_K(E)$ over a field K associated to a directed graph E, which was introduced by Ara, Moreno and Pardo in [8] and by Abrams and Aranda Pino in [3], is the algebraic analogue of $C^*(E)$. Later, algebraic Cuntz-Pimsner rings (see [11]) and Steinberg algebras (see [15, 52]) were introduced as analogues of Cuntz-Pimsner C^*-algebras (see [32, 47]) and groupoid C^*-algebras (see [48]), respectively.

Another significant development in the study of C^*-algebras was the introduction of the notion of a partial action by Exel in [27]. It was realized that important classes of C^*-algebras could naturally be realized as crossed products by partial actions. Algebraic analogues of these notions were developed during the last two decades (see [9, 24, 25]). Among the class of algebraic partial crossed products originally considered, the *unital partial crossed products* (see e.g. [42, p. 2]) were shown to be especially well-behaved.

For more historical details regarding graph algebras, we refer the reader to a survey article by Abrams [1]. The following is an important conjecture regarding the relationship between Leavitt path algebras and graph C^*-algebras:

Conjecture 1.0.1 (Abrams-Tomforde Isomorphism Conjecture [4])**.** Let E and F be directed graphs. If $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ as rings, then $C^*(E) \cong C^*(F)$ as C^*-algebras.

The general belief is that Conjecture 1.0.1 should hold. However, a complete proof of Conjecture 1.0.1 remains elusive. Important special cases of Conjecture 1.0.1 have, however, been proved (see e.g. [26, Thm. 14.7]). Emphasizing this connection, we note that for many of the results in the literature there is a graph C^*-algebra version and a corresponding algebraic Leavitt path algebra version. It is conjectured (see e.g. [54, p. 17]) that there might be a 'Rosetta stone' that allows us to translate between the analytic and algebraic sides (cf. Figure 1) but the exact nature of this relation remains unknown. In this thesis, we continue to develop a unified approach to the algebraic analogues (right hand side of Figure 1) which was introduced by Nystedt, Öinert and Pinedo (see [41, 42]). Our aim is to improve the general theory of graded rings to include the algebraic analogues.

C^*-algebra	Algebra
Cuntz C^*-algebra ([19])	Leavitt algebra ([35])
Graph C^*-algebra ([33])	Leavitt path algebra ([3, 8])
Cuntz-Pimsner C^*-algebra ([32, 47])	Algebraic Cuntz-Pimsner ring ([11, 12])
Groupoid C^*-algebra ([48])	Steinberg algebra ([15, 52])
Crossed products by partial actions ([27])	Unital partial crossed products ([9, 24, 25])

FIGURE 1. C^*-algebras with their corresponding 'algebraizations'.

We will see (in Section 1.5.1) that the Leavitt path algebras are examples of so-called *group graded rings* (see Definition 1.5.1). This type of ring comes equipped with an additional *graded structure* described by a discrete group. This thesis revolves around three special classes of group graded rings:

- *strongly graded rings* (see Definition 1.5.1),
- *epsilon-strongly graded rings* (see Definition 1.6.1), and,
- *nearly epsilon-strongly graded rings* (see Definition 1.7.1).

The class of epsilon-strongly graded rings was introduced by Nystedt, Öinert and Pinedo in [42]. The generalization to nearly epsilon-strongly graded rings was introduced later by Nystedt and Öinert in [41]. Regarding the graded structure of Leavitt path algebras over a unital ring R, Nystedt, Öinert [41] and Hazrat [30] have obtained the following characterizations (see Theorem 1.5.16, Theorem 1.6.14, Theorem 1.7.14 and Proposition 1.7.19).

$$\begin{array}{ccccc} E \text{ finite with no sinks} & \Rightarrow & E \text{ finite} & \Rightarrow & E \text{ a graph} \\ \Updownarrow & & \Updownarrow & & \Updownarrow \\ L_R(E) \text{ unital strong} & \Rightarrow & L_R(E)\ \epsilon\text{-strong} & \Rightarrow & L_R(E) \text{ nearly } \epsilon\text{-strong} \end{array}$$

FIGURE 2. The graded structure of Leavitt path algebras.

The class of strongly graded rings is well-understood and has many connections to geometry and mathematical physics (see e.g. [21, 51] and the references therein). The foundation for the general theory of strongly group graded rings was laid by Dade in his seminal paper [22]. Since then, many structural properties of strongly graded rings have been established (see e.g. the monograph [39] by Năstăsescu and van Oystaeyen). The results in Figure 2 together with Nystedt, Öinert and Pinedo's result that unital partial crossed products are epsilon-strongly graded (see [42]) motivate us to develop a theory of (nearly) epsilon-strongly graded rings. We will also study the objects on the right hand side of Figure 1 and see that the framework of (nearly) epsilon-strongly graded rings help us understand their structure.

1.1. Outline

Here is a detailed outline of the rest of this thesis:

Chapter 1 of Part I is devoted to introducing the general area of graded ring theory together with the particular problems studied in this thesis. We review the theory and examples required to understand Nystedt, Öinert and Pinedo's definition of epsilon-strongly graded rings. We also give a few original results of a preliminary nature. In Section 1.2 of Chapter 1, we treat matters of notation and convention. In Section 1.3, we recall the definition of a Leavitt path algebra and consider some examples. These examples will motivate the general definitions in subsequent sections. In Section 1.4, we recall some classical results about rings with local units. In Section 1.5, we consider group graded rings and some special classes of group graded rings: strongly group graded rings, algebraic crossed products, symmetrically graded rings and non-degenerately graded rings. In particular, we will present some classical results about strongly graded rings which we will generalize in Part II. In Section 1.6 and Section 1.7, we present the definitions of epsilon-strong and nearly epsilon-strong group gradings respectively. We conclude Chapter 1 by recalling the definition of an algebraic Cuntz-Pimsner ring and make a few remarks about its construction.

In Chapter 2 of Part I, we give a detailed summary of the five papers that are included in this thesis.

Part II consists of the five papers (A, B, C, D, E) themselves:

In Paper A, we consider the induced quotient group grading (see Definition 1.5.34) of epsilon-strongly graded rings.

In Paper B, we establish a Hilbert Basis Theorem for epsilon-strongly graded rings. As an application, we extend the well-known classifications of noetherian, artinian and semisimple Leavitt path algebras with coefficients in a field (see e.g. [2, Cor. 4.2.13-14]) to also include coefficients in a general non-commutative unital ring.

In Paper C, we investigate algebraic Cuntz-Pimsner rings. The Leavitt path algebras can be realized as a special family of Cuntz-Pimsmer rings. Motivated by the results in Figure 2 for Leavitt path algebras, we consider the graded structure of algebraic Cuntz-Pimsner rings.

In Paper D, we prove that a graded ring is graded von Neumann regular if and only if it is nearly epsilon-strongly graded and its principal component is von Neumann regular (see Theorem 2.4.3).

In Paper E, we consider prime nearly epsilon-strongly graded rings. This paper builds upon work by Passman [45] and can be seen as a generalization of Connell's

Theorem [18]. Applying our results, we extend a well-known characterization of prime Leavitt path algebras (see e.g. [2, Prop. 4.1.4]).

1.2. Notation and conventions

All rings are assumed to be associative but not necessarily equipped with a multiplicative identity element. A ring R is called *unital* if it is equipped with a non-zero multiplicative identity element. A *subring* R of a ring S is a subset $R \subseteq S$ which is a ring with the operations of S. Let X and Y be non-empty subsets of R. We let XY denote the set of all finite sums of elements of the form xy where $x \in X$ and $y \in Y$. An *idempotent* $f \in R$ is an element such that $f^2 = f$. The set of idempotents of R is denoted by $E(R)$.

Let R be a ring. A *left R-module* ${}_RM$ is an abelian group $(M, +)$ equipped with a ring homomorphism $R \to \mathrm{End}(M)$ defining a left multiplication of R. A *right R-module* M_R is defined analogously. The *left annihilator* of ${}_RM$ is the set,

$$l.\mathrm{Ann}_R({}_RM) = \{r \in R \mid r \cdot m = 0 \quad \forall m \in M\}.$$

The right annihilator of M_R is defined similarly. A ring S is said to be an *R-algebra* if S is also a left R-module and $(r \cdot s)s' = r \cdot (ss')$ for all $r \in R$ and $s, s' \in S$.

We use the symbols $\mathbb{R}, \mathbb{C}$ and $\mathbb{Z}$ to denote the field of real numbers, the field of complex numbers and the ring of integers, respectively. We also use $\mathbb{Z}$ to denote the infinite cyclic group $(\mathbb{Z}, 0, +)$.

The symbol $\delta_{x,y}$ will denote the Kronecker delta function, i.e. for elements x, y of some set, we have that $\delta_{x,y} = 1$ if $x = y$ and $\delta_{x,y} = 0$ if $x \neq y$.

1.3. Leavitt path algebras

The Leavitt path algebras will be a constant source of examples throughout this thesis. Many of the more general definitions are motivated by examples from the class of Leavitt path algebras. We also work with this class of rings directly in the research papers. The Leavitt path algebra associated to a directed graph was introduced by Ara, Moreno and Pardo [8], and independently, using a different approach, by Abrams and Aranda Pino [3]. For a thorough account of the theory of Leavitt path algebras, we refer the reader to the monograph by Abrams, Ara, and Siles Molina [2].

Definition 1.3.1. A *directed graph* E is a tuple (E^0, E^1, s, r) where E^0 is a set of vertices, E^1 is a set of edges and $s\colon E^1 \to E^0$ and $r\colon E^1 \to E^0$ are maps specifying the *source* respectively *range* of each edge.

Given a directed graph E and a coefficient ring R, we define an R-algebra called the Leavitt path algebra. We follow Hazrat [30] and Nystedt-Öinert [41], and let R be a general (possibly non-commutative) unital ring.

Definition 1.3.2. Let E be a directed graph and let R be a unital ring. The *Leavitt path algebra of the graph E with coefficients in R.* denoted by $L_R(E)$, is the algebra over R generated by the symbols $\{v \mid v \in E^0\} \cup \{f \mid f \in E^1\} \cup \{f^* \mid f \in E^1\}$, subject to the following relations:

(a) $uv = \delta_{u,v}u$ for all $u, v \in E^0$,

(b) $s(f)f = fr(f) = f$ for all $f \in E^1$,
(c) $r(f)f^* = f^*s(f) = f^*$ for all $f \in E^1$,
(d) $f^*f' = \delta_{f,f'}r(f)$ for all $f, f' \in E^1$,
(e) $\sum_{f\in E^1, s(f)=v} ff^* = v$ for all $v \in E^0$ for which $0 < |s^{-1}(v)| < \infty$.

We let every element of R commute with the generators.

Remark 1.3.3. We make some remarks regarding Definition 1.3.2.

(a) Let $F_R(E)$ be the free associate algebra over R generated by the symbols $\{v \mid v \in E^0\} \cup \{f \mid f \in E^1\} \cup \{f^* \mid f \in E^1\}$. The Leavitt path algebra $L_R(E)$ is defined as the quotient ring $L_R(E) = F_R(E)/I$ where I is the ideal of $F_R(E)$ generated by the relations (a)-(e). Let $\pi\colon F_R(E) \to L_R(E)$ be the natural quotient map. We will use the common convention of simply writing v, f and f^* for the elements $\pi(v), \pi(f), \pi(f^*)$ in the quotient ring $L_R(E)$. It can be proved that $v \neq 0$, $f \neq 0$ and $f^* \neq 0$ hold in $L_R(E)$ for all $v \in E^0, f \in E^1$.
(b) The elements $f \in L_R(E)$ for $f \in E^1$ are called the *real edges*. The elements $f^* \in L_R(E)$ for $f \in E^1$ are called the *ghost edges*.

We will now consider some concrete examples of Leavitt path algebras. Interestingly, we will see that Laurent polynomial rings and full matrix rings may be realized as Leavitt path algebras. We will later refer back to these examples.

$$A_1 : \qquad \bullet_v$$

Figure 3. Graph with a single vertex.

Example 1.3.4. Let R be a unital ring and let $A_1 = (E^0, E^1, s, r)$ be the directed graph in Figure 3 consisting of a single vertex without any edges. Note that $E^0 = \{v\}$ and $E^1 = \emptyset$. In this case, $L_R(A_1)$ is generated over R by the idempotent v. Hence, $L_R(A_1) \cong Rv \cong R$.

Next, we show that the Laurent polynomial ring is realizable as a Leavitt path algebra.

$E_1 :$ 

Figure 4. Rose with one petal.

Example 1.3.5 (cf. [2, Prop. 1.3.4])**.** Let R be a unital ring and let $E_1 = (E^0, E^1, s, r)$ be the directed graph given in Figure 4. This graph is sometimes called 'a rose with one petal'. Note that $E^0 = \{v\}, E^1 = \{f\}$ and that $s\colon E^1 \to E^0, r\colon E^1 \to E^0$ are given by $s(f) = v = r(f)$. We show that $L_R(E_1) \cong R[x, x^{-1}]$. Define a ring homomorphism

$\phi\colon L_R(E_1) \to R[x, x^{-1}]$ by R-linearly extending $\phi(v) = 1_R, \phi(f) = x$, $\phi(f^*) = x^{-1}$. A routine check shows that ϕ is well-defined. Moreover, note that $R[x, x^{-1}]$ is generated by $1_R, x, x^{-1}$ over R and that $\ker \phi = \{0\}$. Thus, ϕ is a ring isomorphism.

$$A_2 : \qquad \bullet_{v_1} \xrightarrow{f} \bullet_{v_2}$$

FIGURE 5. Two vertices with one connecting edge.

Example 1.3.6. Let R be a unital ring and let A_2 be the graph in Figure 5. Let $M_2(R)$ denote the ring of 2×2-matrices with coefficients in R. We show that $L_R(A_2) \cong M_2(R)$. First note that $M_2(R)$ is generated over R by following matrices:

$$m_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad m_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad m_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Note that m_1 and m_2 are idempotents and m_3 and m_4 are nilpotents. Furthermore, the following relations hold between the generators:

$$\begin{gathered} m_1 m_2 = m_2 m_1 = 0 \\ m_3 m_4 = m_1, \quad m_4 m_3 = m_2 \\ m_1 m_3 = m_3, \quad m_3 m_1 = 0 \\ m_2 m_3 = 0, \quad m_3 m_2 = m_3 \\ m_1 m_4 = 0, \quad m_4 m_1 = m_4 \end{gathered}$$

On the other hand, consider the Leavitt path algebra $L_R(A_2)$. It is generated over R by the elements v_1, v_2, f, f^* where v_1, v_2 are idempotents and f, f^* are nilpotents. The following relations hold (see Definition 1.3.2):

$$\begin{gathered} v_1 v_2 = v_2 v_1 = 0 \\ f f^* = v_1, \quad f^* f = v_2 \\ v_1 f = f, \quad f v_1 = 0 \\ v_2 f = 0, \quad f v_2 = f \\ v_1 f^* = 0, \quad f^* v_1 = f^* \end{gathered}$$

It follows that $M_2(R) \cong L_R(A_2)$ via the ring isomorphism $\phi\colon M_2(R) \to L_R(A_2)$ defined by $\phi(m_1) = v_1, \phi(m_2) = v_2, \phi(m_3) = f$ and $\phi(m_4) = f^*$.

The previous example can be generalized to the ring of $n \times n$-matrices:

$$A_n : \qquad \bullet_{v_1} \longrightarrow \bullet_{v_2} \longrightarrow \cdots\cdots \longrightarrow \bullet_{v_{n-1}} \longrightarrow \bullet_{v_n}$$

FIGURE 6. Line graph with n vertices.

Example 1.3.7 (cf. [2, Prop. 1.3.5])**.** Let R be a unital ring. Let $n \geq 2$ be an arbitrary integer and consider the line graph A_n given in Figure 6. Using a generalization of the argument in Example 1.3.6, it can be proved that $L_R(A_n) \cong M_n(R)$ where $M_n(R)$ denotes the full matrix ring of $n \times n$-matrices with coefficients in R.

1.4. Rings with local units

In this section we present some basic theory about rings possessing local units. This section is based on a survey article by Nystedt [40].

Definition 1.4.1. Let S and T be rings. An *S–T-bimodule* ${}_SM_T$ is a left S-module and a right T-module such that the left and right multiplications satisfies the following condition: for all $s \in S$, $m \in M$ and $t \in T$,

$$(s \cdot m) \cdot t = s \cdot (m \cdot t).$$

If $S = T = R$ we say that M is an *R-bimodule*.

Definition 1.4.2. Let S and T be rings.

(a) A left S-module ${}_SM$ is called *left unital* if there exists some $s \in S$ such that $sm = m$ for all $m \in M$.
(b) A right T-module M_T is called *right unital* if there exists some $t \in T$ such that $mt = m$ for all $m \in M$.
(c) A left S-module ${}_SM$ is called *left s-unital* if $x \in Sx$ for every $x \in M$.
(d) A right T-module M_T is called *right s-unital* if $x \in xT$ for every $x \in M$.
(e) An S–T-bimodule ${}_SM_T$ is called *unital* if ${}_SM$ is left unital and M_T is right unital.
(f) An S–T-bimodule ${}_SM_T$ is called *s-unital* if ${}_SM$ is left s-unital and M_T is right s-unital.
(g) A ring R is called an *s-unital ring* if ${}_RR_R$ is an s-unital R-bimodule.

Remark 1.4.3. We write down some immediate consequences of Definition 1.4.2.

(a) A ring R is s-unital if and only if $x \in xR \cap Rx$ for all $x \in R$. This is equivalent to: for each $x \in R$ there exist some $e, e' \in R$ such that $ex = x$ and $xe' = x$.
(b) A non-trivial ring R is unital if and only if ${}_RR_R$ is unital as an R-bimodule.
(c) A left (right) unital R-module is left (right) s-unital. The class of unital rings is contained in the class of s-unital rings.

We give an example of an s-unital ring that does not admit a multiplicative identity element. This shows that the inclusion in Remark 1.4.3(c) is strict.

Example 1.4.4. Let $\mathrm{Fun}_{\mathrm{fin}}(\mathbb{R})$ denote the set of real-valued functions of $\mathbb{R}$ with finite support. Then $\mathrm{Fun}_{\mathrm{fin}}(\mathbb{R})$ becomes a ring with pointwise multiplication and addition. Take an arbitrary $f \in \mathrm{Fun}_{\mathrm{fin}}(\mathbb{R})$, let $S_f = \mathrm{Supp}(f) \subseteq \mathbb{R}$ be the support of f and consider the indicator function $1_{S_f} \colon \mathbb{R} \to \{0, 1\}$ of the set S_f. Then, $1_{S_f}(x)f(x) = f(x) = f(x)1_{S_f}(x)$ for all $x \in S_f$. Hence, $1_{S_f} f = f = f 1_{S_f}$. Thus, $\mathrm{Fun}_{\mathrm{fin}}(\mathbb{R})$ is an s-unital ring. On the other hand, suppose to get a contradiction that $\chi \in \mathrm{Fun}_{\mathrm{fin}}(\mathbb{R})$ is a multiplicative identity element. For any $x \in \mathbb{R}$, consider the function $f_x \in \mathrm{Fun}_{\mathrm{fin}}(\mathbb{R})$ defined by $f_x(y) = \delta_{x,y}$. Then, we must have $\chi(x) = \chi(x)1 = \chi(x)f_x(x) = f_x(x) = 1$ for every $x \in \mathbb{R}$. This implies that $\mathrm{Supp}(\chi)$ is infinite, which is a contradiction.

Definition 1.4.5. A ring R is called idempotent if $R^2 = R$.

Note that $R^2 \subseteq R$ since the multiplication is a closed binary operation. Hence, to prove that a ring R is idempotent, it is enough to show that $R \subseteq R^2$.

Remark 1.4.6. If R is a left (right) s-unital ring, then for each $r \in R$ there is some $x \in R$ such that $r = xr$ ($r = rx$). Hence, $R \subseteq R^2$ and thus R is idempotent. In other words, the following implication holds:

$$\text{left (right) s-unital} \implies \text{idempotent.}$$

We recall an example showing that idempotent rings are not necessarily s-unital.

Example 1.4.7 ([40, Expl. 6b])**.** Let A be a unital ring with a multiplicative identity element $1_A \neq 0$. Let B denote the set $A \times A$ equipped with componentwise addition and multiplication defined by,

$$(a, b)(c, d) = (ac, ad),$$

for $a, b, c, d \in A$. A short calculation shows that B is associative. Note that $(1, b)$ is a left multiplicative identity element for every $b \in A$. In particular, B is a left s-unital B-module. By Remark 1.4.6, B is an idempotent ring. On the other hand, since $(0, 1) \notin (0, 1)B = \{(0, 0)\}$, it follows that B is not a right s-unital B-module. Thus, B is not an s-unital ring.

Many of the rings considered in this thesis will have local units in some sense. An important example is the Leavitt path algebra associated to a directed graph. It can be proved that all Leavitt path algebras are s-unital rings, but in fact a stronger statement can be obtained. This motivates us to consider local unit properties that are stronger than s-unitality. We will now consider rings possessing sets of local units. This notion is due to Abrams (see [5]). First, we recall the classical idempotent ordering defined on the set of idempotents of a ring.

Definition 1.4.8. Let R be a ring and consider the following partial order on the set of idempotents of R. For $f_1, f_2 \in E(R)$,

$$f_1 \leq f_2 \iff f_1 = f_1 f_2 = f_2 f_1.$$

Let $f_1 \vee f_2$ and $f_1 \wedge f_2$ denote the least upper bound of f_1, f_2 and greatest lower bound of f_1, f_2 (when they exist) with respect to this partial order.

The following is an alternative characterization:

Lemma 1.4.9. Let $e, f \in E(R)$ be idempotents of R. Then $e \leq f$ is equivalent to the following condition: for every $x \in R$,

(a) $x = ex \implies x = fx$, and,
(b) $x = xe \implies x = xf$.

Proof. Let $e, f \in E(R)$ be idempotents. Assume that the implications (a) and (b) hold for every $x \in R$. Taking $x = e$ in (a), we see that $e = e^2$ implies $e = fe$. Similarly, taking $x = e$ in (b), we get that $e = ef$. Hence, $e \leq f$.

Conversely, assume that $e \leq f$. Take an arbitrary $x \in R$. If $x = ex$, then $fx = f(ex) = (fe)x = ex = x$. Similarly, if $x = xe$, then $xf = (xe)f = x(ef) = xe = x$. $\square$

Let $f_1, f_2 \in E(R)$. One way of thinking about the previous lemma is that $f_1 \leq f_2$ if and only if f_2 can be used in place of f_1 as a 'local multiplicative identity element'. If f_1, f_2 commute, then it can be proved that $f_1 \wedge f_2 = f_1 f_2$ and $f_1 \vee f_2 = f_1 + f_2 - f_1 f_2$. If $f_1 f_2 = f_2 f_1 = 0$, then f_1 and f_2 are called *orthogonal*. In particular, note that if f_1, f_2 commute and are orthogonal, then $f_1 \vee f_2 = f_1 + f_2$. For an arbitrary subset of idempotents $F \subseteq E(R)$, we let $\bigvee F$ denote the closure of F with respect to the $\vee$-operator. In other words, if $f_1, f_2 \in \bigvee F$ and $f_1 \vee f_2$ exists, then $f_1 \vee f_2 \in \bigvee F$. It can be seen that $\bigvee F$ consists of the elements of the form $f_1 \vee f_2 \vee \cdots \vee f_n$ with $f_i \in F$.

Definition 1.4.10 (Abrams [5])**.** A ring R is said to have a *set of local units* F if $F \subseteq E(R)$ is a set of idempotents satisfying the following assertions:

(a) For all $f_1, f_2 \in F$, we have $f_1 \vee f_2 \in F$ if the least upper bound $f_1 \vee f_2$ exists.
(b) For every finite set of elements $r_1, r_2, \ldots, r_n \in R$, there exists an idempotent $f \in F$ such that $fr_i = r_i = r_i f$ holds for $1 \leq i \leq n$.

By Remark 1.4.3(a) and Definition 1.4.10(b), it follows that a ring with local units is s-unital.

Remark 1.4.11. We make two remarks regarding Definition 1.4.10.

(a) The condition in Definition 1.4.10(a) was added by Nystedt [40, Def. 21] after private communication with Abrams.
(b) Note that Definition 1.4.10(b) is equivalent to the condition that every finite subset $X \subseteq R$ is contained in the unital subring fRf for some $f \in F$.

Before continuing, we need the following general definition and notation:

Definition 1.4.12. Let R be a ring and let $\{X_i\}_{i \in I}$ be a family of additive subgroups of R. We let $\sum_{i \in I} X_i$ denote the additive subgroup of R consisting of all finite sums $x_{i_1} + \ldots + x_{i_n}$ where $i_1, \ldots, i_n \in I$ and $x_{i_k} \in X_{i_k}$ for $k \in \{1, \ldots, n\}$. If $R = \sum_{i \in I} X_i$, then every $r \in R$ decomposes as

$$r = x_{i_1} + x_{i_2} + \ldots + x_{i_n} \tag{1}$$

for some $i_1, \ldots, i_n \in I$ and some family of elements $x_{i_k} \in X_{i_k}$ ($k \in \{1, \ldots, n\}$). If $R = \sum_{i \in I} X_i$ and $X_i \cap \sum_{j \neq i} X_j = \{0\}$ for all $i \in I$, then we write $R = \bigoplus_{i \in I} X_i$ and say that the sum is *direct*. Equivalently, $R = \bigoplus_{i \in I} X_i$ if and only if the decomposition in (1) is unique for every $r \in R$.

We can now state the following definition:

Definition 1.4.13 (Fuller [28])**.** A ring R is said to have *enough idempotents* if there exists a set of commuting, pairwise orthogonal idempotents $F \subseteq E(R)$ (called a *complete set of idempotents of* R) such that,

$$R = \bigoplus_{f \in F} Rf = \bigoplus_{f \in F} fR. \tag{2}$$

Remark 1.4.14. If R is unital, then we can take $F = \{1_R\}$ in Definition 1.4.13. Hence,

$$\text{unital} \implies \text{enough idempotents.}$$

The motivation behind the name of the concept is clear from Definition 1.4.13. In other contexts, the following characterization is more useful:

Proposition 1.4.15. A ring R has enough idempotents if and only if there exists a set of commuting, pairwise orthogonal idempotents $F \subseteq E(R)$ such that $\bigvee F$ is a set of local units of R.

Proof. Suppose that R has enough idempotents. In other words, let F be a set of commuting, pairwise orthogonal idempotents such that (2) holds. Let $r \in R$. By the first equality in (2),

$$r = r_1 f_1 + r_2 f_2 + \cdots + r_n f_n,$$

for some $r_1, \ldots, r_n \in R$ and $f_1, \ldots, f_n \in F$. Let,

$$f := f_1 \vee f_2 \vee \cdots \vee f_n = f_1 + f_2 + \cdots + f_n,$$

where the equality follows since F is assumed to consist of commuting, pairwise orthogonal idempotents. Moreover note that $f_i f_j = \delta_{i,j} f_i$ for all $i, j \in \{1, \ldots, n\}$. Hence,

$$rf = \Big(\sum_{i=1}^{n} r_i f_i\Big)\Big(\sum_{j=1}^{n} f_i\Big) = \sum_{i,j=1}^{n} r_i f_i f_j = \sum_{i=1}^{n} r_i f_i f_i = \sum_{i=1}^{n} r_i f_i = r.$$

Similarly, there exists some $f' \in \bigvee F$ such that $f'r = r$. Thus, $\bigvee F$ is a set of local units for R. This establishes the 'only if' direction.

Conversely, suppose that $F \subseteq E(R)$ is a set of commuting and pairwise orthogonal idempotents such that $\bigvee F$ is a set of local units of R. In other words, for every $r \in R$ there is some $f \in \bigvee F$ such that $fr = r = rf$. However, $f = f_1 + f_2 + \cdots + f_k$ for some $f_1, \ldots, f_k \in F$. Hence, $r = rf = rf_1 + rf_2 + \cdots + rf_k = fr = f_1 r + f_2 r + \cdots + f_k r$. This proves that $R = \sum_{f \in F} Rf = \sum_{f \in F} fR$. Next, we prove that these sums are direct. Suppose that $r = rf_1 + rf_2 + \cdots + rf_n$ and $r = rf_1' + \cdots + rf_m'$ for some $f_1, \ldots, f_n, f_1', \ldots, f_m' \in F$. Let $f := f_1 + \cdots + f_n$ and $f' := f_1' + \cdots + f_m'$. We assume without loss of generality that $n \geq m$ and $rf_i \neq 0$ and $rf_j' \neq 0$ for all $i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$. Seeking a contradiction, suppose that $f \neq f'$. Then there is some $i \in \{1, \ldots, n\}$ such that $f_i \neq f_j'$ for every $j \in \{1, \ldots, m\}$. Thus $ff_i = f_i$ but $f'f_i = 0$. Hence, $rf_i = r(ff_i) = (rf)f_i = (rf')f_i = r(f'f_i) = 0$, which contradicts the assumption that $rf_i \neq 0$. Thus, $R = \bigoplus_{f \in F} Rf$. Using a similar argument, we also get that $R = \bigoplus_{f \in F} fR$. Hence, R has enough idempotents. $\square$

Remark 1.4.16. For a general ring R, the following implications of properties hold:

unital $\Rightarrow$ enough idempotents $\Rightarrow$ set of local units $\Rightarrow$ s-unital $\Rightarrow$ idempotent.

We now give an example showing that there are non-unital rings with enough idempotents.

Example 1.4.17. Let $\{R_i\}_{i \in I}$ be a family of unital rings for some infinite index set I. Let 1_{R_i} denote the multiplicative identity element of the ring R_i for every $i \in I$. The external direct sum $R := \bigoplus_{i \in I} R_i$ becomes a ring with componentwise multiplication. Moreover, $R = \bigoplus_{i \in I} R 1_{R_i} = \bigoplus_{i \in I} 1_{R_i} R$. Hence, R has enough idempotents. On the other hand, since I is infinite, R does not admit a multiplicative identity element.

The following proposition characterizes the local unit properties of Leavitt path algebras. The proof in [2] generalizes verbatim to coefficients in a general unital ring.

Proposition 1.4.18 ([2, Lem. 1.2.12])**.** Let R be a unital ring and let E be a directed graph. Consider the Leavitt path algebra $L_R(E)$. The following assertions hold:

(a) The set $\{v \mid v \in E^0\}$ is a complete set of idempotents for $L_R(E)$. In other words, $L_R(E)$ has enough idempotents;

(b) $L_R(E)$ is a unital ring if and only if E has finitely many vertices. In this case,

$$1_{L_R(E)} = \sum_{v \in E^0} v.$$

We now give another example of a ring that has enough idempotents but is not unital:

$$\bullet_{v_1} \quad \bullet_{v_2} \quad \bullet_{v_3} \quad \bullet_{v_4} \quad \bullet_{v_5} \quad \bullet_{v_6} \quad \bullet_{v_7} \quad \bullet_{v_8} \quad \bullet_{v_9} \quad \bullet_{v_{10}} \cdots$$

FIGURE 7. Infinite graph.

Example 1.4.19. Let R be a unital ring and let E be the discrete graph with infinitely many vertices (see Figure 7). By Proposition 1.4.18, $L_R(E)$ is an example of a non-unital ring with enough idempotents.

Remark 1.4.20. There are more examples (see [40]) showing that the implications in Remark 1.4.16 can not be reversed. In other words,

$$\begin{aligned} \text{unital} &\not\Leftarrow \text{enough idempotents} \\ &\not\Leftarrow \text{set of local units} \\ &\not\Leftarrow \text{s-unital} \\ &\not\Leftarrow \text{idempotent.} \end{aligned}$$

1.5. Group graded rings

We are going to study rings equipped with additional structural information. The additional structure will be a direct sum decomposition that connects the ring multiplication with the group operation of a discrete group.

Definition 1.5.1. Let G be a group and let S be a ring. A G*-grading* of S is a family $\{S_g\}_{g \in G}$ of additive subgroups of S such that,

$$S = \bigoplus_{g \in G} S_g, \tag{3}$$

and $S_g S_h \subseteq S_{gh}$ for all $g, h \in G$. If the stronger condition $S_g S_h = S_{gh}$ holds for all $g, h \in G$, then we say that the grading is *strong*. If S is equipped with a G-grading, then S is called a G*-graded* ring. The subsets S_g are called the *homogeneous components* of the grading. The *principal component* is the component S_e associated to the neutral element $e \in G$. If $0 \neq r \in S_g$, for some $g \in G$, then r is said to be *homogeneous* and we write $\deg(r) = g$.

Note that strong G-gradings are a special type of G-gradings. We will later (see Section 1.6) consider epsilon-strong G-gradings, which similarly are a special type of G-gradings. We consider some examples before moving on:

Example 1.5.2 (The polynomial ring)**.** Let R be a ring and consider the polynomial ring $R[x]$ which is $\mathbb{Z}$-graded by putting $(R[X])_i = Rx^i$ for $i \geq 0$ and $(R[X])_i = \{0\}$ for $i < 0$. In other words, $\deg(rx^i) = i$ for all $i \geq 0$ and $0 \neq r \in R$. This gives a $\mathbb{Z}$-grading that is not strong.

Example 1.5.3 (The Laurent polynomial ring)**.** Let R be a ring. The Laurent polynomial ring $R[x, x^{-1}]$ is $\mathbb{Z}$-graded by putting $(R[x, x^{-1}])_i = Rx^i$ for $i \in \mathbb{Z}$. If R is idempotent, then this $\mathbb{Z}$-grading is strong.

Example 1.5.4 (Every ring is graded by the trivial group)**.** Let R be a ring and let $G = \{e\}$ be the trivial group. A grading of R is obtained by putting $(R)_e = R$. Hence, every ring R is graded by the trivial group. If R is idempotent, then it follows that $(R)_e(R)_e = R^2 = R = (R)_e$. In fact, a ring R is strongly graded by the trivial group if and only if R is idempotent.

Example 1.5.5 (The trivial grading)**.** Let G be a group and let R be a ring. Put $(R)_e = R$ and $(R)_g = \{0\}$ for every $g \neq e$. This gives a G-grading of R called *the trivial grading*. In other words, every ring is trivially graded by every group. Note that if G is the trivial group, then we obtain the grading considered in Example 1.5.4.

Note that a ring in general admits a multitude of different gradings. For example, we have seen that the Laurent polynomial ring is graded by the trivial group (see Example 1.5.4) and by $\mathbb{Z}$ in two different ways (see Example 1.5.3 and Example 1.5.5). This list is not exhaustive. There are more gradings of the Laurent polynomial ring (see e.g. Example 1.5.35). A grading should be viewed as some additional structural data that is provided on top of the ring structure. However, for many rings there is a natural choice for a group grading. For example, we consider the Laurent polynomial ring to be naturally $\mathbb{Z}$-graded by the grading in Example 1.5.3. We will later see that the Leavitt path algebras are naturally $\mathbb{Z}$-graded.

Remark 1.5.6. We make some further observations from Definition 1.5.1.

(a) Note that it is possible that $S_g = \{0\}$ for some $g \in G$. Given a G-grading of S, we define the *support* of the grading to be the set
$$\mathrm{Supp}(S) = \{g \in G \mid S_g \neq \{0\}\} \subseteq G.$$
As illustrated by Example 1.5.2, $\mathrm{Supp}(S)$ need not be a subgroup of G.

(b) Equation (3) implies that every element $s \in S$ decomposes uniquely as a finite sum $s = \sum_{g \in G} s_g$ where $s_g \in S_g$ are homogeneous elements (cf. Definition 1.4.12). We define the *support* of an element $s \in S$ to be the set $\mathrm{Supp}(s) = \{g \in G \mid s_g \neq 0\}$.

(c) The relation $S_g S_h \subseteq S_{gh}$ expresses that the multiplication of homogeneous elements is compatible with the group operation.

The group ring is an important example of a strongly graded ring.

Example 1.5.7. Let G be a group and let R be a unital ring. Let $\{\delta_g \mid g \in G\}$ be a copy of G (as a set) where the δ_g's are formal symbols. The group ring $R[G]$ is defined to be the free left R-module with basis $\{\delta_g \mid g \in G\}$. In other words, $R[G]$ consists of elements of the form $\sum r_g \delta_g$ where $r_g \in R$ and the sum is finite. We immediately get a decomposition $R[G] = \bigoplus_{g \in G} S_g$ by putting $S_g := R\delta_g$ for every $g \in G$. Moreover, a multiplication is defined on $R[G]$ by linearly extending the rule $r\delta_g \cdot r'\delta_h = rr'\delta_{gh}$ for all $g, h \in G$ and $r, r' \in R$. This turns $R[G]$ into an associative and unital ring. Furthermore, it follows immediately from the definition of the multiplication that $S_g S_h = S_{gh}$ for all $g, h \in G$. Hence, $R[G]$ is strongly G-graded.

The following result will be useful later:

Proposition 1.5.8. Let G be a group with neutral element e and let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. Then the following assertions hold:
(a) The principal component S_e is a subring of S;
(b) The homogeneous component S_g is an S_e-bimodule for each $g \in G$;
(c) If S is non-trivial and strongly G-graded, then $\mathrm{Supp}(S) = G$.

PROOF. (a): Note that $S_e S_e \subseteq S_e$. Hence, S_e is a subring of S.

(b): Note that $S_e S_g \subseteq S_{eg} = S_g$ and $S_g S_e \subseteq S_{ge} = S_g$. Furthermore, for $r_1, r_2 \in S_e$ and $s_g \in S_g$, we have $(r_1 s_g) r_2 = r_1 (s_g r_2)$. Hence, S_g is an S_e–S_e-bimodule.

(c): Seeking a contradiction, suppose that there is some $g \in G$ such that $g \notin \mathrm{Supp}(S)$. It then follows that $S_e = S_g S_{g^{-1}} = \{0\}$. Hence, for any $h \in H$, we have $S_h = S_e S_h = \{0\}$. Thus, $S = \{0\}$, which contradicts our assumption that S is a non-trivial ring. $\square$

If S is a unital ring, then we can say even more:

Proposition 1.5.9 ([39, Prop. 1.1.1])**.** Let G be a group with neutral element e and let $S = \bigoplus_{g \in G} S_g$ be a unital G-graded ring. Then the following assertions hold:
(a) The principal component S_e is a unital subring of S, i.e. $1_S \in S_e$;
(b) The grading $\{S_g\}_{g \in G}$ is strong if and only if $1_S \in S_g S_{g^{-1}}$ for each $g \in G$ if and only if $S_e = S_g S_{g^{-1}}$ for each $g \in G$;
(c) If $r \in S_g$ is an invertible element, then $r^{-1} \in S_{g^{-1}}$.

Remark 1.5.10. We make two remarks about the terminology before moving forward.
(a) Unless otherwise stated, G will be an arbitrary group. With the phrase 'let S be a G-graded ring', we mean that S is an arbitrary ring that comes equipped with a fixed but arbitrary G-grading.
(b) We will use the term 'unital strongly G-graded' for strongly G-graded rings equipped with a non-zero multiplicative identity element. There are strongly G-graded rings which do not admit a multiplicative identity element (see Example 1.6.10).

1.5.1. The canonical $\mathbb{Z}$-grading of Leavitt path algebras. In this section, we recall that every Leavitt path algebra (see Definition 1.3.2) is naturally $\mathbb{Z}$-graded.

Let $E = (E^0, E^1, s, r)$ be a directed graph. A *path* is a series of edges $f_1 f_2 \dots f_n$ such that $r(f_i) = s(f_{i+1})$ for $1 \leq i \leq n-1$. The set of all paths in E is denoted by $\mathrm{Path}(E)$. The *length* of a path is defined by $\mathrm{len}(f_1 f_2 \dots f_n) = n$. We let the source and range of a path be defined by $s(f_1 \dots f_n) = s(f_1)$ and $r(f_1 \dots f_n) = r(f_n)$. By convention, we consider a single vertex $v \in E^0$ to be a path of length 0. A *ghost path* is a series of ghost edges $f_n^* f_{n-1}^* \dots f_1^*$ such that $r(f_i) = s(f_{i+1})$ for $1 \leq i \leq n-1$. If $\alpha = f_1 f_2 \dots f_n$ is a real path, then we write $\alpha^* = f_n^* f_{n-1}^* \dots f_1^*$ for the corresponding ghost path. By convention, we have $s(\alpha^*) = r(\alpha)$ and $r(\alpha^*) = s(\alpha)$. The elements on the form $\alpha\beta^*$, for $\alpha, \beta \in \mathrm{Path}(E)$, are called *monomials*.

Proposition 1.5.11 ([2, Lem. 1.2.12(ii)])**.** The R-algebra $L_R(E)$ is spanned by the following set of monomials:

$$\{\alpha\beta^* \mid \alpha, \beta \in \mathrm{Path}(E), r(\alpha) = r(\beta)\}. \tag{4}$$

In other words, every element in $L_R(E)$ may be written as a finite sum $\sum r_i \alpha_i \beta_i^*$ where $r_i \in R$ and $\alpha_i, \beta_i \in \mathrm{Path}(E)$. In general, this decomposition is not unique, i.e. the set (4) is not a basis for $L_R(E)$ over R. Next, let $\alpha, \beta \in \mathrm{Path}(E)$ such that $r(\alpha) \neq r(\beta)$. Then, by Definition 1.3.2,

$$\alpha\beta^* = (\alpha r(\alpha))(s(\beta^*)\beta^*) = \alpha r(\alpha) r(\beta)\beta^* = \alpha(r(\alpha)r(\beta))\beta^* = 0.$$

We recall the following definition:

Definition 1.5.12. Let R be a unital ring and let E be a directed graph. The *canonical $\mathbb{Z}$-grading* of $L_R(E)$ is defined by,

$$(L_R(E))_n = \mathrm{Span}_R\{\alpha\beta^* \mid \alpha, \beta \in \mathrm{Path}(E), \mathrm{len}(\alpha) - \mathrm{len}(\beta) = n\},$$

for every $n \in \mathbb{Z}$.

By Proposition 1.5.11, it follows that $L_R(E) = \bigoplus_{i \in \mathbb{Z}} (L_R(E))_i$. Furthermore, by the monomial multiplication formula (see [2, Lem. 1.2.12(i)]), it follows that,

$$(L_R(E))_i (L_R(E))_j \subseteq (L_R(E))_{i+j},$$

for all $i, j \in \mathbb{Z}$. Hence, this gives a $\mathbb{Z}$-grading of $L_R(E)$. Next, we give some examples of what the canonical $\mathbb{Z}$-grading looks like.

Example 1.5.13. Let R be a unital ring and let A_2 be the directed graph in Figure 5 (see also Example 1.3.6). In this case we see that,

$$(L_R(A_2))_0 = \mathrm{Span}_R\{v_1, v_2\}, \; (L_R(A_2))_1 = \mathrm{Span}_R\{f\}, (L_R(A_2))_{-1} = \mathrm{Span}_R\{f^*\}.$$

For $|i| > 0$, we see that $(L_R(A_2))_i = \{0\}$. Note that the grading has finite support.

Example 1.5.14. Let R be a unital ring and let E_1 be the graph given in Figure 4 (the rose with one petal). In this case:

$$\begin{aligned}
&(L_R(E_1))_0 = \mathrm{Span}_R\{v\}, \\
&(L_R(E_1))_i = \mathrm{Span}_R\{f^i\}, \; (L_R(E_1))_{-i} = \mathrm{Span}_R\{(f^*)^i\}, \qquad i > 0.
\end{aligned}$$

Note that the grading has infinite support.

We mention here that Hazrat [30] gave a criterion on the finite graph E for the Leavitt path algebra $L_R(E)$ to be strongly $\mathbb{Z}$-graded (see Theorem 1.5.16). We first recall the following definitions:

Definition 1.5.15. Let $E = (E^0, E^1, s, r)$ be a directed graph.

(a) The graph E is *finite* if E^0 and E^1 are finite sets.

(b) For an arbitrary $v \in E^0$, the set $s^{-1}(v) = \{e \in E^1 \mid s(e) = v\}$ is the set of edges which are emitted from v. If $s^{-1}(v) = \emptyset$, then v is called a *sink*. If $s^{-1}(v)$ is an infinite set, then v is called an *infinite emitter*.
(c) For an arbitrary $v \in E^0$, the set $r^{-1}(v) = \{e \in E^1 \mid r(e) = v\}$ is the set of edges which are received by v. If $r^{-1}(v) = \emptyset$, then v is called a *source*.

The following characterization appeared in Figure 2:

Theorem 1.5.16 (Hazrat [30, Thm. 3.11])**.** Let R be a unital ring and let E be a finite directed graph. The canonical $\mathbb{Z}$-grading of $L_R(E)$ is strong if and only if E does not have a sink.

Remark 1.5.17. A result similar to Theorem 1.5.16 was proved for graph C^*-algebras by Szymanski [53] already in 2002. Hazrat appears to have been unaware of Szymanski's paper when he proved Theorem 1.5.16 in 2010. Szymanski, in particular, proved that the gauge action of the graph C^*-algebra attached to the graph E is free (analogous to the canonical $\mathbb{Z}$-grading being strong) if the following assertions hold:
(a) E is row-finite (meaning that every vertex only emits finitely many edges);
(b) E contains no sinks;
(c) E contains no sources.

By the discussion in [13, Sect. 2.3], condition (c) is not necessary for the gauge action of the graph C^*-algebra to be free. Moreover, Hazrat [30, Expl. 3.13] gave an example that implies that (c) is not a necessary condition for the canonical $\mathbb{Z}$-grading of a Leavitt path algebras to be strong (see Example 1.6.12 and Figure 9).

Remark 1.5.18. Note that Theorem 1.5.16 only holds for Leavitt path algebras associated to finite graphs. Recently, Lundström and Öinert [37] gave a complete characterization of when the canonical $\mathbb{Z}$-grading of a Leavitt path algebra is strong.

Example 1.5.19. Since the graph in Example 1.5.13 has a sink, it follows from Theorem 1.5.16 that the canonical $\mathbb{Z}$-grading of $L_R(E)$ is not strong. Indeed, we saw in Example 1.5.13 that $L_R(E)$ has finite support which by Proposition 1.5.8(c) implies that $L_R(E)$ cannot be strongly $\mathbb{Z}$-graded.

The canonical $\mathbb{Z}$-grading is just a special case of a more general method of assigning a grading to the Leavitt path algebra $L_R(E)$. Let G be a group with neutral element e. Put $\deg(v) = e$ for each $v \in E^0$. For every $f \in E^1$, choose a $g \in G$ and put $\deg(f) = g$ and $\deg(f^*) = g^{-1}$. This gives a G-grading of $L_R(E)$ (see [41, Sect. 4]). This G-grading is called a *standard G-grading* of $L_R(E)$. Note that this construction depends on which elements $g \in G$ we assign to the generators of $L_R(E)$. For $G = \mathbb{Z}$, the natural choice is to put $\deg(v) = 0$, $\deg(f) = 1$ and $\deg(f^*) = -1$ for all $v \in E^0$ and $f \in E^1$. In this special case, $\deg(\alpha) = \mathrm{len}(\alpha)$ for any real path α. The resulting grading is the canonical $\mathbb{Z}$-grading of $L_R(E)$.

1.5.2. The category of graded rings. Let G be a group. We now consider the category of G-graded rings: G-RING. The objects are pairs $(S, \{S_g\}_{g\in G})$ where S is a ring and $\{S_g\}_{g\in G}$ is a G-grading of S.

E_2 :

E_3 :

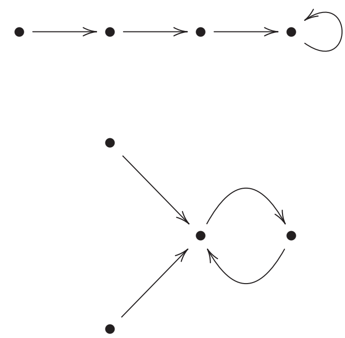

FIGURE 8. Ring isomorphism vs. graded isomorphism (see [30, p. 1]).

Definition 1.5.20. Let G be a group and let $S = \bigoplus_{g\in G} S_g$ and $T = \bigoplus_{g\in G} T_g$ be G-graded rings. A ring homomorphism such that,

$$\phi(S_g) \subseteq T_g \qquad \forall g \in G,$$

is called a *G-graded homomorphism.*

The maps in the category of G-RING are exactly the G-graded ring homomorphisms. In other words, $\phi \in \mathrm{Hom}((S, \{S_g\}_{g\in G}), (T, \{T_g\}_{g\in G}))$ if and only if $\phi\colon S \to T$ is a ring homomorphism satisfying $\phi(S_g) \subseteq T_g$ for each $g \in G$. If $\phi\colon (S, \{S_g\}_{g\in G}) \to (T, \{T_g\}_{g\in G})$ is an isomorphism, we write $S \cong_{\mathrm{gr}} T$ and say that S and T are *graded isomorphic.* Note that $S \cong_{\mathrm{gr}} T$ implies that $S \cong T$ but the reverse implication does not hold in general. Hazrat [30] gave the following example of two Leavitt path algebras which are isomorphic as rings but not as $\mathbb{Z}$-graded rings:

Example 1.5.21 ([30, p. 1])**.** Consider the graphs in Figure 8. It can be shown that $L_R(E_2) \cong L_R(E_3)$ as rings, but $L_R(E_2) \not\cong_{\mathrm{gr}} L_R(E_3)$ when considered with the canonical $\mathbb{Z}$-grading. In other words, $L_R(E_2)$ and $L_R(E_3)$ are isomorphic in the category of rings but different in $\mathbb{Z}$-RING. This example illustrates the power of considering the additional graded structure that is present.

Next, we give an example of a $\mathbb{Z}$-graded ring isomorphism:

Example 1.5.22. Let R be a unital ring and let A_2 be the graph in Figure 5. Moreover, let $M_2(R)$ denote the ring of 2×2-matrices with coefficients in R. Let m_1, m_2, m_3, m_4

be the matrices defined in Example 1.3.6. We can define a $\mathbb{Z}$-grading of $M_2(R)$ by:

$$\begin{aligned}
(M_2(R))_0 &= \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} = \left\{ \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \mid r_1, r_2 \in R \right\} = \mathrm{Span}_R\{m_1, m_2\}, \\
(M_2(R))_1 &= \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \mid r \in R \right\} = \mathrm{Span}_R\{m_3\}, \\
(M_2(R))_{-1} &= \begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix} \mid r \in R \right\} = \mathrm{Span}_R\{m_4\}, \\
(M_2(R))_i &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \qquad |i| > 1.
\end{aligned}$$

A routine check shows that this is in fact a $\mathbb{Z}$-grading. Consider $M_2(R)$ equipped with the $\mathbb{Z}$-grading above and consider $L_R(A_2)$ equipped with the canonical $\mathbb{Z}$-grading (see Example 1.5.13). We now show that the ring isomorphism $\phi\colon M_2(R) \xrightarrow{\sim} L_R(A_2)$ given in Example 1.3.6 is a $\mathbb{Z}$-graded ring isomorphism. Recall that $\phi(m_1) = v_1$, $\phi(m_2) = v_2$, $\phi(m_3) = f$ and $\phi(m_4) = f^*$. It follows that (see Example 1.5.13):

$$\begin{aligned}
\phi((M_2(R))_0) &= \mathrm{Span}_R\{\phi(m_1), \phi(m_2)\}) = \mathrm{Span}_R\{v_1, v_2\} = (L_R(A_2))_0, \\
\phi((M_2(R))_1) &= \mathrm{Span}_R\{\phi(m_3)\}) = \mathrm{Span}_R\{f\} = (L_R(A_2))_1, \\
\phi((M_2(R))_{-1}) &= \mathrm{Span}_R\{\phi(m_4)\}) = \mathrm{Span}_R\{f^*\} = (L_R(A_2))_{-1}.
\end{aligned}$$

Thus, ϕ is a $\mathbb{Z}$-graded ring isomorphism and $M_2(R) \cong_{\mathrm{gr}} L_R(A_2)$.

Note that we will often simply talk about a 'G-graded ring' by which we will mean an object in the category G-RING. Moreover, we will often let the grading on a particular ring be implicitly defined. This has the potential of becoming confusing. We will therefore use the more explicit notation $(S, \{S_g\}_{g \in G})$ on occasion to avoid confusion. This notation will be especially useful in Paper A.

1.5.3. Algebraic crossed products. The prototype example for this section is the skew group ring which generalizes the group ring (see Example 1.5.7).

Example 1.5.23 (The skew group ring)**.** Let R be a unital ring, let G be a group and let $\gamma\colon G \to \mathrm{Aut}(R)$ be a group homomorphism. The *skew group ring* $R \star_\gamma G$ has the same additive structure as the group ring $R[G]$. In other words, $R \star_\gamma G$ is the free left R-module with basis $\{\delta_g \mid g \in G\}$ where the δ_g's are formal symbols. The multiplication is defined by linearly extending the following rule: for $g, h \in G$ and $a, b \in R$, we define

$$(a\delta_g)(b\delta_h) = a\gamma(g)(b)\delta_{gh}.$$

In other words, we are 'skewing' the multiplication with a group action. Note that if we let $\gamma(g) = \mathrm{id}_R$ for every $g \in G$, then we get back the group ring. With this multiplication, $R \star_\gamma G$ becomes an associative and unital ring.

In the theory of C^*-algebras, the crossed product $A \rtimes G$ is a C^*-algebra associated to a group G acting on a C^*-algebra A (see e.g. [10, Sect. 10]). The skew group ring (see Example 1.5.23) can be seen as the algebraic analogue of the C^*-crossed product. The *(algebraic) crossed product* can be defined by considering so-called crossed systems. The precise definition of a crossed system (see e.g. [39, Sect. 1.4]) is a bit technical and therefore omitted. Luckily, the following definition of crossed products is more direct and equivalent to considering crossed systems (see [39, Prop. 1.4.2]).

Definition 1.5.24. Let $S = \bigoplus_{g \in G} S_g$ be a unital G-graded ring. If S_g contains an invertible element for every $g \in G$, then S is called a *crossed product*.

Special cases of crossed products include group rings (see Example 1.5.7), skew group rings (see Example 1.5.23) and twisted group rings (see e.g. [39, p. 12]).

Example 1.5.25. Recall that the group ring $R[G] = \bigoplus_{g \in G} R\delta_g$ is strongly G-graded and has multiplicative identity element $1_R\delta_e$. For every $g \in G$, note that $\delta_g \in (R[G])_g = R\delta_g$ is invertible since $\delta_g \delta_{g^{-1}} = \delta_e = 1_R \delta_e$. Thus, $R[G]$ is a crossed product.

The crossed products constitute a subclass of the strongly graded rings:

Proposition 1.5.26. If $S = \bigoplus_{g \in G} S_g$ is a crossed product, then S is strongly G-graded.

Proof. By definition, S is unital. It follows by Proposition 1.5.9(b) that it is enough to show that $1_S \in S_g S_{g^{-1}}$ for every $g \in G$. Take an arbitrary $g \in G$. By definition, there is some invertible element $r \in S_g$. By Proposition 1.5.9(c), it follows that $r^{-1} \in S_{g^{-1}}$. Thus, $1_S = rr^{-1} \in S_g S_{g^{-1}}$. $\square$

Remark 1.5.27. The category of strongly G-graded rings is denoted by G-STRG and the category of algebraic crossed products is denoted by G-CROSS. Note that G-CROSS $\subseteq$ G-STRG $\subseteq$ G-RING as categories. Furthermore, G-CROSS and G-STRG are full subcategories of G-RING.

Dade was the first to give an example of a strongly graded ring that is not an algebraic crossed product.

Example 1.5.28 (cf. [23, Expl. 2.9(ii)])**.** We define a $\mathbb{Z}/2\mathbb{Z}$-grading of the matrix ring $M_3(\mathbb{C})$. Put,

$$S_0 = \begin{pmatrix} \mathbb{C} & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} & 0 \\ 0 & 0 & \mathbb{C} \end{pmatrix}, \qquad S_1 = \begin{pmatrix} 0 & 0 & \mathbb{C} \\ 0 & 0 & \mathbb{C} \\ \mathbb{C} & \mathbb{C} & 0 \end{pmatrix}.$$

It can be checked that this gives a strong $\mathbb{Z}/2\mathbb{Z}$-grading of $M_3(\mathbb{C})$. However, since S_1 only contains singular matrices, $M_3(\mathbb{C}) = S_0 \oplus S_1$ is not a crossed product. Hence,

$$\mathbb{Z}/2\mathbb{Z}\text{-CROSS} \subsetneq \mathbb{Z}/2\mathbb{Z}\text{-STRG}.$$

The above example by Dade can be considered to be somewhat exotic. The class of strongly graded rings has historically been referred to as 'generalized crossed products' (see e.g. [56]) because of the importance of the subclass of crossed products. However, in the context of Leavitt path algebras, the situation is different. Indeed, there are a lot of natural examples of strongly graded Leavitt path algebras that are not crossed products.

Example 1.5.29 ([30, Expl. 4.21])**.** Let R be a unital ring and consider the graph E_3 in Figure 8. It can be shown that $L_R(E_3)$ is strongly $\mathbb{Z}$-graded but not a crossed product.

1.5.4. Functors on the category of graded rings. We describe two important functors defined on the category G-RING. The second of these functors will be studied further in Paper A. Let G be a group and let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. For a subset X of G, put $S_X := \bigoplus_{g \in X} S_g$.

Lemma 1.5.30. If H is a subgroup of G, then S_H is a subring of S and

$$(S_H, \{S_h\}_{h \in H}) \in H\text{-RING}.$$

Proof. By definition, $(S_H, +, 0)$ is an abelian subgroup of $(S, +, 0)$. Take some arbitrary elements $a, b \in S_H$. Then, $a = \sum_{s \in H} a_s$ and $b = \sum_{t \in H} b_t$ for some finite sums. Note that $ab = \sum_{s \in H, t \in H} a_s b_t \in S_H$ because $a_s b_t \in S_s S_t \subseteq S_{st}$ and $st \in H$. Thus, S_H is a subring of S and $\{S_h\}_{h \in H}$ is an H-grading of S_H. $\square$

Let $\phi\colon S \to T$ be a G-graded ring homomorphism and define a map $\phi_H\colon S_H \to T_H$ by putting $\phi_H(s) := \phi(s)$ for every $s \in S_H$. Note that, in particular, $\phi_H(S_h) \subseteq T_h$ for every $h \in H$. In other words, ϕ_H is an H-graded ring homomorphism. This means the following definition makes sense:

Definition 1.5.31. Let G be a group and let H be a subgroup of G. Define a functor by,

$$\begin{aligned} (-)_H\colon G\text{-RING} &\to H\text{-RING}, \\ (S, \{S_g\}_{g \in G}) &\mapsto (S_H, \{S_g\}_{g \in H}), \\ \mathrm{Hom}((S, \{S_g\}_{g \in G}), (T, \{T_g\}_{g \in G})) \ni \phi &\mapsto \phi_H \in \mathrm{Hom}((S_H, \{S_g\}_{g \in H}), (T_H, \{T_g\}_{g \in H})). \end{aligned}$$

Example 1.5.32. Let S be a $\mathbb{Z}$-graded ring. We take $G = \mathbb{Z}$ and $H = \{0\}$ in the above definition. Note that in this case, the image of S under the functor $(-)_H$ is the principal component S_0 equipped with the trivial group grading (see Example 1.5.4).

We now describe the second functor. Let G be a group and let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. For a normal subgroup N of G, we will see that there is a naturally defined G/N-grading of S. This construction is called the *induced quotient group grading* of S. For each class $C \in G/N$, put,

$$S_C := \bigoplus_{g \in C} S_g.$$

Lemma 1.5.33. The family $\{S_C\}_{C \in G/N}$ is a G/N-grading of S. In other words,

$$(S, \{S_C\}_{C \in G/N}) \in G/N\text{-RING}.$$

Proof. It is clear that $S = \bigoplus_{C \in G/N} S_C$. It remains to show that $S_C S_{C'} \subseteq S_{CC'}$ for all $C, C' \in G/N$. Take two arbitrary classes $C, C' \in G/N$ and two arbitrary elements $g \in C, g' \in C'$. Then $S_g S_{g'} \subseteq S_{gg'} \subseteq S_{CC'}$ because $gg' \in CC'$. Hence, $S_C S_{C'} \subseteq S_{CC'}$. Thus, $(S, \{S_C\}_{C \in G/N}) \in G/N$-RING. $\square$

Let $\phi\colon S \to T$ be a G-graded ring homomorphism. Then, $\phi(S_g) \subseteq T_g$ for every $g \in G$. Since ϕ is a ring homomorphism, it follows that $\phi(S_C) \subseteq T_C$ for every class $C \in G/N$. Hence, ϕ is G/N-graded with respect to the induced G/N-grading. Now, we define the second functor:

Definition 1.5.34. Let G be a group and let N be a normal subgroup of G. We define a functor by,

$$U_{G/N}\colon G\text{-RING} \to G/N\text{-RING},$$
$$(S, \{S_g\}_{g\in G}) \mapsto (S, \{S_C\}_{C\in G/N}),$$
$$\mathrm{Hom}((S, \{S_g\}_{g\in G}), (T, \{T_g\}_{g\in G})) \ni \phi \mapsto \phi \in \mathrm{Hom}((S, \{S_C\}_{C\in G/N}), (T, \{T_C\}_{C\in G/N}))$$

Note that $U_{G/N}$ maps the underlying ring to itself but equipped with a different grading.

Example 1.5.35. Let R be a ring and consider the Laurent polynomial ring $R[x, x^{-1}]$ with its standard $\mathbb{Z}$-grading. That is,

$$R[x, x^{-1}] = \bigoplus_{i\in\mathbb{Z}} Rx^i. \tag{5}$$

Consider the quotient group $\mathbb{Z}/2\mathbb{Z}$ of $\mathbb{Z}$. The image of (5) under the functor $U_{\mathbb{Z}/2\mathbb{Z}}$ is,

$$R[x, x^{-1}] = \Big(\bigoplus_{i\in 2\mathbb{Z}} Rx^i\Big) \oplus \Big(\bigoplus_{i\in 1+2\mathbb{Z}} Rx^i\Big) = S_{[0]} \oplus S_{[1]},$$

where $[0]$ denotes the class $0 + 2\mathbb{Z}$ and $[1]$ denotes the class $1 + 2\mathbb{Z}$.

It is natural to ask if the induced quotient group ring of a strongly graded ring is in turn strongly graded. We recall that this property holds for both strongly graded rings and crossed products.

Proposition 1.5.36. Let G be a group and let N be a normal subgroup of G. The functor $U_{G/N}$ restricts to the subcategories G-STRG and G-CROSS. More precisely, the following assertions hold:

(a) If $(S, \{S_g\}_{g\in G}) \in G$-STRG, then $U_{G/N}((S, \{S_g\}_{g\in G})) \in G/N$-STRG.
(b) If $(S, \{S_g\}_{g\in G}) \in G$-CROSS, then $U_{G/N}((S, \{S_g\}_{g\in G})) \in G/N$-CROSS.

Proof. (a): It is enough to show that $S_{CC'} \subseteq S_C S_{C'}$ for all classes $C, C' \in G/N$. Take two arbitrary classes $C, C' \in G/N$. Any element in the coset CC' can be written as a product gh where $g \in C$ and $h \in C'$. By the assumption that S is strongly G-graded, we get that $S_{gh} = S_g S_h \subseteq S_C S_{C'}$. Thus, $S_{CC'} \subseteq S_C S_{C'}$.

(b): Assume that $(S, \{S_g\}_{g\in G}) \in G$-CROSS. Take an arbitrary $g \in G$. Then there are some elements $a \in S_g$ and $b \in S_{g^{-1}}$ such that $ab = 1_S$. Let $[g]$ denote the class of g in G/N. Recall that $[g^{-1}] = [g]^{-1}$. It follows that $a \in S_g \subseteq S_{[g]}$ and $b \in S_{g^{-1}} \subseteq S_{[g^{-1}]} = S_{[g]^{-1}}$. This proves that $(S, \{S_C\}_{C\in G/N}) \in G/N$-CROSS. □

In Paper A, we will see that the property in Proposition 1.5.36 does not hold for general epsilon-strongly graded rings.

1.5.5. Graded rings with involution and symmetrically graded rings. In this section, we consider the important class of symmetrically graded rings. These were initially introduced in the study of Steinberg algebras (see [14]). We use the concept of a graded ring with involution to show that both Leavitt path algebras and Steinberg algebras are symmetrically graded.

Definition 1.5.37 ([14, Def. 4.5])**.** Let G be a group and let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. If $S_g = S_g S_{g^{-1}} S_g$ for every $g \in G$, then we say that S is *symmetrically G-graded*.

We give some immediate examples:

Example 1.5.38. Let R be an idempotent ring and consider the standard $\mathbb{Z}$-grading of the polynomial ring $R[x] = \bigoplus_{i \in \mathbb{Z}} S_i$ by putting $S_i = Rx^i$ for $i \geq 0$ and $S_i = \{0\}$ for $i < 0$. Note that $S_1 S_{-1} S_1 = \{0\} \neq S_1$, hence the grading is not symmetrical. On the other hand, it is straightforward to show that the Laurent polynomial ring with the standard $\mathbb{Z}$-grading $R[x, x^{-1}] = \bigoplus_{i \in \mathbb{Z}} Rx^i$ is symmetrically $\mathbb{Z}$-graded.

Remark 1.5.39. Assume that S is a symmetrically G-graded ring. If $S_g = \{0\}$ for some $g \in G$, then $S_{g^{-1}} = S_{g^{-1}} S_g S_{g^{-1}} = \{0\}$. In other words,

$$g \in \mathrm{Supp}(S) \iff g^{-1} \in \mathrm{Supp}(S).$$

We will give more interesting examples later on. For now, we consider the notion of a graded ring with involution:

Definition 1.5.40 ([29, Sect. 1.9])**.** Let G be a group and let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. Let $*\colon S \to S$ be an involution of S, i.e. an anti-isomorphism of order 2. If $a \in S_g \implies a^* \in S_{g^{-1}}$ for all $g \in G$ and $a \in S_g$, then S is called a *graded $*$-ring* and $*\colon S \to S$ is called a *graded involution*.

The group rings and Leavitt path algebras are examples of graded $*$-rings we have already encountered:

Example 1.5.41. Let G be a group and let R be a unital ring. Consider the additive map $*\colon R[G] \to R[G]$ defined by R-linearly extending the rules $\delta_g \mapsto \delta_{g^{-1}}$ for each $g \in G$. Then, for all $g, h \in G$,

$$(\delta_g \delta_h)^* = (\delta_{gh})^* = \delta_{(gh)^{-1}} = \delta_{h^{-1}g^{-1}} = \delta_{h^{-1}} \delta_{g^{-1}} = (\delta_h)^* (\delta_g)^*.$$

It follows that $*$ is a graded involution. Thus, $R[G]$ is a G-graded $*$-ring.

Example 1.5.42. Let R be a unital ring and let E be a directed graph. Consider the Leavitt path algebra $L_R(E)$ equipped with the canonical $\mathbb{Z}$-grading. For every path $\alpha = f_1 f_2 \dots f_n$ there is a ghost path $\alpha^* = f_n^* f_{n-1}^* \dots f_1^*$. It is clear that $(\alpha^*)^* = \alpha$ and furthermore $(\alpha\beta)^* = \beta^* \alpha^*$ for all paths α and β. Let $i \in \mathbb{Z}$ be an arbitrary integer. If $\alpha\beta^* \in (L_R(E))_i$, then $\mathrm{len}(\alpha) - \mathrm{len}(\beta) = i$. This means that $(\alpha\beta^*)^* = \beta\alpha^* \in (L_R(E))_{-i}$ since $\mathrm{len}(\beta) - \mathrm{len}(\alpha) = -(\mathrm{len}(\alpha) - \mathrm{len}(\beta)) = -i$. Hence we get a graded involution by R-linearly extending the rules $\alpha\beta^* \mapsto \beta\alpha^*$ for all $\alpha, \beta \in \mathrm{Path}(E)$. In other words, $L_R(E)$ is a $\mathbb{Z}$-graded $*$-ring.

The *Steinberg algebra* associated to a topological groupoid $\mathcal{G}$ was introduced by Steinberg [52] and independently by Clark, Farthing, Sims, and Tomforde [15]. Notably, the Leavitt path algebra $L_R(E)$ is graded isomorphic to the Steinberg algebra of a groupoid associated to the graph E (see e.g. [50]). Note also that the Steinberg algebras are the algebraic analogues of groupoid C^*-algebras (see [49]). We briefly recall the construction of the Steinberg algebra. For more details we refer the reader to a survey article by Rigby [50]. Recall that a *groupoid* $\mathcal{G}$ is a small category in which every morphism is an isomorphism. The *unit space* $\mathcal{G}^{(0)}$ can be identified with the objects of this category. We also have maps $d\colon \mathcal{G} \to \mathcal{G}^{(0)}$ and $c\colon \mathcal{G} \to \mathcal{G}^{(0)}$ specifying the domain and codomain respectively for each morphism in $\mathcal{G}$. Now, let $\mathcal{G}$ be a topological groupoid. An *open bissection* of $\mathcal{G}$ is an open set $U \subseteq \mathcal{G}$ such that $d|U$ and $c|U$ are continuous maps. A topological groupoid $\mathcal{G}$ is called *ample Hausdorff* (see [50, Def. 1.5]) if the topology on $\mathcal{G}$ is Hausdorff and has a base consisting of open compact bissections. Let R be a unital ring and let $\mathcal{G}$ be an ample Hausdorff groupoid. The *Steinberg algebra* $A_R(\mathcal{G})$ is the R-linear span of characteristic functions $1_B : \mathcal{G} \to R$ where B is a compact open bissection of $\mathcal{G}$ (see [15, Lem. 3.4]). The multiplication is given by convolution. Let G be an arbitrary discrete group which we may consider as a groupoid with a single object. A continuous homomorphism $c\colon \mathcal{G} \to G$ is called a *cocycle*. Given a cocycle, it is possible to define a so-called G-grading of $\mathcal{G}$ by putting $\mathcal{G}_g := c^{-1}(g)$ for each $g \in G$. We get that $\mathcal{G} = \bigsqcup_{g \in G} \mathcal{G}_g$ and $\mathcal{G}_g \mathcal{G}_h \subseteq \mathcal{G}_{gh}$ for all $g, h \in G$ (cf. [16, p. 1]). By [14, Prop. 5.1], the G-grading of $\mathcal{G}$ induces a G-grading of $A_R(\mathcal{G})$. More precisely, given a G-graded ample Hausdorff groupoid there is a natural G-grading of the Steinberg algebra $A_R(\mathcal{G})$ with homogeneous components,

$$A_R(\mathcal{G})_g = \{f \in A_R(\mathcal{G}) \mid f(\gamma) \neq 0 \implies c(\gamma) = g\} = A_R(\mathcal{G}_g), \tag{6}$$

for every $g \in G$.

We prove that these types of Steinberg algebras are group graded $*$-rings.

Proposition 1.5.43. Let R be a unital ring and let $\mathcal{G}$ be an ample Hausdorff G-graded groupoid. Then the Steinberg algebra $A_R(\mathcal{G})$ is a G-graded $*$-ring.

Proof. The Steinberg algebra $A_R(\mathcal{G})$ is equipped with an involution $f \mapsto f^*$ defined by,

$$f^*(\gamma) = f(\gamma^{-1}),$$

for all $f \in A_R(\mathcal{G})$ and $\gamma \in \mathcal{G}$ (see [50, p. 12] and cf. [52, Prop. 3.6]). Furthermore, take an arbitrary $g \in G$ and let $f \in (A_R(\mathcal{G}))_g$. Then, by (6), we have $\mathrm{Supp}(f) \subseteq \mathcal{G}_g$. By the definition of the involution, we see that $\mathrm{Supp}(f^*) = \{\gamma^{-1} \mid \gamma \in \mathrm{Supp}(f)\}$. Let $c\colon \mathcal{G} \to G$ be the underlying cocycle defining the G-grading of $\mathcal{G}$. Take an arbitrary $\gamma \in \mathrm{Supp}(f)$ and suppose that $\gamma\colon x \to y$ where $x, y \in \mathrm{ob}(\mathcal{G})$. Then $\gamma^{-1}\colon y \to x$ and $\gamma^{-1} \circ \gamma = \mathrm{id}_x$. Moreover, since c is a functor, $c(\gamma^{-1}) \circ c(\gamma) = c(\mathrm{id}_x) = \mathrm{id}_{c(x)}$. But since G is a group it only contains a single object. This means that $\mathrm{id}_{c(x)}$ is the neutral element of the group. Furthermore, we have by assumption that $c(\gamma) = g$. Hence, it follows that $c(\gamma^{-1}) = g^{-1}$ and thus $\mathrm{Supp}(f^*) \subseteq \mathcal{G}_{g^{-1}}$. In other words, $f^* \in A_R(\mathcal{G}_{g^{-1}}) = (A_R(\mathcal{G}))_{g^{-1}}$. Thus, $A_R(\mathcal{G})$ is a G-graded $*$-ring. □

We now give sufficient conditions for a graded $*$-ring to be symmetrically graded. This generalizes both Nystedt and Öinert's proof that the Leavitt path algebras are symmetrically graded (see [41, Prop. 4.5]) and Clark, Exel and Pardo's proof that the Steinberg algebras are symmetrically graded (see [14, Prop. 5.1]).

Proposition 1.5.44. Let R be a ring and let $S = \bigoplus_{g \in G} S_g$ be a G-graded $*$-ring which is also a left R-module. Moreover, suppose that there is a set $M_g \subseteq S_g$ such that $S_g = \mathrm{Span}_R(M_g)$ for each $g \in G$. If $mm^*m = m$ for all $m \in M_g$ and $g \in G$, then S is symmetrically G-graded.

PROOF. Take an arbitrary $g \in G$. It is enough to show that $S_g \subseteq S_g S_{g^{-1}} S_g$. Since S_g is the R-linear span of M_g, it is in fact enough to show that $m \in S_g S_{g^{-1}} S_g$ for every $m \in M_g$. But, by assumption $m = mm^* m \in S_g S_{g^{-1}} S_g$. Hence, we have $S_g = S_g S_{g^{-1}} S_g$ and thus S is symmetrically G-graded. $\square$

We apply Proposition 1.5.44 to some important special cases:

Corollary 1.5.45. Let G be a group and let R be a unital ring. Then the following assertions hold:

(a) The group ring $R[G]$ is symmetrically G-graded;
(b) The Leavitt path algebra $L_R(E)$ of any graph E is symmetrically $\mathbb{Z}$-graded;
(c) The Steinberg algebra $A_R(\mathcal{G})$ of an ample Hausdorff G-graded groupoid $\mathcal{G}$ is symmetrically G-graded.

PROOF. (a): Consider the group ring $R[G]$. Note that,

$$\delta_g(\delta_g)^*\delta_g = \delta_g\delta_{g^{-1}}\delta_g = (\delta_g\delta_{g^{-1}})\delta_g = (\delta_e)\delta_g = \delta_g.$$

For each $g \in G$, we see that $(R[G])_g = R\delta_g$ is the R-linear span of the set $\{\delta_g\}$.

(b): Consider the Leavitt path algebra $L_R(E)$ together with the canonical $\mathbb{Z}$-grading and the involution defined above. Note that,

$$(\alpha\beta^*)(\alpha\beta^*)^*(\alpha\beta^*) = \alpha(\beta^*\beta)(\alpha^*\alpha)\beta^* = \alpha\beta^*,$$

for all paths α, β. Furthermore, recall that that $(L_R(E))_n$ is the R-linear span of all monomials $\alpha\beta^*$ with $\operatorname{len}(\alpha) - \operatorname{len}(\beta) = n$ (see Definition 1.5.12).

(c): ([14, Prop. 5.1]) Let $\mathcal{G}$ be an ample Hausdorff G-graded groupoid. Take an arbitrary $g \in G$ and let B be a compact open bissection of $\mathcal{G}_g$. Then $1_B \in (A_R(\mathcal{G}))_g$ and $(1_B)^* = 1_{B^{-1}} \in (A_R(\mathcal{G}))_{g^{-1}}$. Moreover, recall that the convolution multiplication reduces to $1_B 1_D = 1_{BD}$ for the characteristic functions of compact open bissections (see [14, Eqn. 2.4]). Then it follows that,

$$1_B(1_B)^*1_B = 1_B 1_{B^{-1}} 1_B = 1_{BB^{-1}B} = 1_B,$$

where $BB^{-1}B = B$ follows from the fact that BB^{-1} contains id_y for every $y \in c(B)$. Furthermore, recall that $(A_R(\mathcal{G}))_g$ is the R-linear span of functions 1_B where B is a compact open bissection of $\mathcal{G}_g$. $\square$

Next, we prove that a strong grading is symmetric:

Proposition 1.5.46. If S is a strongly G-graded ring, then S is symmetrically G-graded.

PROOF. Suppose that S is strongly G-graded. Take an arbitrary $g \in G$. Then,

$$S_g = S_{eg} = S_e S_g = (S_g S_{g^{-1}}) S_g = S_g S_{g^{-1}} S_g.$$

Hence, S is symmetrically G-graded. $\square$

Note that Proposition 1.5.46 implies that the group ring $R[G]$ is symmetrically G-graded. We give an example of a ring that is symmetrically graded but not strongly graded:

Example 1.5.47. Let R be a unital ring and let A_2 be the graph in Figure 5. Since A_2 contains a sink, it follows by Theorem 1.5.16 that $L_R(A_2)$ is not strongly $\mathbb{Z}$-graded. On the other hand, by Corollary 1.5.45(b), $L_R(A_2)$ is symmetrically $\mathbb{Z}$-graded.

We will later make use of the following propositions:

Proposition 1.5.48 ([14, p. 8])**.** If $S = \bigoplus_{g \in G} S_g$ is a symmetrically G-graded ring, then S_e is an idempotent ring.

Proof. Since the grading is symmetric, we see that $S_e = S_e S_e S_e = S_e^3$. Hence, by the grading, $S_e = S_e^3 \subseteq S_e^2 \subseteq S_e$. Thus, $S_e^2 = S_e$. □

Proposition 1.5.49. Let $S = \bigoplus_{g \in G} S_g$ be a symmetrically G-graded ring. The following assertions hold:
(a) If S_e is the trivial ring, then S is the trivial ring;
(b) If S_e is a unital ring, then S is a unital ring.

Proof. (a): Suppose that $S_e = \{0\}$. Then $S_g = S_g S_{g^{-1}} S_g \subseteq S_e S_g = \{0\}$ for every $g \in G$. Hence, $S = \{0\}$.

(b): Suppose that S_e is a unital ring and that 1 is the multiplicative identity element of S_e. Take $g \in G$ and $s_g \in S_g$. Then $s_g = \sum_{i=1}^n r_i s_i$ for some elements $r_1, \ldots, r_n \in S_g S_{g^{-1}} \subseteq S_e$ and $s_1, \ldots, s_n \in S_g$. Thus, $1 s_g = 1(\sum_{i=1}^n r_i s_i) = \sum_{i=1}^n (1 r_i) s_i = \sum_{i=1}^n r_i s_i = s_g$. Similarly, we show that $s_g 1 = s_g$. Since every element s of S is a finite sum $s = \sum s_g$ where $s_g \in S_g$ $(g \in G)$, it follows that $1s = s = s1$. Thus, S also has multiplicative identity element 1. □

1.5.6. Non-degenerately graded rings. We now consider yet another type of group graded rings. Passman seems to be the first to give the following definition (see also [17, 43]):

Definition 1.5.50 ([46, p. 32])**.** A G-graded ring $S = \bigoplus_{g \in G} S_g$ is said to be *non-degenerately G-graded* if for every $g \in G$ and $0 \neq s_g \in S_g$ we have $s_g S_{g^{-1}} \neq \{0\}$ and $S_{g^{-1}} s_g \neq \{0\}$.

We show that unital strongly graded rings are non-degenerately graded:

Proposition 1.5.51. If S is a unital strongly G-graded ring, then S is non-degenerately G-graded.

Proof. Let S be a unital strongly G-graded ring. Suppose that $s_g S_{g^{-1}} = \{0\}$ for some $g \in G$ and $s_g \in S_g$. Then $s_g S_{g^{-1}} S_g = \{0\}$. But since S is unital strongly G-graded, we have $1 \in S_{g^{-1}} S_g$. Hence, $s_g = s_g \cdot 1 \in s_g S_{g^{-1}} S_g = \{0\}$, i.e. $s_g = 0$. Similarly, we get that $S_g s_{g^{-1}} = \{0\}$ implies that $s_{g^{-1}} = 0$. Thus, S is non-degenerately G-graded. □

Remark 1.5.52. We make two remarks regarding Definition 1.5.50:
(a) The definition of non-degenerately graded rings relates to Passman's notion of component-regular graded rings (see [46, p. 32]). A G-graded ring $S = \bigoplus_{g \in G} S_g$ is called *component-regular* if $l.\mathrm{Ann}_S(S_g) = r.\mathrm{Ann}_S(S_g) = \{0\}$ for every $g \in G$. It can be proved that a non-degenerately G-graded ring S is component-regular if and

only if every homogeneous component S_g is faithful as an S_e-bimodule (see [46, p. 32]). Unital strongly G-graded rings are component-regular.

(b) Năstăsescu and van Oystaeyen [39, p. 39] call non-degenerately G-graded rings *e-faithful*.

The following example shows that neither non-unital strongly graded rings nor symmetrically graded rings need to be non-degenerately graded:

Example 1.5.53. Let K be a field and let $\mathcal{S} = \{e, g\}$ be the semigroup defined by

$$e \cdot e = e, \quad e \cdot g = g, \quad g \cdot e = g \quad \text{and} \quad g \cdot g = g.$$

Furthermore, let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Consider the twisted semigroup ring $(K \times K)[\mathcal{S}]$ where multiplication is defined by the following rule:

$$(x_1 + x_2 g)(y_1 + y_2 g) = x_1 y_1 + (x_2 y_1 e_2 + x_1 y_2 e_1)g,$$

for all $x_1, x_2, y_1, y_2 \in K \times K$. It can be proved that $(K \times K)[\mathcal{S}]$ becomes a ring with this multiplication and furthermore that $(K \times K)[\mathcal{S}]$ is idempotent (see [40, Expl. 5]). Now, let $G = \{e\}$ be the trivial group and consider the trivial G-grading of $(K \times K)[\mathcal{S}]$ defined by $((K \times K)[\mathcal{S}])_e = (K \times K)[\mathcal{S}]$. Note that this G-grading is strong since,

$$((K \times K)[\mathcal{S}])_e((K \times K)[\mathcal{S}])_e = ((K \times K)[\mathcal{S}])^2 = (K \times K)[\mathcal{S}] = ((K \times K)[\mathcal{S}])_e.$$

By Proposition 1.5.46, we get that $(K \times K)[\mathcal{S}]$ is symmetrically G-graded.

Next, we show that $(K \times K)[\mathcal{S}]$ is not non-degenerately G-graded. Let $r := (0, 1)g \in (K \times K)[\mathcal{S}]$. Then, for all $x_1, x_2 \in K \times K$,

$$(x_1 + x_2 g)r = x_1(0, 1)e_1 g = 0.$$

In other words, $(K \times K)[\mathcal{S}]r = \{0\}$. Since $r \neq 0$, this shows that $(K \times K)[\mathcal{S}]$ is not non-degenerately G-graded.

1.6. Epsilon-strongly graded rings

The notion of an epsilon-strongly graded ring was introduced by Nystedt, Öinert and Pinedo [42, Def. 4]. Recall that an ideal is called unital if it is equipped with a (possibly zero) multiplicative identity element.

Definition 1.6.1 ([42, Def. 4])**.** Let G be a group and let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. We call S *epsilon-strongly G-graded* if $S_g S_{g^{-1}}$ is a unital ideal of S_e for each $g \in G$ and,

$$S_g S_h = S_g S_{g^{-1}} S_{gh} \qquad \forall g, h \in G, \tag{7}$$

$$S_g S_h = S_{gh} S_{h^{-1}} S_h \qquad \forall g, h \in G. \tag{8}$$

Remark 1.6.2. Let S be an epsilon-strongly G-graded ring. We follow Nystedt, Öinert and Pinedo's convention of letting ϵ_g denote the multiplicative identity element of the ideal $S_g S_{g^{-1}}$ for every $g \in G$. It might happen that $\epsilon_g = 0$ for some $g \in G$.

The following characterization was also given by Nystedt, Öinert and Pinedo:

Proposition 1.6.3 ([41, Prop. 3.1], [42, Prop. 7])**.** Let S be a G-graded ring. The following assertions are equivalent:

(a) S is epsilon-strongly G-graded;
(b) S_g is a unital $S_gS_{g^{-1}}$–$S_{g^{-1}}S_g$-bimodule for each $g \in G$;
(c) S is symmetrically G-graded and $S_gS_{g^{-1}}$ is a unital ideal of S_e for each $g \in G$;
(d) For every $g \in G$, the left R-module S_g is finitely generated projective and the map $n_g\colon (S_g)_R \to \mathrm{Hom}_R({}_RS_{g^{-1}}, R)_R$, defined by $n_g(s)(t) = ts$ for $s \in S_g$ and $t \in S_{g^{-1}}$ is an isomorphism.

Remark 1.6.4. We make some remarks regarding Proposition 1.6.3.

(a) If $S = \bigoplus_{g \in G} S_g$ is an epsilon-strongly G-graded ring, then it follows from Proposition 1.6.3(c) that S is symmetrically G-graded. In other words,
$$\text{epsilon-strongly graded} \implies \text{symmetrically graded.}$$
(b) Note that if $S = \bigoplus_{g \in G} S_g$ is a unital strongly G-graded, then $S_gS_{g^{-1}} = S_e$ is a unital ideal for each $g \in G$ by Proposition 1.5.9. Furthermore, S is symmetrically G-graded by Proposition 1.5.46. Hence, by Proposition 1.6.3(c), S is epsilon-strongly G-graded. In other words,
$$\text{unital strongly graded} \implies \text{epsilon-strongly graded.}$$

Nystedt, Öinert and Pinedo's original motivation for introducing epsilon-strong gradings was to unify the treatment of unital strongly graded rings and so-called unital partial crossed products. Partial crossed products were introduced by Dokuchaev, Exel and Simón [24] as a generalization of crossed products (see Section 1.5.3).

Definition 1.6.5. A *unital twisted partial action of G on a unital ring R* (see e.g. [42, p. 2]) is a triple
$$(\{\alpha_g\}_{g \in G}, \{D_g\}_{g \in G}, \{w_{g,h}\}_{(g,h) \in G \times G}),$$
where for all $g, h \in G$, D_g is a (possibly zero) unital ideal of R, $\alpha_g\colon D_{g^{-1}} \to D_g$ is a ring isomorphism and $w_{g,h}$ is an invertible element in D_gD_{gh}. Let $1_g \in Z(R)$ denote the (not necessarily nonzero) multiplicative identity element of the ideal D_g. We require that the following conditions hold for all $g, h \in G$:

(P1) $\alpha_e = \mathrm{id}_R$;
(P2) $\alpha_g(D_{g^{-1}}D_h) = D_gD_{gh}$;
(P3) if $r \in D_{h^{-1}}D_{(gh)^{-1}}$, then $\alpha_g(\alpha_h(r)) = w_{g,h}\alpha_{gh}(r)w_{g,h}^{-1}$;
(P4) $w_{e,g} = w_{g,e} = 1_g$;
(P5) if $r \in D_{g^{-1}}D_hD_{hl}$, then $\alpha_g(rw_{h,l})w_{g,hl} = \alpha_g(r)w_{g,h}w_{gh,l}$.

Given a unital twisted partial action of G on R, we can form the *unital partial crossed product* $R \star_\alpha^w G := \bigoplus_{g \in G} D_g\delta_g$ where the δ_g's are formal symbols. For $g, h \in G, r \in D_g$ and $r' \in D_h$ the multiplication is defined by the rule:
$$(r\delta_g)(r'\delta_h) = r\alpha_g(r'1_{g^{-1}})w_{g,h}\delta_{gh}$$

The unital ring $R \star_\alpha^w G$ is associative (see e. g. [24, Thm. 2.4]). Moreover, Nystedt, Öinert and Pinedo [42, Thm. 35] showed that its natural G-grading is epsilon-strong.

Remark 1.6.6. The class of epsilon-strongly graded rings can be seen as an extension of unital strongly graded rings to include the unital partial crossed products. From this perspective, we can think of epsilon-strongly graded rings as 'generalized unital partial crossed products', similarly to how the class of strongly graded rings can be considered to be 'generalized crossed products'.

Next, we give a simple example of an epsilon-strongly $\mathbb{Z}$-graded ring that is not strongly $\mathbb{Z}$-graded.

Example 1.6.7. Let R be a unital ring and consider the $\mathbb{Z}$-grading of the ring $M_2(R)$ given in Example 1.5.22. Recall that,

$$(M_2(R))_0 = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \quad (M_2(R))_{-1} = \begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix}, \quad (M_2(R))_1 = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix},$$

and $(M_2(R))_i = \left\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right\}$ for $|i| > 1$. Note that this grading is not strong since the support is finite (see Proposition 1.5.8(c)). However,

$$(M_2(R))_1 (M_2(R))_{-1} = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}, \quad (M_2(R))_{-1} (M_2(R))_1 = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix},$$

are unital ideals of $(M_2(R))_0$ with multiplicative identity elements:

$$\epsilon_1 = \begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix}, \quad \epsilon_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1_R \end{pmatrix}.$$

Moreover, a routine check shows that (7) and (8) hold. Thus, $M_2(R)$ is epsilon-strongly $\mathbb{Z}$-graded by Definition 1.6.1. Furthermore, since $M_2(R) \cong_{\text{gr}} L_R(A_2)$ (see Example 1.5.22), this is an example of a Leavitt path algebra such that the canonical $\mathbb{Z}$-grading is epsilon-strong but not strong.

We show that the class of non-unital strongly graded rings is not contained in the class of epsilon-strongly graded rings. To this end, we show that epsilon-strongly graded rings are necessarily unital.

Proposition 1.6.8. If S is a non-trivial epsilon-strongly G-graded ring, then S_e is a unital ring.

PROOF. By Proposition 1.6.3(c), S is symmetrically G-graded and $S_g S_{g^{-1}}$ is a unital ideal of S_e for each $g \in G$. In particular, it follows that $S_e S_e$ is a unital ideal. Since S is symmetrically G-graded, it follows by Proposition 1.5.48 that $S_e S_e = S_e$. Moreover, by Proposition 1.5.49(a) and the assumption that S is a non-trivial ring, it follows that S_e is a non-trivial ring. Hence, S_e is a unital ring. □

Corollary 1.6.9. If S is a non-trivial epsilon-strongly G-graded ring, then S is a unital ring.

PROOF. By Proposition 1.6.8, S_e is a unital ring. From Proposition 1.6.3(c), it follows that S is symmetrically G-graded. Now by Proposition 1.5.49(b), we get that S is a unital ring. □

We can now construct an example of a non-unital strongly graded ring which is not epsilon-strongly graded:

Example 1.6.10. Let R be an idempotent but non-unital ring (for instance Example 1.4.7). Consider the grading by the trivial group $G = \{e\}$ of R given by $R_e := R$ (see Example 1.5.4). Since $R_e R_e = R_e$ it follows that this is a strong G-grading. However, since R is not a unital ring, the grading is not epsilon-strong by Corollary 1.6.9.

Remark 1.6.11. Note that there are many examples (see e.g. Example 1.6.10 and Example 1.6.12) of non-unital rings equipped with a strong G-grading. In other words, *the class of strongly graded rings is not contained in the class of epsilon-strongly graded rings* when we consider non-unital rings. In this sense the name 'epsilon-strong' might be confusing.

For a more interesting example of a non-unital strongly graded ring that is not epsilon-strongly graded, consider the following example given by Hazrat:

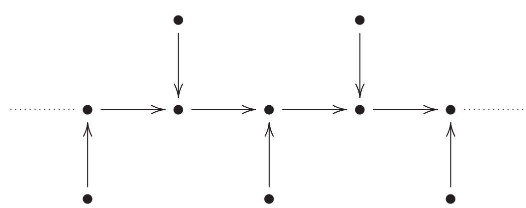

FIGURE 9. Infinite graph (see [30, Expl. 3.13]).

Example 1.6.12 ([30, Expl. 3.13])**.** Let R be a unital ring and consider the graph E in Figure 9. Note that E has infinitely many vertices and infinitely many edges. However, the Leavitt path algebra $L_R(E)$ is strongly $\mathbb{Z}$-graded (see [30, Expl. 3.13]). Since E^0 is infinite, $L_R(E)$ is not unital (see Proposition 1.4.18) and hence not epsilon-strongly $\mathbb{Z}$-graded by Corollary 1.6.9.

The following is a characterization of when an epsilon-strongly G-graded ring is strongly G-graded:

Proposition 1.6.13 ([42, Prop. 8])**.** Let $S = \bigoplus_{g \in G} S_g$ be an epsilon-strongly G-graded ring. The grading is strong if and only if $\epsilon_g = 1_S$ for every $g \in G$.

PROOF. Suppose that S is epsilon-strongly G-graded. By Corollary 1.6.9, S is unital. Moreover, by Proposition 1.5.9(b)(iii), a unital ring is strongly graded if and only if $S_e = S_g S_{g^{-1}}$ for each $g \in G$. Take an arbitrary $g \in G$. Since ϵ_g is the the multiplicative identity element of the ideal $S_g S_{g^{-1}} \subseteq S_e$, the condition $S_g S_{g^{-1}} = S_e$ is equivalent to $\epsilon_g = 1_{S_e} = 1_S$. Hence, S is strongly G-graded if and only if $\epsilon_g = 1_S$ for each $g \in G$. $\square$

Our interest in Leavitt path algebras is partially motivated by the following theorem proved by Nystedt and Öinert [41]. Note that this can be seen as an extension of Theorem 1.5.16 by Hazrat (cf. Figure 2).

Theorem 1.6.14 ([41, Thm. 4.1])**.** Let R be a unital ring and let E be a finite directed graph. Then the canonical $\mathbb{Z}$-grading of $L_R(E)$ is epsilon-strongly $\mathbb{Z}$-graded.

Remark 1.6.15. Nystedt and Öinert, in fact, proved the above result for every standard G-grading of $L_R(E)$.

1.6.1. A functorial characterization of epsilon-strongly graded rings. Let $S = \bigoplus_{g \in G} S_g$ be a unital G-graded ring. We recall the definition of two functors, called Ind and Coind respectively, defined on certain categories of modules. Next, we state Dade's famous characterization of unital strongly graded rings (see [22, Thm. 2.8]). We also recall that S is epsilon-strongly G-graded if and only if the functors Ind and Coind are naturally isomorphic. The latter result was given by Martínez, Pinedo and Soler [38, Prop. 3.9], and follows directly from a previous result by Năstăsescu and van Oystaeyen [39, Thm. 2.6.9].

The category of left S-modules, denoted by S-mod, consists of left S-modules and S-linear maps. We now consider gradings of modules:

Definition 1.6.16. Let $S = \bigoplus_{g \in G} S_g$ be a unital G-graded ring, let M be a unital left S-module and let $\{M_x\}_{x \in G}$ be a family of additive subgroups of M satisfying $M = \bigoplus_{x \in G} M_x$ and $S_g M_x \subseteq M_{gx}$ for all $g, x \in G$. Then the pair $(M, \{M_x\}_{x \in G})$ is called a *graded left S-module*.

Example 1.6.17. Note that $({}_S S, \{S_g\}_{g \in G})$ is a graded left S-module.

Example 1.6.18. Let K be a field and consider the polynomial ring $S := K[x]$. Recall that $K[x]$ is $\mathbb{Z}$-graded by putting $S_n := Kx^n$ for $n \geq 0$ and $S_n := \{0\}$ for $n < 0$. Consider the ideal I generated by the polynomial $p(x) = x$, i.e. $I := (x)$. Note that I consists of all polynomials with zero constant term. Clearly, $I = \bigoplus_{n \in \mathbb{Z}} I_n$ with $I_n := \{0\}$ for $n \leq 0$ and $I_n := Kx^n$ for $n > 0$. Moreover, $S_n I_m = (Kx^n)(Kx^m) = Kx^{n+m} = I_{n+m}$ for all $n, m > 0$. Hence, $S_n I_m \subseteq I_{n+m}$ for all $n, m \in \mathbb{Z}$. We conclude that I is a graded left S-module.

We now define the category of graded modules:

Definition 1.6.19. Let $S = \bigoplus_{g \in G}$ be a unital G-graded ring.

(a) Let $(M, \{M_x\}_{g \in G}), (N, \{N_x\}_{x \in G})$ be graded left S-modules. An S-linear map $f \colon M \to N$ is called *graded* if $f(M_x) \subseteq N_x$ for every $x \in G$. We write:
$$\mathrm{Hom}_{S\text{-gr}}(M, N) = \{f \in \mathrm{Hom}_S(M, N) \mid f(M_x) \subseteq N_x \quad \forall x \in G\}. \tag{9}$$
(b) The category of graded left S-modules, denoted by S-gr, consists of graded left S-modules and graded S-linear maps. More precisely, the objects of S-gr are graded left S-modules and the morphisms between $M, N \in S$-gr are given by (9).

We now recall the definition of $\mathrm{Ind} \colon S_e\text{-mod} \to S\text{-gr}$.

Definition 1.6.20. Let $N \in S_e$-mod. Put,
$$\mathrm{Ind}(N) := (S \otimes_{S_e} N, \{S_g \otimes_{R_e} N\}_{g \in G}).$$
For $N_1, N_2 \in S_e$-mod and $f \in \mathrm{Hom}_{R_e}(N_1, N_2)$, we define,
$$\mathrm{Ind}(f) := \mathrm{id}_S \otimes_{R_e} f.$$

A routine check shows that $\mathrm{Ind}(N) \in S$-gr. Moreover, $\mathrm{Ind}(f)$ is a graded S-linear morphism. It is also straightforward to show that $\mathrm{Ind}\colon S_e\text{-mod} \to S\text{-gr}$ is a covariant functor. We can now state Dade's famous characterization:

Theorem 1.6.21 (Dade's Theorem [22])**.** Let $S = \bigoplus_{g \in G} S_g$ be a unital G-graded ring. Then S is strongly G-graded if and only if the functor Ind is an equivalence of categories.

Remark 1.6.22. If S is unital strongly G-graded, then Theorem 1.6.21 implies that $S_e\text{-mod} \simeq S\text{-gr}$ (i.e. the categories are equivalent). However, the converse is not true. In other words, $S_e\text{-mod} \simeq S\text{-gr}$ does not in general imply that S is strongly G-graded (cf. [39, Expl. 3.2.4]).

Next, we recall the definition of the functor $\mathrm{Coind}\colon S_e\text{-mod} \to S\text{-gr}$. Let $N \in S_e\text{-mod}$. We equip $\mathrm{Hom}_{S_e}(S, N)$ with a left S-module structure by defining

$$(af)(x) = f(xa)$$

for all $a \in S$ and $f \in \mathrm{Hom}_{S_e}(S, N)$. Moreover, it can be shown (cf. [39, p. 34]) that

$$\mathrm{Hom}_{S_e}(S, N) = \bigoplus_{g \in G} \mathrm{Hom}_{S_e}(S_{g^{-1}}, N).$$

Thus, we see that $\mathrm{Hom}_{S_e}(S, N)$ is a graded left S-module.

Definition 1.6.23. Let $N \in S_e$-mod. Put,

$$\mathrm{Coind}(N) := (\mathrm{Hom}_{S_e}(S, N), \{\mathrm{Hom}_{S_e}(S_{g^{-1}}, N)\}_{g \in G}).$$

For $N_1, N_2 \in S_e$-mod and $f \in \mathrm{Hom}_{S_e}(N_1, N_2)$, the map

$$\mathrm{Coind}(f) := \mathrm{Hom}_{S_e}(S, f)\colon \mathrm{Hom}_{S_e}(S, N_1) \to \mathrm{Hom}_{S_e}(S, N_2)$$

is given by post-composition, i.e.,

$$\mathrm{Hom}_{S_e}(S, N_1) \ni \phi \mapsto f \circ \phi \in \mathrm{Hom}_{S_e}(S, N_2).$$

It is straightforward to see that $\mathrm{Coind}\colon S_e\text{-mod} \to S\text{-gr}$ is a well-defined covariant functor. Finally, we state the following characterization of epsilon-strongly graded rings:

Proposition 1.6.24 (Martínez, Pinedo, Soler [38])**.** Let $S = \bigoplus_{g \in G} S_g$ be a unital G-graded ring. Then S is epsilon-strongly G-graded if and only if the functors Ind and Coind are naturally isomorphic.

Proof. It can be shown (see [39, Thm. 2.6.9]) that the functors Ind and Coind are naturally isomorphic if and only if the condition (d) in Proposition 1.6.3 holds. □

Having the characterization in Proposition 1.6.24 at our disposal, we now see that already Năstăsescu and van Oystaeyen [39] gave a number of results on this class of rings (i.e. the class which we now call epsilon-strongly graded). We refer the reader to Theorem 2.6.9 – Corollary 2.6.13 in [39].

1.7. Nearly epsilon-strongly graded rings

The requirement that E is a finite graph in Theorem 1.6.14 is essential for a Leavitt path algebra to be epsilon-strongly $\mathbb{Z}$-graded (see Example 1.7.17). In this section, we will explain how Nystedt and Öinert removed this condition and were still able to say something substantial about the canonical $\mathbb{Z}$-grading of a Leavitt path algebra.

Definition 1.7.1 ([41, Def. 3.3])**.** Let G be a group and let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. If S_g is an s-unital $S_g S_{g^{-1}}$–$S_{g^{-1}} S_g$-bimodule for each $g \in G$, then we call S *nearly epsilon-strongly* G-graded.

The following characterization is similar to Proposition 1.6.3 for epsilon-strongly graded rings:

Proposition 1.7.2 ([41, Prop. 3.3])**.** Let S be a G-graded ring. The following assertions are equivalent:

(a) The ring S is nearly epsilon-strongly G-graded;
(b) The ring S is symmetrically graded and $S_g S_{g^{-1}}$ is an s-unital ring for each $g \in G$;
(c) For each $g \in G$ and each $s \in S_g$ there exist some $\epsilon_g(s) \in S_g S_{g^{-1}}$ and $\epsilon_g(s)' \in S_{g^{-1}} S_g$ such that $\epsilon_g(s) s = s = s \epsilon_g(s)'$.

Note that for an epsilon-strongly G-graded ring, the ideals $S_g S_{g^{-1}}$ are unital for each $g \in G$. For a nearly epsilon-strongly G-graded ring, the ideals $S_g S_{g^{-1}}$ are s-unital for each $g \in G$. Recall that the unital rings constitute a subclass of s-unital rings.

Proposition 1.7.3. If S is an epsilon-strongly G-graded ring, then S is nearly epsilon-strongly G-graded.

PROOF. Suppose that S is epsilon-strongly G-graded. By Proposition 1.6.3(c), S is symmetrically G-graded and $S_g S_{g^{-1}}$ is unital for each $g \in G$. In particular, $S_g S_{g^{-1}}$ is s-unital for each $g \in G$. By Proposition 1.7.2, S is nearly epsilon-strongly G-graded. $\square$

Remark 1.7.4. Consider an arbitrary G-grading $\{S_g\}_{g \in G}$ of a ring S. By Remark 1.6.4, Proposition 1.7.3 and Proposition 1.7.2(b), the following implications hold for $\{S_g\}_{g \in G}$:

unital strong $\implies$ epsilon-strong $\implies$ nearly epsilon-strong $\implies$ symmetrical

Next, we will prove that only s-unital rings admit a nearly epsilon-strong grading. First we recall the following lemma:

Lemma 1.7.5 (Tominaga [55, Thm. 1])**.** Let T be a ring and let M be a left (right) T-module. Take a finite subset $X \subseteq M$ and assume that for each $x \in X$ there is some $u_x \in T$ such that $u_x x = x$ ($x u_x = x$). Then, there is some $u \in T$ such that for each $x \in X$, it holds that $ux = x$ ($xu = x$).

The following proposition is analogous to Proposition 1.5.49.

Proposition 1.7.6. Let S be a symmetrically G-graded ring. If S_e is an s-unital ring, then S is an s-unital ring. More precisely: if S_e is s-unital, then for every $s \in S$ there are some $u, u' \in S_e$ such that $us = s = su'$.

Proof. Let $s \in S$ with homogeneous decomposition $s = \sum_{g \in G} s_g$. Note that the set $M := \text{Supp}(s) = \{g \in G \mid s_g \neq 0\} \subseteq G$ is finite. Fix an arbitrary $g \in M$. By the assumption that the grading is symmetrical, there exist $a_1, \dots, a_l \in S_g S_{g^{-1}} \subseteq S_e$, $b_1, \dots, b_k \in S_{g^{-1}} S_g \subseteq S_e$ and $s_1, \dots, s_l, s'_1, \dots, s'_k \in S_g$ such that

$$s_g = \sum_{i=1}^{l} a_i s_i = \sum_{j=1}^{k} s'_j b_j.$$

Since S_e is assumed to be s-unital it follows from Lemma 1.7.5 that there are some $u_g, u'_g \in S_e$ such that $u_g a_i = a_i$ and $b_j e'_u = b_j$ for all $i \in \{1, \dots, l\}$ and $j \in \{1, \dots k\}$. Then, $u_g s_g = s_g = s_g u'_g$. Since $g \in M$ was chosen arbitrarily, we have some $u_g, u'_g \in S_e$ such that $u_g s_g = s_g = s_g u'_g$ for every $g \in M$. By Lemma 1.7.5 applied to the finite set $\{s_g \mid g \in M\}$, there are some $u, u' \in S_e$ such that $u s_g = s_g = s_g u'$ for every $g \in M$. Hence, $us = u \sum_{g \in G} s_g = \sum_{g \in G} u s_g = \sum_{g \in G} s_g = s$ and similarly $su' = s$. In particular, S is an s-unital ring. □

We now show that only s-unital rings admit a nearly epsilon-strong grading:

Corollary 1.7.7. If S is a nearly epsilon-strongly G-graded ring, then S is s-unital.

Proof. Suppose that S is nearly epsilon-strongly G-graded. By Proposition 1.7.2, S is symmetrically G-graded and $S_g S_{g^{-1}}$ is s-unital for each $g \in G$. In particular, $S_e S_e$ is s-unital. But by Proposition 1.5.48, $S_e S_e = S_e$. Hence, S_e is an s-unital ring and thus it follows from Proposition 1.7.6 that S is an s-unital ring. □

Next, we will show that the class of s-unital strongly graded rings is included in the class of nearly epsilon-strongly graded rings. This first lemma is analogous to Proposition 1.5.9(a).

Lemma 1.7.8. If S is an s-unital G-graded ring, then S_e is an s-unital subring of S.

Proof. Take an arbitrary element $s \in S_e$. By assumption there are $t, t' \in S$ such that $ts = s = st'$. Let $t = \sum_{g \in G} t_g$ be the decomposition of t with $t_g \in S_g$. Then $s = ts = \sum_{g \in G} t_g s$ is a decomposition of s. Note that s is a homogeneous element because we assumed that $s \in S_e$. By the uniqueness of the decomposition we must therefore have that $t_g s = 0$ for every $g \neq e$. Hence, $s = ts = t_e s$. Similarly, it follows that $s = st' = st'_e$. Hence, S_e is an s-unital ring. □

Proposition 1.7.9. If S is an s-unital strongly G-graded ring, then S is nearly epsilon-strongly G-graded.

Proof. Since S is strongly G-graded, it is symmetrically G-graded by Proposition 1.5.46. By Proposition 1.7.2, it is enough to show that $S_g S_{g^{-1}}$ is s-unital for each $g \in G$. Since S is strong, $S_g S_{g^{-1}} = S_e$ for every $g \in G$. Moreover, S_e is an s-unital ring by Lemma 1.7.8. □

However, similarly to the epsilon-strongly graded case, there are strongly graded rings which are not nearly epsilon-strongly graded.

Example 1.7.10. Let R be an idempotent ring that is not s-unital (for instance, consider the ring in Example 1.4.7). The trivial group grading of R is defined by $R_e = R$ (see Example 1.5.4). This grading is strong and hence also symmetrical by Proposition 1.5.46. Seeking a contradiction, suppose that this grading is nearly epsilon-strongly graded. By Corollary 1.7.7, we get that $R_e = R$ is s-unital, which contradicts our assumption on R. Thus, this grading is not nearly epsilon-strong.

By combining our previous results, we obtain the following:

Corollary 1.7.11. Let S be a strongly G-graded ring. Then the following assertions are equivalent:

(a) S is nearly epsilon-strongly G-graded;
(b) S is s-unital strongly G-graded;
(c) S_e is s-unital.

Proof. $(a) \Rightarrow (b)$: Suppose that S is nearly epsilon-strongly G-graded. By Corollary 1.7.7 we get that S is an s-unital strongly G-graded ring.

$(b) \Rightarrow (c)$: Apply Lemma 1.7.8.

$(c) \Rightarrow (a)$: Suppose that S_e is s-unital. Note that since S is assumed to be strongly G-graded, it is also symmetrically G-graded (see Proposition 1.5.46). Thus it follows by Proposition 1.7.6 that S is s-unital strongly G-graded. Now, by Proposition 1.7.9, S is nearly epsilon-strongly G-graded. □

Next, we recall that nearly epsilon-strongly graded rings are non-degenerately graded:

Proposition 1.7.12 ([41, Prop. 3.4])**.** If S is nearly epsilon-strongly G-graded, then S is non-degenerately G-graded.

Proof. Suppose that $s_g S_{g^{-1}} = \{0\}$ for some $g \in G$ and some $s_g \in S_g$. Then $s_g S_{g^{-1}} S_g = \{0\}$. Since S is nearly epsilon-strongly G-graded, it follows by Proposition 1.7.2(c) that there exists some $\epsilon_g(s_g)' \in S_{g^{-1}} S_g$ such that $s_g \epsilon_g(s_g)' = s_g$. But then $s_g = s_g \epsilon_g(s_g)' \in s_g S_{g^{-1}} S_g = \{0\}$ and thus $s_g = 0$. Similarly, we show that $S_g s_{g^{-1}} = \{0\}$ implies that $s_{g^{-1}} = 0$. Thus S is non-degenerately G-graded. □

Nearly epsilon-strong gradings are symmetrical by Proposition 1.7.2(b). However, symmetrical gradings need not be nearly epsilon-strong:

Example 1.7.13. Let G be the trivial group. Consider the symmetrically G-graded ring $(K \times K)[\mathcal{S}]$ given in Example 1.5.53 which is not non-degenerately G-graded. By Proposition 1.7.12 this implies that the ring $(K \times K)[\mathcal{S}]$ cannot be nearly epsilon-strongly G-graded. Hence, $(K \times K)[\mathcal{S}]$ is an example of a symmetrically G-graded ring which is not nearly epsilon-strongly G-graded.

The main motivation for introducing nearly epsilon-strongly graded rings comes from Leavitt path algebras. In fact, Nystedt and Öinert showed the following theorem:

Theorem 1.7.14 ([41, Thm. 4.2])**.** Let R be a unital ring and let E be a directed graph. The canonical $\mathbb{Z}$-grading of $L_R(E)$ is nearly epsilon-strong.

Note that it is not assumed that the graph E is finite in Theorem 1.7.14. The next examples show what happens when we have infinitely many vertices and infinitely many edges, respectively.

Example 1.7.15. Consider the infinite graph given in Example 1.4.19. It follows by Theorem 1.7.14 that $L_R(E)$ is nearly epsilon-strongly $\mathbb{Z}$-graded. Moreover, note that,

$$\begin{aligned}(L_R(E))_0 &= \mathrm{Span}\{v_1, v_2, v_3, \dots\},\\ (L_R(E))_i &= \{0\} \quad \forall i \neq 0.\end{aligned}$$

We see that $(L_R(E))_0$ is too large to admit a multiplicative identity element but is still an s-unital ring. Hence $L_R(E)$ is not unital (see Proposition 1.4.18) and does not admit an epsilon-strong $\mathbb{Z}$-grading by Corollary 1.6.9. Thus this is an example of a $\mathbb{Z}$-grading that is nearly epsilon-strong but not epsilon-strong.

Remark 1.7.16. Remark 1.7.4, Example 1.6.7, Example 1.7.15 and Example 1.7.13 show that unital strongly graded rings, epsilon-strongly graded rings, nearly epsilon-strongly graded rings and symmetrically graded rings are proper classes of each other. In other words, we have the following chain of proper class inclusions:

$$\begin{aligned}\text{unital strong graded rings } &\subsetneq \text{epsilon-strong graded rings } \subsetneq\\ &\subsetneq \text{nearly epsilon-strong graded rings } \subsetneq\\ &\subsetneq \text{symmetrically graded rings}\end{aligned}$$

Note that unital nearly epsilon-strongly graded rings are not necessarily epsilon-strong:

Example 1.7.17 ([41, Expl. 4.2])**.** Let E consist of two vertices v_1, v_2 with a countably infinite number of edges $f_1, f_2, \dots$ (see Figure 10). By Proposition 1.4.18, $L_R(E)$ is a unital ring and, by Theorem 1.7.14, $L_R(E)$ is nearly epsilon-strongly $\mathbb{Z}$-graded. Note that,

$$\begin{aligned}(L_R(E))_0 &= \mathrm{Span}\{v_1, v_2, f_i f_j^* \mid i, j > 0\},\\ (L_R(E))_1 &= \mathrm{Span}\{f_1, f_2, f_3, \dots\},\\ (L_R(E))_{-1} &= \mathrm{Span}\{f_1^*, f_2^*, f_3^*, \dots\}.\end{aligned}$$

Seeking a contradiction, suppose that $L_R(E)$ is epsilon-strongly $\mathbb{Z}$-graded. By Proposition 1.6.3, $(L_R(E))_1$ has a left multiplicative identity $\epsilon_1 \in (L_R(E))_1(L_R(E))_{-1}$. By assumption, $\epsilon_1 f_i = f_i$ for all $i \geq 1$. This means that we must have $\epsilon_1 = v_1$. On the other hand, $v_1 \notin (L_R(E))_1(L_R(E))_{-1}$. Thus, $L_R(E)$ is not epsilon-strongly $\mathbb{Z}$-graded. This is an example of a unital nearly epsilon-strongly $\mathbb{Z}$-graded ring that is not epsilon-strongly $\mathbb{Z}$-graded.

$$\bullet_{v_1} \xrightarrow{(\infty)} \bullet_{v_2}$$

FIGURE 10. Two vertices, infinitely many edges (see [41, p. 2]).

In fact, the situation in Example 1.7.15 and Example 1.7.17 can be formulated more generally.

Lemma 1.7.18 (cf. [30, Lem. 3.2(4)], [37, Lem. 3])**.** Let R be a unital ring and let E be a directed graph. A vertex $v \in E^0$ is a sink or an infinite emitter if and only if $v \notin (L_R(E))_1(L_R(E))_{-1}$.

PROOF. First assume that $v \in E^0$ is neither a sink nor an infinite emitter. Then, $v = \sum_{f \in s^{-1}(v)} ff^* \in (L_R(E))_1(L_R(E))_{-1}$. Conversely, assume that $v \in E^0$ is a sink. Seeking a contradiction, suppose that $v \in (L_R(E))_1(L_R(E))_{-1}$. Then $v = v^2 \in v(L_R(E))_1(L_R(E))_{-1} = \{0\}$. This is a contradiction, since $v \neq 0$. Hence, $v \notin (L_R(E))_1(L_R(E))_{-1}$. Next, assume that $v \in E^0$ is an infinite emitter. Seeking a contradiction, suppose that $v \in (L_R(E))_1(L_R(E))_{-1}$. Then, $v = \sum_{i=1}^n \alpha_i \beta_i^* \delta_i \gamma_i^*$ for some $\alpha_i \beta_i \in (L_R(E))_1, \delta_i \gamma_i^* \in (L_R(E))_{-1}$. Note that $\text{len}(\gamma_i) > 0$ for every i. Then $f = fv = \sum_{i=1}^n \alpha_i \beta_i^* \delta_i \gamma_i^* f$ for infinitely many edges f. Since n is finite, this yields a contradiction. Thus, $v \notin (L_R(E))_1(L_R(E))_{-1}$. □

We now prove the center upward implication in Figure 2. Let α, β be paths in a directed graph E. We write $\alpha \leq \beta$ if and only if α is an initial subpath of β. This gives a partial order on the set $\text{Path}(E)$.

Proposition 1.7.19. Let R be a unital ring and let E be a directed graph. If E is not a finite graph, then $L_R(E)$ is not epsilon-strongly $\mathbb{Z}$-graded.

PROOF. If E contains infinitely many vertices, then $L_R(E)$ is not a unital ring by Proposition 1.4.18. Hence, $L_R(E)$ is not epsilon-strongly $\mathbb{Z}$-graded by Corollary 1.6.9.

Next, suppose that E contains finitely many vertices but infinitely many edges. This implies that there are two vertices v_1, v_2 (not necessarily distinct) and infinitely many edges, say, $f_1, f_2, f_3, \dots$ such that $s(f_i) = v_1$ and $r(f_i) = v_2$ for every $i > 0$. Seeking a contradiction, suppose that $L_R(E)$ is epsilon-strongly graded. Let ϵ_1 be the multiplicative identity element of $(L_R(E))_1(L_R(E))_{-1}$. Since $\epsilon_1 f_i = f_i$ for $i \geq 0$, it follows that $\epsilon_1 = v_1 + \epsilon'$ for some element $\epsilon' \in (L_R(E))_0$. It can be shown (see [41] and Proposition A.4.3) that if $L_R(E)$ is epsilon-strongly $\mathbb{Z}$-graded, then for each $i \in \mathbb{Z}$, we have $\epsilon_i = \sum \alpha\alpha^*$ where the sum goes over a certain finite set of minimal paths in the subpath ordering. Since v_1 is a minimal path of length 0, it follows that no other path in the sum can start at v_1. In other words, $v_1 \epsilon' = 0$. Hence, $v_1 \epsilon_1 = v_1(v_1 + \epsilon') = v_1 + v_1 \epsilon' = v_1$. Since $(L_R(E))_1(L_R(E))_{-1}$ is an ideal of $(L_R(E))_0$ and $\epsilon_1 \in (L_R(E))_1(L_R(E))_{-1}$, it follows that $v_1 = v_1 \epsilon_1 \in (L_R(E))_1(L_R(E))_{-1}$. But by Lemma 1.7.18, $v_1 \notin (L_R(E))_1(L_R(E))_{-1}$. This contradiction proves that $L_R(E)$ is not epsilon-strongly $\mathbb{Z}$-graded. □

Here it seems appropriate to mention the following general result obtained by Martínez, Pinedo and Soler:

Theorem 1.7.20 ([38, Thm. 3.12])**.** Let S be a unital G-graded ring. Then S is epsilon-strongly G-graded if and only if S is nearly epsilon-strongly G-graded and S_g is a finitely generated left S_e-module for every $g \in G$.

1.8. Algebraic Cuntz-Pimsner rings

The (algebraic) Cuntz-Pimsner rings were introduced by Carlsen and Ortega in [11] using a categorical approach. Their starting point is the following definition:

Definition 1.8.1 ([11, Def. 1.1]). Let R be a ring. An *R-system* is a triple (P, Q, ψ) where P and Q are R-bimodules and $\psi\colon P \otimes_R Q \to R$ is an R-bimodule homomorphism from the balanced tensor product $P \otimes_R Q$ to R.

Carlsen and Ortega [11] considered representations of a given R-system:

Definition 1.8.2 ([11, Def. 1.2, Def. 3.3]). Let R be a ring and let (P, Q, ψ) be an R-system. A *covariant representation* is a tuple (S, T, σ, B) such that the following assertions hold:

(a) B is a ring;
(b) $S\colon P \to B$ and $T\colon Q \to B$ are additive maps;
(c) $\sigma\colon R \to B$ is a ring homomorphism;
(d) $S(pr) = S(p)\sigma(r), S(rp) = \sigma(r)S(p), T(qr) = T(q)\sigma(r), T(rq) = \sigma(r)T(q)$ for every $r \in R$, $q \in Q$ and $p \in P$;
(e) $\sigma(\psi(p \otimes q)) = S(p)T(q)$ for all $p \in P$ and $q \in Q$.

The covariant representation (S, T, σ, B) is called *injective* if the map σ is injective. The covariant representation (S, T, σ, B) is called *surjective* if B is generated as a ring by the set $\sigma(R) \cup S(P) \cup T(Q)$.

A surjective covariant representation (S, T, σ, B) is called *graded* if there is a $\mathbb{Z}$-grading $\{B_i\}_{i\in\mathbb{Z}}$ of B such that $\sigma(R) \subseteq B_0$, $T(Q) \subseteq B_1$ and $S(P) \subseteq B_{-1}$.

Example 1.8.3 ([11, Expl. 1.3(1)]). Let R be any ring and put $P = Q = R$ where we consider R as an R-bimodule. Define $\psi\colon P \otimes Q \to R$ by $\psi(p \otimes q) = pq$. Then (P, Q, ψ) is an R-system. Let $R[x, x^{-1}]$ be the Laurent polynomial ring equipped with the standard $\mathbb{Z}$-grading (see Example 1.5.3) and define T, S, σ by additively extending the relations $T(q) = qx, S(p) = px^{-1}, \sigma(r) = r$ for all $r \in R, q \in Q, p \in P$. Then it can be shown that $(S, T, \sigma, R[x, x^{-1}])$ is a graded and injective representation of (P, Q, ψ).

Carlsen and Ortega [11] introduced a technical condition called *Condition (FS)* (see [11, Def. 3.4]). They then considered the category of covariant representations for a given R-system (P, Q, ψ) satisfying Condition (FS). The algebraic Cuntz-Pimsner rings are defined as certain graded, injective covariant representations satisfying a universal property (see Definition C.2.12). Note that unlike the Cuntz-Pimsner C^*-algebra, the Cuntz-Pimsner ring is not well-defined for every R-system (P, Q, ψ) (see [11, Expl. 4.11]). However, if R is a semiprime ring, then the Cuntz-Pimsner ring is well-defined for every R-system (P, Q, ψ) which satisfies Condition (FS) (see [11, Lem. 5.3]). If the Cuntz-Pimsner ring of the R-system (P, Q, ψ) exists, we denote it by $\mathcal{O}_{(P,Q,\psi)}$.

Besides fitting into the larger program of 'algebrafying' C^*-algebras, the Cuntz-Pimsner rings are also interesting because they generalize some well-known rings. Indeed, Carlsen and Ortega [11] gave the following examples of rings realizable as Cuntz-Pimsner rings:

(a) Let E be a directed graph and K a unital commutative ring. The *Leavitt path algebra* $L_K(E)$ is graded isomorphic to a Cuntz-Pimsner ring (see [11, Expl. 5.8]).

(b) Let R be a unital ring, let $\phi \in \mathrm{Aut}(R)$ be a ring automorphism and let $\gamma\colon \mathbb{Z} \to \mathrm{Aut}(R)$ be the group homomorphism induced by $\gamma(1) = \phi$. The skew group ring $R \star_\gamma \mathbb{Z}$ is graded isomorphic to a Cuntz-Pimsner ring (see [11, Expl. 5.5]). This construction is also called a *crossed product by a single automorphism*.

(c) Let R be a unital ring, let u be an idempotent of R and let α be a ring isomorphism $\alpha\colon R \to uRu$. The *corner skew Laurent polynomial ring* $R[t_+, t_-, \alpha]$ (see [6]) is graded isomorphic to a Cuntz-Pimsner ring (see [11, Expl. 5.7]).

Recently, Clark, Fletcher, Hazrat and Li [44] have shown that a special class of Steinberg algebras are also realizable as Cuntz-Pimsner rings. More precisely, they proved the following theorem:

Theorem 1.8.4 ([44, Cor. 4.6])**.** Let $\mathcal{G}$ be a locally compact Hausdorff ample groupoid and let $c\colon \mathcal{G} \to \mathbb{Z}$ be a cocycle. Suppose that c is unperforated in the sense that if $n > 0$ and $g \in \mathcal{G}_n$, then there exist $g_1, g_2, \dots, g_n \in \mathcal{G}_1$ such that $g = g_1 g_2 \dots g_n$. Then, $A_R(\mathcal{G}) \cong_{\mathrm{gr}} \mathcal{O}_{(A_R(\mathcal{G}_{-1}), A_R(\mathcal{G}_1), \psi)}$.

This is again analogous with the C^*-setting, where similar conditions exist for a groupoid C^*-algebra to be realizable as a Cuntz-Pimsner C^*-algebra (see [49]).

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CHAPTER 2

Summary of the scientific papers

2.1. Summary of Paper A

We consider the functor $U_{G/N}$ and the induced quotient group gradings (see Definition 1.5.34) of epsilon-strongly graded rings. We let G-ϵSTRG denote the category of epsilon-strongly G-graded rings. The main problem of Paper A is the following question:

Question 2.1.1. For which $(S, \{S_g\}_{g \in G}) \in G\text{-}\epsilon\text{STRG}$ do we have,

$$U_{G/N}((S, \{S_g\}_{g \in G})) \in G\text{-}\epsilon\text{STRG}?$$

In particular, does the functor $U_{G/N}$ restrict to the subcategory of epsilon-strongly G-graded rings?

The functor $U_{G/N}$ restricts to the category of strongly G-graded rings (see Proposition 1.5.36). However, we give an example showing that $U_{G/N}$ does not restrict to G-ϵSTRG. Next, recall that there is a partial order $\leq$ defined on the idempotents $E(R)$ of a ring R (see Definition 1.4.8). We obtain sufficient and necessary conditions for the induced quotient group grading of an epsilon-strongly G-graded ring to be epsilon-strong:

Theorem 2.1.2. Let S be an epsilon-strongly G-graded ring and let N be a normal subgroup of G. The induced G/N-grading $\{S_C\}_{C \in G/N}$ is epsilon-strong if and only if, for every $C \in G/N$, there is some $\chi_C \in E(S_N)$ such that $f \leq \chi_C$ for all $f \in \bigvee\{\epsilon_g \mid g \in C\}$.

For nearly epsilon-strongly graded rings, we obtain the following result which will in fact be utilized in Paper E:

Proposition 2.1.3. Let $N \lhd G$ be a normal subgroup of G. If S is nearly epsilon-strongly G-graded, then the induced G/N-grading $\{S_C\}_{C \in G/N}$ is nearly epsilon-strong.

2.2. Summary of Paper B

Let $S = \bigoplus_{g \in G} S_g$ be an epsilon-strongly G-graded ring. In Paper B, we show that noetherianity and artianity of the principal component S_e can be lifted to S under certain conditions. The following result is our generalization of the Hilbert Basis Theorem for strongly graded rings (see [5]).

Theorem 2.2.1. Let G be an arbitrary polycyclic-by-finite group and let $S = \bigoplus_{g \in G} S_g$ be an epsilon-strongly G-graded ring. Then S is left (right) noetherian if and only if the principal component S_e is left (right) noetherian.

As for artinianity, we obtain the following result:

Theorem 2.2.2. Let G be an arbitrary torsion-free group and let $S = \bigoplus_{g \in G} S_g$ be an epsilon-strongly G-graded ring. Then S is left (right) artinian if and only if S_e is left (right) artinian and $\mathrm{Supp}(S)$ is finite.

Applying this to the special case of Leavitt path algebras, we obtain a characterization of noetherian, artinian and semisimple Leavitt path algebras with coefficients in a general non-commutative unital ring. Our result generalizes a similar characterization for Leavitt path algebras with coefficients in a field (see [1, Cor. 4.2.13-14] and [16, Thm. 5.2]) and Steinberg's recent characterization for Leavitt path algebras with coefficients in a commutative unital ring (see [23]).

Theorem 2.2.3. Let E be a directed graph and let R be a unital ring. Consider the Leavitt path algebra $L_R(E)$ with coefficients in R. The following assertions hold:

(a) $L_R(E)$ is a left (right) noetherian unital ring if and only if E is finite and satisfies Condition (NE) and R is left (right) noetherian.
(b) $L_R(E)$ is a left (right) artinian unital ring if and only if E is finite acyclic and R is left (right) artinian.
(c) If $L_R(E)$ is a semisimple unital ring, then E is finite acyclic and R is semisimple. Conversely, if R is semisimple with $n \cdot 1_R$ invertible for every integer $n \neq 0$ and E is finite acyclic, then $L_R(E)$ is a semisimple unital ring.

As a further application, we obtain characterizations of noetherian and artinian unital partial crossed products. Recall (see [17]) that Nystedt, Öinert and Pindeo proved that the unital partial crossed product $R \star_\alpha^\omega G = \bigoplus_{g \in G} D_g \delta_g$ is epsilon-strongly G-graded. Using Theorem 2.2.1, we establish the following result:

Corollary 2.2.4. If G is a polycyclic-by-finite group, then $R \star_\alpha^\omega G$ is left (right) noetherian if and only if R is left (right) noetherian.

Similarly, the following corollary is a consequence of Theorem 2.2.2:

Corollary 2.2.5. Let G be a torsion-free group. Then $R \star_\alpha^\omega G$ is left (right) artinian if and only if R is left (right) artinian and $D_g = \{0\}$ for all but finitely many $g \in G$.

2.3. Summary of Paper C

In Paper C, we obtain partial classifications of epsilon-strongly and nearly epsilon-strongly $\mathbb{Z}$-graded algebraic Cuntz-Pimsner rings up to graded isomorphism (see Theorem 2.3.4). We also obtain a complete classification of unital strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings up to graded isomorphism (see Theorem 2.3.3). As an application, we characterize noetherian and artinian corner skew Laurent polynomial rings (see Corollary 2.3.6) using the results of Paper B. We also recover two results by Hazrat (see Corollary 2.3.7 and Corollary 2.3.8).

We introduce the following special type of R-systems (see Definition 1.8.1):

Definition 2.3.1. Let R be a ring and let (P, Q, ψ) be an R-system. If R is s-unital and P and Q are s-unital R-bimodules, then (P, Q, ψ) is called an *s-unital* R-system. If R is unital and P, Q are unital R-bimodules, then (P, Q, ψ) is called a *unital* R-system.

We show that an s-unital (unital) R-system gives an s-unital (unital) Cuntz-Pimsner ring. Our main technique in characterizing the graded structure of Cuntz-Pimsner rings involves introducing a new type of graded covariant representations. The definition is inspired by the situation for Cuntz-Pimsner C^*-algebras. In fact, Chirvasitu [6] has obtained sufficient and necessary conditions for the gauge action of a Cuntz-Pimsner C^*-algebra to be free. Inspired by his work, we obtain sufficient conditions for a Cuntz-Pimsner ring to be unital strongly $\mathbb{Z}$-graded. We also introduce *faithful covariant representations* and make the following useful definition:

Definition 2.3.2. Let R be ring, let (P, Q, ψ) be an R-system and let (S, T, σ, B) be a graded covariant representation of (P, Q, ψ). For $k \geq 0$ define,

$$I_{\psi,\sigma}^{(k)} = \operatorname{Span}\{\sigma(\psi_k(p \otimes q)) \mid p \in P^{\otimes k}, q \in Q^{\otimes k}\}.$$

We call (S, T, σ, B) a *semi-full* covariant representation if $B_{-k}B_k = I_{\psi,\sigma}^{(k)}$ for each $k \geq 0$.

The condition in Definition 2.3.2 is chosen in such a way that we can obtain sufficient conditions for the algebraic Cuntz-Pimsner ring to be nearly epsilon-strongly graded. A crucial reduction step, based on recent work by Clark, Fletcher, Hazrat and Li [18], tells us that we only need to consider semi-full Cuntz-Pimsner representations for our purposes. Using this technique, we obtain partial classifications of nearly epsilon-strongly and epsilon-strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings. For the unital strongly graded case, we get the following complete classification:

Theorem 2.3.3. Let $\mathcal{O}_{(P,Q,\psi)}$ be an algebraic Cuntz-Pimsner ring of some system (P, Q, ψ). Then, $\mathcal{O}_{(P,Q,\psi)}$ is unital strongly $\mathbb{Z}$-graded if and only if

$$\mathcal{O}_{(P,Q,\psi)} \cong_{\text{gr}} \mathcal{O}_{(P',Q',\psi')}$$

where (P', Q', ψ') is an R'-system satisfying the following assertions:

(a) (P', Q', ψ') is a unital R'-system;
(b) $(\iota_{P'}^{CP}, \iota_{Q'}^{CP}, \iota_{R'}^{CP}, \mathcal{O}_{(P',Q',\psi')})$ is a semi-full and faithful covariant representation of (P', Q', ψ');
(c) ψ' is surjective.

We also get similar results for nearly epsilon-strongly and epsilon-strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings. However, in these cases an additional technical assumption is needed in the 'only if' direction. For nearly epsilon-strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings, we obtain the following result:

Theorem 2.3.4. Let $\mathcal{O}_{(P,Q,\psi)}$ be a Cuntz-Pimsner ring of some system (P, Q, ψ). If $\mathcal{O}_{(P,Q,\psi)}$ is nearly epsilon-strongly $\mathbb{Z}$-graded and $\operatorname{Ann}_{\mathcal{O}_0}(\mathcal{O}_1) \cap (\operatorname{Ann}_{\mathcal{O}_0}(\mathcal{O}_1))^{\perp} = \{0\}$, then $\mathcal{O}_{(P,Q,\psi)} \cong_{\text{gr}} \mathcal{O}_{(P',Q',\psi')}$ where (P', Q', ψ') is an R'-system and the following assertions are satisfied:

(a) (P', Q', ψ') is an s-unital R'-system;
(b) $(\iota_{P'}^{CP}, \iota_{Q'}^{CP}, \iota_{R'}^{CP}, \mathcal{O}_{(P',Q',\psi')})$ is a semi-full covariant representation of (P', Q', ψ');
(c) (P', Q', ψ') satisfies Condition (FS);
(d) $I_{\psi',\iota_{\mathcal{O}_0}^{CP}}^{(k)}$ is s-unital for $k \geq 0$.

Conversely, if (P', Q', ψ') is an R'-system satisfying assertions (a)-(d), then $\mathcal{O}_{(P',Q',\psi')}$ is nearly epsilon-strongly $\mathbb{Z}$-graded.

Remark 2.3.5. It is not clear to the author if there are nearly epsilon-strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings satisfying $\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1) \cap (\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1))^{\perp} \neq \{0\}$. It might be possible to remove this condition from Theorem 2.3.4.

As a consequence of our classification results and our Hilbert Basis Theorem for epsilon-strongly graded rings in Paper B (see Theorem 2.2.1), we obtain the following characterization of noetherian and artinian corner skew Laurent polynomial rings.

Corollary 2.3.6. Let R be a unital ring and let $\alpha\colon R \to uRu$ be a ring isomorphism where u is an idempotent of R. Consider the corner skew Laurent polynomial ring $R[t_+, t_-; \alpha]$. The following assertions hold:
(a) $R[t_+, t_-; \alpha]$ is left (right) noetherian if and only if R is left (right) noetherian;
(b) $R[t_+, t_-; \alpha]$ is neither left nor right artinian.

Furthermore, we recover two results by Hazrat. The first result gives sufficient conditions for a corner skew Laurent polynomial ring to be strongly graded:

Corollary 2.3.7 (cf. [9, Prop. 1.6.6])**.** Let R be a unital ring and let $\alpha\colon R \to uRu$ be a ring isomorphism where u is an idempotent of R. Then the fractional skew group ring $R[t_+, t_-; \alpha]$ is strongly $\mathbb{Z}$-graded if u is a full idempotent.

Secondly, we recover one direction of Hazrat's characterization of strongly graded Leavitt path algebras of finite graphs (see Figure 2 and Theorem 1.5.16).

Corollary 2.3.8 (cf. [11, Thm. 3.15])**.** Let E be a finite graph without any sinks and let R be a unital ring. Then the Leavitt path algebra $L_R(E)$ is strongly $\mathbb{Z}$-graded.

2.4. Summary of Paper D

A ring R (possibly non-unital) is called *von Neumann regular* if $x \in xRx$ for every $x \in R$. In other words: for every $x \in R$, we can find some 'weak inverse' $y \in R$ such that $x = xyx$. If R is unital, this property is equivalent to every left R-module being flat (see [13, Thm. 4.21]). Note that e.g. fields are von Neumann regular. On the other hand, the ring of integers $\mathbb{Z}$ is not von Neumann regular. The class of von Neumann regular rings is well-studied (see e.g. [8]).

We consider the graded version of the von Neumann regularity condition:

Definition 2.4.1 (Năstăsescu and van Oystaeyen [15])**.** Let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. Then S is called *graded von Neumann regular* if for every homogeneous element $x_g \in S_g$ $(g \in G)$ there exists some $y \in S$ such that $x_g = x_g y x_g$.

This definition was thoroughly investigated in the context of unital strongly G-graded rings. Notably, Năstăsescu and van Oystaeyen [15, Cor. C.I.5.3] obtained the following as a consequence of Dade's Theorem:

Theorem 2.4.2 (Năstăsescu and van Oystaeyen [15])**.** Let $S = \bigoplus_{g \in G} S_g$ be a unital strongly G-graded ring. Then S is graded von Neumann regular if and only if S_e is von Neumann regular.

Proof. Recall that S is graded von Neumann regular if and only if every module $M \in S$-gr is gr-flat (see [15, Prop. I.5.4.3]). However, gr-flat is equivalent to flat (see [15, Prop. I.2.18]). By Dade's theorem (see Theorem 1.6.21), we have an equivalence of categories $S\text{-gr} \simeq S_e\text{-mod}$. Since flatness is a categorical property, it follows that every S_e-module is flat if and only if every graded S-module is flat. In other words, S_e is von Neumann regular if and only if S is graded von Neumann regular. □

The main result of Paper D is the following generalization of Theorem 2.4.2:

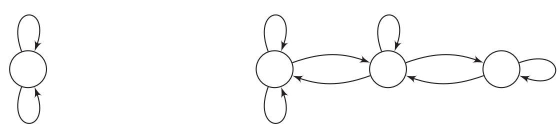
Theorem 2.4.3. Let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. Then S is graded von Neumann regular if only if the ring S_e is von Neumann regular and S is nearly epsilon-strongly G-graded.

Remark 2.4.4. Note that S is not necessarily unital in Theorem 2.4.3.

Let us now consider Leavitt path algebras. Let K be a field. Abrams and Rangaswamy [4] established that the Leavitt path algebra $L_K(E)$ is von Neumann regular if and only if E is acyclic. However, considering $L_K(E)$ equipped with its standard $\mathbb{Z}$-grading, Hazrat [10] proved that $L_K(E)$ is always graded von Neumann regular. Utilizing Theorem 2.4.3, the following extension of Hazrat's result is shown in Paper D:

Theorem 2.4.5. Let R be a unital ring and let E be a directed graph. Then $L_R(E)$ is graded von Neumann regular if and only if R is von Neumann regular.

We explain the context and significance of Theorem 2.4.5. Recall that there is a proposed program to classify Leavitt path algebras by using techniques similar to what Rørdam and Kirchberg–Phillips used to established their celebrated classification results for graph C^*-algebras (see e.g. [1, Sect. 7.3.1]). This approach utilizes certain moves defined on graphs similar to Reidemeister moves for knots. One of these moves is the so-called Cuntz splice. The Morita invariance of this move is an important open question. Even in the special cases $K = \mathbb{C}$ or $K = \mathbb{F}_2$, this is still an open question. (cf. [25, p. 16] and Tomforde's web site [24])



(A) E_2 (B) E_2^-

Figure 1. The Cuntz splice

Question 2.4.6 (Ruiz-Tomforde [22])**.** The Cuntz splice transforms the graph E_2 to E_2^- (see Figure 1). Are $L_R(E_2)$ and $L_R(E_2^-)$ Morita equivalent?

Rørdam [21] showed that $C^*(E_2) \cong C^*(E_2^-)$ as C^*-algebras. The following result provides some empirical evidence that the answer to Question 2.4.6, in the special case when $R = \mathbb{Z}$, might be 'no'.

Theorem 2.4.7 (Johansen and Sørensen [12])**.** The Leavitt path algebras $L_{\mathbb{Z}}(E_2)$ and $L_{\mathbb{Z}}(E_2^-)$ are not $*$-isomorphic.

Using Theorem 2.4.5 we can now make an observation regarding Theorem 2.4.7. Note that $L_{\mathbb{Z}}(E_2)$ and $L_{\mathbb{Z}}(E_2^-)$ are not graded von Neumann regular. On the other hand, $L_{\mathbb{C}}(E_2)$ and $L_{\mathbb{C}}(E_2^-)$ are graded von Neumann regular. This is likely not enough to fully explain the result in Theorem 2.4.7, but our observation demonstrates that the coefficient ring R does in fact matter with regards to the algebraic structure of the Leavitt path algebra $L_R(E)$.

Next, we outline some further applications obtained in Paper D. Recall that a ring R is called *semiprime* if $aRa = 0$ implies $a = 0$. Moreover, a ring R is called *semiprimitive* if the Jacobson radical $J(R)$ is zero. Abrams and Aranda Pino [3] showed that Leavitt path algebras over fields are semiprimitive and semiprime. In Paper D, we obtain the following generalization of Theorem 2.4.5:

Corollary 2.4.8. Let R be a von Neumann regular unital ring and let E be a directed graph. Then $L_R(E)$ is semiprimitive and semiprime.

We also recover the following result which Hazrat [10] previously established using other methods:

Corollary 2.4.9 (cf. [10, Prop. 8])**.** Let R be a unital ring and let $\alpha\colon R \to uRu$ be a ring isomorphism where u is an idempotent of R. The corner skew Laurent polynomial ring $R[t_+, t_-; \alpha]$ is a graded von Neumann regular ring if and only if R is von Neumann regular.

Finally, we obtain the following sufficient condition for a corner skew Laurent polynomial ring to be semiprimitive and semiprime:

Corollary 2.4.10. Let R be a unital ring and let $\alpha\colon R \to uRu$ be a ring isomorphism where u is an idempotent of R. If R is von Neumann regular, then the corner skew Laurent polynomial ring $R[t_+, t_-; \alpha]$ is semiprimitive and semiprime.

2.5. Summary of Paper E

In Paper E, we generalize a result by Passman [20] on the primeness of unital strongly graded rings to (not necessarily unital) nearly epsilon-strongly graded rings. Recall that a proper ideal P is called *prime* if for all ideals A, B such that $AB \subseteq P$ we have $A \subseteq P$ or $B \subseteq P$. A ring R is called *prime* if the zero ideal is prime. A commutative ring is prime if and only it is an integral domain. In this sense, we can view prime rings to be a non-commutative generalization of integral domains.

In 1963, Connell famously established the following characterization:

Theorem 2.5.1 (Connell [7])**.** Let R be a unital ring and let G be a group. The group ring $R[G]$ is prime if and only if R is prime and G has no non-trivial finite normal subgroups.

We will see that there are generalizations of Connell's Theorem to large classes of group graded rings. First, we need the following definition:

Definition 2.5.2 (cf. Passman [19]). Let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. For a subset $I \subseteq S$ and $x \in G$, we consider the subset $I^x := S_{x^{-1}} I S_x$. Let H be a subgroup of G. If $I^x \subseteq I$ for every $x \in H$, then I is called *H-invariant*. Let N be a normal subgroup of H. Then I is called *H/N-invariant* if $S_{C^{-1}} I S_C \subseteq I$ for every $C \in H/N$.

Example 2.5.3. In this example, we let g denote the generator of the infinite cyclic group $\mathbb{Z}$. Let $S := M_2(\mathbb{C})$ be equipped with the $\mathbb{Z}$-grading given in Example 1.5.22. Recall that:

$$S_e = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix}, \quad S_{g^{-1}} = \begin{pmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{pmatrix}, \quad S_g = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix},$$

and $S_{g^i} = \left\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \right\}$ for $|i| > 1$. Consider the following ideals of S_e:

$$(0) = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right), \qquad I_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right), \qquad \text{and} \qquad I_2 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

Then $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in S_{g^{-1}} I_1 S_g = (I_1)^g$, but $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \notin I_1$. Hence, I_1 is not $\mathbb{Z}$-invariant. Similarly, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in S_g I_2 S_{g^{-1}} = (I_2)^{g^{-1}}$, but $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \notin I_2$, which proves that I_2 is not $\mathbb{Z}$-invariant. However, (0) is a $\mathbb{Z}$-invariant ideal of $(M_2(\mathbb{C}))_0$.

Using an involved bookkeeping device, Passman obtained the following generalization of Connell's Theorem:

Theorem 2.5.4 (Passman [20]). If S is a unital strongly G-graded ring, then S is not prime if and only if there exist:

(a) subgroups $N \lhd H \subseteq G$ with N finite,
(b) an H-invariant ideal I of S_e with $I^x I = \{0\}$ for every $x \in G \setminus H$, and
(c) nonzero H-invariant ideals $\tilde{A}, \tilde{B}$ of S_N with $\tilde{A}, \tilde{B} \subseteq I S_N$ and $\tilde{A}\tilde{B} = \{0\}$.

Our main result is a generalization of Passman's Theorem 2.5.4 to (not necessarily unital) nearly epsilon-strongly graded rings. Our decision to work with nearly epsilon-strongly graded rings, rather than epsilon-strongly graded rings, is mainly motivated by our application to prime Leavitt path algebras. However, there is also a technical reason. As we discovered in Paper A, the category of nearly epsilon-strongly graded rings has the crucial property that it is closed under the induced quotient group grading construction (see Proposition 2.1.3). This allows us to adapt Passman's methods from the unital strongly graded setting, but not without non-trivial technical difficulties.

We obtain the following characterization:

Theorem 2.5.5. Let S be a nearly epsilon-strongly G-graded ring. Then S is not prime if and only if there exists:

(a) subgroups $N \lhd H \subseteq G$ with N finite,
(b) an H-invariant ideal I of S_e such that $I^x I = \{0\}$ for every $x \in G \setminus H$, and
(c) nonzero H/N-invariant ideals $\tilde{A}, \tilde{B}$ of S_N such that $\tilde{A}, \tilde{B} \subseteq I S_N$ and $\tilde{A}\tilde{B} = \{0\}$.

Remark 2.5.6. We make two remarks regarding Theorem 2.5.5:

(a) In Paper E, we give several equivalent versions of the conditions (a)-(c).
(b) Note that H/N-invariance implies H-invariance (see Lemma E.3.13). For strongly graded rings, the reverse implication holds (see Proposition 2.5.11). Hence, we recover Passman's Theorem 2.5.4 as a special case of Theorem 2.5.5.

We now summarize the applications given in Paper E. For unital partial crossed products, we obtain the following result:

Corollary 2.5.7. Let $R \star_\alpha^w G$ be a unital partial crossed product where G is a torsion-free group. Then $R \star_\alpha^w G$ is prime if and only if R is G-prime.

We obtain an analogous result for s-unital partial skew group rings (see Theorem E.13.5). We also consider prime Leavitt path algebras. Recall the following:

Definition 2.5.8. Let E be a directed graph. The graph E is said to satisfy *Condition (MT-3)* if for every pair of vertices $u, v \in E^0$, there is some vertex $w \in E^0$ such that there are paths from u to w and from v to w.

In the case of Leavitt path algebras over a field K, it was shown by Abrams, Bell and Rangaswamy that $L_K(E)$ is prime if and only if E satisfies Condition (MT-3) (see [2, Thm. 1.4]). In the case of Leavitt path algebras with coefficients in a commutative unital ring, a generalization was obtained by Larki (see [14, Prop. 4.5]). We obtain the following further generalization:

Theorem 2.5.9. Let $L_R(E)$ be a Leavitt path algebra over a unital ring R. Then $L_R(E)$ is prime if and only if R is prime and E satisfies Condition (MT-3).

Moreover, we show a Connell-type result for non-unital group rings. Often the group ring $R[G]$ is considered only for unital R. However, this construction generalizes in a straightforward manner to non-unital coefficient rings. If R is s-unital, then $R[G]$ can be proven to be nearly epsilon-strongly G-graded. We obtain the following result:

Theorem 2.5.10. Let R be an s-unital ring and let G be a group. Then the group ring $R[G]$ is prime if and only if R is prime and G has no non-trivial finite normal subgroups.

For completeness, we include a proof of the following insight by Lundström:

Proposition 2.5.11. Let $N \lhd H$ be groups and let S be a strongly H-graded ring. An ideal I of S_N is H/N-invariant if I is H-invariant.

Proof. Note that $S_N S_y = S_{Ny} = S_{yN} = S_y S_N$ for every $y \in H$. Suppose that I is H-invariant and take $x \in H$. Then $S_{x^{-1}} I S_x \subseteq I$. We thus get $S_{x^{-1}N} I S_{xN} = S_N (S_{x^{-1}} I S_x) S_N \subseteq S_N I S_N \subseteq I$ where the last inclusion follows since I is an ideal of S_N. Hence, I is H/N-invariant. $\square$

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Part II

Scientific papers

PAPER A

Induced quotient group gradings of epsilon-strongly graded rings

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Daniel Lännström

Let G be a group and let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. Given a normal subgroup N of G, there is a naturally induced G/N-grading of S. It is well-known that if S is strongly G-graded, then the induced G/N-grading is strong for any N. The class of epsilon-strongly graded rings was recently introduced by Nystedt, Öinert and Pinedo as a generalization of unital strongly graded rings. We give an example of an epsilon-strongly graded partial skew group ring such that the induced quotient group grading is not epsilon-strong. Moreover, we give necessary and sufficient conditions for the induced G/N-grading of an epsilon-strongly G-graded ring to be epsilon-strong. Our method involves relating different types of rings equipped with local units (s-unital rings, rings with sets of local units, rings with enough idempotents) with generalized epsilon-strongly graded rings.

A.1. Introduction

Let G be an arbitrary group with neutral element e. Our rings will be associative but not necessarily unital. Recall that a *G-grading* of a ring S is a family of additive subgroups $\{S_g\}_{g \in G}$ of S such that (i) $S = \bigoplus_{g \in G} S_g$ and (ii) $S_g S_h \subseteq S_{gh}$ for all $g, h \in G$. If the stronger condition $S_g S_h = S_{gh}$ holds for all $g, h \in G$, then the G-grading is called *strong*. The S_g's are called *homogeneous components*. The *principal component* S_e, i.e. the homogeneous component corresponding to the neutral element e, is a subring of S. In general, there are many different G-gradings of a fixed ring S. However, as is common in the literature, we will write 'S is a G-graded ring' when we consider S together with some implicit G-grading.

We now recall the construction of the induced quotient grading. Let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring and let N be a normal subgroup of G. There is a natural way to define a new, induced G/N-grading of S. Let,

$$S_C := \bigoplus_{g \in C} S_g, \qquad \forall C \in G/N.$$

It is straightforward to check that (i) $S = \bigoplus_{C \in G/N} S_C$ and (ii) $S_C S_{C'} \subseteq S_{CC'}$ for any $C, C' \in G/N$. The G/N-grading $\{S_C\}_{C \in G/N}$ of S is called the *induced G/N-grading*. Note that the principal component of this G/N-grading is $S_{eN} = S_N$. It is well-known that the 'strongness' of the grading is preserved under this construction: If the original G-grading $\{S_g\}_{g \in G}$ is strong, then the induced G/N-grading $\{S_C\}_{C \in G/N}$ is strong (cf. Proposition A.2.1). This property is of particular importance for the theory of strongly group graded rings initiated by Dade (see Section A.2).

The class of *epsilon-strongly G-graded rings* was introduced by Nystedt, Öinert and Pinedo [17] as a generalization of unital strongly G-graded rings. A G-grading $\{S_g\}_{g \in G}$ of S is *epsilon-strong* if any only if, for every $g \in G$, there is an element $\epsilon_g \in S_g S_{g^{-1}}$ such that for all $s \in S_g$ the equations $\epsilon_g s = s = s\epsilon_{g^{-1}}$ hold (see [17, Prop. 7]). In that case we say that S is *epsilon-strongly G-graded*. It is natural to ask if 'epsilon-strongness' is preserved by the induced quotient grading. This question was the starting point of this paper.

Question A.1.1. If $\{S_g\}_{g \in G}$ is epsilon-strong, is then $\{S_C\}_{C \in G/N}$ epsilon-strong?

We will give an example of an epsilon-strongly graded unital partial skew group ring such that the induced quotient group grading is not epsilon-strong (Example A.7.2). Our main result is a characterization of when the induced quotient group grading is epsilon-strong in terms of idempotents of the principal component (Theorem A.1.2).

To be able to formulate our main theorem, we first need some notation. For any ring R, let $E(R)$ denote the set of idempotents of R. There is a partial order on $E(R)$ defined by $a \leq b \iff a = ab = ba$. Let $\vee$ and $\wedge$ denote the least upper bound and greatest lower bound with respect to this ordering. In the case where $a, b \in E(R)$ commute, it can be proven that $a \vee b = a + b - ab$ and $a \wedge b = ab$. For any set $F \subseteq E(R)$, we write $\bigvee F$ for the $\vee$-closure of F; i.e. $a, b \in \bigvee F \implies a \vee b \in \bigvee F$ provided that the upper bound $a \vee b$ exists. Recall (see [17, Prop. 5]) that if $\{S_g\}_{g \in G}$ is epsilon-strong, then the elements ϵ_g are central idempotents of the principal component S_e. Our main result reads as follows:

Theorem A.1.2. Let S be an epsilon-strongly G-graded ring and let N be a subgroup of G. The induced G/N-grading $\{S_C\}_{C \in G/N}$ is epsilon-strong if and only if, for every $C \in G/N$, there is some element $\chi_C \in E(S_N)$ such that $f \leq \chi_C$ for all $f \in \bigvee\{\epsilon_g \mid g \in C\}$.

Our method to prove Theorem A.1.2 involves generalizations of epsilon-strongly graded rings, which we call *Nystedt-Öinert-Pinedo graded rings* (see Section A.3). The first generalization is the class of *nearly epsilon-strongly G-graded rings* introduced by Nystedt and Öinert in [16] (see also [15]). Inspired by their definition, we will introduce two additional families of graded rings: *essentially and virtually epsilon-strongly graded* *rings* (see Definition A.3.3). The proof of Theorem A.1.2 will amount to showing that any induced quotient group grading of an epsilon-strongly graded ring is essentially epsilon-strong (Theorem A.5.15).

This paper will hopefully be the beginning of a larger effort to develop a theory of epsilon-strongly graded rings similar to the theory of strongly graded rings (cf. Section A.2). Seemingly unrelated to the present investigation, the induced quotient group grading construction has been studied independently in [10] and [18].

Below is an outline of the rest of this paper:

In Section A.2, we give some additional background and a precise formulation of the problems considered in this paper.

In Section A.3, we define and prove some basic results about Nystedt-Öinert-Pinedo graded rings. In particular, we prove that only unital rings admit epsilon-strong gradings (Proposition A.3.8).

In Section A.4, we prove that Leavitt path algebras are virtually epsilon-strongly G-graded (Proposition A.4.3). This class will be an important source of examples.

In Section A.5, we investigate the induced quotient group gradings of Nystedt-Öinert-Pinedo graded rings. We prove that the induced G/N-grading of a nearly epsilon-strongly G-graded ring is also nearly epsilon-strong (Proposition A.5.8). Under certain assumptions, the same conclusion also holds for essentially epsilon-strongly graded rings (Corollary A.5.11) and virtually epsilon-strongly graded rings (Proposition A.5.13). As a special case, we get that the induced G/N-grading of a Leavitt path algebra is also virtually epsilon-strong (Corollary A.5.14). Finally, we establish our main results: Theorem A.5.15 and Theorem A.1.2.

In Section A.6, we introduce a special type of epsilon-strong G-gradings called epsilon-finite gradings. This class has the property that for any normal subgroup N of G, the induced G/N-grading is epsilon-finite (Proposition A.6.3). Moreover, one-sided noetherianity of the principal component S_e is a sufficient condition for S to be epsilon-strongly G-graded (Theorem A.6.5).

In Section A.7, we give an example of an epsilon-strongly $\mathbb{Z}$-graded ring such that the induced $\mathbb{Z}/2\mathbb{Z}$-grading is not epsilon-strong (Example A.7.2).

In Section A.8, we consider induced quotient group gradings of epsilon-crossed products (see Section A.2.2). We give sufficient conditions for the induced G/N-grading of a unital partial skew group ring to give an epsilon-crossed product (Proposition A.8.3).

A.2. The induced quotient group grading functor

Broadly speaking, we aim to investigate how the theory of strongly group graded rings relates to epsilon-strongly graded ring. The systematic study of strongly group graded rings began with Dade's seminal paper [6]. In that paper, he took a functorial approach and studied certain functors defined on the categories of strongly group graded rings and modules. Most notable is the celebrated Dade's Theorem, which asserts that a G-graded ring S is strongly graded if and only if the category of graded modules over S is equivalent to the category of modules over S_e. The following introductory example from [6] shows the relation to classical Clifford theory. Let G be an arbitrary group and recall that the complex group ring $\mathbb{C}[G] = \bigoplus_{g \in G} \mathbb{C}\delta_g$ is strongly G-graded. Furthermore, let N

be a normal subgroup of G. The induced quotient group grading $\mathbb{C}[G] = \bigoplus_{C \in G/N} S_C$ is strong. Note that $S_N = \bigoplus_{g \in N} \mathbb{C}\delta_g = \mathbb{C}[N]$. By Dade's Theorem, there is an equivalence of categories between graded $\mathbb{C}[G]$-modules and $\mathbb{C}[N]$-modules. This example motivates us to take a functorial approach and to study the induced quotient group gradings of epsilon-strongly graded rings.

In the remainder of this section, we will introduce further notation to precisely formulate the problems considered in this paper.

A.2.1. Category of epsilon-strongly graded rings. Let G-RING denote the category of rings equipped with a G-grading. More precisely, the objects of G-RING are pairs $(S, \{S_g\}_{g \in G})$ where S is a ring and $\{S_g\}_{g \in G}$ is a G-grading of S. The morphisms are ring homomorphisms that respect the gradings. To make this more precise, let $(S, \{S_g\}_{g \in G})$ and $(T, \{T_g\}_{g \in G})$ be objects in G-RING. The ring homomorphism $\phi \colon S \to T$ is called *G-graded* if $\phi(S_g) \subseteq T_g$ for each $g \in G$. The class $\hom((S, \{S_g\}_{g \in G}), (T, \{T_g\}_{g \in G}))$ consists of the G-graded ring homomorphisms $S \to T$. By the definition of G-RING, it is straightforward to see that the category of strongly G-graded rings, which we denote by G-STRG, is a full subcategory of G-RING. We will later work with other subclasses of G-gradings, which similarly corresponds to full subcategories of G-RING.

Next, we recall (see e.g. [13, pg. 3]) the definition of the *induced quotient grading functor*. With the notation above, let N be a normal subgroup of G and let $\{S_C\}_{C \in G/N}$, $\{T_C\}_{C \in G/N}$ be the induced G/N-gradings of S and T respectively. If $\phi \colon S \to T$ is a G-graded homomorphism, then $\phi(S_C) \subseteq T_C$ for each $C \in G/N$. Hence, $\phi \colon S \to T$ is G/N-graded with respect to the induced G/N-gradings. This implies that the induced quotient group grading construction defines a functor,

$$\begin{gathered} U_{G/N} \colon G\text{-RING} \to G/N\text{-RING}, \\ (S, \{S_g\}_{g \in G}) \mapsto (S, \{S_C\}_{C \in G/N}), \\ \hom((S, \{S_g\}_{g \in G}), (T, \{T_g\}_{g \in G})) \ni \phi \mapsto \phi \in \hom((S, \{S_C\}_{C \in G/N}), (T, \{T_C\}_{C \in G/N})). \end{gathered}$$

It is well-known that 'strongness' is preserved by the induced quotient group grading:

Proposition A.2.1. (cf. [13, Prop. 1.2.2]) Let G be an arbitrary group and let N be any normal subgroup of G. The functor $U_{G/N}$ restricts to the subcategory G-STRG. In other words, the functor,

$$U_{G/N} \colon G\text{-STRG} \to G/N\text{-STRG},$$

is well-defined.

We denote the category of epsilon-strongly G-graded rings by G-ϵSTRG. The objects of this category are epsilon-strongly G-graded rings and the morphisms are G-graded ring homomorphisms. The basic problem of this paper (cf. Question A.1.1) can be reformulated as:

Question A.2.2. For which objects $(S, \{S_g\}_{g \in G}) \in \mathrm{ob}(G\text{-}\epsilon\mathrm{STRG})$ do we have that,

$$U_{G/N}((S, \{S_g\}_{g \in G})) = (S, \{S_C\}_{C \in G/N}) \in \mathrm{ob}(G/N\text{-}\epsilon\mathrm{STRG})? \tag{10}$$

In particular, is the restriction of $U_{G/N}$ to G-ϵSTRG well-defined?

We will give an example of an epsilon-strongly G-graded ring such that (10) does not hold (Example A.7.2). In other words, the functor $U_{G/N}$ does not restrict to G-ϵSTRG. Note that Theorem A.1.2 provides a complete answer to Question A.2.2, i.e. a characterization of $(S, \{S_g\}_{g \in G}) \in \text{ob}(G\text{-}\epsilon\text{STRG})$ such that (10) holds.

A.2.2. Epsilon-crossed products. Let S be a strongly G-graded ring. Recall (see e.g. [13, pg. 2]) that S is called an *algebraic crossed product* if S_g contains an invertible element for each $g \in G$. The class of epsilon-crossed products was introduced by Nystedt, Öinert and Pinedo in [17] as an 'epsilon-analogue' of the classical algebraic crossed products. Let G be an arbitrary group and let S be an arbitrary epsilon-strongly G-graded ring. In other words, take an arbitrary object $(S, \{S_g\}_{g \in G}) \in \text{ob}(G\text{-}\epsilon\text{STRG})$. Recall (see [17, Def. 30]) that an element $s \in S_g$ is called *epsilon-invertible* if there exists some element $t \in S_{g^{-1}}$ such that $st = \epsilon_g$ and $ts = \epsilon_{g^{-1}}$. Furthermore, recall (see [17, Def. 32]) that $(S, \{S_g\}_{g \in G})$ is called an *epsilon-crossed product* if there is an epsilon-invertible element in S_g for all $g \in G$. Let G-ϵCROSS denote the category of epsilon-crossed products. The morphisms are G-graded ring homomorphisms. It is straightforward to show that G-ϵCROSS is a full subcategory of G-ϵSTRG.

For an algebraic crossed-product, the induced quotient group grading gives an algebraic crossed product (see e.g. [13, Prop. 1.2.2]). It is natural to ask when the induced quotient grading of an epsilon-crossed product gives an epsilon-crossed product. This is better formulated in terms of the functor $U_{G/N}$:

Question A.2.3. For which objects $(S, \{S_g\}_{g \in G}) \in \text{ob}(G\text{-}\epsilon\text{CROSS})$ do we have that,

$$U_{G/N}((S, \{S_g\}_{g \in G}) = (S, \{S_C\}_{C \in G/N}) \in \text{ob}(G/N\text{-}\epsilon\text{CROSS})?$$

The author has not been able to answer Question A.2.3 in full generality. However, in Section 7, we will provide examples of epsilon-crossed products $(S, \{S_g\}_{g \in G}) \in \text{ob}(G\text{-}\epsilon\text{CROSS})$ such that $(S, \{S_C\}_{C \in G/N}) \in \text{ob}(G/N\text{-}\epsilon\text{CROSS})$ and examples such that $(S, \{S_C\}_{C \in G/N}) \notin \text{ob}(G/N\text{-}\epsilon\text{CROSS})$.

A.3. Nystedt-Öinert-Pinedo graded rings

The purpose of this section is to introduce two new generalizations of epsilon-strongly graded rings. To this end, we recall several different ways in which a non-unital ring might have approximate multiplicative identity elements (also known as local units). We refer the reader to [14] for a detailed survey of these definitions. A ring R has a *set of local units* E if E is a $\vee$-closed subset of $E(R)$ consisting of commuting idempotents such that for every $r \in R$ there exists some $f \in E$ such that $fr = rf = r$ (cf. [3] and [14, Def. 21]). A ring R is said to have *enough idempotents* if there is a set F of pairwise orthogonal, commuting idempotents such that $\bigvee F$ is a set of local units for R (cf. e.g. [14, Def. 27]). Finally, recall that a ring R is called *s-unital* if $x \in xR \cap Rx$ for each $x \in R$. This is equivalent to that there, for every positive integer n and elements $s_1, s_2, \ldots, s_n \in R$, exists some $f \in R$ satisfying $fs_i = s_i f = s_i$ for all $1 \leq i \leq n$ (see [20, Thm. 1]). These definitions relate to each other in the following way.

Proposition A.3.1. ([14]) The following strict inclusions hold between the classes of rings.

$$\begin{aligned}\{\text{unital rings}\} &\subsetneq \{\text{rings with enough idempotents}\}\\ &\subsetneq \{\text{rings with sets of local units}\}\\ &\subsetneq \{\text{s-unital rings}\}.\end{aligned}$$

Before defining Nystedt-Öinert-Pinedo graded rings, we recall the following:

Definition A.3.2. ([5, Def. 4.5]) A G-graded ring S is called *symmetrically graded* if,

$$S_g S_{g^{-1}} S_g = S_g, \qquad \forall g \in G.$$

Consider a G-graded ring $S = \bigoplus_{g \in G} S_g$ where the principal component is S_e. Note that $S_g S_{g^{-1}} \subseteq S_e$ is an S_e-ideal for each $g \in G$. The classes of Nystedt-Öinert-Pinedo graded rings correspond to symmetrically G-graded rings with local unit properties on the rings $S_g S_{g^{-1}}$ according to the following:

class	$S_g S_{g^{-1}}$
epsilon-strongly (see [17])	unital ring
virtually epsilon-strongly	ring with enough idempotents
essentially epsilon-strongly	ring with sets of local units
nearly epsilon-strongly (see [16])	s-unital ring

FIGURE 1. Nystedt-Öinert-Pinedo classes of graded rings

We also explicitly write down these crucial definitions.

Definition A.3.3. Let $S = \bigoplus_{g \in G} S_g$ be a symmetrically G-graded ring.

(a) If $S_g S_{g^{-1}}$ is a unital ring for each $g \in G$, then S is called *epsilon-strongly graded* (cf. [17, Prop. 7]).
(b) If $S_g S_{g^{-1}}$ is a ring with enough idempotents for each $g \in G$, then S is called *virtually epsilon-strongly graded.*
(c) If $S_g S_{g^{-1}}$ is a ring with a set of local units for each $g \in G$, then S is called *essentially epsilon-strongly graded.*
(d) If $S_g S_{g^{-1}}$ is an s-unital ring for each $g \in G$, then S is called *nearly epsilon-strongly graded* (cf. [16, Prop. 10]).

The notion of an essentially epsilon-strong grading will be central to our study of the induced quotient group gradings of epsilon-strongly graded rings (see Theorem A.5.15). Moreover, virtually epsilon-strong gradings relate to Leavitt path algebras (see Proposition A.4.3).

Remark A.3.4. We make some remarks concerning Definition A.3.3.

(a) By Proposition A.3.1, we have the following strict inclusions between the classes of Nystedt-Öinert-Pinedo graded rings:

$$\begin{aligned}\{\text{epsilon-strongly graded rings}\} &\subsetneq \{\text{virtually epsilon-strongly graded rings}\}\\ &\subsetneq \{\text{essentially epsilon-strongly graded rings}\}\\ &\subsetneq \{\text{nearly epsilon-strongly graded rings}\}.\end{aligned}$$

(b) Let R be a ring that has a set of local unit but does not have enough idempotents (see [14, Expl. 28]). Then R is graded by the trivial group by putting $S_e := R$. Note that $RR = R$ is a ring with a set of local units. Hence, R is trivially essentially epsilon-strongly graded. However, since R does not have enough idempotents, this grading cannot be virtually epsilon-strong. In fact, this is our only example distinguishing essentially epsilon-strong gradings from virtually epsilon-strong gradings (see Remark A.5.16).

We recall the following characterization, which will be used implicitly in the rest of this paper. Note that Proposition A.3.5(a) implies that the definition of epsilon-strong given in the introduction is equivalent to Definition A.3.3(a).

Proposition A.3.5. ([16, Prop. 7, Prop. 10]) Let $S = \bigoplus_{g\in G} S_g$ be a G-graded ring. The following assertions hold:

(a) S is epsilon-strongly graded if and only if, for each $g \in G$, there exist $\epsilon_g \in S_g S_{g^{-1}}$ and $\epsilon_{g^{-1}} \in S_{g^{-1}} S_g$ such that $\epsilon_g s = s = s\epsilon_{g^{-1}}$ for every $s \in S_g$;
(b) S is nearly epsilon-strongly graded if and only if, for each $g \in G$ and $s \in S_g$, there exist $\epsilon_g(s) \in S_g S_{g^{-1}}$ and $\epsilon'_g(s) \in S_{g^{-1}} S_g$ such that $\epsilon_g(s)s = s = s\epsilon'_g(s)$.

For unital rings, the class of strongly graded rings is a subclass of epsilon-strongly graded rings. However, for general non-unital rings, this is not the case.

Proposition A.3.6. A unital strongly G-graded ring $S = \bigoplus_{g\in G} S_g$ is epsilon-strongly G-graded.

PROOF. Note that $S_g S_{g^{-1}} = S_e$ is a unital ring for every $g \in G$. □

Example A.3.7. (Example of a strongly graded ring that is not epsilon-strongly graded)

Let R be an idempotent ring without multiplicative identity and consider R graded by the trivial group by putting $S_e := R$. Since $S_e S_e = RR = R = S_e$, the grading is strong. On the other hand, $S_e S_e = R$ is not unital. Hence, by definition, the grading cannot be epsilon-strong.

In fact, it turns out that every epsilon-strongly G-graded ring is unital. For this purpose, we recall that if R is a ring with a left multiplicative identity element ϵ and a right multiplicative identity element ϵ', then $\epsilon = \epsilon'$ is a multiplicative identity element of R.

Proposition A.3.8. If S is an epsilon-strongly G-graded ring, then S is unital with multiplicative identity element ϵ_e. In that case, S_e is a unital subring of S.

Proof. Let R denote the principal component of S. By Proposition A.3.5, there are two, a priori distinct, elements $\epsilon_e, \epsilon'_e \in S_e^2 \subseteq R$ such that $\epsilon_e r = r = r\epsilon'_e$ for any $r \in R$. This means that $\epsilon_e = \epsilon'_e$ is a multiplicative identity element of R. Take an arbitrary element $g \in G$. There are $\epsilon_g \in S_g S_{g^{-1}} \subseteq R$ and $\epsilon'_g \in S_{g^{-1}} S_g \subseteq R$ such that $\epsilon_g s_g = s_g = s_g \epsilon'_g$ for all $s_g \in S_g$. Fix an arbitrary $s_g \in S_g$. Then,

$$\epsilon_e s_g = \epsilon_e(\epsilon_g s_g) = (\epsilon_e \epsilon_g) s_g = \epsilon_g s_g = s_g,$$

and similarly, $s_g \epsilon_e = s_g$. Since a general element $s \in S$ is a finite sum $s = \sum s_g$ of elements $s_g \in S_g$, it follows that ϵ_e is a multiplicative identity element of S. □

Remark A.3.9. Note that by Proposition A.3.8, only unital rings admit an epsilon-strong grading. However, there are a lot of virtually epsilon-strongly graded rings which are not unital. In the next section we will give an example of such a ring (Example A.4.5).

For the remainder of this section, we briefly consider gradings of the factor ring S/I. If S is G-graded, then S/I inherits a G-grading for certain ideals I. More precisely, let G be an arbitrary group and let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. Recall that an ideal I of S is called *homogeneous* (or graded) if $I = \bigoplus_{g \in G}(I \cap S_g)$. If I is a homogeneous ideal, then the factor ring is naturally G-graded by,

$$S/I = \bigoplus_{g \in G} S_g/(I \cap S_g) = \bigoplus_{g \in G} (S_g + I)/I. \tag{11}$$

We will show that if S is epsilon-strongly G-graded, then S/I is epsilon-strongly G-graded.

Lemma A.3.10. Let $\phi \colon A \to B$ be a G-graded epimorphism of G-graded rings A and B. If A is epsilon-strongly G-graded, then B is epsilon-strongly G-graded.

Proof. Suppose $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{g \in G} B_g$. For every $g \in G$, let ϵ_g be the multiplicative identity element of $A_g A_{g^{-1}}$. Using that A is epsilon-strongly G-graded, we have that $A_g A_{g^{-1}} A_g = A_g$ and $A_g A_{g^{-1}} = \epsilon_g A_e$ for every $g \in G$. Since ϕ is a G-graded epimorphism, we have that $\phi(A_g) = B_g$ for every $g \in G$. Applying ϕ to both equations we get $B_g B_{g^{-1}} B_g = B_g$ and $B_g B_{g^{-1}} = \phi(\epsilon_g) B_e$ for every $g \in G$. Hence, B is epsilon-strongly G-graded. □

Proposition A.3.11. Let S be an epsilon-strongly G-graded ring. If I is a homogeneous ideal of S, then the natural G-grading of S/I is epsilon-strong.

Proof. Follows by Lemma A.3.10 since the natural epimorphism $\pi \colon S \to S/I$ is G-graded. □

A.4. Leavitt path algebras

In this section, we will show that the Leavitt path algebra associated to any directed graph is virtually epsilon-strongly graded. Let R be an arbitrary unital ring and let $E = (E^0, E^1, s, r)$ be a directed graph. Here, E^0 denotes the vertex set, E^1 denotes the set of edges and the maps $s \colon E^1 \to E^0, r \colon E^1 \to E^0$ specify the *source* and the *range*, respectively, of each edge $f \in E^1$. The Leavitt path algebra $L_R(E)$ of E with coefficients in R is an algebraic analogue of the graph C^*-algebra associated to E. For more details about Leavitt path algebras, we refer the reader to the monograph by Abrams, Ara, and Siles Molina [1]. In the case of R being a field, Leavitt path algebras were first considered by Ara, Moreno and Pardo [4] and Abrams and Aranda Pino in [2]. Later, Tomforde [19] considered Leavitt path algebras with coefficients in a commutative ring. We follow Hazrat [9] and let R be a general (possibly non-commutative) unital ring.

Definition A.4.1. For a directed graph $E = (E^0, E^1, s, r)$ and a unital ring R, the *Leavitt path algebra with coefficients in* R is the R-algebra $L_R(E)$ generated by the symbols $\{v \mid v \in E^0\}$, $\{f \mid f \in E^1\}$ and $\{f^* \mid f \in E^1\}$ subject to the following relations:

(a) $uv = \delta_{u,v}u$ for all $u, v \in E^0$,
(b) $s(f)f = fr(f) = f$ and $r(f)f^* = f^*s(f) = f^*$ for all $f \in E^1$,
(c) $f^*f' = \delta_{f,f'}r(f)$ for all $f, f' \in E^1$,
(d) $\sum_{f\in E^1, s(f)=v} ff^* = v$ for all $v \in E^0$ for which $s^{-1}(v)$ is non-empty and finite.

We let R commute with the generators.

A *path* is a sequence of edges $\alpha = f_1 f_2 \dots f_n$ such that $r(f_i) = s(f_{i+1})$ for $1 \leq i \leq n-1$. We write $s(\alpha) = s(f_1)$ and $r(\alpha) = r(f_n)$. Using the relations in Definition A.4.1, it can be shown that a general element of $L_R(E)$ has the form of a finite sum $\sum r_i \alpha_i \beta_i^*$ where $r_i \in R$, α_i and β_i are paths such that $r(\alpha_i) = s(\beta_i^*) = r(\beta_i)$ for every i. There is an anti-graded involution on $L_R(E)$ defined by $f \mapsto f^*$ for every $f \in E^1$. The image of a path $\alpha = f_1 f_2 \dots f_n$ under the involution is $\alpha^* = f_n^* f_{n-1}^* \dots f_1^*$. Note that for any elements $\alpha, \beta \in L_R(E)$ we have that $(\alpha\beta)^* = \beta^*\alpha^*$.

Next, we recall (see e.g. [16, Sect. 4]) a process to assign a G-grading to $L_R(E)$ for an arbitrary group G. Let $F_R(E) = R\langle v, f, f^* \mid v \in E^0, f \in E^1 \rangle$ denote the free R-algebra generated by all symbols of the form v, f, f^*. Put $\deg(v) = e$ for each $v \in E^0$. For every $f \in E^1$, choose a $g \in G$ and put $\deg(f) = g$ and $\deg(f^*) = g^{-1}$ This extends to a G-grading of $F_R(E)$ in the obvious way. Next, let J be the ideal generated by the relations (a)-(d) in Definition A.4.1. It is easy to check that J is homogeneous and thus the factor algebra $L_R(E) = F_R(E)/J$ is G-graded by the factor G-grading (see (11)). This G-grading is called a *standard* G*-grading* of G. Note that this construction depends on which elements $g \in G$ we assign to the generators. In the special case of $G = \mathbb{Z}$, the natural choice is to put $\deg(f) = 1$ and $\deg(f^*) = -1$ for all $f \in E^1$. In this special case, $\deg(\alpha) = \mathrm{len}(\alpha)$ for any real path α. The resulting grading is called the *canonical* $\mathbb{Z}$*-grading* of $L_R(E)$. For a general standard G-grading, the homogeneous component of degree $h \in G$ can be expressed as,

$$(L_R(E))_h = \mathrm{Span}_R\{\alpha\beta^* \mid \deg(\alpha)\deg(\beta)^{-1} = h, r(\alpha) = r(\beta)\}.$$

The canonical $\mathbb{Z}$-grading was investigated by Hazrat [9]. Among other results, he gave a criterion on the finite graph E for the Leavitt path algebra $L_R(E)$ to be strongly $\mathbb{Z}$-graded (see [9, Thm. 3.11]). Continuing the investigation of the group graded structure of Leavitt path algebras, Nystedt and Öinert introduced the class of nearly epsilon-strongly graded rings and proved the following.

Proposition A.4.2. ([16, Thm. 28]) Let G be an arbitrary group, let R be a unital ring and let E be a directed graph. Then any standard G-grading of $L_R(E)$ is nearly epsilon-strong. Note that, in particular, $L_R(E)$ is symmetrically G-graded.

Having introduced the notion of virtually epsilon-strongly graded rings, the author of the present paper realized that Proposition A.4.2 could be made more precise. We recall that $L_R(E)$ is a ring with enough idempotents (see e.g. [1, Lem. 1.2.12(v)]). In line with this property, it turns out that any standard G-grading of $L_R(E)$ is virtually epsilon-strong.

Proposition A.4.3. Let G be an arbitrary group. If R is a unital ring and E is any graph, then every standard G-grading of $L_R(E)$ is virtually epsilon-strong.

Proof. To save space we denote $(L_R(E))_g$ by S_g. Since the standard G-grading is symmetric by Proposition A.4.2, it is enough to show that $S_g S_{g^{-1}}$ is a ring with enough idempotents for each $g \in G$.

Take $g \in G$. We will show that $S_g S_{g^{-1}}$ has enough idempotents by constructing a set M of pairwise orthogonal, commuting idempotents of $S_g S_{g^{-1}}$ such that $E_g = \bigvee M$ is a set of local units for $S_g S_{g^{-1}}$. Let $A = \{\alpha \mid \alpha\beta^* \in S_g S_{g^{-1}}\}$ and define a partial order of A by letting $\alpha \leq \beta$ if and only if α is an initial subpath of β. Next, let,

$$M = \{\alpha\alpha^* \mid \alpha \text{ minimal in } (A, \leq)\}.$$

By construction, if $\alpha\alpha^* \in M$ there exists some β such that $\alpha\beta^* \in S_g S_{g^{-1}}$. Then, $\beta\alpha^* \in S_{g^{-1}} S_g$. Hence, $\alpha\alpha^* = (\alpha\beta^*)(\beta\alpha^*) \in (S_g S_{g^{-1}})(S_{g^{-1}} S_g) \subseteq (S_g S_{g^{-1}}) S_e = S_g S_{g^{-1}}$. Thus, $M \subseteq S_g S_{g^{-1}}$.

Recall (see [1, Lem. 1.2.12]) that the following equation holds for any paths $\gamma, \delta, \lambda, \rho$:

$$(\gamma\delta^*)(\lambda\rho^*) = \begin{cases} \gamma\kappa\rho^* & \text{if } \lambda = \delta\kappa \text{ for some path } \kappa \\ \gamma\sigma^*\rho^* & \text{if } \delta = \lambda\sigma \text{ for some path } \sigma \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

In particular, for any paths α, β:

$$(\alpha\alpha^*)(\beta\beta^*) = (\beta\beta^*)(\alpha\alpha^*) = \begin{cases} \alpha\alpha^* & \alpha = \beta \\ \beta\beta^* & \alpha \leq \beta \\ \alpha\alpha^* & \beta \leq \alpha \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

It follows from (13) that the set M consists of pairwise orthogonal, commuting idempotents.

Note that for each $\alpha\beta^* \in S_g S_{g^{-1}}$ there exists a unique element $\delta\delta^* \in M$ such that $\delta \leq \alpha$. Hence, we can define a function by $\mu(\alpha\beta^*) = \delta\delta^*$. Since $\alpha = \delta\delta'$ for some path δ', it follows by (12) that $\mu(\alpha\beta^*)\alpha\beta^* = (\delta\delta^*)(\alpha\beta^*) = \delta\delta'\beta^* = \alpha\beta^*$ for any $\alpha\beta^* \in S_g S_{g^{-1}}$. On the other hand, let $\alpha_1\beta_1^*, \alpha_2\beta_2^* \in S_g S_{g^{-1}}$ such that,

$$\delta_1\delta_1^* = \mu(\alpha_1\beta_1^*) \neq \mu(\alpha_2\beta_2^*) = \delta_2\delta_2^*, \tag{14}$$

for some paths δ_1, δ_2 satisfying $\delta_1 \leq \alpha_1$ and $\delta_2 \leq \alpha_2$. Note that $\delta_1 \not\leq \alpha_2$ and $\delta_2 \not\leq \alpha_1$ as otherwise (14) would not hold. Moreover, it follows by (14) and the definition of M that

$\alpha_2 \not\leq \delta_1$ and $\alpha_1 \not\leq \delta_2$. Thus, by (12), we have that $\mu(\alpha_1\beta_1^*)\alpha_2\beta_2^* = (\delta_1\delta_1^*)(\alpha_2\beta_2^*) = 0$ and similarly $\mu(\alpha_2\beta_2^*)\alpha_1\beta_1^* = (\delta_2\delta_2^*)(\alpha_1\beta_1^*) = 0$.

Consider an arbitrary $s = \sum_{i\in I} r_i\alpha_i\beta_i^* \in S_g S_{g^{-1}}$ for some finite index set I. Let $J \subseteq I$ the subset of elements $\alpha_i\beta_i^*$ with unique images under the map μ, i.e. for all $i \in I$ there is some $j \in J$ such that $\mu(\alpha_i\beta_i^*) = \mu(\alpha_j\beta_j^*)$. Put $e = \bigvee_{j\in J} \mu(\alpha_j\beta_j^*) = \sum_{j\in J} \mu(\alpha_j\beta_j^*)$. Then,

$$\begin{aligned} es &= \Big(\sum_{j\in J} \mu(\alpha_i\beta_i^*)\Big)\Big(\sum_{i\in I} r_i\alpha_i\beta_i^*\Big) = \sum_{i\in I, j\in J} \mu(\alpha_j\beta_j^*) r_i\alpha_i\beta_i^* \\ &= \sum_{i\in I} \mu(\alpha_i\beta_i^*) r_i\alpha_i\beta_i^* = \sum_{i\in I} r_i\mu(\alpha_i\beta_i^*)\alpha_i\beta_i^* = \sum_{i\in I} r_i\alpha_i\beta_i^* = s. \end{aligned}$$

Note that $s^* = \sum_{i\in I}(\alpha_i\beta_i^*)^* = \sum_{i\in I}\beta_i\alpha_i^* \in S_g S_{g^{-1}}$. Hence, by the above argument there exists some element $f \in \bigvee M \subseteq S_g S_{g^{-1}}$ such that $fs^* = s^*$. But by construction $f = \sum \alpha\alpha^*$ for some finite sum, implying that $f^* = \sum(\alpha\alpha^*)^* = \sum \alpha\alpha^* = f$. Thus, $s = (s^*)^* = (fs^*)^* = (s^*)^* f^* = sf^* = sf$.

Finally, note that $e \vee f \in \bigvee M$ and $(e \vee f)s = s(e \vee f) = s$. Hence, $E_g = \bigvee M$ is a set of local units for $S_g S_{g^{-1}}$. □

We end this section with two examples that distinguish the class of virtually epsilon-strongly graded rings from strongly graded rings and epsilon-strongly graded rings.

Example A.4.4. (Virtually epsilon-strongly graded but not strongly graded) Let R be a unital ring and let E be the finite graph in Figure 2. Note that $L_R(E)$ is not strongly $\mathbb{Z}$-graded since v_2 is a sink (see [9, Thm. 3.15]). However, $L_R(E)$ is epsilon-strongly $\mathbb{Z}$-graded since E is finite (see [16, Thm. 24]). Recall that epsilon-strongly graded implies virtually epsilon-strongly graded (see Remark A.3.4(a)). Thus, $L_R(E)$ is virtually epsilon-strongly $\mathbb{Z}$-graded but not strongly $\mathbb{Z}$-graded.

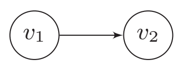

FIGURE 2. Finite graph with a sink

Example A.4.5. (Virtually epsilon-strongly graded but not epsilon-strongly graded) Let R be a unital ring and let E be a graph consisting of infinitely many disjoint vertices, i.e. $E^0 = \{v_n \mid n \geq 0\}$ and $E^1 = \emptyset$ (see Figure 3). Consider $L_R(E)$ with the canonical $\mathbb{Z}$-grading. Since E^0 is infinite, $(L_R(E))_0$ does not admit a multiplicative identity element. Hence, by Proposition A.3.8, $L_R(E)$ cannot be epsilon-strongly graded. Furthermore, for any integer k such that $|k| > 0$, $(L_R(E))_k = \{0\}$. Hence, $(L_R(E))_k(L_R(E))_{-k} = \{0\} \neq (L_R(E))_0$ for $|k| > 0$ implying that $L_R(E)$ is not strongly $\mathbb{Z}$-graded. On the other hand, $L_R(E)$ is virtually epsilon-strongly $\mathbb{Z}$-graded by Proposition A.4.3.

A.5. Induced quotient group gradings of Nystedt-Öinert-Pinedo rings

In this section, we will derive our main results about the induced quotient group gradings of epsilon-strongly graded rings (see Theorem A.5.15 and Theorem A.1.2).

FIGURE 3. Discrete infinite graph

This involves a systematic study of induced quotient group gradings of Nystedt-Öinert-Pinedo rings. Recall that for idempotents $e, f \in E(R)$ we write $e \leq f$ if and only if $e = ef = fe$, i.e. e absorbs f. We will think about $e \leq f$ as expressing that f can be used as a "local unit" in place of e. More precisely, we have the following.

Lemma A.5.1. Let $e, f \in E(R)$ be idempotents of R. Then, $e \leq f$ is equivalent to the following: For any $x \in R$,
(a) $x = ex \implies x = fx$, and,
(b) $x = xe \implies x = xf$.

Proposition A.5.2. Let R be a ring and let E be a set of local units for R. Then R is a unital ring if and only if there exists some $e' \in E(R)$ such that $e \leq e'$ for all $e \in E$. If such an element exists, then it is the multiplicative identity element of R.

PROOF. Assume that $e' \in E(R)$ satisfies $e \leq e'$ for all $e \in E$. Let $x \in R$ be any element. Then, there is some $e_x \in E$ such that $x = e_x x = x e_x$. But by assumption, $e_x \leq e'$. By Lemma A.5.1, $x = e'x = xe'$. Hence e' is the multiplicative identity element of R.

Conversely, assume that R is unital with multiplicative identity 1. Then, $e \leq 1$ for all $e \in E$. □

Example A.5.3. Consider $R_1 := \bigoplus_{\mathbb{Z}} \mathbb{Q}$ and $R_2 := \prod_{\mathbb{Z}} \mathbb{Q}$. For $i \in \mathbb{Z}$, let $\delta_i \colon \mathbb{Z} \to \mathbb{Q}$ be the function such that for each integer j, $\delta_i(j) = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker delta. Since δ_i has finite support, $\delta_i \in R_1 \subset R_2$ for each $i \in \mathbb{Z}$. It is easy to see that the set $E = \bigvee\{\delta_i \mid i \in \mathbb{Z}\}$ is a set of local units for both R_1 and R_2. Note that R_2 is unital with multiplicative identity $1_{R_2} = \bigvee_{i \in \mathbb{Z}} \delta_i$ while R_1 is not unital.

The following characterization becomes very useful when E is a finite set.

Corollary A.5.4. Let R be a ring with a set of local units E. If E is a finite set, then $(E, \leq)$ has a greatest element and thus R is a unital ring.

PROOF. Assume that $e_1, \ldots, e_n$ are the elements of E. Then $e_1 \vee \cdots \vee e_n \in E$ exists since it is a finite join. Furthermore, $e_1 \vee \cdots \vee e_n \in R$ is the greatest element of E with respect to the ordering $\leq$. Thus, R is unital by Proposition A.5.2. □

Before considering induced quotient group gradings of Nystedt-Öinert-Pinedo rings, we show that the functor $U_{G/N}$ restricts to the category of symmetrically G-graded rings.

Proposition A.5.5. Let G be an arbitrary group and let N be a normal subgroup of G. If S is symmetrically G-graded, then the induced G/N-grading is symmetric.

PROOF. We need to show that for any class $C \in G/N$, we have that $S_C S_{C^{-1}} S_C = S_C$. The inclusion $S_C S_{C^{-1}} S_C \subseteq S_{CC^{-1}C} = S_C$ holds for any $C \in G/N$. Let $[g] =$

$gN \in G/N$ denote the coset of $g \in G$. It remains to prove that $S_{[g]} \subseteq S_{[g]}S_{[g]^{-1}}S_{[g]}$ for all $g \in G$. But $S_{[g]} = \bigoplus_{n \in N} S_{gn}$, hence it is enough to show $S_{gn} \subseteq S_{[g]}S_{[g]^{-1}}S_{[g]}$ for all $n \in N$ and $g \in G$. But, since S is symmetrically G-graded, it holds that $S_{gn} \subseteq S_{gn}S_{(gn)^{-1}}S_{gn}$. Furthermore, $(gn)^{-1} = n^{-1}g^{-1} = g^{-1}n'$ for some $n' \in N$ since N is normal. Thus, $S_{gn} \subseteq S_{[g]}S_{[g]^{-1}}S_{[g]}$. □

Note that Proposition A.5.5 implies that the induced quotient group grading of any Nystedt-Öinert-Pinedo ring is symmetric. Hence, we will focus on deciding which of the local unit properties of Definition A.3.3 are preserved.

Before considering nearly epsilon-strongly graded rings, we recall the following useful but somewhat obscure result. For the convenience of the reader, we include a proof.

Lemma A.5.6. ([20, Thm. 1]) Let T be a ring and let M be a left (right) T-module. Take a finite subset $X \subseteq M$ and assume that for each $x \in X$ there is some $e_x \in T$ such that $e_x x = x$ ($xe_x = x$). Then, there is some $e \in T$ such that $ex = x$ ($xe = x$) for all $x \in X$.

Proof. We only prove the left case as the right case is treated analogously. The proof goes by induction on the size of the set $X = \{x_1, x_2, \ldots, x_n\}$.

The base case $n = 1$ is clear.

Assume that the lemma holds for $n = k$ for some $k > 0$ and consider the subset,

$$\{x_1, x_2, \ldots x_k, x_{k+1}\} \subseteq X.$$

Then, for all $1 \leq i \leq k+1$, there is some $e_i \in T$ such that $e_i x_i = x_i$. For $1 \leq i \leq k$, let $v_i = x_i - e_{k+1}x_i$. By the induction hypothesis, there is some $e' \in T$ such that $e'v_i = v_i$ for $1 \leq i \leq k$. Put $e = e' + e_{k+1} - e'e_{k+1}$. It is clear that $e \in T$. Moreover,

$$ex_{k+1} = e'x_{k+1} + e_{k+1}x_{k+1} - e'e_{k+1}x_{k+1} = e'x_{k+1} + x_{k+1} - e'x_{k+1} = x_{k+1},$$

and, for $1 \leq i \leq k$,

$$\begin{aligned} ex_i &= e'x_i + e_{k+1}x_i - e'e_{k+1}x_i = e'(x_i - e_{k+1}x_i) + e_{k+1}x_i = e'v_i + e_{k+1}x_i \\ &= v_i + e_{k+1}x_i = x_i. \end{aligned}$$

Hence, the lemma follows by the induction principle. □

Throughout the rest of this section, let N be a normal subgroup of G and let $S = \bigoplus_{g \in G} S_g$ be a nearly epsilon-strongly G-graded ring.

Lemma A.5.7. Consider a fixed class $C \in G/N$. For any positive integer n and elements $s_{g_1}, s_{g_2}, \ldots, s_{g_n}$ such that $s_{g_i} \in S_{g_i}$ and $g_i \in C$ for $1 \leq i \leq n$, there exists some $e \in S_C S_{C^{-1}}$ ($e' \in S_{C^{-1}}S_C$) such that $es_{g_i} = s_{g_i}$ ($s_{g_i}e' = s_{g_i}$) for all $1 \leq i \leq n$.

Proof. We only prove the left case as the right case is treated analogously. Let $T = S_C S_{C^{-1}}$, $M = S_C$ and $X = \{s_{g_1}, s_{g_2}, \ldots, s_{g_n}\}$. Take an arbitrary integer $1 \leq i \leq n$. Since S is nearly epsilon-strongly graded, there is some $e \in S_{g_i}S_{(g_i)^{-1}} \subseteq T$ such that $es_{g_i} = s_{g_i}$. The statement now follows from Lemma A.5.6. □

We can now prove that the functor $U_{G/N}$ restricts to the category of nearly epsilon-strongly G-graded rings.

Proposition A.5.8. Let G be an arbitrary group and let N be a normal subgroup of G. If $S = \bigoplus_{g \in G} S_g$ is a nearly epsilon-strongly G-graded ring, then the induced G/N-grading is nearly epsilon-strong.

Proof. Take an arbitrary $C \in G/N$ and $s \in S_C$. We need to show that there exist some $\epsilon_C(s) \in S_C S_{C^{-1}}$ and $\epsilon_C(s)' \in S_{C^{-1}} S_C$ such that $\epsilon_C(s)s = s\epsilon_C(s)' = s$. Note that s can be written as $s = s_{g_1} + s_{g_2} + \cdots + s_{g_n}$ for some positive integer n and elements $s_{g_i} \in S_{g_i}$ such that $g_i \in C$ for all $1 \leq i \leq n$. Hence, by Lemma A.5.7, there exist $e \in S_C S_{C^{-1}}$ and $e' \in S_{C^{-1}} S_C$ (depending on C and s) such that $es = se' = s$. □

We now consider essentially epsilon-strongly graded rings.

Lemma A.5.9. Let $S = \bigoplus_{g \in G} S_g$ be essentially epsilon-strongly G-graded where for each $g \in G$, E_g is a set of local units for the ring $S_g S_{g^{-1}}$. Then for any $g \in G$ and any $s \in S_g$ there exist some $e \in E_g$ and $e' \in E_{g^{-1}}$ such that $es = s = se'$.

Proof. Take an arbitrary $g \in G$. Since S is symmetrically graded, $S_g = S_g S_{g^{-1}} S_g = (S_g S_{g^{-1}}) S_g$. Hence, for any $s \in S_g$, we can write $s = rs'$ for some $r \in S_g S_{g^{-1}}$ and $s' \in S_g$. But since E_g is a set of local units of $S_g S_{g^{-1}}$ there exists some $e \in E_g$ such that $er = r = re$. This implies that,

$$es = e(rs') = (er)s' = rs' = s.$$

The existence of $e' \in E_{g^{-1}}$ is proved similarly. □

Let N be a normal subgroup of G and let $S = \bigoplus_{g \in G} S_g$ be an essentially epsilon-strongly G-graded ring. By assumption $S_g S_{g^{-1}}$ has set of local units E_g for each $g \in G$. We shall show that taking joins of these elements will be enough to construct local units for the rings $S_C S_{C^{-1}}$. For a family of sets $\{E_i\}_{i \in I}$ we let $\bigvee_{i \in I} E_i$ denote the set of finite joins of elements in $\bigcup_{i \in I} E_i$. Note that $\bigvee_{i \in I} E_i$ is the $\vee$-closure of $\bigcup_{i \in I} E_i$.

Proposition A.5.10. Consider a fixed class $C \in G/N$. Assume that for any $g, h \in C$ and any $e \in E_g, f \in E_h$ we have that $e \vee f$ exists. Then, $E_C = \bigvee_{g \in C} E_g$ is a set of local units for $S_C S_{C^{-1}}$.

Proof. Let $z \in S_C S_{C^{-1}}$. We shall construct an element $e \in E_C$ (depending on C and z) such that $ez = ze = z$. Note that $z = \sum_i x_i y_i$ where the sum is finite and $x_i \in S_C$ and $y_i \in S_{C^{-1}}$ for each i. Next, consider a fixed i. Then, x_i decomposes uniquely as a finite sum $x_i = \sum s_g$ where $s_g \in S_g$. Let $\mathrm{Supp}(x_i) := \{g \in G \mid s_g \neq 0\}$ and note that $\mathrm{Supp}(x_i)$ is a finite subset of C. Similarly, y_i decomposes uniquely as a finite sum $y_i = \sum t_h$ where $t_h \in S_h$. Let $\mathrm{Supp}(y_i) := \{h \in G \mid t_h \neq 0\}$ and note that $\mathrm{Supp}(y_i)$ is a finite subset of C^{-1}. Hence, the sets,

$$A = \bigcup_i \mathrm{Supp}(x_i), \quad B = \bigcup_i \mathrm{Supp}(y_i), \tag{15}$$

are both finite. By substituting the expressions for x_i and y_i in the definition of z we see that z is a finite sum of elements $s_g t_h$ with $g \in A$, $h \in B$, $s_g \in S_g$ and $t_h \in S_h$.

Take $g \in A, h \in B$ and corresponding elements $s_g \in S_g, t_h \in S_h$. By Lemma A.5.9, there are $\epsilon_g(s_g) \in E_g \subseteq E_C$ and $\epsilon'_h(t_h) \in E_{h^{-1}} \subseteq E_C$ such that $\epsilon_g(s_g)s_g = s_g$ and $t_h\epsilon'_h(t_h) = t_h$. From Lemma A.5.1 it follows that,

$$s_g t_h = (\epsilon_g(s_g) \vee \epsilon'_h(t_h))s_g t_h = s_g t_h(\epsilon_g(s_g) \vee \epsilon'_h(t_h)).$$

Now, let

$$e = \bigvee_{g\in A} \epsilon_g(s_g) \vee \bigvee_{h\in B} \epsilon'_h(t_h). \tag{16}$$

Since the sets in (15) are finite, the upper bounds on the right hand side of (16) exist by our assumptions. Furthermore, $e \in E_C$ by construction. Moreover, by (16) and noting that $\epsilon_g(s_g) \vee \epsilon'_h(t_h) \leq e$ for all $g \in A, h \in B$,

$$ez = e\sum s_g t_h = \sum (e s_g t_h) = \sum s_g t_h = z.$$

Similarly, $ze = z$. Hence, E_C is a set of local units for $S_C S_{C^{-1}}$. □

We can now finish the essentially epsilon-strongly graded case, with the following conclusion.

Corollary A.5.11. Let N be a normal subgroup of G and let $S = \bigoplus_{g\in G} S_g$ be essentially epsilon-strongly G-graded where for each $g \in G$, E_g is a set of local units for $S_g S_{g^{-1}}$. For every class $C \in G/N$, assume that for any $g, h \in C$ and any $e \in E_g, f \in E_h$ the join $e \vee f$ exists. Then, the induced G/N-grading is essentially epsilon-strong.

Proof. By Proposition A.5.5 the induced G/N-grading is symmetric. Furthermore, by Proposition A.5.10, it follows that for each $C \in G/N$, the ring $S_C S_{C^{-1}}$ has a set of local units. Hence, the induced G/N-grading is essentially epsilon-strong. □

We now consider the virtually epsilon-strongly graded case. It turns out that we will need some further assumptions in order to prove that the induced quotient group grading is virtually epsilon-strong.

Lemma A.5.12. Let $S = \bigoplus_{g\in G} S_g$ be virtually epsilon-strongly G-graded where for each $g \in G$, M_g is a set of pairwise orthogonal, commuting idempotents such that $E_g = \bigvee M_g$ is a set of local units for $S_g S_{g^{-1}}$. Then, for any $g \in G$ and $s \in S_g$ there exist $m_1, m_2, \ldots, m_i \in M_g$ and $m'_1, m'_2, \ldots, m'_j \in M_{g^{-1}}$ such that,

$$(m_1 \vee m_2 \vee \cdots \vee m_i)s = s,$$

and,

$$s(m'_1 \vee m'_2 \vee \cdots \vee m'_j) = s.$$

Proof. Follows from Lemma A.5.9 and the fact that $E_g = \bigvee M_g$ is a set of local units for $S_g S_{g^{-1}}$. □

Proposition A.5.13. Let N be a normal subgroup of G and let $S = \bigoplus_{g\in G} S_g$ be virtually epsilon-strongly G-graded where for each $g \in G$, M_g is a set of commuting orthogonal idempotents such that $E_g = \bigvee M_g$ is a set of local units for $S_g S_{g^{-1}}$. For each class $C \in G/N$, let $A_C := \bigvee_{g\in C} M_g$ and assume that the following statements hold:

(a) For any $g, h \in C$ and $e \in M_g, f \in M_h$ we have that $ef = fe$;
(b) The maximal elements of $(A_C, \leq)$ are pairwise orthogonal idempotents;
(c) Every chain in the poset $(A_C, \leq)$ is finite.

Then, the induced G/N-grading is virtually epsilon-strong.

Proof. Consider the induced G/N-grading and take an arbitrary $C \in G/N$. To establish the proposition we need to show that $S_C S_{C^{-1}}$ is a ring with enough idempotents. Put, $A_C = \bigvee_{g \in C} M_g$ and note that $ef = fe \in A_C$ for any $e, f \in A_C$. In other words, A_C is a set of commuting idempotents that is closed with regards to the join operator $\vee$. We want to choose the maximal elements from A_C with respect to the idempotent ordering. More precisely, let,

$$D_C = \{e \in A_C \mid e \text{ is maximal in } (A_C, \leq)\}.$$

By assumptions (a) and (b) we get that D_C is a set of pairwise orthogonal, commuting idempotents. Furthermore, we claim that that $\bigvee D_C$ is a set of local units for $S_C S_{C^{-1}}$. To this end, we define a function $\mu \colon A_C \to D_C$ by sending $m \in A_C$ to an element $\mu(m) \in D_C$ such that $m \leq \mu(m)$. Note that this is well-defined because of (c).

Now, let $z \in S_C S_{C^{-1}}$ be a general element. We shall construct an element $e \in \bigvee D_C$ such that $ez = z = ze$. Note that $z = \sum_i x_i y_i$ where the sum is finite and $x_i \in S_C$ and $y_i \in S_{C^{-1}}$ for each i. Next, if we consider a fixed i, x_i decomposes as a finite sum $x_i = \sum s_g$ where $0 \neq s_g \in S_g$ for a finite number of elements $g \in C$. Similarly, $y_i = \sum t_h$ where $0 \neq t_h \in S_h$ for a finite number of elements $h \in C^{-1}$. More precisely, the sets,

$$A = \bigcup_i \mathrm{Supp}(x_i), \quad B = \bigcup_i \mathrm{Supp}(y_i),$$

are both finite. By substituting the expressions for x_i and y_i in the definition of z we see that z is a finite sum of elements $s_g t_h$ with $g \in A$, $h \in B$, $s_g \in S_g$ and $t_h \in S_h$.

Take $g \in A, h \in B$ and let $a_g b_h$ be any element such that $a_g \in S_g, b_h \in S_h$. By Lemma A.5.9 there exist $m_1, m_2, \dots, m_i \in M_g \subseteq A_C$ and $m'_1, m'_2, \dots, m'_j \in M_{h^{-1}} \subseteq A_C$ such that, $(m_1 \vee m_2 \vee \dots \vee m_i)a_g = a_g$ and $b_h(m'_1 \vee m'_2 \vee \dots \vee m'_j) = b_h$. Hence,

$$(m_1 \vee m_2 \vee \dots \vee m_i \vee m'_1 \vee m'_2 \vee \dots \vee m'_j) a_g b_h = a_g b_h (m_1 \vee m_2 \vee \dots \vee m_i \vee m'_1 \vee m'_2 \vee \dots \vee m'_j).$$

Let $f_g = \mu(m_1) \vee \mu(m_2) \vee \dots \vee \mu(m_i)$ and $f'_h = \mu(m'_1) \vee \mu(m'_2) \vee \dots \vee \mu(m'_j)$. Note that $f_g \vee f'_h \in \bigvee D_C$ and $m \leq \mu(m)$ for any $m \in A_C$. Hence,

$$a_g b_h = (f_g \vee f'_h) a_g b_h = a_g b_h (f_g \vee f'_h). \tag{17}$$

Let $e = \bigvee_{g \in A} f_g \vee \bigvee_{h \in B} f'_h$. Then, by (17) we get that $z = ez = ze$. □

As a side note, we notice that the assumptions (a), (b) and (c) in Proposition A.5.13 are in fact satisfied in the special case of Leavitt path algebras. By studying the proof of Proposition A.4.3 it can be seen that any element of E_g, for any $g \in G$, is of the form $\sum_i \alpha_i \alpha_i^*$ for some paths α_i. It is straightforward to use (13) to show that any two elements of this form commute, hence (a) is satisfied. For any subset $A \subseteq G$ consider the maximal elements M of the set $\bigvee_{g \in A} M_g$ with respect to the idempotent ordering. Again by using (13), we see that M is the minimal elements with regards to the initial subpath ordering, i.e. $\alpha\alpha^* \leq \beta\beta^*$ if and only if α is an initial subpath of β. Hence, for any $\alpha\alpha^*, \beta\beta^* \in M$ such that $\alpha \neq \beta$, we have $(\alpha\alpha^*)(\beta\beta^*) = 0$ by (13). Moreover, any given path α only has finitely many initial subpaths, i.e. any chain in the idempotent ordering containing $\alpha\alpha^*$ is finite. Thus (b) and (c) are satisfied in the case of Leavitt path algebras. Hence, as a corollary to Proposition A.5.13 and Proposition A.4.3, we obtain the following.

Corollary A.5.14. Let G be an arbitrary group, let R be a unital ring and let E be a directed graph. Consider the Leavitt path algebra $L_R(E)$ equipped with any standard G-grading. For every normal subgroup N of G, the induced G/N-grading is virtually epsilon-strong.

We now return to the main track and apply our investigation to the induced quotient group gradings of epsilon-strongly graded rings. The following theorem is one of our main results:

Theorem A.5.15. Let G be an arbitrary group and let $S = \bigoplus_{g \in G} S_g$ be an epsilon-strongly G-graded ring. Consider the induced G/N-grading of S. The following assertions hold:

(a) For each class $C \in G/N$, the set $E_C = \bigvee\{\epsilon_g \mid g \in C\}$ is a set of local units for the ring $S_C S_{C^{-1}}$. In particular, the induced G/N-grading of S is essentially epsilon-strong;
(b) Suppose that for each class $C \in G/N$, (i) the poset $(E_C, \leq)$ contains no infinite chain and (ii) the maximal elements of $(E_C, \leq)$ are pairwise orthogonal. Then, the induced G/N-grading of S is virtually epsilon-strong.

PROOF. (a): Take an arbitrary $g \in G$. Since $S_g S_{g^{-1}}$ is a unital ring with multiplicative identity element ϵ_g, it is a ring with a set of local units $E_g = \{\epsilon_g\}$. Furthermore, $\epsilon_g \in Z(S_e)$ (see [17, Prop. 5]). This means that $\epsilon_g \epsilon_h = \epsilon_h \epsilon_g$ for every $g, h \in G$. Hence, $\epsilon_g \vee \epsilon_h$ exists for every $g, h \in G$. Thus, by Proposition A.5.10, for each $C \in G/N$, the ring $S_C S_{C^{-1}}$ has a set of local units,

$$E_C = \bigvee_{g \in C} E_g = \bigvee\{\epsilon_g \mid g \in C\}.$$

(b): Note that S is virtually epsilon-strongly G-graded by letting $M_g = \{\epsilon_g\}$ for each $g \in G$. Condition (a) in Proposition A.5.13 is satisfied since the ϵ_g's are central idempotents. Moreover, condition (b) and (c) are in this special case equivalent to $(E_C, \leq)$ having no infinite chains and $(E_C, \leq)$ having pairwise orthogonal maximal elements for each $C \in G/N$. $\square$

Remark A.5.16. We make no claim about the necessity of the conditions in Theorem A.5.15(b). Is it true that any induced quotient group grading of an arbitrary epsilon-strongly graded ring is virtually epsilon-strong? The author has not been able to find any example where the induced quotient grading is essentially epsilon-strong but not virtually epsilon-strong. The difference between 'virtual' and 'essential' seems subtle.

Finally, we establish our main result:

Proof of Theorem A.1.2. By Definition A.3.3, the induced G/N-grading is epsilon-strong if and only if (i) the induced G/N-grading is symmetric, and (ii) $S_C S_{C^{-1}}$ is unital for each $C \in G/N$. By Proposition A.5.5, (i) holds. Moreover, by Proposition A.5.2, (ii) holds if and only if E_C has an upper bound in $E(S_N)$ for each $C \in G/N$. □

A.6. Epsilon-finite gradings

In this section, we introduce a subclass of epsilon-strong G-gradings with the special property that any induced G/N-grading is epsilon-strong.

Definition A.6.1. Let G be an arbitrary group and let S be an epsilon-strongly G-graded ring. Put $F = \bigvee\{\epsilon_g \mid g \in G\}$. We call the G-grading *epsilon-finite* if F is a finite set. In that case, we say that S is *epsilon-finitely G-graded*.

Remark A.6.2. We make some remarks regarding Definition A.6.1.

(a) Note that the set $\{\epsilon_g \mid g \in G\}$ is finite if and only if $\bigvee\{\epsilon_g \mid g \in G\}$ is finite.
(b) A unital strongly G-graded ring $S = \bigoplus_{g\in G} S_g$ is epsilon-finite since $\epsilon_g = 1_S$ for every $g \in G$ (see [17, Prop. 8]).
(c) An epsilon-strongly G-graded ring S with finite support, i.e. $|\mathrm{Supp}(S)| < \infty$, is necessarily epsilon-finite. However, the converse does not hold in general. Consider e.g. the complex group ring $\mathbb{C}[\mathbb{Z}]$, which is epsilon-finite but not finitely supported.

Proposition A.6.3. Let S be an epsilon-finitely G-graded ring. For any normal subgroup N of G, the induced G/N-grading is epsilon-finite.

Proof. We begin by showing that the induced G/N-grading is epsilon-strong. By Theorem A.5.15(a), the induced G/N-grading is essentially epsilon-strong and it holds for every $C \in G/N$ that $E_C = \bigvee\{\epsilon_g \mid g \in C\}$ is a set of local units for $S_C S_{C^{-1}}$. To prove that the induced G/N-grading is epsilon-strong, we need to show that $S_C S_{C^{-1}}$ is unital for each class $C \in G/N$. Fix an arbitrary class $C \in G/N$. Then, $E_C = \bigvee\{\epsilon_g \mid g \in C\} \subseteq \bigvee\{\epsilon_g \mid g \in G\}$, where the latter set is finite by assumption. Hence, E_C is finite and by Corollary A.5.4, E_C has a greatest element ϵ_C which is the multiplicative identity element of $S_C S_{C^{-1}}$ by Proposition A.5.2. Thus, the induced G/N-grading is epsilon-strong.

Let $C \in G/N$ again be an arbitrary class. Note that $\epsilon_C = \bigvee_{g\in C} \epsilon_g = \bigvee_{i\in I_C} \epsilon_i$ for some finite set $I_C \subseteq C$ which completely determines ϵ_C. But, $\{\epsilon_i \mid i \in \bigcup_{C\in G/N} I_C\} \subseteq \{\epsilon_g \mid g \in G\}$ is a finite set. Thus, $\{\epsilon_C \mid C \in G/N\}$ is finite. □

Remark A.6.4. At this point we make two remarks.

(a) By Proposition A.6.3, the functor $U_{G/N}$ restricts to the category of epsilon-finitely G-graded rings.
(b) In the next section, we we will give an example of an epsilon-strongly G-graded ring $(S, \{S_g\}_{g\in G}) \in \mathrm{ob}(G\text{-}\epsilon\mathrm{STRG})$ which is not epsilon-finite but for which there is a normal subgroup N of G such that $(S, \{S_C\}_{C\in G/N}) \in \mathrm{ob}(G/N\text{-STRG})$ (Example A.7.4). In particularly, note that $(S, \{S_C\}_{C\in G/N})$ is epsilon-finite by Remark A.6.2(b).

We continue by proving that having a noetherian principal component is a sufficient condition for a grading to be epsilon-finite. Recall (see e.g. [11, Ch. 7.22]) that a ring R is called *block decomposable* if $1 \in R$ decomposes into $1 = c_1 + c_2 + \cdots + c_r$ where c_i are central primitive idempotents. If this holds then any central idempotent can be expressed as a finite sum $c = \sum_i c_i$ where the sum is taken over all i such that $cc_i \neq 0$. In particular, this implies that the set of central idempotents is finite.

Theorem A.6.5. Let S be an epsilon-strongly G-graded ring. The following assertions hold:

(a) If S_e is block decomposable, then S is epsilon-finitely G-graded;
(b) If S_e is left or right noetherian, then S is epsilon-finitely G-graded.

Hence, in particular, if S_e is left or right noetherian and N is any normal subgroup of G, then the induced G/N-grading of S is epsilon-strong.

Proof. (a): Note that ϵ_g is a central idempotent of R (see [17, Prop. 5]). This implies that $\bigvee\{\epsilon_g \mid g \in G\}$ is a subset of the set of central idempotents of R. But since R is assumed to be block decomposable, there are only finitely many central idempotents in R. Hence, the G-grading is epsilon-finite.

(b): By [11, Prop. 22.2], one-sided noetherianity is a sufficient condition for R to be block decomposable. □

A.7. Examples

The class of unital partial skew group rings is an important type of rings with a natural epsilon-strong group grading. In this section, we will consider two concrete examples of unital partial skew group rings. Throughout, let G be an arbitrary group with neutral element e and let R be an arbitrary ring equipped with a multiplicative identity element.

Partial actions of groups were first introduced in the study of C^*-algebras by Exel [8]. A later development by Dokuchaev and Exel [7] was to consider partial actions on a ring in a purely algebraic context. Given a partial action on a ring R, they constructed the *partial skew group ring* of the partial action, generalizing the classical skew group ring of an action on a ring. The partial skew group ring of a general partial action is not necessarily associative (cf. [7, Expl. 3.5]). However, the subclass of *unital partial actions* (see Definition A.7.1) give rise to associative partial skew group rings. In fact, it is enough to assume that the partial action is idempotent (see [7, Cor. 3.2]).

Definition A.7.1. A *unital partial action* of G on R is a collection of unital ideals $\{D_g\}_{g \in G}$ and ring isomorphisms $\alpha_g \colon D_{g^{-1}} \to D_g$ such that,

(1) $D_e = R$ and $\alpha_e = id_R$.
(2) $\alpha_g(D_{g^{-1}} D_h) = D_g D_{gh}$ for all $g, h \in G$.
(3) For all $x \in D_{h^{-1}} D_{(gh)^{-1}}$, $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$.

We let 1_g denote the multiplicative identity element of the ideal D_g. Note that the 1_g's are central idempotents in R.

Recall (cf. e.g. [12, Sect. 5]) that a unital partial action α of a group G on a ring R gives us a unital, associative algebra called the *unital partial skew group ring*

$R \star_\alpha G = \bigoplus_{g \in G} D_g \delta_g$, where the δ_g's are formal symbols. The multiplication is defined by linearly extending the relations,

$$(a_g \delta_g)(b_h \delta_h) = a_g \alpha_g(b_h 1_{g^{-1}}) \delta_{gh},$$

for $g, h \in G$ and $a_g \in D_g, b_h \in D_g$. The relations in Definition A.7.1 essentially tell us that this multiplication is well-defined. Moreover, $R \star_\alpha G$ is epsilon-strongly G-graded with $\epsilon_g = 1_g \delta_0$ (see [17, pg. 2]).

Next, we will give an example of a unital partial action such that the unital partial skew group ring has a quotient grading which is not epsilon-strong.

Example A.7.2. Let $R := \text{Fun}(\mathbb{Z} \to \mathbb{Q}) = \prod_{\mathbb{Z}} \mathbb{Q}$ be the algebra of bi-infinite sequences with component-wise addition and multiplication. Define a partial action of $(\mathbb{Z}, +, 0)$ on R in the following way. Let $D_0 = R$ and $D_i = e_i R$, $i \neq 0$ where e_i is the Kronecker delta sequence. More precisely, $e_i(j) = \delta_{i,j}$ for all $i, j \in \mathbb{Z}$. Moreover, define $\alpha_i \colon D_{-i} \to D_i$ by,

$$f \mapsto (i \mapsto f(-i)).$$

Note that $D_g D_h = (0)$ if $g, h \neq 0$ and $g \neq h$. Moreover, $D_g D_0 = D_0 D_g = D_g$ for all $g \in G$. This means, condition (2) in Definition A.7.1 is satisfied for the case $g \neq 0, h \neq -g$. But on the other hand if $g \neq 0, h = -g$ then the condition reads $\alpha_g(D_{-g} D_{-g}) = D_g D_0$ which holds since α_g is an isomorphism. The case $g = 0$ is also clear. Hence, condition (2) holds. Next consider condition (3). If $h \neq 0, g \neq -h$, then $D_{-h} D_{-g-h} = (0)$ hence the condition is trivially satisfied. On the other hand, if $h \neq 0, g = -h$, then the condition reads $\alpha_{-h}(\alpha_h(x)) = \alpha_0(x)$ for all $x \in D_{-h} D_0 = D_{-h}$, which also holds for our definition of α. Finally, if $h = 0$, then condition (3) just reads $\alpha_g(x) = \alpha_g(x)$ for all $x \in D_{-g}$ since $\alpha_0 = \text{id}$ by definition. This proves that α is a unital partial action.

Consider the partial skew group ring $R \star_\alpha \mathbb{Z}$ with the canonical grading $R \star_\alpha \mathbb{Z} = \bigoplus_{n \in \mathbb{Z}} D_n \delta_n$. This grading is epsilon-strong with $\epsilon_i = e_i \delta_0$ (see [17, pg. 1]). In particular, note that the set $\{\epsilon_i \mid i \in \mathbb{Z}\}$ is infinite and contains infinitely many orthogonal central idempotents.

Next, note that the induced $\mathbb{Z}/2\mathbb{Z}$-grading on $R \star_\alpha \mathbb{Z}$ has components,

$$S_0 = \bigoplus_{n \in \mathbb{Z}} D_{2n} \delta_{2n}, \quad S_1 = \bigoplus_{n \in \mathbb{Z}} D_{2n+1} \delta_{2n+1}.$$

We will show that $S_1 S_1$ does not admit a multiplicative identity element, which implies that the induced $\mathbb{Z}/2\mathbb{Z}$-grading is not epsilon-strong.

Now, by Theorem A.5.15(a),

$$E = \bigvee \{\epsilon_{2n+1} \mid n \in \mathbb{Z}\},$$

is a set of local units for $S_1 S_1$. By Proposition A.5.2, $S_1 S_1$ is unital if and only if E has an upper bound in $E(S_1 S_1) \subseteq S_1 S_1$. A moments thought gives that, if such an element exists, it must be $d\delta_0$ where $d \in R$, $d(2n+1) = 1$ and $d(2n) = 0$ for all $n \in \mathbb{Z}$. To conclude we prove that $d\delta_0 \notin S_1 S_1$. Note that any element $x\delta_0 \in S_1 S_1$ is a finite sum of elements of the form,

$$(a\delta_{2n+1})(b\delta_{-2n-1}) = \alpha_{2n+1}(\alpha_{-2n-1}(a) b) \delta_0 = a_i b_i' \delta_0,$$

where $ab' \in R$ is a function satisfying $\mathrm{Supp}(ab') = \{2n+1\}$. This implies that, $\mathrm{Supp}(x) < \infty$ for any element of the form $x\delta_0 \in S_1S_1$. On the other hand, $\mathrm{Supp}(d) = \infty$. Hence, $d\delta_0 \notin S_1S_1$. Thus, the induced $\mathbb{Z}/2\mathbb{Z}$-grading is not epsilon-strong.

Remark A.7.3. Note that Example A.7.2 shows that the functor $U_{G/N}$ does not restrict to the category G-ϵSTRG (see Question A.2.2). Also note that the induced quotient group grading in Example A.7.2 is an example of a grading that is essentially epsilon-strong (in fact, virtually epsilon-strong) but not epsilon-strong (cf. Theorem A.5.15).

The next example shows that being epsilon-finite is not a necessary condition for the induced quotient group grading to be epsilon-finite (cf. Remark A.6.4(b)).

Example A.7.4. Let the group $(\mathbb{Z}, +, 0)$ act (globally) on $R = \mathrm{Fun}(\mathbb{Z} \to \mathbb{Q})$ by bilateral shifting. That is, 1 acts by mapping,

$$\ldots \quad a_{-3} \quad a_{-2} \quad a_{-1} \quad \underline{a_0} \quad a_1 \quad a_2 \quad a_3 \quad \ldots$$

to

$$\ldots \quad a_{-4} \quad a_{-3} \quad a_{-2} \quad \underline{a_{-1}} \quad a_0 \quad a_1 \quad a_2 \quad \ldots,$$

and -1 similarly shifts the sequence one step in the other direction.

Let $\beta\colon \mathbb{Z} \to \mathrm{Aut}(R)$ be the group homomorphism corresponding to this action. By restricting the (global) action to an ideal of R we get a partial action (see e.g. [7, Section 4]). More precisely, let $A := \{f \in R \mid f(0) = 0\}$. Then A is a unital ideal of R. Moreover, the natural partial action $\{\alpha_n\}_{n\in\mathbb{Z}}$ is defined on the ideals,

$$D_n := A \cap \beta_n(A) = \{f \in R \mid f(0) = f(n) = 0\},$$

for $n \in \mathbb{Z}$. For example $\alpha_2\colon D_{-2} \to D_2$ maps,

$$\ldots \quad a_{-3} \quad 0 \quad a_{-1} \quad \underline{0} \quad a_1 \quad a_2 \quad a_3 \quad \ldots$$

to

$$\ldots \quad a_{-5} \quad a_{-4} \quad a_{-3} \quad \underline{0} \quad a_{-1} \quad 0 \quad a_1 \quad \ldots$$

Consider the partial skew group ring $A \star_\alpha \mathbb{Z} = \bigoplus_{n\in\mathbb{Z}} D_n\delta_n$. For $n \in \mathbb{Z}$, the multiplicative identity element of D_n is given by,

$$1_n(i) = \begin{cases} 0 & i = 0 \\ 0 & i = n \\ 1 & \text{otherwise} \end{cases}$$

for all $i, j \in \mathbb{Z}$. Note that the set $\{\epsilon_n \mid n \in \mathbb{Z}\} = \{1_n\delta_0 \mid n \in \mathbb{Z}\}$ is infinite but the epsilons are not orthogonal!

Next, consider the induced $\mathbb{Z}/2\mathbb{Z}$-grading of $A\star_\alpha\mathbb{Z} = S_0 \oplus S_1$ where $S_0 = \bigoplus_{n\in\mathbb{Z}} D_{2n}\delta_{2n}$ and $S_1 = \bigoplus_{n\in\mathbb{Z}} D_{2n+1}\delta_{2n+1}$. Note that S_0 is a subring of $A \star_\alpha \mathbb{Z}$ with multiplicative identity $1_0\delta_0$. We will show that $1_0\delta_0 \in S_1S_1$, proving that the induced grading is strong (see e.g. [13, Prop. 1.1.1]). Indeed, let $\gamma(i) = 0$ for $i \neq -1$ and $\gamma(-1) = 1$. Then,

$$(\gamma\delta_{-3})(1_3\delta_3) + (1_{-1}\delta_{-1})(1_1\delta_1) = \gamma\delta_0 + 1_{-1}\delta_0 = 1_0\delta_0.$$

Thus, the induced $\mathbb{Z}/2\mathbb{Z}$-grading is strong.

A.8. Induced quotient group gradings and epsilon-crossed products

The class of epsilon-crossed product is defined analogously to the classical algebraic crossed product (see [17, Def. 32]). Moreover, the category of epsilon-crossed products is denoted by G-ϵCROSS (see Section A.2.2). On the other hand, recall (see [17, pg. 1]) the notion of a *unital partial crossed product*. This construction is a priori unrelated to the epsilon-crossed product and generalizes the classical algebraic crossed product by a twisted action. The full definition is rather technical. However, for our purposes it suffices to note that a unital partial skew group (see Definition A.7.1) is a special type of unital partial crossed product. The relationship between epsilon-crossed products and unital partial crossed products is described in the following theorem by Nystedt, Öinert and Pinedo:

Theorem A.8.1. ([17, Thm. 33]) If $(S, \{S_g\}_{g \in G}) \in \text{ob}(G\text{-}\epsilon\text{CROSS})$, then S can be presented as a unital partial crossed product by a unital twisted partial action of G on S_e. Conversely, if $S = \bigoplus_{g \in G} D_g \delta_g$ is a unital partial crossed product, then $(S, \{D_g \delta_g\}_{g \in G}) \in \text{ob}(G\text{-}\epsilon\text{CROSS})$.

With this characterization in mind, we will now consider Question A.2.3. Unfortunately, it does not hold in general that the induced G/N-grading of an epsilon-crossed product gives an epsilon-crossed product. The following example shows that the functor $U_{G/N}$ does not restrict to G-ϵCROSS.

Example A.8.2. Consider the unital partial skew group ring $(R \star_\alpha \mathbb{Z}, \{D_i \delta_i\}_{i \in \mathbb{Z}})$ given in Example A.7.2. Note that $(R \star_\alpha \mathbb{Z}, \{D_i \delta_i\}_{g \in \mathbb{Z}})$ is an epsilon-crossed product by Theorem A.8.1. However, by Example A.7.2, the induced $\mathbb{Z}/2\mathbb{Z}$-grading $U_{\mathbb{Z}/2\mathbb{Z}}((R \star_\alpha \mathbb{Z}, \{D_i \delta_i\}_{i \in \mathbb{Z}})) \notin \text{ob}(\mathbb{Z}/2\mathbb{Z}\text{-}\epsilon\text{STRG})$. Thus, in particular, $U_{\mathbb{Z}/2\mathbb{Z}}((R \star_\alpha \mathbb{Z}, \{D_i \delta_i\}_{i \in \mathbb{Z}})) \notin \text{ob}(\mathbb{Z}/2\mathbb{Z}\text{-}\epsilon\text{CROSS})$.

The following proposition will allow us to find examples where the condition in Question A.2.3 is true.

Proposition A.8.3. Let G be an arbitrary group, let R be a unital ring and let $\{\alpha_g\}_{g \in G}$ be a unital partial action on R where 1_g denotes the multiplicative identity element of the ideal D_g. Assume that,

(a) $D_g D_h = (0)$ for all $g, h \in G$ such that $g \neq h$ and $g, h \neq e$,
(b) the set $\{1_g \mid g \in G\}$ is finite.

Let N be any normal subgroup of G. Then the induced G/N-grading of $R \star_\alpha G$ gives an epsilon-crossed product. In other words, $U_{G/N}((R \star_\alpha G, \{D_g \delta_g\}_{g \in G})) \in \text{ob}(G/N\text{-}\epsilon\text{CROSS})$.

Proof. We first show that the G/N-grading of $R \star_\alpha G$ is epsilon-strong. Write $R \star_\alpha G = \bigoplus_{C \in G/N} S_C$ and take a class $C \in G/N$. By Theorem A.5.15, we need to show that $E_C = \bigvee \{1_g \delta_0 \mid g \in C\}$ has an upper bound. Note that assumption (a) implies that $1_g 1_h = 0$ for $g \neq h \neq e$. Hence, by (b) and Proposition A.5.2,

$$\epsilon_C = \bigvee_{g \in C} 1_g \delta_0 = 1_{g_1} \delta_0 + 1_{g_2} \delta_0 + \ldots 1_{g_n} \delta_0,$$

for some choice of representatives $g_i \in C$ for $1 \leq i \leq n$ satisfying,

$$\{1_{g_i} \mid 1 \leq i \leq n\} = \{1_g \mid g \in C\}.$$

Hence, $S_C S_{C^{-1}}$ is unital. Since $C \in G/N$ was chosen arbitrarily, it follows that the induced G/N-grading of $R \star_\alpha G$ is epsilon-strong.

Next, we show that for each $C \in G/N$ there is an epsilon-invertible element in S_C. For $g, h \in G$ and $x \in D_g, y \in D_h$, we have that $(x\delta_g)(y\delta_h) = \alpha_g(\alpha_{g^{-1}}(x)y)\delta_{gh} = 0$ if $h \neq g^{-1}$ since $D_{g^{-1}} D_h = (0)$ by assumption. Moreover, it is easy to check that $1_{g_i}\delta_0 = (1_{g_i}\delta_{g_i})(1_{g_i^{-1}}\delta_{g_i^{-1}})$ for $1 \leq i \leq n$. Let $s = (1_{g_1}\delta_{g_1} + \cdots + 1_{g_n}\delta_{g_n}) \in S_C$ and $t = (1_{g_1^{-1}}\delta_{g_1^{-1}} + \cdots + 1_{g_n^{-1}}\delta_{g_n^{-1}}) \in S_{C^{-1}}$. Then,

$$st = \sum_{1 \leq i,j \leq n} (1_{g_i}\delta_{g_i})(1_{g_j^{-1}}\delta_{g_j^{-1}}) = \sum_{1 \leq i \leq n} (1_{g_i}\delta_{g_i})(1_{g_i^{-1}}\delta_{g_i^{-1}}) = \sum_{1 \leq i \leq n} 1_{g_i}\delta_0 = \epsilon_C.$$

Thus, $(R \star_\alpha G, \{S_C\}_{C \in G/N}) = U_{G/N}((R \star_\alpha G, \{D_g\delta_g\}_{g \in G}))$ is an epsilon-crossed product. $\square$

It is not clear to the author if the conditions in Proposition A.8.3 are necessary. The following is an explicit example where the induced quotient group grading gives an epsilon-crossed product. In other words, an example of a unital partial skew group ring (which is an epsilon-crossed product by Theorem A.8.1) such that the condition in Question A.2.3 is true.

Example A.8.4. Let $R = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ and let G be the cyclic group of order 4. Then G acts on R by shifting. Let e_i denote the Kronecker sequence, i.e. $e_i(j) = \delta_{i,j}$ for all integers i, j. Consider the unital ideal $A = e_1R + e_2R$ and the induced partial action on A. We get that $D_0 = A, D_1 = e_2\mathbb{Q}, D_2 = 0, D_3 = e_1\mathbb{Q}$. Now, consider the unital partial skew group ring $A \star_\alpha G = D_0\delta_0 \oplus D_1\delta_1 \oplus D_2\delta_2 \oplus D_3\delta_3$.

Let N be the cyclic group of order 2. By Proposition A.8.3, the induced G/N-grading gives an epsilon-crossed product. That is, $U_{G/N}((R \star_\alpha G, \{D_g\delta_g\}_{g \in G})) \in \mathrm{ob}(G/N\text{-}\epsilon\mathrm{CROSS})$. Note that,

$$S_{[0]} = D_0\delta_0 \oplus D_2\delta_2 = D_0\delta_0, \quad S_{[1]} = D_1\delta_1 \oplus D_3\delta_3.$$

Furthermore,

$$\epsilon_{[1]} = (1,1)\delta_0 = ((0,1)\delta_1 + (1,0)\delta_3)((0,1)\delta_1 + (1,0)\delta_3),$$

where $(0,1)\delta_1 + (1,0)\delta_3 \in S_{[1]}$ is an epsilon-invertible element.

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PAPER B

Chain conditions for epsilon-strongly graded rings with applications to Leavitt path algebras

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Daniel Lännström

Let G be a group with neutral element e and let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. A necessary condition for S to be noetherian is that the principal component S_e is noetherian. The following partial converse is well-known: If S is strongly-graded and G is a polycyclic-by-finite group, then S_e being noetherian implies that S is noetherian. We will generalize the noetherianity result to the recently introduced class of epsilon-strongly graded rings. We will also provide results on the artinianity of epsilon-strongly graded rings.

As our main application we obtain characterizations of noetherian and artinian Leavitt path algebras with coefficients in a general unital ring. This extends a recent characterization by Steinberg for Leavitt path algebras with coefficients in a commutative unital ring and previous characterizations by Abrams, Aranda Pino and Siles Molina for Leavitt path algebras with coefficients in a field. Secondly, we obtain characterizations of noetherian and artinian unital partial crossed products.

B.1. Introduction

Let R be an associative ring equipped with a multiplicative identity $1 \neq 0$ and let G be a group with neutral element e. It is straightforward to show that if the group ring $R[G]$ is right (left) noetherian, then R is right (left) noetherian and G is a right (left) noetherian group, i.e. satisfies the ascending chain condition on right (left) subgroups. In 1954, Hall [19] proved that if G is polycyclic-by-finite and R is right (left) noetherian, then $R[G]$ is right (left) noetherian. Motivated by this partial result, Bovdi

[18, Sect. 1.148] asked for a complete converse: If R and G are right (left) noetherian, is $R[G]$ right (left) noetherian? This question was settled in 1989 when Ivanov [22] gave an example of a right noetherian group G such that the group ring $R[G]$ is not right noetherian for any right noetherian base ring R, proving that the converse does not hold for general noetherian groups. On the other hand, it is not known if polycylic-by-finite groups is the largest class for which the converse holds. Determining for which groups G, noetherianity of R is a sufficient condition for the group ring $R[G]$ to be noetherian is still an open question (see [10, Sect. 2]). For artinianity, a more definitive answer was obtained by Connell [13] in 1963. Namely, $R[G]$ is left (right) artinian if and only if R is left (right) artinian and G is finite.

Let $\alpha\colon G \to \mathrm{Aut}(R)$ be a group homomorphism. Recall that the *skew group ring* $R \star_\alpha G$ has the same additive structure as $R[G]$ but the multiplication is skewed by the group action. More precisely, the multiplication of monomials is defined by $(ae_g)(be_h) = a\alpha_g(b)e_{gh}$. It was proved by Park [31] that $R\star_\alpha G$ is left (right) artinian if and only if R is left (right) artinian and G is finite. Skew group rings are examples of so-called algebraic crossed products. Unfortunately, the characterization by Park does not generalize to general crossed products. Passman [32] has given examples of artinian twisted group rings by an infinite p-group showing that G being finite is not necessary for a crossed product to be artinian.

A generalization of crossed products is the notion of a strongly graded ring. Recall that a ring S is *graded* by the group G if $S = \bigoplus_{g\in G} S_g$ for additive subsets S_g of S and $S_g S_h \subseteq S_{gh}$ for all $g, h \in G$. If $S_g S_h = S_{gh}$ for all $g, h \in G$, then S is *strongly graded* by G. There is a rigid relation between the *principal component* $R = S_e$ and the whole ring S. In particular, it turns out that the characterization of noetherianity by Hall [19], generalizes to strongly graded rings. Namely, let $S = \bigoplus_{g\in G} S_g$ be a strongly graded ring where G is a polycyclic-by-finite group and let $R = S_e$ be the principal component. Then S is left (right) noetherian if and only if R is left (right) noetherian. This can be proved using Dade's Theorem (see [27, Thm. 5.4.8]) or with a Hilbert Basis Theorem argument (see [9, Prop. 2.5]). For artinian strongly graded rings, even though Passman's example makes a full generalization impossible, Saorín [33] has obtained some results similar to those of Connell and Park, but with heavy conditions on the principal component.

Loosely speaking, all of these classical objects: group rings, skew group rings and crossed products are derived from some kind of action of a group G acting on a ring R. A recent development has been to consider the partial analogues of these objects, coming from a partial action of G on R. We will elaborate more on these objects later in the introduction.

The class of *epsilon-strongly* graded rings was introduced by Nystedt, Öinert and Pinedo in [30]. For our purposes, this class is the correct partial analogue of strongly graded rings. We aim to investigate noetherianity and artinianity on the right-hand side in Figure 1. In this article, we will prove the following characterization of noetherianity.

Theorem B.1.1. Let G be a polycyclic-by-finite group and let S be an epsilon-strongly G-graded ring. Then S is left (right) noetherian if and only if $R = S_e$ is left (right) noetherian.

Classical objects	Generalizations
actions	partial actions
skew group rings	partial skew group rings
crossed products	unital partial crossed products
strongly graded rings	epsilon-strongly graded rings

FIGURE 1. Classical objects and their partial analogues considered in this paper.

For artinianity, we obtain the following result:

Theorem B.1.2. Let G be a torsion-free group and let $S = \bigoplus_{g \in G} S_g$ be an epsilon-strongly G-graded ring. Then S is left (right) artinian if and only if $R = S_e$ is left (right) artinian and $S_g = \{0\}$ for all but finitely many $g \in G$.

Partial actions were first introduced by Exel [17] in the early 1990s to study C^*-algebras. For a survey of the history of partial actions, see [14, 8]. Later, partial actions on a ring R were considered by Dokuchaev and Exel in [16] to construct partial skew group rings as analogues of the classical skew group rings. The partial skew group ring of an arbitrary partial action is not necessarily associative (see [16, Expl. 3.5]). Analogously, twisted partial actions were considered in [15] and shown to give rise to partial crossed products.

Among the partial crossed products originally considered by Dokuchaev and Exel, the subclass coming from so-called unital twisted partial actions (see Section 5 for a definition) was considerd in [7] and shown to be especially well-behaved. For instance, these crossed products are always associative and unital algebras (see Section 5). In the sequel, this type of crossed products will simply be called *unital partial crossed products*.

Studying partial skew group rings, Carvalho, Cortes and Ferrero [11] obtained the following generalization from the classical setting.

Theorem B.1.3. ([11, Cor. 3.4]) Let α be a unital partial action of a polycyclic-by-finite group G on a ring R. If R is right (left) noetherian, then the partial skew group ring $R \star_\alpha G$ is right (left) noetherian.

Using the techniques in this paper, we will extend their result to unital partial crossed products (see Corollary B.5.1).

B.1.1. Leavitt path algebras. The Leavitt path algebra is an algebra attached to a directed graph E. Surprisingly, the Leavitt path algebra of a finite graph is naturally epsilon-strongly $\mathbb{Z}$-graded (see [28]). Leavitt path algebras were first introduced in [4] as an algebraic analogue of graph C^*-algebras. The monograph [2] by Abrams, Ara and Siles Molina is a comprehensive general reference. The main focus of the research has been Leavitt path algebras with coefficients in a field. In [35], Leavitt path algebras over a unital commutative ring were considered. They have been further studied in [26] and [23]. In this paper we will follow [21], [28] and consider Leavitt path algebras with coefficients in an arbitrary unital ring.

In the case of Leavitt path algebras with coefficients in a field, there has been much research connecting properties of the underlying graph with algebraic properties of the corresponding algebra. One particular question is how finiteness conditions on the algebra relates to finiteness conditions on the graph. In [3] it is shown that the Leavitt path algebra of a finite graph E is noetherian if and only if the graph E does not have any cycles with exits. Furthermore, it was proven in [1] that the Leavitt path algebra is artinian if and only if the graph is finite acyclic. Another proof of this result is given in [29].

Noetherianity and artianity of Leavitt path algebras with coefficients in a commutative ring has been characterized by Steinberg [34]. Using a different technique, we will extend his characterization to Leavitt path algebras with cofficients in a general unital ring. This also generalizes previous results by Abrams, Aranda Pino and Siles Molina, cf. [3, 5, 1, 2].

Theorem B.1.4. Let E be a directed graph and let R be a unital ring. Consider the Leavitt path algebra $L_R(E)$ with coefficients in R. The following assertions hold:

(a) $L_R(E)$ is a left (right) noetherian unital ring if and only if E is finite and satisfies Condition (NE) and R is left (right) noetherian.
(b) $L_R(E)$ is a left (right) artinian unital ring if and only if E is finite acyclic and R is left (right) artinian.
(c) If $L_R(E)$ is a semisimple unital ring, then E is finite acyclic and R is semisimple. Conversely, if R is semisimple with $n \cdot 1_R$ invertible for every integer $n \neq 0$ and E is finite acyclic, then $L_R(E)$ is a semisimple unital ring.

Remark B.1.5. Steinberg [34] proves a complete characterization of semisimple Leavitt path algebras with coefficients in a commutative ring. Our assumption in Theorem B.1.4(c) that $n \cdot 1_R$ is invertible for every $n \neq 0$ is not a necessary condition.

Most of these previous studies do not consider the additional structure on the Leavitt path algebra coming from the grading. Here we will follow the approach first taken by Hazrat [21] and focus on the graded structure. The key point realized by Nystedt and Öinert [28] is that the Leavitt path algebra of a finite graph has a canonical epsilon-strong $\mathbb{Z}$-grading. Since $\mathbb{Z}$ is both torsion-free and polycyclic-by-finite, we can apply our general theorems for epsilon-strongly graded rings.

In Section B.2, we give miscellaneous preliminaries that are needed later on.

In Section B.3, we prove our main results: Theorem B.1.1 and Theorem B.1.2. In the first part of the section, Theorem B.1.1 is proved by considering the special cases: finite G and infinitely cyclic G. Furthermore, Theorem B.1.2 is essentially a consequence of Bergman's famous observation that the Jacobson radical of a $\mathbb{Z}$-graded ring is a graded ideal.

In Section B.4, we apply the investigation of chain conditions on epsilon-strongly graded rings to Leavitt path algebras with coefficients in a unital ring. Indeed, Corollary B.4.8, Proposition B.4.14 and Corollary B.4.19 generalize previous characterizations of noetherian, artinian and semisimple Leavitt path algebras of a finite graph with coefficients in a field, cf. [3, 5, 1, 2]. Finally, we prove Theorem B.1.4, generalizing the characterization given by Steinberg [34].

In Section B.5, we apply Theorem B.1.1 and Theorem B.1.2 to unital partial crossed products and obtain characterizations of noetherian and artinian unital partial crossed products: Corollary B.5.1 and Corollary B.5.3. We will show that Corollary B.5.1 generalizes Theorem B.1.3 (cf. [11, Cor. 3.4]).

B.2. Preliminaries

In the following we fix a group G with neutral element e. Our rings, unless otherwise stated, will be associative and equipped with a multiplicative identity $1 \neq 0$. A ring S is called *G-graded* or *graded of type G*, if there exists a family $\{S_g\}_{g\in G}$ of additive subgroups of S such that (i) $S = \bigoplus_{g\in G} S_g$ and (ii) $S_g S_h \subseteq S_{gh}$ for all $g, h \in G$. The additive subgroups S_g are called *homogeneous components* and the elements of S contained in some S_g, are called the *homogeneous elements*. The homogeneous component S_e is called the *principal component* of S and will usually be denoted by R. The *support* of a G-grading $\mathrm{Supp}(S)$ is the set of $g \in G$ such that $S_g \neq \{0\}$. An element in $x \in S$ decomposes uniquely as a finite sum $x = \sum s_g$ where the s_g's are homogeneous elements. The *support* of an element $x \in S$, denoted by $\mathrm{Supp}(x)$, is the finite set such that $g \in \mathrm{Supp}(x) \iff s_g \neq 0$ in the decomposition of x.

The following definition was introduced [12, Def. 4.5] in the study of Steinberg algebras.

Definition B.2.1. A G-graded ring $S = \bigoplus_{g\in G} S_g$ is called *symmetrically graded* if,

$$S_g S_{g^{-1}} S_g = S_g, \qquad \forall g \in G. \tag{18}$$

We point out that strong gradings are symmetric. Interestingly, the Leavitt path algebra of any graph is symmetrically graded (see [28, Prop. 3.2]). An easy example of a grading that is not symmetric is the standard $\mathbb{Z}$-grading on the polynomial ring $R[X]$, i.e. $S_n = \{0\}$ for $n < 0$ and $S_n = RX^n$ for $n \geq 0$.

Remark B.2.2. Let S be a symmetrically graded ring. If $S_g S_{g^{-1}} = \{0\}$ for some $g \in G$, then $\{0\} = S_{g^{-1}} S_g S_{g^{-1}} = S_g S_{g^{-1}} S_g$, hence by (18), we get that $S_g = S_{g^{-1}} = \{0\}$. Thus, $g \in \mathrm{Supp}(S) \iff g^{-1} \in \mathrm{Supp}(S)$.

Example B.2.3. Let R be a ring. An easy check shows that $R[X^2, X^{-2}] = \bigoplus_{n\in\mathbb{Z}} S_n$ where $S_{2k+1} = \{0\}$ and $S_{2k} = RX^{2k}$ is a symmetrical grading. Since $S_{2k+1}S_{-2k-1} = \{0\} \neq R$, the grading is not strong.

If S is a G-graded ring with principal component R, then $S_g S_{g^{-1}}$ is an R-ideal for each $g \in G$. The following definition was introduced by Nystedt, Öinert and Pinedo.

Definition B.2.4. ([30, Def. 4 and Prop. 7]) A G-graded ring $S = \bigoplus_{g\in G} S_g$ with principal component $R = S_e$ is called *epsilon-strongly graded* if,

(a) the grading is symmetrically graded, and,
(b) $S_g S_{g^{-1}}$ is a unital R-ideal for each $g \in G$.

It is straightforward to see that the ring in Example B.2.3 is epsilon-strongly graded. We point out that Definition B.2.4 is equivalent to the following conditions: (i) S being

G-graded and (ii) for each $g \in G$, there exists some $\epsilon_g \in S_g S_{g^{-1}}$ such that for all $x \in S_g$, $x = \epsilon_g x = x\epsilon_{g^{-1}}$ (see [30, Prop. 7]).

We can restrict an epsilon-strong grading to a subgroup of the grading group. This process gives us both a new grading and a new ring. The proof of the following proposition is straightforward and left to the reader.

Proposition B.2.5. Let $S = \bigoplus_{g\in G} S_g$ be an epsilon-strongly G-graded ring with principal component $R = S_e$. Let H be a subgroup of G. Then $S(H) = \bigoplus_{g\in H} S_g$ is an epsilon-strongly H-graded ring with principal component $S = S_e$.

On the other hand, if N is a normal subgroup of G, then there is a way to assign a new, induced G/N-grading to the same ring S. Indeed, let $S = \bigoplus_{g\in G} S_g$ be a G-graded ring with principal component R. The induced G/N-grading is given by $S = \bigoplus_{C\in G/N} S_C$ where $S_C = \bigoplus_{g\in C} S_g$. In particular, the new principal component is $S_{[e]} = \bigoplus_{g\in N} S_g = S(N)$. One can show that the induced G/N-grading is strong if the G-grading of S is strong (see [27, Prop. 1.2.2]).

In the proof of Theorem B.1.1 we will need the following proposition for the reduction step.

Proposition B.2.6. ([25, Prop. 5.4], cf. [9, Prop. 2.3]) Let S be an epsilon-strongly G-graded ring with principal component $R = S_e$. If R is left or right noetherian, then for any normal subgroup N of G, the induced G/N-grading of S is epsilon-strong.

Finally, we recall the following definition.

Definition B.2.7. A group G is *polycyclic-by-finite* if it has a subnormal series,

$$1 = G_0 \lhd G_1 \lhd \dots G_{n-1} \lhd G_n \lhd G_{n+1} = G,$$

such that G/G_n is finite and $G_{i+1}/G_i, 1 \le i \le n-1$ is a cyclic group. In other words, a polycyclic-by-finite is a group containing a polycyclic subgroup with finite index.

For example, the groups $\mathbb{Z}^r$ for $r > 0$ are polycyclic-by-finite.

B.3. Noetherian and artinian epsilon-strongly graded rings

Recall that a ring is called *left (right) noetherian* if every ascending sequence of left (right) ideals of the ring stabilizes. A left (right) R-module ${}_RM$ (M_R) is *left (right) noetherian* if it satisfies the ascending chain condition on left (right) R-submodules. A ring R is left (right) noetherian if and only if R is left (right) noetherian as a left (right) module over itself. Another well-known characterization is that a ring R is left (right) noetherian if and only if every finitely generated left (right) R-module is left (right) noetherian. Artinian rings and modules have similar characterizations but instead with respect to the descending chain condition.

B.3.1. The Ascending Chain Condition. We first recall the following well-known result (see e.g. [27, Prop. 5.4.2]).

Proposition B.3.1. Let G be an arbitrary group and let $S = \bigoplus_{g\in G} S_g$ be a G-graded ring with principal component R. If S is left (right) noetherian, then R is left (right) noetherian.

The converse of Proposition B.3.1 does not hold in general as the following example shows.

Example B.3.2. ([20, Expl. 1.1.22]) Let S be the ring of Laurent polynomials (over some field K) in infinitely many variables $X_1, X_2, \ldots$. Then, S is strongly graded by extending the standard one variable $\mathbb{Z}$-grading to an infinite variable $\prod_{\mathbb{N}} \mathbb{Z}$-grading. More precisely, an element $g \in \prod_{\mathbb{N}} \mathbb{Z}$ can be identified with a sequence where $g(i)$ is the degree in the variable X_i.

Notice that S is not noetherian but $R = S_e = K$ is noetherian since it is a field.

In the case of epsilon-strongly graded rings there is a converse of Proposition B.3.1 for a certain class of groups. For this purpose, we will generalize the proof for strongly graded rings given in [9]. Some changes are needed, but otherwise the proof is similar and uses a version of Hilbert's Basis Theorem.

To establish the general result we will first consider the special cases of G finite and $G = \mathbb{Z}$. Then, we reduce the general case to these two special cases.

We recall that if M is a finitely generated projective left R-module, then $\mathrm{Hom}_R(M, R)$ is a finitely generated projective right R-module.

Proposition B.3.3. Let G be an arbitrary group and let $S = \bigoplus_{g \in G} S_g$ be an epsilon-strongly G-graded ring. If $\mathrm{Supp}(S) < \infty$, then S is finitely generated as both a left and right R-module.

Proof. By [30, Prop. 7], for each $g \in G$, S_g is a finitely generated projective left R-module isomorphic to $\mathrm{Hom}_R(S_{g^{-1}}, R)$ as a right R-module. In particular, S_g is finitely generated as both a left and right R-module. Hence, $S = \bigoplus_{g \in G} S_g$ is finitely generated since the support is finite. □

Proposition B.3.4. Let G be a finite group and let $S = \bigoplus_{g \in G} S_g$ be an epsilon-strongly G-graded ring with principal component R. If R is left (right) noetherian, then S is left (right) noetherian.

Proof. Since the support of $S = \bigoplus_{g \in G} S_g$ is finite, Proposition B.3.3 implies that ${}_R S$ is finitely generated. Therefore, ${}_R S$ is left (right) noetherian and, in particular, ${}_S S$ is left (right) noetherian. □

Next, we shall prove the special case of $G = \mathbb{Z}$. Indeed, let $S = \bigoplus_{g \in G} S_g$ be an epsilon-strongly $\mathbb{Z}$-graded ring with principal component R. We can define a *length* for each element $s \in S$ in the following way. Any element $s \in S$ decomposes as a finite sum $s = \sum_{i \in \mathrm{Supp}(s)} s_i$ where $0 \neq s_i \in S_i$. We note that the set $\mathrm{Supp}(s) \subset \mathbb{Z}$ is finite, hence we can let a and b be the least and greatest integers of $\mathrm{Supp}(s)$ respectively. Then, $s = \sum_{i=a}^{b} s'_i$ where $s'_i = s_i$ if $i \in \mathrm{Supp}(s)$ and $s'_i = 0$ otherwise. We note that $s'_a \neq 0$ and $s'_b \neq 0$. The strictly positive integer $b - a + 1$ is called the *length* of s.

Now, let I be a right ideal of S. For $n \geq 1$, let,

$$\mathrm{Id}_n(I) = \Big\{ r \in R \mid r + \sum_{i=-n+1}^{-1} s_i \in I, s_i \in S_i \Big\},$$

where the sum from $i = -n+1$ to -1 is to be understood as empty for $n = 1$. Informally, $\mathrm{Id}_n(I)$ is the leading coefficients of elements of I with length at most n.

Lemma B.3.5. Let S be an epsilon-strongly $\mathbb{Z}$-graded ring. With the notation as above, the following assertions hold.

(a) For $n \geq 1$, $\mathrm{Id}_n(I)$ is a right ideal of R and $\mathrm{Id}_n(I) \subseteq \mathrm{Id}_m(I)$ if $1 \leq n \leq m$.
(b) Let $J \subseteq I$ be right ideals of S such that $\mathrm{Id}_n(I) = \mathrm{Id}_n(J)$ for all $n \geq 1$. Then, $J = I$.

Proof. (a): Straightforward.

(b): Seeking a contradiction, suppose that $I \neq J$. Then, we can choose,

$$x = \sum_{i=a}^{b} s_i \in I \setminus J,$$

with $s_i \in S_i$ such that the length $b - a + 1$ is minimal. For the moment let us assume that we can require $b = 0$ without loss of generality. Write $x = s_0 + \sum_{i=c}^{-1} s_i$ and note that x has length $1 - c$. Then $s_0 \in \mathrm{Id}_{1-c}(I) = \mathrm{Id}_{1-c}(J)$, which implies that there exists some $y = s_0 + \sum_{i=c}^{-1} \tilde{s}_i \in J$. Note that $x - y = \sum_{i=c}^{-1}(s_i - \tilde{s}_i)$ has length $-c$. Since $s_i - \tilde{s}_i \in S_i$ and by the choice of x as the element of minimal length in $I \setminus J$, we have $x - y \in J$. Hence, $x \in J$, which is a contradiction.

To finish the proof we need to show that we can assume $b = 0$. Again, let $x = \sum_{i=a}^{b} s_i$ be a fixed element in $I \setminus J$ of minimal length. Using that I is a right ideal of S, we get that $xS_{-b} \subseteq I$. For the moment assume that $xS_{-b} \not\subseteq J$. This implies that there exists some $y_{-b} \in S_{-b}$ such that $xy_{-b} \in I \setminus J$. Therefore, $xy_{-b} = \sum_{i=a}^{b} s_i y_{-b}$ with $s_i y_{-b} \in S_i S_{-b} \subseteq S_{i-b}$. Hence, letting $s_i' = s_{i+b} y_{-b} \in S_i$ we see that $xy_{-b} = \sum_{i=a-b}^{0} s_i' = s_0' + \sum_{i=a-b}^{-1} s_i'$ has length $b - a + 1$. In other words, we can choose x with $b = 0$ in the decomposition.

Finally, we prove that $xS_{-b} \not\subseteq J$ for any $x \in I \setminus J$ of minimal length. Suppose to get a contradiction that $xS_{-b} \subseteq J$. Then $xS_{-b}S_b \subseteq J$. Furthermore, since, $\epsilon_{-b} \in S_{-b}S_b$,

$$x\epsilon_{-b} = \sum_{i=a}^{b} s_i \epsilon_{-b} = s_b + \sum_{i=a}^{b-1} s_i \epsilon_{-b} \in J \subseteq I.$$

Thus,

$$x - x\epsilon_{-b} = \left(s_b + \sum_{i=a}^{b-1} s_i\right) - \left(s_b + \sum_{i=a}^{b-1} s_i \epsilon_{-b}\right) = \sum_{i=a}^{b-1} s_i(1 - \epsilon_{-b}) \in I.$$

Since $(1 - \epsilon_{-b}) \in R$ we have that $s_i(1 - \epsilon_{-b}) \in S_i$. But since the length of $x - x\epsilon_{-b}$ is $b - a$, which is strictly less than the the length of x, the choice of x implies that $x - x\epsilon_{-b} \in J$. Hence $x \in J$, which is a contradiction.

$\square$

Proposition B.3.6. Let $S = \bigoplus_{i \in \mathbb{Z}} S_i$ be an epsilon-strongly $\mathbb{Z}$-graded ring and let R be the principal component of S. If R is right (left) noetherian, then S is right (left) noetherian.

PROOF. We will only show the right-handed case as the left-handed case can be proved analogously.

Assume that $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ is an ascending sequence of right ideals in S. Then the diagonal sequence,

$$\mathrm{Id}_1(I_1) \subseteq \mathrm{Id}_2(I_2) \subseteq \mathrm{Id}_3(I_3) \subseteq \dots,$$

is an ascending sequence of right ideals in R. Since R is right noetherian by assumption there exists some n such that $\mathrm{Id}_n(I_n) = \mathrm{Id}_k(I_k)$ for any $k \geq n$.

Moreover, for $1 \leq i \leq n-1$, consider the sequence,

$$\mathrm{Id}_i(I_1) \subseteq \mathrm{Id}_i(I_2) \subseteq \mathrm{Id}_i(I_3) \subseteq \dots.$$

For each i there exists n_i such that $\mathrm{Id}_i(I_{n_i}) = \mathrm{Id}_i(I_m)$ for $m \geq n_i$. Taking,

$$n' = \max_{1 \leq i \leq n-1}(n, n_i),$$

we have that $\mathrm{Id}_i(I_{n'}) = \mathrm{Id}_i(I_m)$ for $m \geq n'$ and all i. By Lemma B.3.5, $I_m = I_{n'}$. Hence, the original sequence stabilizes and S is right noetherian. $\square$

We now consider the general case of polycyclic-by-finite groups.

Theorem B.3.7. Let G be a polycyclic-by-finite group and let $S = \bigoplus_{g \in G} S_g$ be an epsilon-strongly G-graded ring with principal component R. If R is right (left) noetherian, then S is right (left) noetherian.

PROOF. We only prove the right-handed case since the left-handed case can be treated analogously. Suppose R is right noetherian. We reduce the general proof to the cases where $G = \mathbb{Z}$ or G is finite. The proof goes by induction on the length n of the subnormal series.

For $n = 1$, we have $1 = G_0 \lhd G_1 = G$, i.e. $G = G/1$ is finite. Thus, S is right noetherian by Proposition B.3.4.

Next, assume that the theorem holds for subnormal series of length k for some $k > 0$. Let G be a polycyclic-by-finite group with subnormal series,

$$1 = G_0 \lhd G_1 \lhd \dots \lhd G_k \lhd G_{k+1} = G,$$

as in Definition B.2.7. By Proposition B.2.5, $S(G_k)$ is epsilon-strongly G_k-graded with principal component R. By the induction hypothesis, $S(G_k)$ is right noetherian. Furthermore, $S(G_{k+1})$ is epsilon-strongly G_{k+1}-graded with principal component R. Since R is right noetherian, Proposition B.2.6 implies that, the induced G_{k+1}/G_k-grading is epsilon-strong with principal component $S(G_k)$. By the assumption on G, G_{k+1}/G_k is either (i) isomorphic to $\mathbb{Z}$ or (ii) finite. In the first case, Proposition B.3.6 implies that $S(G_{k+1})$ is right noetherian. In the second case, Proposition B.3.4 implies that $S(G_{k+1})$ is right noetherian.

Hence, $S = S(G) = S(G_{k+1})$ is right noetherian and the theorem follows by the induction principle. $\square$

We can now establish our characterization of noetherian epsilon-strongly graded rings.

Proof of Theorem B.1.1. Assume that S is left (right) noetherian. Then, Proposition B.3.1 implies that $R = S_e$ is left (right) noetherian. Conversely, let the principal component $R = S_e$ be left (right) noetherian. Then, Theorem B.3.7 implies that S is left (right) noetherian. $\square$

B.3.2. The Descending Chain Condition. Given a torsion-free group G we characterize when an epsilon-strongly G-graded ring is one-sided artinian.

The following well-known result gives a necessary condition for a group graded ring to be one-sided artinian (see e.g. [27, Prop. 5.4.2]).

Proposition B.3.8. Let G be an arbitrary group and let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring with principal component R. If S is left (right) artinian, then R is left (right) artinian.

First, we prove the sufficiency of the conditions in Theorem B.1.2.

Proposition B.3.9. Let G be an arbitrary group and let $S = \bigoplus_{g \in G} S_g$ be an epsilon-strongly G-graded ring. If $\mathrm{Supp}(S) < \infty$ and R is left (right) artinian, then S is left (right) artinian.

Proof. By Proposition B.3.3, ${}_R S$ is finitely generated and thus ${}_R S$ is left artinian. In particular, ${}_S S$ is left artinian. $\square$

Considering the case where G is torsion-free we can use the following general theorem (see [27, Thm. 9.6.1]) to get a converse to Proposition B.3.9.

Theorem B.3.10. Let G be a torsion-free group and let $S = \bigoplus_{g \in G} S_g$ be a G-graded group. If S is a left (right) artinian, then $\mathrm{Supp}(S) < \infty$.

Remark B.3.11. We show that the right case of Theorem B.3.10 follows from the left case. Letting S be a right artinian ring, the opposite ring S^o is left artinian. Furthermore, the ring S^o is G-graded by $(S^o)_g = S_{g^{-1}}$ (cf. [27, Rmk. 1.2.4]). With this grading, $|\mathrm{Supp}(S^o)| = |\mathrm{Supp}(S)|$. But the left case of Theorem B.3.10 implies $\mathrm{Supp}(S^o) < \infty$ and thus $\mathrm{Supp}(S) < \infty$.

We are now ready to prove our characterization of artinian epsilon-strongly graded rings.

Proof of Theorem B.1.2. The 'if' direction follows from Proposition B.3.9. For the other direction, assume S is a left (right) artinian epsilon-strongly graded ring. First note that by Proposition B.3.8, the principal component $R = S_e$ is left (right) artinian. Secondly, by Theorem B.3.10, $\mathrm{Supp}(S) < \infty$, which is equivalent to $S_g = \{0\}$ for all but finitely many $g \in G$. $\square$

Remark B.3.12. Passman's example [32] of an artinian twisted group rings by an infinite p-group shows that Theorem B.1.2 and Theorem B.3.10 do not hold for arbitrary groups.

B.4. Applications to Leavitt path algebras

A *directed graph* $E = (E^0, E^1, s, r)$ consists of a set of vertices E^0, a set of edges E^1 and maps $s\colon E^1 \to E^0$, $r\colon E^1 \to E^0$ specifying the source $s(f)$ and range $r(f)$ vertex for each edge $f \in E^1$. We call a graph E *finite* if E^0 and E^1 are finite sets. A *path* is a sequence of edges $\alpha = f_1 f_2 \dots f_n$ such that $s(f_{i+1}) = r(f_i)$ for $1 \leq i \leq n-1$. We write $s(\alpha) = s(f_1)$ and $r(\alpha) = r(f_n)$. A *cycle* is a path such that $s(f_1) = r(f_n)$ and $s(f_i) \neq s(f_1)$ for $2 \leq i \leq n$.

Definition B.4.1. For a directed graph $E = (E^0, E^1, s, r)$ and a ring R, the *Leavitt path algebra with coefficients in* R is the R-algebra generated by the symbols $\{v \mid v \in E^0\}$, $\{f \mid f \in E^1\}$ and $\{f^* \mid f \in E^1\}$ subject to the following relations,

(a) $v_i v_j = \delta_{i,j} v_i$ for all $v_i, v_j \in E^0$,
(b) $s(f)f = fr(f) = f$ and $r(f)f^* = f^* s(f) = f^*$ for all $f \in E^1$,
(c) $f^* f' = \delta_{f,f'} r(f)$ for all $f, f' \in E^1$,
(d) $\sum_{f \in E^1, s(f)=v} ff^* = v$ for all $v \in E^0$ for which $s^{-1}(v)$ is non-empty and finite.

We let R commute with the generators.

The symbols $f \in E^1$ are called *real edges* and the symbols f^* for $f \in E^1$ are called *ghost edges*. For a real path $\alpha = f_1 f_2 f_3 \dots f_n$ the corresponding *ghost path* is $\alpha^* = f_n^* f_{n-1}^* \dots f_3^* f_2^* f_1^*$. In particular, $s(\alpha) = r(\alpha^*)$ and $r(\alpha) = s(\alpha^*)$. Using the relations above it can be proved that a general element in $L_R(E)$ has the form of a finite sum $\sum r_i \alpha_i \beta_i^*$ where $r_i \in R$, α_i and β_i are real paths such that $r(\alpha_i) = s(\beta_i^*) = r(\beta_i)$ for every i. For a path $\alpha = f_1 f_2 f_3 \dots f_n$, let $\text{len}(\alpha) = n$ denote the length of n.

For any group G there is a class of G-gradings called the standard gradings of $L_R(E)$ (see for example [28]). Taking $G = \mathbb{Z}$ we obtain the so-called canonical $\mathbb{Z}$-grading of $L_R(E)$. In this case, the homogeneous components are given by,

$$(L_R(E))_n = \Big\{ \sum r_i \alpha_i \beta_i^* \mid r_i \in R, \text{len}(\alpha_i) - \text{len}(\beta_i) = n \Big\}, \tag{19}$$

for $n \in \mathbb{Z}$. It is proved in [28, Thm. 3.3] that every standard G-grading of $L_R(E)$ is symmetric and furthermore when E is finite, every standard G-grading is epsilon-strong. In the sequel, we will consider the Leavitt path algebra $L_R(E)$ equipped with the canonical $\mathbb{Z}$-grading. If E is finite, then $L_R(E)$ is an epsilon-strongly $\mathbb{Z}$-graded ring.

In this section, we will characterize when $L_R(E)$ is noetherian and artinian under the assumption that E is a finite graph.

Definition B.4.2. A graph E is said to satisfy *Condition (NE)* if there is no cycle with an exit.

This condition is in fact a large restriction on the paths in the graph.

Lemma B.4.3. ([3, Lem. 1.2]) If E is a finite graph satisfying Condition (NE), then any path μ with $|\mu| > |E^0|$ ends in a cycle.

We will begin by showing that Condition (NE) allows us to lift algebraic properties of the base ring R to the principal component $(L_R(E))_0$.

Following [3, Thm. 1.4], for an integer $m \geq 0$, let,

$$C_m = \mathrm{Span}_R\{\alpha\beta^* \mid \mathrm{len}(\alpha) = \mathrm{len}(\beta) = m\}.$$

Note that by (19), $(L_R(E))_0 = \bigcup_{m=0}^{\infty} C_m$ for any graph E. If E is a finite graph, then there are only finitely many paths of a fixed length. Hence, if E is a finite graph, then C_m is a finitely generated R-module for each non-negative integer m.

Lemma B.4.4. (cf. [3, Thm. 1.4]) If for some integer $t \geq 0$, $C_{t+1} \subseteq \bigcup_{m=0}^{t} C_m$, then $\bigcup_{m=0}^{\infty} C_m = \bigcup_{m=0}^{t} C_m$.

Proof. Assume that $C_{t+1} \subseteq \bigcup_{m=0}^{t} C_m$. We show that $C_{t+i} \subseteq \bigcup_{m=0}^{t} C_m$ for any i by induction.

Now, assume that $C_{t+k} \subseteq \bigcup_{m=0}^{t} C_m$ for some k and let

$$\alpha = e_1 e_2 \dots e_{t+k+1} f_{t+k+1}^* f_k^* \dots f_2^* f_1^*.$$

Letting $\beta = e_2 \dots e_{t+k+1} f_{t+k+1}^* \dots f_2^*$ we have that,

$$\alpha = e_1 \beta f_1^* \in e_1 C_{t+k} f_1^* \subseteq e_1 \Big(\bigcup_{m=0}^{t} C_m \Big) f_1^* \subseteq \bigcup_{m=1}^{t+1} C_m \subseteq \bigcup_{m=0}^{t} C_m,$$

where the last inclusion is a consequence of the assumption $C_{t+1} \subseteq \bigcup_{m=0}^{t} C_m$. □

Proposition B.4.5. If E is a finite graph satisfying Condition (NE), then there exists some non-negative integer k such that $(L_R(E))_0 = \bigcup_{m=0}^{k} C_m$. In particular, $(L_R(E))_0$ is a finitely generated R-module.

Proof. Assume that E satisfies condition (NE). We will show that the condition in Lemma B.4.4 holds. By Lemma B.4.3 there is an integer $k > 1$ such that any path μ in E with $\mathrm{len}(\mu) \geq k$ ends in a cycle. Let $\alpha\beta^* \neq 0 \in C_k$. Then since $r(\alpha) = r(\beta)$ and condition (NE) holds, α and β both ends in the same cycle. Let, $\alpha = \alpha' e_1 e_2 \dots e_n$ and $\beta = \beta' e_1 e_2 \dots e_n$ for some edges $e_1, \dots, e_n$. Furthermore, since the cycle does not have an exit, it follows from Definition 4.1(d) that $e_n e_n^* = s(e_n)$. Hence,

$$\alpha\beta^* = \alpha' e_1 e_2 \dots e_{n-1}(e_n e_n^*) e_{n-1}^* \dots e_1^* (\beta')^* = \alpha' e_1 \dots e_{n-1} e_{n-1}^* \dots e_1^* (\beta')^* \in C_{k-1}.$$

This proves that $C_k \subseteq C_{k-1}$.

Now, Lemma B.4.4 yields, $(L_R(E))_0 = \bigcup_{m=0}^{k-1} C_m$. In particular, $(L_R(E))_0$ is finitely generated since each C_m is finitely generated. □

As a corollary we obtain that noetherianity lifts from the base ring R to $(L_R(E))_0$.

Corollary B.4.6. Let E be a finite graph and let R be a left (right) noetherian ring. If E satisfies Condition (NE), then $(L_R(E))_0$ is left (right) noetherian as a ring.

Proof. Assume that E satisfies Condition (NE). By Proposition B.4.5, $(L_R(E))_0$ is finitely generated. Since R is assumed to be left (right) noetherian, we get that ${}_R(L_R(E))_0, (((L_R(E))_0)_R)$ are noetherian and, in particular $(L_R(E))_0$ is a left (right) noetherian ring. □

Next, we will show that Condition (NE) in some sense restricts the number of idempotents in $(L_R(E))_0$. In particular, we show that Condition (NE) is a necessary condition for $(L_R(E))_0$ to be one-sided noetherian.

Proposition B.4.7. If E does not satisfy Condition (NE), then there exists an infinite sequence of distinct, pairwise orthogonal idempotents $\{\mu_n\mu_n^*\}_{n\in\mathbb{N}}$ in $(L_R(E))_0$. If this is the case, then $(L_R(E))_0$ is not left or right noetherian.

Proof. Following [6, Prop. 3.1(ii)], assume that there is a cycle γ is a with an exit path α. By shifting the cycle, we may assume that the base vertex of the cycle is the starting point of the exit. More precisely, we may assume that $\gamma = e_1e_2\dots e_n$ for edges e_i such that $s(\gamma) = s(e_1) = s(\alpha)$ and $e_i \neq e_1$ for $2 \leq i \leq n$. Consider the set $\{\gamma^n\alpha\alpha^*(\gamma^*)^n \mid n \in \mathbb{N}\}$. It is straightforward to prove these are orthogonal idempotents in $(L_R(E))_0$. To see that they are distinct, suppose to get a contradiction that,

$$\gamma^n\alpha\alpha^*(\gamma^*)^n = \gamma^m\alpha\alpha^*(\gamma^*)^m,$$

for $n > m$. Then, multiplying with $\alpha^*(\gamma^*)^m$ on the left,

$$0 = (\alpha^*\gamma^{n-m})\alpha\alpha^*(\gamma^*)^n = \alpha^*(\gamma^*)^m\gamma^n\alpha\alpha^*(\gamma^*)^n = \alpha^*(\gamma^*)^m\gamma^m\alpha\alpha^*(\gamma^*)^m = \alpha^*(\gamma^*)^m,$$

which is a contradiction.

The last statement is clear since a one-sided noetherian ring does not contain infinitely many orthogonal idempotents. □

We can now prove our main characterization of noetherian Leavitt path algebras, generalizing [3, Thm. 3.10].

Corollary B.4.8. Let E be a finite graph and let R be a left (right) noetherian ring. Let the Leavitt path algebra $L_R(E)$ be graded with the canonical $\mathbb{Z}$-grading. Then the following are equivalent:
(a) E satisfies Condition (NE),
(b) $(L_R(E))_0$ is left (right) noetherian,
(c) $L_R(E)$ is left (right) noetherian.

Proof. $(a) \implies (b)$: Assume that E has no cycle with an exit. Since R is left (right) noetherian, by Corollary B.4.6, $(L_R(E))_0$ is left (right) noetherian.

$(b) \implies (c)$: Since the canonical $\mathbb{Z}$-grading of $L_R(E)$ is epsilon-strong (see [28, Thm. 3.3]), this follows from Theorem B.1.1.

$(c) \implies (a)$: Assume that $L_R(E)$ is left (right) noetherian. Then by Theorem B.1.1, $(L_R(E))_0$ is left (right) noetherian. Therefore, by Proposition B.4.7, E satisfies Condition (NE). □

Remark B.4.9. Note that if R is two-sided noetherian in Corollary B.4.8, then we get equivalences between the following assertions:
(a) E satisfies Condition (NE),
(b) $(L_R(E))_0$ is left noetherian,
(c) $(L_R(E))_0$ is right noetherian,
(d) $L_R(E)$ is left noetherian,
(e) $L_R(E)$ is right noetherian.

Remark B.4.10. Taking R to be a field we obtain a new proof of Abrams, Aranda Pino and Siles Molina's characterization of noetherian Leavitt path algebras of finite graphs with coefficients in a field (cf. [3]).

Next, we will similarly apply the characterization of artinian epsilon-strongly graded rings. We first show that Condition (NE) will allow us to lift additional algebraic properties from R to $(L_R(E))_0$. This follows as a corollary to a more general structure theorem for the principal component $(L_R(E))_0$. Define a filtration of $(L_R(E))_0$ as follows. For $n \geq 0$, put,

$$D_n = \operatorname{Span}_R\{\alpha\beta^* \mid \operatorname{len}(\alpha) = \operatorname{len}(\beta) \leq n\}.$$

It is straightforward to show that D_n is an R-subalgebra of $(L_R(E))_0$. For $v \in E^0$ and $n > 0$ let $P(n, v)$ denote the set of paths γ with $\operatorname{len}(\gamma) = n$ and $r(\gamma) = v$. Let $\operatorname{Sink}(E)$ denote the set of sinks in E.

Theorem B.4.11. ([2, Cor. 2.1.16]) For a non-negative integer n, let $M_n(R)$ denote the full $n \times n$-matrix ring. Then, for a finite graph E,

$$D_0 \simeq \prod_{v \in E^0} R,$$

$$D_n \simeq \prod_{\substack{0 \leq i \leq n-1 \\ v \in \operatorname{Sink}(E)}} M_{|P(i,v)|}(R) \times \prod_{v \in E^0} M_{|P(n,v)|}(R),$$

as R-algebras.

We now show that we can lift even more properties from the base ring R to $(L_R(E))_0$ when Condition (NE) is satisfied.

Corollary B.4.12. Let R be a ring and let E be a finite graph that satisfies Condition (NE). Then $(L_R(E))_0$ is Morita equivalent to R^k for some $k > 0$ and, in particular, the following assertions hold:

(a) R is semisimple if and only if $(L_R(E))_0$ is semisimple.
(b) R is von Neumann regular if and only if $(L_R(E))_0$ is von Neumann regular.
(c) R is left (right) noetherian if and only if $(L_R(E))_0$ is left (right) noetherian.
(d) R is left (right) artinian if and only if $(L_R(E))_0$ is left (right) artinian.

Proof. Assuming that E is finite and satisfies Condition (NE), Proposition B.4.5 implies that $(L_R(E))_0 = \bigcup_{m=0}^{k} C_m = D_k$, which is a finite direct product of full matrix rings by Theorem B.4.11. That is, $(L_R(E))_0 = M_{n_1}(R) \times M_{n_2}(R) \times \cdots \times M_{n_k}(R)$ for some positive integers $n_1, n_2, \ldots, n_k$.

Recall that R and the full matrix ring $M_m(R)$ are Morita equivalent for each m (see for example [24, page 18.6]), i.e. there is an equivalence of categories $M_m(R)\text{-}\mathsf{Mod} \approx R\text{-}\mathsf{Mod}$. In particular, we have equivalences $M_{n_i}(R)\text{-}\mathsf{Mod} \approx R\text{-}\mathsf{Mod}$ for each $1 \leq i \leq k$. Forming the product categories, it is not hard to see that,

$$M_{n_1}(R)\text{-}\mathsf{Mod} \times M_{n_2}(R)\text{-}\mathsf{Mod} \times \cdots \times M_{n_k}(R)\text{-}\mathsf{Mod} \approx R\text{-}\mathsf{Mod} \times R\text{-}\mathsf{Mod} \times \cdots \times R\text{-}\mathsf{Mod}.$$

For any rings R, S it can be shown that R-$\mathsf{Mod} \times S$-$\mathsf{Mod} \approx (R \times S)$-$\mathsf{Mod}$. Hence,

$$(M_{n_1}(R) \times \cdots \times M_{n_k}(R))\text{-}\mathsf{Mod} \approx R^k\text{-}\mathsf{Mod},$$

and thus $(L_R(E))_0$ and R^k are Morita equivalent.

Furthermore, it is well-known that R^k is semisimple/von Neumann regular/one-sided artinian/one-sided noetherian if and only if R is semisimple/von Neumann regular/one-sided artinian/one-sided noetherian. Moreover, these properties are Morita invariant, which is enough to establish the corollary. □

We will begin by showing a partial result concerning the characterization of artinian Leavitt path algebras.

Lemma B.4.13. Let E be a finite graph. Then E is acyclic if E satisfies Condition (NE) and contains no infinite paths.

Proof. Assume that E satisfies Condition (NE). Then any infinite path must end in a cycle, hence E contains infinite paths if and only if E contains a cycle. □

The following is our first result concerning artinian Leavitt path algebras.

Proposition B.4.14. Let E be a finite graph and let R be a left (right) artinian ring. Let $L_R(E)$ be the Leavitt path algebra graded by the canonical $\mathbb{Z}$-grading. Then the following assertions are equivalent:

(a) E is acyclic,

(b) $(L_R(E))_0$ is left (right) artinian and $\mathrm{Supp}(L_R(E))$ is finite,

(c) $L_R(E)$ is left (right) artinian.

Proof. $(a) \implies (b)$: Assume that E is acyclic. Then, in particular, E trivially satisfies Condition (NE), so by Corollary B.4.12, $(L_R(E))_0$ is left (right) artinian. We claim that there exists some n satisfying $(L_R(E))_k = \{0\}$ for all $k \in \mathbb{Z}$ such that $|k| > n$. Since E is finite and acyclic there is a maximal length of paths in E. By (19), a monomial $\alpha\beta^* \in (L_R(E))_k$ if and only if $\mathrm{len}(\alpha) - \mathrm{len}(\beta) = k$. Taking n larger than the maximal path length in E, it follows that $(L_R(E))_k = \{0\}$ for all k such that $|k| > n$, thus proving the claim.

$(b) \implies (c)$: Since the canonical $\mathbb{Z}$-grading on $L_R(E)$ is epsilon-strong (see [28, Thm. 3.3]), Theorem B.1.2 implies that $L_R(E)$ is left (right) artinian.

$(c) \implies (a)$: Assume that $L_R(E)$ is left (right) artinian. Then Theorem B.1.2 implies that (i) $(L_R(E))_0$ is left (right) artinian and (ii) there is n such that $(L_R(E))_k = \{0\}$ for all $k \in \mathbb{Z}$ such that $k > n$. In particular, (i) implies that $(L_R(E))_0$ is left (right) noetherian, so E satisfies Condition (NE) by Corollary B.4.8. We claim that (ii) implies that E does not contain any infinite paths. Assuming that the claim hold, Lemma B.4.13 proves that E is acyclic. Suppose to get a contradiction that E contains an infinite path. Then, we can construct finite paths of arbitrary length by taking initial subpaths of the infinite path. Hence, $(L_R(E))_k \neq \{0\}$ for arbitrarily large k, contradicting (ii). □

Remark B.4.15. By making the stronger assumption that R is semisimple we see by Corollary B.4.12 that condition (b) can be replaced by,

(b') $(L_R(E))_0$ is semisimple and $\mathrm{Supp}(L_R(E))$ finite.

To get our full characterization we will apply a theorem by Nystedt, Öinert and Pinedo, which will allow us to lift semisimplicity from the principal component to the whole ring. Let $S = \bigoplus_{i\in\mathbb{Z}} S_i$ be an arbitrary epsilon-strongly $\mathbb{Z}$-graded ring and let $Z(S_0)_{\text{fin}}$ denote the elements $r \in Z(S_0)$ such that $r\epsilon_i = 0$ for all but finitely many integers i. There is a way to define a trace function $\text{tr}_\gamma\colon Z(S_0)_{\text{fin}} \to Z(S_0)$ (see [30, Def. 14]) such that the following theorem holds.

Theorem B.4.16. ([30, Thm. 23]) Let S be an epsilon-strongly $\mathbb{Z}$-graded ring. Assume that $\epsilon_i = 0$ for all but finitely many integers i and that $\text{tr}_\gamma(1)$ is invertible in S_0. If S_0 is semisimple, then S is semisimple.

Remark B.4.17. Note that Nystedt, Öinert and Pinedo show Theorem B.4.16 for an epsilon-strongly G-graded ring, where G is an arbitrary group.

The theorem only gives sufficient conditions: $\text{tr}_\gamma(1)$ is not necessarily invertible in S_0 if S is semisimple. We will need the following to be able to apply the theorem.

Lemma B.4.18. Let E be a finite graph and R be a ring such that $n \cdot 1_R$ is invertible for every integer $n \neq 0$. If $\text{Supp}(L_R(E))$ is finite, then $\text{tr}_\gamma(\sum_{v\in E^0} v)$ is invertible in $(L_R(E))_0$.

Proof. Assume that $\text{Supp}(L_R(E))$ is finite. Note that the multiplicative identity of $(L_R(E))_0$ is $\epsilon_0 = \sum_{v\in E^0} v$ where the sum is well-defined since E is a finite graph. By the definition of the trace ([30, Def. 14]), $\text{tr}_\gamma(\epsilon_0) = \sum_{i\in\mathbb{Z}} \epsilon_i$. Indeed, this expression is valid since $\epsilon_i = 0$ for all but finitely many integers i by the assumption that the support is finite.

By [28, Thm. 3.3], every non-zero ϵ_i can be written as a finite sum,

$$\epsilon_i = \sum_i v_i + \sum_i \alpha_i\alpha_i^*,$$

where α_i are paths. Let $v_1, \dots, v_{|E^0|}$ be the vertices of E. Since ϵ_0 is a term in the trace, there are two sequences of positive integer $(n_i)_{i=1}^{|E^0|}, (n_i')_{i=1}^{c'}$ and a sequence of paths $(\alpha_i)_{i=1}^{c'}$ such that,

$$\text{tr}_\gamma(\epsilon_0) = \sum_{i=1}^{|E^0|} n_i v_i + \sum_{i=1}^{c'} n_i'\alpha_i\alpha_i^*.$$

We will show that the right-hand side is invertible in $(L_R(E))_0$. Recall that $v_iv_j = \delta_{i,j}v_i$ and $v_i\alpha_j = \delta_{v_i,s(\alpha_j)}\alpha_j$ for all indices i, j. For $1 \le i, j \le c'$, write $\alpha_i \le \alpha_j$ if α_i is an initial subpath of α_j. A moment's thought yields that $(\alpha_i\alpha_i^*)(\alpha_j\alpha_j^*) = (\alpha_j\alpha_j^*)(\alpha_i\alpha_i^*) = \alpha_j\alpha_j^*$ if $\alpha_i \le \alpha_j$ and 0 otherwise. Let T be the set of triples (i, j, t) of indices such that $\alpha_i \le \alpha_j$ or $\alpha_j \le \alpha_i$ and let t be the index of the longer path. Finally, let $f\colon \{1, 2, \dots, c'\} \to \{1, 2, \dots, |E^0|\}$ be the function satisfying $s(\alpha_i) = v_{f(i)}$ for all $1 \le i \le c'$. In other words, f maps the path index i to the index of the start vertex $s(\alpha_i)$.

Now, making an Ansatz for a right inverse and calculating gives,

$$\Big(\sum_{i=1}^{|E^0|} n_i v_i + \sum_{i=1}^{c'} n_i'\alpha_i\alpha_i^*\Big)\Big(\sum_{i=1}^{|E^0|} m_i v_i + \sum_{i}^{c'} m_i'\alpha_i\alpha_i^*\Big)$$
$$= \sum_{i=1}^{|E^0|} n_i m_i v_i + \sum_{i=1}^{c'} (n_{f(i)}m_i' + n_i' m_{f(i)})\alpha_i\alpha_i^* + \sum_{(i,j,t)\in T} n_i' m_j' \alpha_t\alpha_t^*.$$

Letting $m_i = 1/n_i$ for all i we see that the first sum equals $\epsilon_0 = \sum_{i=1}^{|E^0|} v_i$. Next, we will solve for m_i' with the aim to kill the second and third sums. Consider the coefficient of $\alpha_t\alpha_t^*$ for some index t. We require that,

$$\begin{aligned} 0 &= n_{f(t)}m_t' + n_t' m_{f(t)} + \sum_{(i,j,t)\in T} n_i' m_j' \\ &= n_{f(t)}m_t' + n_t' m_{f(t)} + \Big(\sum_{\substack{(t,j,t)\in T \\ j\neq t}} n_i' m_j' + \sum_{(i,t,t)\in T} n_i' m_t'\Big) \\ &= \Big(n_{f(t)} + \sum_{(i,t,t)\in T} n_i'\Big) m_t' + n_t' m_{f(t)} + \sum_{\substack{(t,j,t)\in T \\ j\neq t}} n_i' m_j'. \end{aligned}$$

Rearranging, this means we need to solve for $(m_i')_{i=1}^{c'}$ in the following set of simultaneous equations,

$$m_t' = -\Big(n_{f(t)} + \sum_{(i,t,t)\in T} n_i'\Big)^{-1}\Big(\frac{n_t'}{n_{f(t)}} + \sum_{\substack{(t,j,t)\in T \\ j\neq t}} n_t' m_j'\Big), \tag{20}$$

where $1 \leq t \leq c'$.

We will find a solution to this system of equations by solving a set of independent subsystems. Consider the chain decomposition of the poset $A = \{\alpha_1, \alpha_2, \dots, \alpha_{c'}\}$ equipped with the ordering defined above. More precisely, let S be the minimal elements of A and for each $\alpha_k \in S$, let $P_k : \alpha_k \leq \alpha_{k_1} \leq \cdots \leq \alpha_{k_n}$ be the longest chain starting at α_k. Note that the family of chains $(P_k)_{\alpha_k \in S}$ is a partition of A.

Now, consider a fixed chain $P_{k_0} : \alpha_{k_0} \leq \alpha_{k_1} \leq \cdots \leq \alpha_{k_n}$ and put $I = \{k_0, k_1, \dots, k_n\}$. We claim that the set of equations (20) where $t \in I$ is an independent subsystem. Indeed, a moment's thought yields that this is a system in the variables $\{m_t' \mid t \in I\}$ and conversely these variables only appear in the equations for which $t \in I$. More precisely, since α_{k_0} is minimal, there is no triple $(k_0, j, k_0) \in T$ such that $j \neq k_0$. Hence, taking $t = k_0$ in (20), the second sum is empty and we can solve directly for m_{k_0}'. Moreover, for $0 \leq i \leq n$ note that $(k_i, j, k_i) \in T$ if and only if $j \in \{k_0, k_1, \dots, k_i\}$. Hence, taking $t = k_i$ in (20), we can solve for m_{k_i}' in terms of $m_{k_0}', m_{k_1}', \dots, m_{k_{i-1}}'$. Going along the chain and successively solving the equations, we obtain a solution to the subsystem (20) for the indices in P_{k_0}. Repeating this process for each chain, we obtain a solution to the whole system.

Thus, we obtain a right inverse of $\operatorname{tr}_\gamma(\epsilon_0)$. Since $\operatorname{tr}_\gamma(\epsilon_0) \in Z((L_R(E))_0)$, this is enough to establish the lemma. $\square$

The following is one of our main result:

Corollary B.4.19. Let E be a finite graph and let R be a semisimple ring such that $n \cdot 1_R$ is invertible for every integer $n \neq 0$. Let $L_R(E)$ be the corresponding Leavitt path algebra graded by the canonical $\mathbb{Z}$-grading. Then the following assertions are equivalent:

(a) E is acyclic,
(b) $(L_R(E))_0$ is semisimple and $\mathrm{Supp}(L_R(E))$ is finite,
(c) $L_R(E)$ is left artinian,
(d) $L_R(E)$ is right artinian,
(e) $L_R(E)$ is semisimple,

Proof. (a) $\iff$ (b) $\iff$ (c): Proposition B.4.14.

(b) $\implies$ (e): Assume that $(L_R(E))_0$ is semisimple and $\mathrm{Supp}(L_R(E))$ is finite. By Lemma B.4.18 and Theorem B.4.16, $L_R(E)$ is semisimple.

(d) $\implies$ (c): Assume that $L_R(E)$ is right artinian. By the Hopkins–Levitzki theorem, $L_R(E)$ is left artinian if and only if $L_R(E)$ is left noetherian. Since R is semisimple, it is two-sided noetherian. Hence, since $L_R(E)$ is right noetherian, Corollary B.4.8 implies $L_R(E)$ is also left noetherian.

(e) $\implies$ (c), (e) $\implies$ (d): This follows by Artin-Wedderburn theorem. □

Remark B.4.20. As a special case of Corollary B.4.19 we obtain a characterization of Leavitt path algebras over a field of characteristic 0. This is consistent with Abrams, Aranda Pino and Siles Molina's characterization of artinian and semisimple Leavitt path algebras with coefficients in a field, cf. [1, 5, 2]. However, their characterization does not require that the base field has characteristic 0. In particular, this means that the assumption on the base ring R in Corollary B.4.19 is not a necessary condition.

Finally, we will prove Theorem B.1.4, which is a generalization of Steinberg's characterization of Leavitt path algebras (cf. [34]). We shall first show that noetherian unital Leavitt path algebras come from finite graphs.

Lemma B.4.21. If $(L_R(E))_0$ is left (right) noetherian, then E is finite and satisfies Condition (NE).

Proof. Assume that $(L_R(E))_0$ is left (right) noetherian. If we can prove that E is finite, then the statement follows from Proposition B.4.7.

Suppose to get a contradiction that E is not finite. If E^0 is an infinite set, we can find an infinite sequence of pairwise orthogonal idempotents $\{v_i\}_{i \in \mathbb{N}}$ in $(L_R(E))_0$ contradicting that $(L_R(E))_0$ is left (right) noetherian. Similarly, if E^1 is infinite, it is straightforward to check that $\{ee^*\}_{e \in E^1}$ is an infinite sequence of orthogonal idempotents. Hence, E is finite. □

Before finishing the proof we recall the following well-known property of group graded rings. Let G be an arbitrary group and let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring with principal component R. If S is semisimple, then R is semisimple.

Proof of Theorem B.1.4. (a) : Assume that $L_R(E)$ is left (right) noetherian. By Theorem B.1.1, $(L_R(E))_0$ is left (right) noetherian and hence by Lemma B.4.21, E

is finite and satisfies Condition (NE). Furthermore, by Corollary B.4.12, R is left (right) noetherian. The converse follows directly from Corollary B.4.8.

(b) : Assume that $L_R(E)$ is left (right) artinian. By Theorem B.1.2, in particular, $(L_R(E))_0$ is left (right) artinian and therefore also left (right) noetherian. By Lemma B.4.21, E is finite and satisfies Condition (NE). This means we can apply Corollary B.4.12 to get that R is left (right) artinian. Hence, Proposition B.4.14 proves E is acyclic. The converse follows directly from Proposition B.4.14.

(c) : Assume that $L_R(E)$ is semisimple. This implies that $(L_R(E))_0$ is semisimple. On the other hand, $L_R(E)$ is, in particular, left artinian, hence by (b), E is finite acyclic and hence satisfies Condition (NE). Thus, Corollary B.4.12 implies that R is semisimple. The converse follows directly from Corollary B.4.19. □

B.5. Applications to unital partial crossed products

Let R be an associative, non-trivial unital ring and let G be a group with neutral element e. A *unital twisted partial action of G on R* (see [30, pg. 2]) is a triple,

$$(\{\alpha_g\}_{g\in G}, \{D_g\}_{g\in G}, \{w_{g,h}\}_{(g,h)\in G\times G}),$$

where for each $g \in G$, the D_g's are unital ideals of R, $\alpha_g \colon D_{g^{-1}} \to D_g$ are ring isomorphisms and for each $(g,h) \in G \times G$, $w_{g,h}$ is an invertible element in $D_g D_{gh}$. Let $1_g \in Z(R)$ denote the (not necessarily non-zero) multiplicative identity of the ideal D_g. We require that the following conditions hold for all $g, h \in G$:

(P1) $\alpha_e = \mathrm{id}_R$;
(P2) $\alpha_g(D_{g^{-1}} D_h) = D_g D_{gh}$;
(P3) if $r \in D_{h^{-1}} D_{(gh)^{-1}}$, then $\alpha_g(\alpha_h(r)) = w_{g,h}\alpha_{gh}(r)w_{g,h}^{-1}$;
(P4) $w_{e,g} = w_{g,e} = 1_g$;
(P5) if $r \in D_{g^{-1}} D_h D_{hl}$, then $\alpha_g(rw_{h,l})w_{g,hl} = \alpha_g(r)w_{g,h}w_{gh,l}$.

Given a unital twisted partial action of G on R, we can form the *unital partial crossed product* $R \star_\alpha^w G = \bigoplus_{g\in G} D_g\delta_g$ where the δ_g's are formal symbols. For $g, h \in G, r \in D_g$ and $r' \in D_h$ the multiplication is defined by the rule:

(P6) $(r\delta_g)(r'\delta_h) = r\alpha_g(r'1_{g^{-1}})w_{g,h}\delta_{gh}$.

Directly from the definition of the multiplication it follows that $1_R\delta_e$ is the multiplicative identity of $R\star_\alpha^w G$. It can also be proved that $R\star_\alpha^w G$ is an associative R-algebra (see [15, Thm. 2.4]). Moreover, as mentioned in the introduction, Nystedt, Öinert and Pinedo [30] shows that the natural G-grading is epsilon-strong. Furthermore, note that the principal component of $R \star_\alpha^\omega G$ can be identified with R.

Our characterization of noetherian epsilon-strongly graded rings (Theorem B.1.1) gives the following generalization of Theorem B.1.3.

Corollary B.5.1. If G is a polycyclic-by-finite group, then $R \star_\alpha^\omega G$ is left (right) noetherian if and only if R is left (right) noetherian.

Proof. Since $R \star_\alpha^\omega G$ is epsilon-strongly G-graded (see [30]) by the polycyclic-by-finite group G, the statement follows from Theorem B.1.1. □

Remark B.5.2. We show that Theorem B.1.3 (cf. [11, Cor. 3.4]) follows as a special case of Corollary B.5.1. Let $\alpha = \{\alpha_g \colon D_{g^{-1}} \to D_g\}_{g\in G}$ be a partial action of G on R such that each ideal D_g of R is unital. Taking $w_{g,h} = 1_g 1_{gh}$ it becomes a unital twisted partial action. The above theorem yields the required statement for the skew group ring $R \star_\alpha G$.

Applying our characterization of artinian epsilon-strongly graded rings (Theorem B.1.2) we obtain the following:

Corollary B.5.3. Let G be a torsion-free group. Then $R \star_\alpha^\omega G$ is left (right) artinian if and only if R is left (rigth) artinian and $D_g = \{0\}$ for all but finitely many $g \in G$.

Proof. Since $R \star_\alpha^\omega G$ is epsilon-strongly G-graded (see [30]) by the torsion-free group G, the statement follows from Theorem B.1.2. □

Remark B.5.4. Note that Passman's example [32] of an artinian twisted group ring by an infinite p-group shows that Corollary B.5.3 does not hold for an arbitrary group G.

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PAPER C

The graded structure of algebraic Cuntz-Pimsner rings

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Daniel Lännström

Algebraic Cuntz-Pimsner rings are naturally $\mathbb{Z}$-graded rings that generalize corner skew Laurent polynomial rings, Leavitt path algebras and unperforated $\mathbb{Z}$-graded Steinberg algebras. In this article, we characterize strongly, epsilon-strongly and nearly epsilon-strongly $\mathbb{Z}$-graded algebraic Cuntz-Pimsner rings up to graded isomorphism. We recover two results by Hazrat on when corner skew Laurent polynomial rings and Leavitt path algebras are strongly graded. As a further application, we characterize noetherian and artinian corner skew Laurent polynomial rings.

C.1. Introduction

The Cuntz-Pimsner C^*-algebras were first introduced by Pimsner in [21] and further studied by Katsura in [12]. The Cuntz-Pimsner algebra is constructed from a C^*-correspondence and comes equipped with a natural gauge action. In a recent article, Chirvasitu [7] obtained necessary and sufficient conditions for the gauge action to be free. The *(algebraic) Cuntz-Pimsner rings* were introduced by Carlsen and Ortega in [5] as algebraic analogues of the Cuntz-Pimsner algebras, and simplicity of Cuntz-Pimsner rings were studied in [6]. These rings are interesting to us since they generalize some very famous families of rings. Indeed, Carlsen and Ortega originally gave two important examples of rings realizable as Cuntz-Pimsner rings: *Leavitt path algebras* (see [5, Expl. 5.8] and Section C.2.4) and *corner skew Laurent polynomial rings* (see [5, Expl. 5.7] and Section C.2.5). Recently, Clark, Fletcher, Hazrat and Li [19] showed that unperforated $\mathbb{Z}$-graded Steinberg algebras are also realizable as Cuntz-Pimsner rings. The Cuntz-Pimsner rings do not come with a gauge action but instead a natural $\mathbb{Z}$-grading. This grading is the main object of study in this article.

In the case of Leavitt path algebras, the natural $\mathbb{Z}$-grading was systematically investigated by Hazrat [11]. In particular, he obtained necessary and sufficient conditions for the Leavitt path algebra of a finite graph to be strongly $\mathbb{Z}$-graded (see [11, Thm. 3.15]). The class of *epsilon-strongly graded rings* was first introduced by Nystedt, Öinert and Pinedo in [18] as a generalization of unital strongly graded rings. This subclass of graded rings has been investigated further by the author in [13, 14]. Interestingly, the Leavitt path algebra of a finite graph was proved to be epsilon-strongly $\mathbb{Z}$-graded by Nystedt and Öinert (see [17, Thm. 1.2]). Seeking to extend their result, they introduced the notion of a *nearly epsilon-strongly graded ring* (see Definition C.2.2) and proved that every Leavitt path algebra (even for infinite graphs) is nearly epsilon-strongly $\mathbb{Z}$-graded (see [17, Thm. 1.3]). In other words, there are sufficient conditions in the literature for the natural $\mathbb{Z}$-grading of a Leavitt path algebra to be strong, epsilon-strong and nearly epsilon-strong respectively. These types of gradings have certain structural properties that help us understand the Leavitt path algebras. The present work began as an effort to generalize the previously mentioned results about Leavitt path algebras to a larger class of Cuntz-Pimsner rings. It turns out that we can obtain partial characterizations of nearly epsilon-strongly and epsilon-strongly graded Cuntz-Pimsner rings (see Theorem C.6.1 and Theorem C.6.2). For unital strongly graded Cuntz-Pimsner rings we obtain a complete characterization (see Theorem C.6.3). For that purpose, we obtain sufficient conditions for a Cuntz-Pimsner ring to be strongly graded (see Corollary C.4.12). In particular, we recover Hazrat's results on Leavitt path algebras (see Corollary C.4.15) and corner skew Laurent polynomial ring (see Corollary C.4.16) as special cases.

Carlsen and Ortega [5] constructed the Cuntz-Pimsner rings using a categorical approach. Let R be an associative but not necessarily unital ring. Recall (see [5, Def. 1.1]) that an *R-system* is a triple (P, Q, ψ) where P and Q are R-bimodules and $\psi \colon P \otimes_R Q \to R$ is an R-bimodule homomorphism where $P \otimes_R Q$ denotes the balanced tensor product. A technical assumption called Condition (FS) (see Definintion C.2.8) is generally imposed on the R-system (P, Q, ψ). We will introduce two special types of R-systems called *s-unital* and *unital R-systems* (see Definition C.3.6). Given an R-system, Carlsen and Ortega considered representations of that system. This is the key definition in their construction:

Definition C.1.1. ([5, Def. 1.2, Def. 3.3]) Let R be a ring and let (P, Q, ψ) be an R-system. A *covariant representation* is a tuple (S, T, σ, B) such that the following assertions hold:

(a) B is a ring;
(b) $S \colon P \to B$ and $T \colon Q \to B$ are additive maps;
(c) $\sigma \colon R \to B$ is a ring homomorphism;
(d) $S(pr) = S(p)\sigma(r), S(rp) = \sigma(r)S(p), T(qr) = T(q)\sigma(r), T(rq) = \sigma(r)T(q)$ for all $r \in R$, $q \in Q$ and $p \in P$;
(e) $\sigma(\psi(p \otimes q)) = S(p)T(q)$ for all $p \in P$ and $q \in Q$.

The covariant representation (S, T, σ, B) is *injective* if the map σ is injective. The covariant representation (S, T, σ, B) is *surjective* if B is generated as a ring by $\sigma(R) \cup S(P) \cup T(Q)$.

A surjective covariant representation (S, T, σ, B) is called *graded* if there is a $\mathbb{Z}$-grading $\{B_i\}_{i\in\mathbb{Z}}$ of B such that $\sigma(R) \subseteq B_0$, $T(Q) \subseteq B_1$ and $S(P) \subseteq B_{-1}$.

Remark C.1.2. Let (S, T, σ, B) be a covariant representation and assume that B is $\mathbb{Z}$-graded. Note that (S, T, σ, B) is a graded covariant representation if and only if the grading of B is compatible with the representation structure.

Carlsen and Ortega [5] then considered the category of surjective covariant representations of (P, Q, ψ) denoted by $\mathcal{C}_{(P,Q,\psi)}$. The maps between (S, T, σ, B) and (S', T', σ', B') are ring homomorphisms $\phi \colon B \to B'$ such that $\phi \circ S = S'$, $\phi \circ T = T'$ and $\phi \circ \sigma = \sigma'$. We write $(S, T, \sigma, B) \cong_{\mathrm{r}} (S', T', \sigma', B')$ if the covariant representations are isomorphic as objects in $\mathcal{C}_{(P,Q,\psi)}$. In the case when (P, Q, ψ) satisfies Condition (FS) (see Definition C.2.8), they obtained a complete characterization of injective, graded, surjective covariant representations up to isomorphism in $\mathcal{C}_{(P,Q,\psi)}$ (see [5, Sect. 7]). The *Cuntz-Pimsner rings* are defined as certain universal covariant representations (see Definition C.2.12). Unlike in the C^*-setting, the Cuntz-Pimsner ring is not well-defined for all R-systems (P, Q, ψ) (see [5, Expl. 4.11]).

Let both R and (P, Q, ψ) vary. If a $\mathbb{Z}$-graded ring B shows up in a graded covariant representation (S, T, σ, B) of some R-system (P, Q, ψ), then we call B a *representation ring*. Following Clark, Fletcher, Hazrat and Li [19], we then say that B is *realized by* the representation (S, T, σ, B) of the R-system (P, Q, ψ).

The key new technique of this article is to consider a special type of graded covariant representations (cf. Section C.2.2 for notation):

Definition C.1.3. Let R be a ring, let (P, Q, ψ) be an R-system and let (S, T, σ, B) be a graded covariant representation of (P, Q, ψ). For $k \geq 0$, let $I_{\psi,\sigma}^{(k)}$ be the B_0-ideal generated by the set $\{\sigma(\psi_k(p \otimes q)) \mid p \in P^{\otimes k}, q \in Q^{\otimes k}\} \subseteq B_0$. We call (S, T, σ, B) a *semi-full* covariant representation if $B_{-k}B_k = I_{\psi,\sigma}^{(k)}$ for every $k \geq 0$.

Remark C.1.4. A C^*-correspondence (A, E, ϕ) is called *full* if the closure of $\langle x, y\rangle$ for $x, y \in E$ spans A. One way to generalize this to the algebraic setting is to require that ψ be surjective. Semi-fullness is a weaker condition. Indeed, if R is unital and ψ is surjective, then every graded covariant representation of (P, Q, ψ) is semi-full.

Below is an outline of the rest of this article:

In Section C.2, we recall the definitions of nearly epsilon-strongly graded rings and algebraic Cuntz-Pimsner rings.

In Section C.3, we prove that certain nearly epsilon-strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings can be realized from semi-full covariant representations (see Corollary C.3.12). This is based on recent work by Clark, Fletcher, Hazrat and Li [19] and is the crucial reduction step in the characterization.

In Section C.4, we find sufficient conditions for an injective and graded covariant representation to be strongly $\mathbb{Z}$-graded (see Proposition C.4.10). Using our general theorems, we recover two results by Hazrat as special cases (see Corollary C.4.15 and Corollary C.4.16).

In Section C.5, we obtain sufficient conditions for an injective and semi-full covariant representation ring to be nearly epsilon-strongly $\mathbb{Z}$-graded and epsilon-strongly $\mathbb{Z}$-graded respectively (see Proposition C.5.6 and Proposition C.5.7).

In Section C.6, we obtain partial characterizations of nearly epsilon-strongly and epsilon-strongly graded Cuntz-Pimsner rings (see Theorem C.6.1 and Theorem C.6.2). For unital strongly graded Cuntz-Pimsner rings we obtain a complete characterization (see Theorem C.6.3).

In Section C.7, we collect some important examples. Notably, we give an example of a Leavitt path algebra realizable as a Cuntz-Pimsner ring in two different ways (see Example C.7.3). We also give an example of a trivial Cuntz-Pimsner ring that is not nearly epsilon-strongly $\mathbb{Z}$-graded (see Example C.7.1).

In Section C.8, we apply our results to characterize noetherian and artinian corner skew Laurent polynomial rings (see Corollary C.8.3).

C.2. Preliminaries

All rings are assumed to be associative but not necessarily equipped with a multiplicative identity element. Let R be a ring and let $A \subseteq R$ be a subset. The R-ideal generated by A is denoted by (A). Let ${}_RM$ be a left R-module and let $B \subseteq M$ be a subset. The *R-linear span of B*, denoted by $\operatorname{Span}_R B$, is the R-submodule of ${}_RM$ generated by B. More precisely, $\operatorname{Span}_R B = \Big\{ \sum b_i + \sum r_j \cdot b_j \mid b_i, b_j \in B, r_j \in R \Big\}$, where the sums are finite.

C.2.1. Nearly epsilon-strongly graded rings. Recall that a ring S is called *$\mathbb{Z}$-graded* if there exists a family of additive subsets $\{S_i\}_{i \in \mathbb{Z}}$ of S such that $S = \bigoplus_{i \in \mathbb{Z}} S_i$ and $S_m S_n \subseteq S_{m+n}$ for all $m, n \in \mathbb{Z}$. If the stronger condition $S_m S_n = S_{m+n}$ holds for all $m, n \in \mathbb{Z}$, then the $\mathbb{Z}$-grading $\{S_i\}_{i \in \mathbb{Z}}$ is called *strong*. The subsets S_i are called the *homogeneous components* of S. The *support* of S is defined to be the set $\operatorname{Supp}(S) = \{ i \in \mathbb{Z} \mid S_i \neq \{0\} \}$. The component S_0 is called the *principal component* of S. It is straightforward to show that S_0 is a subring of S. Next, let $S = \bigoplus_{i \in \mathbb{Z}} S_i$ and $T = \bigoplus_{i \in \mathbb{Z}} T_i$ be two $\mathbb{Z}$-graded rings. A ring homomorphism $\phi \colon S \to T$ is called *graded* if $\phi(S_i) \subseteq T_i$ for each $i \in \mathbb{Z}$. If $\phi \colon S \xrightarrow{\sim} T$ is a graded ring isomorphism, then we write $S \cong_{\mathrm{gr}} T$ and say that S and T are *graded isomorphic*.

Let R be a ring. A left (right) R-module ${}_RM$ is called *left (right) s-unital* if for every $x \in M$ there exists some $r_x \in R$ such that $r_x \cdot x = x$ ($x \cdot r_x = x$). A left (right) R-module ${}_RM$ is called *left (right) unital* if there exists some $r \in R$ such that $r \cdot x = x$ ($x \cdot r = x$) for every $x \in M$. Note that our definition of a unital module is stronger than the standard definition. Let R, S be rings. A bimodule ${}_RM_S$ is called *s-unital* (*unital*) if ${}_RM$ is left s-unital (unital) and M_S is right s-unital (unital). In particular, an ideal I of R is called *s-unital* (*unital*) if ${}_RI_R$ is s-unital (unital).

Remark C.2.1. Let R be a ring. It follows from [22, Thm. 1] that if M is a left (right) s-unital R-module, then for any positive integer n and elements $x_1, x_2, \ldots, x_n \in M$ there exists some $r \in R$ such that $r \cdot x_i = x_i$ ($x_i \cdot r = x_i$) for all $i \in \{1, \ldots, n\}$.

If S is a $\mathbb{Z}$-graded ring, then S_i is an S_0-bimodule for every $i \in \mathbb{Z}$ (see [15, Rmk. 1.1.2]). Note that $S_i S_{-i}$ is an ideal of S_0 for every $i \in \mathbb{Z}$. Hence, in particular, S_i is an $S_i S_{-i}$–$S_{-i} S_i$-bimodule for each $i \in \mathbb{Z}$. The following definitions were introduced by Nystedt and Öinert:

Definition C.2.2. ([17, Def. 3.1, Def. 3.2, Def. 3.3]) Let $S = \bigoplus_{i\in\mathbb{Z}} S_i$ be a $\mathbb{Z}$-graded ring.

(a) If S_i is an s-unital S_iS_{-i}–$S_{-i}S_i$-bimodule for each $i \in \mathbb{Z}$, then S is called *nearly epsilon-strongly* $\mathbb{Z}$-graded.
(b) If S_i is a unital S_iS_{-i}–$S_{-i}S_i$-bimodule for each $i \in \mathbb{Z}$, then S is called *epsilon-strongly* $\mathbb{Z}$-graded.
(c) (cf. [8, Def. 4.5]) If $S_i = S_iS_{-i}S_i$ for every $i \in \mathbb{Z}$, then S is called *symmetrically* $\mathbb{Z}$-graded.

Remark C.2.3. We make two remarks regarding Definition C.2.2.

(a) Nystedt and Öinert made these definitions for general group graded rings graded by an arbitrary group. However, in this article we will only consider the special case of $\mathbb{Z}$-graded rings.
(b) If S is epsilon-strongly $\mathbb{Z}$-graded, then S is a unital ring (see [14, Prop. 3.8]). In other words, only unital rings admit an epsilon-strong grading.

We recall the following characterizations of nearly epsilon-strongly graded rings and epsilon-strongly graded rings.

Proposition C.2.4. ([17, Prop. 3.1, Prop. 3.3]) Let $S = \bigoplus_{i\in\mathbb{Z}} S_i$ be a $\mathbb{Z}$-graded ring. The following assertions hold:

(a) S is nearly epsilon-strongly $\mathbb{Z}$-graded if and only if S is symmetrically $\mathbb{Z}$-graded and S_iS_{-i} is an s-unital ideal for each $i \in \mathbb{Z}$;
(b) S is epsilon-strongly $\mathbb{Z}$-graded if and only if S is symmetrically $\mathbb{Z}$-graded and S_iS_{-i} is a unital ideal for each $i \in \mathbb{Z}$.

Moreover, the following implications hold (see [14, Rem. 3.4(a)]):

$$\text{unital strongly graded} \Rightarrow \text{ epsilon strongly graded} \Rightarrow \text{nearly epsilon-strongly graded.} \quad (21)$$

C.2.2. The Toeplitz representation. Let (P, Q, ψ) be an R-system. Put $P^{\otimes 0} = Q^{\otimes 0} = R$ and $\psi_0(r_1 \otimes r_2) = r_1r_2$. Let $\psi_1 = \psi$. For $n > 1$, recursively define $Q^{\otimes n} = Q^{\otimes n-1} \otimes Q$ and $P^{\otimes n} = P \otimes P^{\otimes n-1}$. Let $\psi_n \colon P^{\otimes n} \otimes Q^{\otimes n} \to R$ be defined by,

$$\psi_n((p_1 \otimes p_2) \otimes (q_2 \otimes q_1)) = \psi(p_1 \cdot \psi_{n-1}(p_2 \otimes q_2), q_1),$$

for $p_1 \in P, p_2 \in P^{\otimes n-1}, q_1 \in Q$, and $q_2 \in Q^{\otimes n-1}$. Then, $(P^{\otimes n}, Q^{\otimes n}, \psi_n)$ is an R-system for each $n \geq 0$. Furthermore, by [5, Lem. 1.5], if (S, T, σ, B) is a covariant representation of (P, Q, ψ), then (S^n, T^n, σ, B) is a covariant representation of $(P^{\otimes n}, Q^{\otimes n}, \psi_n)$ where $S^n \colon P^{\otimes n} \to B$ and $T^n \colon Q^{\otimes n} \to B$ are maps satisfying the equations $S^n(p_1 \otimes \cdots \otimes p_n) = S(p_1)S(p_2)\ldots S(p_n)$ and $T^n(q_1 \otimes \cdots \otimes q_n) = T(q_1)T(q_2)\ldots T(q_n)$ for $q_i \in Q$ and $p_j \in P$.

Carlsen and Ortega proved (see [5, Thm. 1.7]) that there is an injective, surjective and graded covariant representation that satisfies a universal property. This covariant representation is called the *Toeplitz representation* and is denoted by $(\iota_Q, \iota_P, \iota_R, \mathcal{T}_{(P,Q,\psi)})$. The ring $\mathcal{T}_{(P,Q,\psi)}$ is called the *Toeplitz ring*. We recall (see [5, Thm. 1.7, Prop. 3.1]) the canonical $\mathbb{Z}$-grading of the Toeplitz ring. The ring homomorphism $\iota_R \colon R \to \mathcal{T}_{(P,Q,\psi)}$ (cf.

Definition C.1.1(c)), turns the ring $\mathcal{T}_{(P,Q,\psi)}$ into an R-algebra. For every pair (m,n) of non-negative integers, consider the following additive subset of $\mathcal{T}_{(P,Q,\psi)}$,

$$\mathcal{T}_{(m,n)} = \mathrm{Span}_R\{\iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p) \mid q \in Q^{\otimes m}, p \in P^{\otimes n}\}.$$

Carlsen and Ortega showed that $\mathcal{T}_{(P,Q,\psi)} = \bigoplus_{m,n\geq 0} \mathcal{T}_{(m,n)}$ is a semigroup grading of $\mathcal{T}_{(P,Q,\psi)}$ (see [5, Def. 1.6]). For every $i \in \mathbb{Z}$, define,

$$\mathcal{T}_i = \bigoplus_{\substack{i\in\mathbb{Z} \\ m-n=i}} \mathcal{T}_{(m,n)}. \tag{22}$$

The canonical $\mathbb{Z}$-grading of the Toeplitz ring is then given by $\mathcal{T}_{(P,Q,\psi)} = \bigoplus_{i\in\mathbb{Z}} \mathcal{T}_i$. Moreover, the Toeplitz ring satisfies the following universal property:

Theorem C.2.5. ([5, Thm. 1.7, Prop. 3.2]) Let R be a ring and let (P,Q,ψ) be an R-system. Let $\mathcal{T}_{(P,Q,\psi)} = \bigoplus_{i\in\mathbb{Z}} \mathcal{T}_i$ be the Toeplitz ring associated to (P,Q,ψ) and let (S,T,σ,B) be any graded covariant representation of (P,Q,ψ). Then there is a unique $\mathbb{Z}$-graded ring epimorphism $\eta\colon \mathcal{T}_{(P,Q,\psi)} \to B$ such that $\eta \circ \iota_R = \sigma, \eta \circ \iota_Q = T$, and $\eta \circ \iota_P = S$.

We relate morphisms in the category of graded covariant representations to morphisms in the category of $\mathbb{Z}$-graded rings:

Lemma C.2.6. Let R be a ring and let (P,Q,ψ) be an R-system. Suppose that (S,T,σ,B) and (S',T',σ',B') are two graded covariant representations of (P,Q,ψ). If

$$\phi\colon (S,T,\sigma,B) \to (S',T',\sigma',B')$$

is a morphism in the category $\mathcal{C}_{(P,Q,\psi)}$ (see the introduction), then $\phi\colon B \to B'$ is a $\mathbb{Z}$-graded ring homomorphism.

Proof. Applying Theorem C.2.5 to (S,T,σ,B), it follows that $B_i = \eta(\mathcal{T}_i)$ and hence, by (22),

$$B_i = \mathrm{Span}_R\{T(q)S(p) \mid q \in Q^{\otimes m}, p \in P^{\otimes n} \text{ where } m-n=i\},$$

for every $i \in \mathbb{Z}$. Similarly, $B'_i = \mathrm{Span}_R\{T'(q)S'(p) \mid q \in Q^{\otimes m}, p \in P^{\otimes n}$ where $m-n = i\}$, for every $i \in \mathbb{Z}$. Since $\phi \circ T = T'$ and $\phi \circ S = S'$ it follows that $\phi(B_i) \subseteq B'_i$. Thus, ϕ is a $\mathbb{Z}$-graded ring homomorphism. $\square$

The following corollary is straightforward to prove:

Corollary C.2.7. Let R be a ring and let (P,Q,ψ) be an R-system. Suppose that $(S,T,\sigma,B) \cong_{\mathrm{r}} (S',T',\sigma',B')$ are two isomorphic graded covariant representations of (P,Q,ψ). Then, we have that $B \cong_{\mathrm{gr}} B'$.

C.2.3. Adjointable operators, Condition (FS) and Cuntz-Pimsner representations. Let (P,Q,ψ) be an R-system. Recall from the C^*-setting, that finite generation of the Hilbert module E is equivalent to the ring of compact operators $B(E) = K(E)$ being unital. In the algebraic setting, the ring of compact operators $K(E)$ is replaced by $\mathcal{F}_P(Q)$ and $\mathcal{F}_Q(P)$ (see [5, Def. 2.1]). We will later see that if P,Q are finitely generated, then $\mathcal{F}_P(Q)$ and $\mathcal{F}_Q(P)$ are unital (see Proposition

C.4.3). For now, we recall the definition of these rings. A right R-module homomorphism $t\colon Q_R \to Q_R$ is called *adjointable* if there exists a left R-module homomorphism $s\colon {}_RP \to_R P$ such that $\psi(p \otimes t(q)) = \psi(s(p) \otimes q)$ for all $q \in Q$ and $p \in P$. The set of adjointable homomorphisms is denoted by $\mathcal{L}_P(Q)$ and $\mathcal{L}_Q(P)$. Note that $\mathcal{L}_P(Q)$ and $\mathcal{L}_Q(P)$ are subrings of $\mathrm{End}(Q_R)$ and $\mathrm{End}({}_RP)$ respectively. Given fixed elements $q \in Q$ and $p \in P$, define $\theta_{q,p}\colon Q_R \to Q_R$ and $\theta_{p,q}\colon {}_RP \to {}_RP$ by $\theta_{q,p}(x) = q \cdot \psi(p \otimes x)$ and $\theta_{p,q}(y) = \psi(y \otimes q) \cdot p$ for $x \in Q$ and $y \in P$ respectively. The R-linear span of the homomorphisms $\{\theta_{q,p} \mid q \in Q, p \in P\}$ is denoted by $\mathcal{F}_P(Q)$. Similarly, the R-linear span of $\{\theta_{p,q} \mid q \in Q, p \in P\}$ is denoted by $\mathcal{F}_Q(P)$. It can be proved that $\mathcal{F}_P(Q)$ and $\mathcal{F}_Q(P)$ are two-sided ideals of $\mathcal{L}_P(Q)$ and $\mathcal{L}_Q(P)$ respectively (see [5, Lem. 2.3]). The following technical condition was introduced by Carlsen and Ortega:

Definition C.2.8. ([5, Def. 3.4]) Let R be a ring. An R-system (P, Q, ψ) is said to satisfy *Condition (FS)* if for all finite sets $\{q_1, q_2, \ldots, q_n\} \subseteq Q$ and $\{p_1, p_2, \ldots, p_m\} \subseteq P$ there exist some $\Theta \in \mathcal{F}_P(Q)$ and $\Phi \in \mathcal{F}_Q(P)$ such that $\Theta(q_i) = q_i$ and $\Phi(p_j) = p_j$ for all $1 \le i \le n$ and $1 \le j \le m$.

Note that we have the following inclusion of rings:

$$\begin{aligned} \mathcal{F}_P(Q) &\subseteq \mathcal{L}_P(Q) \subseteq \mathrm{End}(Q_R), \\ \mathcal{F}_Q(P) &\subseteq \mathcal{L}_Q(P) \subseteq \mathrm{End}({}_RP). \end{aligned} \tag{23}$$

Carlsen and Ortega (see [5, Def. 3.10]) defined maps $\Delta\colon R \to \mathcal{L}_P(Q)$ and $\Gamma\colon R \to \mathcal{L}_Q(P)$ by $\Delta(r)(q) = rq$ and $\Gamma(r)(p) = pr$ for all $r \in R, q \in Q, p \in P$.

In the C^*-setting, it turns out that there are always injective morphisms

$$\pi_n\colon K(E^{\otimes n}) \to \mathcal{T}_E$$

for each $n > 0$. In the algebraic setting, Carlsen and Ortega obtained something similar under the assumption that the system satisfies Condition (FS). Another way to put it is that if the R-system satisfies Condition (FS), then there are induced representations of $\mathcal{F}_P(Q)$ and $\mathcal{F}_Q(P)$. Recall that the opposite ring R^{op} of a ring R has the same additive structure but with a new multiplication defined by $a \star b = ba$ for all $a, b \in R$.

Proposition C.2.9. ([5, Prop. 3.11]) Let R be a ring, let (P, Q, ψ) be an R-system satisfying Condition (FS) and let (S, T, σ, B) be a covariant representation of (P, Q, ψ). Then there exist unique ring homomorphisms $\pi_{T,S}\colon \mathcal{F}_P(Q) \to B$ and $\chi_{T,S}\colon \mathcal{F}_Q(P) \to B^{\mathrm{op}}$ such that $\pi_{T,S}(\theta_{q,p}) = T(q)S(p)$ and $\chi_{T,S}(\theta_{p,q}) = S(p) \star T(q)$ for all $q \in Q, p \in P$. The maps satisfy the following equations for all $\Theta \in \mathcal{F}_P(Q)$ and $\Phi \in \mathcal{F}_Q(P)$:

$$\begin{aligned} \pi_{T,S}(\Delta(r)\Theta) &= \sigma(r)\pi_{T,S}(\Theta), & \pi_{T,S}(\Theta\Delta(r)) &= \pi_{T,S}(\Theta)\sigma(r) \\ \chi_{T,S}(\Gamma(r)\Phi) &= \sigma(r) \star \chi_{T,S}(\Phi), & \chi_{T,S}(\Phi\Gamma(r)) &= \chi_{T,S}(\Phi) \star \sigma(r) \\ \pi_{T,S}(\Theta)T(q) &= T(\Theta(q)), & \chi_{T,S}(\Phi) \star S(p) &= S(\Phi(p)). \end{aligned} \tag{24}$$

Moreover, $\pi_{T,S}(\mathcal{F}_P(Q)) = \chi_{T,S}(\mathcal{F}_Q(P)) = \mathrm{Span}_R\{T(q)S(p) \mid q \in Q, p \in P\} \subseteq B$. If σ is injective, then the maps $\pi_{T,S}$ and $\chi_{T,S}$ are also injective.

Remark C.2.10. We make two remarks regarding Proposition C.2.9.

(a) The equation $\chi_{T,S}(\Phi) \star S(p) = S(p)\chi_{T,S}(\Phi) = S(\Phi(p))$ is misprinted in [5, Prop. 3.11].

(b) Following Carlsen and Ortega, let π denote the map $\bigcup_m \mathcal{F}_{P^{\otimes m}}(Q^{\otimes m}) \to \mathcal{T}_{(P,Q,\psi)}$.

We now recall the definition of the Cuntz-Pimsner invariant representations. If the R-system (P, Q, ψ) satisfies Condition (FS), then the Cuntz-Pimsner invariant representations exhaust all injective, surjective graded covariant representations of (P, Q, ψ) up to isomorphism in $\mathcal{C}_{(P,Q,\psi)}$ (see [5, Rem. 3.30]).

Definition C.2.11. ([5, Def. 3.15, Def. 3.16]) Let R be a ring and let (P, Q, ψ) be an R-system satisfying Condition (FS). Let J be an ideal of R. If $J \subseteq \Delta^{-1}(\mathcal{F}_P(Q))$, then the ideal J is called *ψ-compatible*. If $\ker \Delta \cap J = \{0\}$, then J is called *faithful*. For a ψ-compatible ideal $J \subseteq R$, let $\mathcal{T}(J)$ be the ideal of $\mathcal{T}_{(P,Q,\psi)}$ generated by the set $\{\iota_R(x) - \pi(\Delta(x)) \mid x \in J\}$. The *Cuntz-Pimsner ring relative to* J is defined as the quotient ring $\mathcal{O}_{(P,Q,\psi)} = \mathcal{T}_{(P,Q,\psi)}/\mathcal{T}(J)$. Let $\rho\colon \mathcal{T}_{(P,Q,\psi)} \to \mathcal{O}_{(P,Q,\psi)}$ be the quotient map. Let $\iota_Q^J = \rho \circ \iota_Q$, $\iota_P^J = \rho \circ \iota_P$ and $\iota_R^J = \rho \circ \iota_R$. The covariant representation $(\iota_Q^J, \iota_P^J, \iota_R^J, \mathcal{O}_{(P,Q,\psi)}(J))$ is called the *Cuntz-Pimsner representation relative to* J.

A covariant representation (S, T, σ, B) is called *invariant relative to* J if $\pi_{T,S}(\Delta(x)) = \sigma(x)$ holds in B for each $x \in J$. The relative Cuntz-Pimsner representation

$$(\iota_Q^J, \iota_P^J, \iota_R^J, \mathcal{O}_{(P,Q,\psi)}(J))$$

is invariant relative to J and satisfies a universal property among invariant representations (see [5, Thm. 3.18]). Finally, we recall the definition of the Cuntz-Pismner ring:

Definition C.2.12. ([5, Def. 5.1]) Let R be a ring and let (P, Q, ψ) be an R-system. Suppose that there exists a unique maximal ψ-compatible, faithful ideal J of R. The *Cuntz-Pimsner ring* is defined as $\mathcal{O}_{(P,Q,\psi)} = \mathcal{O}_{(P,Q,\psi)}(J) = \mathcal{T}_{(P,Q,\psi)}/\mathcal{T}(J)$ and the *Cuntz-Pimsner representation* $(\iota_Q^{CP}, \iota_P^{CP}, \iota_R^{CP}, \mathcal{O}_{(P,Q,\psi)})$ is defined to be

$$(\iota_Q^J, \iota_P^J, \iota_R^J, \mathcal{O}_{(P,Q,\psi)}(J)).$$

C.2.4. Leavitt path algebras. The Leavitt path algebra associated to a directed graph was introduced by Ara, Moreno and Pardo [4] and by Abrams and Aranda Pino [2]. For a thorough account of the theory of Leavitt path algebras, we refer the reader to the monograph by Abrams, Ara, and Siles Molina [1]. We now recall the realization of Leavitt path algebras as Cuntz-Pimsner rings given by Carlsen and Ortega (see [5, Expl. 1.10, Expl. 5.9]). They only considered Leavitt path algebras with coefficients in a commutative unital ring, but their construction also works for non-commutative unital rings. Let K be a unital ring that will serve as the coefficient ring. Let $E = (E^0, E^1, s, r)$ be a directed graph consisting of a vertex set E^0, an edge set E^1 and maps $s\colon E^1 \to E^0$ and $r\colon E^1 \to E^0$ specifying the source vertex $s(f)$ and range vertex $r(f)$ for each edge $f \in E^1$. For vertices $u, v \in E^0$, let $\delta_{u,v} = 1$ if $u = v$ and $\delta_{u,v} = 0$ if $u \neq v$. Moreover, let $\{\eta_v \mid v \in E^0\}$ be a copy of the set E^0 and similarly let $\{\eta_f \mid f \in E^1\}$ and $\{\eta_{f^*} \mid f \in E^1\}$ be copies of the set E^1.

(a) Put $R := \bigoplus_{v \in E^0} K\eta_v$. Define a multiplication on R by K-linearly extending the rules $\eta_u \eta_v = \delta_{u,v}\eta_v$ for all $u, v \in E^0$.

(b) Put $Q := \bigoplus_{f\in E^1} K\eta_f$. Let R act on the left of Q by K-linearly extending the rules $\eta_v \cdot \eta_f = \delta_{v,s(f)}\eta_f$ for all $v \in E^0, f \in E^1$. Let R act on the right of Q by K-linearly extending the rules $\eta_f \cdot \eta_v = \delta_{v,r(f)}\eta_f$.

(c) Put $P := \bigoplus_{f\in E^1} K\eta_{f^*}$. Let R act on the left of P by K-linearly extending the rules $\eta_v \cdot \eta_{f^*} = \delta_{v,r(f)}\eta_{f^*}$ for all $v \in E^0, f \in E^1$. Let R act on the right of P by K-linearly extending the rules $\eta_{f^*} \cdot \eta_v = \delta_{v,s(f)}\eta_{f^*}$ for all $v \in E^0, f \in E^1$.

(d) Define an R-bimodule homomorphism $\psi\colon P \otimes_R Q \to R$ by $\eta_{f^*} \otimes \eta_{f'} \mapsto \delta_{f,f'}\eta_{r(f)}$ for all $f, f' \in E^1$.

We will refer to the above R-system (P, Q, ψ) as the *standard Leavitt path system* associated to the directed graph E (with coefficients in K). Carlsen and Ortega proved (see [5, Expl. 5.8]) that (P, Q, ψ) satisfies Condition (FS), that the Cuntz-Pimsner ring is well-defined and that $\mathcal{O}_{(P,Q,\psi)} \cong_{\text{gr}} L_K(E)$. The covariant representation $(\iota_Q^{CP}, \iota_P^{CP}, \iota_R^{CP}, \mathcal{O}_{(P,Q,\psi)})$ is called the *standard Leavitt path algebra covariant representation*. Clark, Fletcher, Hazrat and Li also obtained these facts using more general methods (see [19, Expl. 3.6]).

C.2.5. Corner skew Laurent polynomial rings. The general construction of fractional skew monoid rings was introduced by Ara, Gonzalez-Barroso, Goodearl and Pardo in [3] as algebraic analogues of certain C^*-algebras introduced by Paschke [20]. Here, we consider the special case of a fractional skew monoid ring by a corner isomorphism which is also called a *corner skew Laurent polynomial ring*. Let R be a unital ring and let $\alpha\colon R \to eRe$ be a corner ring isomorphism where e is an idempotent of R. The corner skew Laurent polynomial ring $R[t_+, t_-; \alpha]$ is defined to be the universal unital ring satisfying the following conditions:

(a) There is a unital ring homomorphism $i\colon R \to R[t_+, t_-; \alpha]$;

(b) $R[t_+, t_-; \alpha]$ is the R-algebra satisfying the following equations for every $r \in R$:

$$t_-t_+ = 1, \qquad t_+t_- = i(e), \qquad i(r)t_- = t_-i(\alpha(r)), \qquad t_+i(r) = i(\alpha(r))t_+.$$

Moreover, $R[t_+, t_-; \alpha]$ is $\mathbb{Z}$-graded with $A_0 = R$, $A_i = Rt_+^i$ for $i < 0$ and $A_i = t_-^i R$ for $i > 0$. Note that $t_- \in A_1$ and $t_+ \in A_{-1}$. Carlsen and Ortega [5, Expl. 5.7] proved that the corner skew Laurent polynomial ring $R[t_+, t_-; \alpha]$ can be realized as a Cuntz-Pimsner ring.

C.3. Nearly epsilon-strongly $\mathbb{Z}$-graded rings as Cuntz-Pimsner rings

In this section, we will see that a recent result by Clark, Fletcher, Hazrat and Li [19] will allow us to derive necessary conditions for certain Cuntz-Pimsner rings to be nearly epsilon-strongly $\mathbb{Z}$-graded. Inspired by Exel we make the following definition:

Definition C.3.1. (cf. [9, Def. 4.9]) Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be a $\mathbb{Z}$-graded ring. If $A_n = (A_1)^n$ and $A_{-n} = (A_{-1})^n$ for $n > 0$, then A is called *semi-saturated*.

We show that the Toeplitz ring and any graded covariant representation is semi-saturated.

Proposition C.3.2. Let R be a ring and let (P, Q, ψ) be an R-system.

(a) The Toeplitz ring $\mathcal{T}_{(P,Q,\psi)} = \bigoplus_{i\in\mathbb{Z}} \mathcal{T}_i$ is semi-saturated.

(b) Let (S,T,σ,B) be any graded covariant representation of (P,Q,ψ). Then $B = \bigoplus_{i\in\mathbb{Z}} B_i$ is semi-saturated.

PROOF. (a): Take an arbitrary integer $t > 0$. It follows from the $\mathbb{Z}$-grading that $(\mathcal{T}_1)^t \subseteq \mathcal{T}_t$. We prove the reverse inclusion. Let $\iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p) \in \mathcal{T}_t$ where $q \in Q^{\otimes m}, p \in P^{\otimes n}$ and $m - n = t$. We need to show that $\iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p) \in (\mathcal{T}_1)^t$. Suppose $q = f_1 \otimes f_2 \otimes \cdots \otimes f_{n+t}$ and $p = g_1 \otimes g_2 \otimes \cdots \otimes g_n$. Then,

$$\iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p) = \iota_Q(f_1)\iota_Q(f_2)\ldots\iota_Q(f_{t-1})\iota_{Q^{\otimes(n+1)}}(f_t \otimes f_{t+1} \otimes f_t \otimes \cdots \otimes f_{n+t})\iota_{P^{\otimes n}}(p),$$

is contained in $(\mathcal{T}_1)^t$. Hence, $\mathcal{T}_t = (\mathcal{T}_1)^t$ for $t > 0$. A similar argument shows that $\mathcal{T}_{-t} = (\mathcal{T}_{-1})^t$ for $t > 0$.

(b): By Theorem C.2.5, there is a $\mathbb{Z}$-graded ring epimorphism $\eta\colon \mathcal{T}_{(P,Q,\psi)} \to B$. Hence, $B_n = \eta(\mathcal{T}_n) = \eta((\mathcal{T}_1)^n) = \eta(\mathcal{T}_1)^n = (B_1)^n$ for any $n > 0$. Similarly, $B_{-n} = (B_{-1})^n$ for any $n > 0$. □

If M is a left R-module, then the left annihilator $\mathrm{Ann}_R(M) = \{r \in R \mid r \cdot m = 0\ \forall m \in M\}$ is an ideal of R. If J is an ideal of R, then $J^\perp = \{r \in R \mid rx = xr = 0\ \forall x \in J\}$. The following result was recently obtained by Clark, Fletcher, Hazrat and Li. Their formulation of the theorem is weaker but they in fact prove the stronger statement below.

Theorem C.3.3. ([19, Cor. 3.2]) Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be a $\mathbb{Z}$-graded ring satisfying the following assertions:

(a) A is semi-saturated;
(b) For $\{a_1, a_2, \ldots, a_n\} \subseteq A_1$ there is $r \in A_1A_{-1}$ such that $ra_l = a_l$ for each $1 \le l \le n$, and for $\{b_1, b_2, \ldots, b_m\} \subseteq A_{-1}$ there is $s \in A_1A_{-1}$ such that $b_ls = b_l$ for each $1 \le l \le m$;
(c) $\mathrm{Ann}_{A_0}(A_1) \cap \mathrm{Ann}_{A_0}(A_1)^\perp = \{0\}$.

Let $\psi\colon A_{-1} \otimes A_1 \to A_0$ be defined by $\psi(a' \otimes a) = a'a$. Then the A_0-system (A_{-1}, A_1, ψ) satisfies Condition (FS). Let $i_{A_{-1}}\colon A_{-1} \to A$, $i_{A_1}\colon A_1 \to A$, $i_{A_0}\colon A_0 \to A$ denote the inclusion maps and let $J = A_1A_{-1}$. Then $(i_{A_{-1}}, i_{A_1}, i_{A_0}, A)$ is a surjective covariant representation of (A_{-1}, A_1, ψ) and,

$$(i_{A_{-1}}, i_{A_1}, i_{A_0}, A) \cong_{\mathrm{r}} (\iota^J_{A_{-1}}, \iota^J_{A_1}, \iota^J_{A_0}, \mathcal{O}_{(A_{-1},A_1,\psi')}(J)). \tag{25}$$

Furthermore, J is faithfully maximal, hence,

$$(\iota^J_{A_{-1}}, \iota^J_{A_1}, \iota^J_{A_0}, \mathcal{O}_{(A_{-1},A_1,\psi')}(J)) = (\iota^{CP}_{A_{-1}}, \iota^{CP}_{A_1}, \iota^{CP}_{A_0}, \mathcal{O}_{(A_{-1},A_1,\psi')}).$$

In particular, we have that $A \cong_{\mathrm{gr}} \mathcal{O}_{(A_{-1},A_1,\psi')}$.

PROOF. Note that (A_{-1}, A_1, ψ) is an A_0-system. Since A is semi-saturated, it follows that A is generated as a ring by $A_{-1} \cup A_1 \cup A_0$. Hence, $(i_{A_1}, i_{A_{-1}}, i_{A_0}, A)$ is a surjective covariant representation. In the proof of [19, Thm. 3.1], they show that (A_{-1}, A_1, ψ) satisfies Condition (FS) and that the ideal $J = A_1A_{-1}$ is the maximal faithful, ψ-compatible ideal of A_0. Hence, the Cuntz-Pimsner representation is well-defined and equal to $(\iota^J_{A_{-1}}, \iota^J_{A_1}, \iota^J_{A_0}, \mathcal{O}_{(A_{-1},A_1,\psi')}(J))$. Moreover, they show that the graded representation $(i_{A_1}, i_{A_{-1}}, i_{A_0}, A)$ is Cuntz-Pimsner invariant with respect to J. By the universal property of relative Cuntz-Pimsner rings (see [5, Thm. 3.18]), there exists a surjective map $\eta\colon (\iota^{CP}_{A_{-1}}, \iota^{CP}_{A_1}, \iota^{CP}_{A_0}, \mathcal{O}_{(A_{-1},A_1,\psi')}) \to (i_{A_1}, i_{A_{-1}}, i_{A_0}, A)$. It follows by Lemma C.2.6, that $\eta\colon \mathcal{O}_{(A_{-1},A_1,\psi)} \to A$ is $\mathbb{Z}$-graded. By the graded uniqueness theorem for Cuntz-Pimsner rings (see [5, Cor. 5.4]), it follows that η is also injective. Thus, (25) holds. Note that $A \cong_{\mathrm{gr}} \mathcal{O}_{(A_{-1},A_1,\psi')})$ follows from Corollary C.2.7. $\square$

Let R be a ring, let (P,Q,ψ) be an R-system and let (S,T,σ,B) be a graded covariant representation of (P,Q,ψ). Recall (see Definition C.1.1) that for every $k \geq 0$ and $q \in Q^{\otimes k}, p \in P^{\otimes k}$ we have that $\sigma(\psi_k(p \otimes q)) = S^{\otimes k}(p)T^{\otimes k}(q)$. Since $S^{\otimes k}(p) \in B_{-k}$ and $T^{\otimes k}(q) \in B_k$, it follows that, $\sigma(\psi_k(p \otimes q)) \in B_{-k}B_k$. Moreover, since $I^{(k)}_{\psi,\sigma}$ is generated as a B_0-ideal by the set $\{\sigma(\psi_k(p \otimes q)) \mid p \in P^{\otimes k}, q \in Q^{\otimes k}\}$, we have that $I^{(k)}_{\psi,\sigma} \subseteq B_{-k}B_k$. Recall (see Definition C.1.3) that we call (S,T,σ,B) semi-full if $I^{(k)}_{\psi,\sigma} = B_{-k}B_k$ for every $k \geq 0$. The following result is one of the key insights of this article:

Proposition C.3.4. The covariant representation

$$(i_{A_{-1}}, i_{A_1}, i_{A_0}, A) \cong_{\mathrm{r}} (\iota^J_{A_{-1}}, \iota^J_{A_1}, \iota^J_{A_0}, \mathcal{O}_{(A_{-1},A_1,\psi')}(J)) = (\iota^{CP}_{A_{-1}}, \iota^{CP}_{A_1}, \iota^{CP}_{A_0}, \mathcal{O}_{(A_{-1},A_1,\psi')})$$

in Theorem C.3.3 is a semi-full covariant representation of (A_{-1}, A_1, ψ).

Proof. Note that A comes equipped with a $\mathbb{Z}$-grading which trivially satisfies $i_{A_{-1}}(A_{-1}) \subseteq A_{-1}$, $i_{A_1}(A_1) \subseteq A_1$ and $i_{A_0}(A_0) \subseteq A_0$. Hence, $(i_{A_{-1}}, i_{A_1}, i_{A_0}, A)$ is a graded representation of (A_{-1}, A_1, ψ). Note that $I^{(k)}_{\psi, i_{A_0}} \subseteq A_{-k}A_k$. Recall that A is semi-saturated by Proposition C.3.2(b). Thus, for any monomial $a'a \in A_{-k}A_k$, we have that $a' = a'_1 a'_2 \ldots a'_k$ and $a = a_1 a_2 \ldots a_k$ for some elements $a'_i \in A_{-1}$ and $a_i \in A_1$. Next, note that by the definition,

$$\psi_k((a'_1 \otimes a'_2 \otimes \cdots \otimes a'_k) \otimes (a_1 \otimes \ldots a_k)) = a'_1 a'_2 \ldots a'_k a_1 \ldots a_k = a'a.$$

Thus, $A_{-k}A_k = I^{(k)}_{\psi, i_{A_0}}$. For $k = 0$, note that $\mathrm{Im}(\psi_0) = A_0^2$ since $\psi_0(r \otimes r') = rr'$ for all $r, r' \in A_0$ by convention. Thus, we have that $A_0A_0 = A_0^2 = i_{A_0}(A_0^2) = I^{(0)}_{\psi, i_{A_0}}$. Hence, it follows that $I^{(k)}_{\psi, i_{A_0}} = A_{-k}A_k$ for every integer $k \geq 0$. $\square$

Remark C.3.5. In particular, Proposition C.3.4 implies that some of the examples Clark, Fletcher, Hazrat and Li gave in [19] are realizable from semi-full representations. More precisely, the corner skew Laurent polynomial rings (see [19, Expl. 3.4]) and the Steinberg algebras associated to unperforated graded groupoids (see [19, Cor. 4.6]) are realizable as the representation ring belonging to a semi-full covariant representation.

We will see that, for our purposes, we only need to consider s-unital and unital R-systems. In the C^*-setting, Chirvasitu [7] only considered unital C^*-correspondences (i.e. the coefficient C^*-algebra A is unital). This assumption guarantees that the Cuntz-Pimsner C^*-algebra is unital. We analogously introduce the following notions for R-systems:

Definition C.3.6. Let R be a ring and let (P,Q,ψ) be an R-system. The R-system (P,Q,ψ) is called *s-unital* if R is an s-unital ring and P, Q are s-unital R-bimodules. The R-system (P,Q,ψ) is called *unital* if R is a unital ring and P, Q are unital R-bimodules.

Remark C.3.7. At this point we make two remarks.

(a) Note that we explicitly require that R is an s-unital (unital) ring for the R-system (P, Q, ψ) to be s-unital (unital). This is needed since the trivial module $\{0\}$ is a unital R-bimodule for any ring R (cf. Example C.7.1).
(b) Let R be a unital ring, let (P, Q, ψ) be a unital R-system and let (S, T, σ, B) be a covariant representation of (P, Q, ψ). If 1_R is the multiplicative identity element of R, then $1_B = \sigma(1_R)$ is the multiplicative identity element of B.

We now show that a certain type of semi-saturated, nearly epsilon-strongly $\mathbb{Z}$-graded rings can be realized as Cuntz-Pimsner rings coming from s-unital R-systems.

Definition C.3.8. If $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is a semi-saturated, nearly epsilon-strongly $\mathbb{Z}$-graded ring that satisfies $\mathrm{Ann}_{A_0}(A_1) \cap (\mathrm{Ann}_{A_0}(A_1))^{\perp} = \{0\}$, then A is called *pre-CP*.

As a special case of Theorem C.3.3, we obtain the following:

Corollary C.3.9. Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a pre-CP ring. Let $\psi \colon A_{-1} \otimes A_1 \to A_0$ be defined by $a \otimes b \mapsto ab$. Then (A_{-1}, A_1, ψ) is an s-unital A_0-system that satisfies Condition (FS) and

$$(i_{A_{-1}}, i_{A_1}, i_{A_0}, A) \cong_{\mathrm{r}} (\iota_{A_{-1}}^{CP}, \iota_{A_1}^{CP}, \iota_{A_0}^{CP}, \mathcal{O}_{(A_{-1}, A_1, \psi)}). \tag{26}$$

In particular, $A \cong_{\mathrm{gr}} \mathcal{O}_{(A_{-1}, A_1, \psi)}$. Furthermore, the covariant representation (26) is semi-full.

Proof. Note that conditions (a) and (c) in Theorem C.3.3 are satisfied by definition. Moreover, by the assumption that A is nearly epsilon-strongly $\mathbb{Z}$-graded (see Definition C.2.2), it follows that A_1 is an s-unital $A_1 A_{-1}$–$A_{-1} A_1$-bimodule. From this, (b) follows directly. Furthermore, we see that (A_{-1}, A_1, ψ) is an s-unital A_0-system. The conclusion now follows by applying Theorem C.3.3 and Proposition C.3.4. $\square$

Next, we give two sets of sufficient conditions for a ring to be pre-CP. Recall that a ring is called *semi-prime* if it has no nonzero nilpotent ideals.

Lemma C.3.10. Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a $\mathbb{Z}$-graded ring. The following assertions hold:

(a) If A_0 is semi-prime, then $\mathrm{Ann}_{A_0}(A_1) \cap (\mathrm{Ann}_{A_0}(A_1))^{\perp} = \{0\}$. If A is semi-saturated, nearly epsilon-strongly $\mathbb{Z}$-graded and A_0 is semi-prime, then A is pre-CP.
(b) If A is unital strongly $\mathbb{Z}$-graded, then A is pre-CP.

Proof. (a): Note that $\mathrm{Ann}_{A_0}(A_1) \cap (\mathrm{Ann}_{A_0}(A_1))^{\perp}$ is a nilpotent ideal of A_0.

(b): Since A is unital strongly $\mathbb{Z}$-graded, it follows that $A_i = (A_1)^i, A_{-i} = (A_{-1})^i$ for $i > 0$. Hence, A is semi-saturated. Moreover, $\mathrm{Ann}_{A_0}(A_1) \subseteq \mathrm{Ann}_{A_0}(A_1 A_{-1}) = \mathrm{Ann}_{A_0}(A_0) = \{0\}$ since A_0 is unital. It follows that $\mathrm{Ann}_{A_0}(A_1) \cap (\mathrm{Ann}_{A_0}(A_1))^{\perp} = \{0\}$. Finally, recall that unital strongly $\mathbb{Z}$-graded rings are nearly epsilon-strongly $\mathbb{Z}$-graded (see (21)). Thus, A is pre-CP. $\square$

Proposition C.3.11. Let K be a unital ring and let E be any directed graph. Then the Leavitt path algebra $L_K(E)$ is pre-CP.

Proof. The Leavitt path algebra $L_K(E)$ is nearly epsilon-strongly $\mathbb{Z}$-graded (see [17, Thm. 1.3]). Moreover, since $L_K(E)$ can be realized as a Cuntz-Pimsner ring (see Section C.2.4), it follows by Proposition C.3.2(b) that $L_K(E)$ is semi-saturated. Next, we prove that,

$$\mathrm{Ann}_{L_K(E)_0}(L_K(E)_1) = \mathrm{Span}_K\{v \in E^0 \mid vE^1 = \{0\}\}. \tag{27}$$

Since $L_K(E)_1 L_K(E)_{-1}$ is s-unital by Proposition C.2.4(a) and,

$$\mathrm{Ann}_{L_K(E)_0}(L_K(E)_1) \subseteq \mathrm{Ann}_{L_K(E)_0}(L_K(E)_1 L_K(E)_{-1}),$$

it follows that,

$$L_K(E)_1 L_K(E)_{-1} \cap \mathrm{Ann}_{L_K(E)_0}(L_K(E)_1) \subseteq \mathrm{Ann}_{L_K(E)_1 L_K(E)_{-1}}(L_K(E)_1 L_K(E)_{-1}) = \{0\}. \tag{28}$$

Furthermore, recall that the natural $\mathbb{Z}$-grading of $L_K(E)$ is given by,

$$L_K(E)_i = \mathrm{Span}_K\{\alpha\beta^* \mid \alpha, \beta \in \mathrm{Path}(E), \mathrm{len}(\alpha) - \mathrm{len}(\beta) = i\},$$

for all $i \in \mathbb{Z}$. By convention, the elements $v \in L_K(E)_0$ are considered to be paths of zero length. This means that $L_K(E)_0$ is generated by the sets E^0 and $B := \{\alpha\beta^* \mid \mathrm{len}(\alpha) = \mathrm{len}(\beta) \geq 1\}$. Any $\alpha\beta^* \in B$ can be written $\alpha\beta^* = f_1\alpha'(\beta')^*(f_2)^* \in L_K(E)_1 L_K(E)_0 L_K(E)_{-1} = L_K(E)_1 L_K(E)_{-1}$ for some $f_1, f_2 \in E^1$ and $\alpha, \beta \in \mathrm{Path}(E)$. Thus, $B \subseteq L_K(E)_1 L_K(E)_{-1}$. By (28), it follows that $\mathrm{Ann}_{L_K(E)_0}(L_K(E)_1) \subseteq \mathrm{Span}_K\{v \in E^0\}$.

To establish (27), it remains to prove that for any $v \in E^0$, we have that $vL_K(E)_1 = \{0\}$ if and only if $vE^1 = \{0\}$. The 'only if' direction is clear since $E^1 \subseteq L_K(E)_1$. On the other hand, let $v \in E^0$ such that $vE^1 = \{0\}$. Note that any $\alpha\beta^* \in L_K(E)_1$ satisfies $\mathrm{len}(\alpha) - \mathrm{len}(\beta) = 1$ which implies that $\mathrm{len}(\alpha) \geq 1$. Hence, we can write $\alpha = f'\alpha'$ for some $f' \in E^1$ and some $\alpha' \in \mathrm{Path}(E)$. It follows that $v\alpha\beta^* = (vf')\alpha'\beta^* = 0$. Hence, $vL_K(E)_1 = \{0\}$.

A moment's thought yields that,

$$(\mathrm{Ann}_{L_K(E)_0}(L_K(E)_1))^\perp \cap \mathrm{Span}_K\{v \in E^0\} = \mathrm{Span}_K\{v \in E^0 \mid vE^1 \neq \{0\}\}.$$

Hence, $\mathrm{Ann}_{L_K(E)_0}(L_K(E)_1) \cap (\mathrm{Ann}_{L_K(E)_0}(L_K(E)_1))^\perp = \{0\}$ and $L_K(E)$ is pre-CP. $\square$

From Corollary C.3.9, we derive necessary conditions for certain Cuntz-Pimsner rings to be nearly epsilon-strongly $\mathbb{Z}$-graded.

Corollary C.3.12. Let (P, Q, ψ) be an R-system such that (i) $\mathcal{O}_{(P,Q,\psi)} = \bigoplus_{i\in\mathbb{Z}} \mathcal{O}_i$ exists and is nearly epsilon-strongly $\mathbb{Z}$-graded and (ii) $\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1) \cap (\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1))^\perp = \{0\}$.

Let $\psi' \colon \mathcal{O}_{-1} \otimes \mathcal{O}_1 \to \mathcal{O}_0$ be defined by $\psi'(a \otimes a') = aa'$. Then $(\mathcal{O}_{-1}, \mathcal{O}_1, \psi')$ is an s-unital $\mathcal{O}_0$-system such that,

$$(i_{\mathcal{O}_{-1}}, i_{\mathcal{O}_1}, i_{\mathcal{O}_0}, \mathcal{O}_{(P,Q,\psi)}) \cong_{\mathrm{r}} (\iota^{CP}_{\mathcal{O}_{-1}}, \iota^{CP}_{\mathcal{O}_1}, \iota^{CP}_{\mathcal{O}_0}, \mathcal{O}_{(\mathcal{O}_{-1},\mathcal{O}_1,\psi')}).$$

In particular, $\mathcal{O}_{(P,Q,\psi)} \cong_{\mathrm{gr}} \mathcal{O}_{(\mathcal{O}_{-1},\mathcal{O}_1,\psi')}$ Furthermore, the following assertions hold:

(a) $(\mathcal{O}_{-1}, \mathcal{O}_1, \psi')$ is an s-unital $\mathcal{O}_0$-system that satisfies Condition (FS);
(b) $(\iota^{CP}_{\mathcal{O}_{-1}}, \iota^{CP}_{\mathcal{O}_1}, \iota^{CP}_{\mathcal{O}_0}, \mathcal{O}_{(\mathcal{O}_{-1},\mathcal{O}_1,\psi')})$ is a semi-full covariant representation of $(\mathcal{O}_{-1}, \mathcal{O}_1, \psi')$;
(c) $I^{(k)}_{\psi', \iota^{CP}_{\mathcal{O}}} = \mathcal{O}_{-k}\mathcal{O}_k$ is s-unital for $k \geq 0$.

PROOF. By Proposition C.3.2, $\mathcal{O}_{(P,Q,\psi)}$ is semi-saturated. Hence, with (i) and (ii), it follows that $\mathcal{O}_{(P,Q,\psi)}$ is pre-CP. Thus, Corollary C.3.9 establishes the isomorphism of covariant representations and the conclusions (a), (b). Since the covariant representation is semi-full we have that $I^{(k)}_{\psi',\iota^{CP}_{\mathcal{O}}} = \mathcal{O}_{-k}\mathcal{O}_k$ for each $k \geq 0$. By (i) and Proposition C.2.4(a), we see that $\mathcal{O}_{-k}\mathcal{O}_k$ is s-unital for every $k \geq 0$. Thus, (c) is established. $\square$

Remark C.3.13. It is not clear to the author if the assumption (ii) in Corollary C.3.12 is needed. No examples of nearly epsilon-strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings that do not satisfy $\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1) \cap (\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1))^{\perp} = \{0\}$ have been found. On the other hand, it follows from Lemma C.3.10 that condition (ii) in Corollary C.3.12 is satisfied if either $\mathcal{O}_0$ is semi-prime or $\mathcal{O}_{(P,Q,\psi)}$ is strongly $\mathbb{Z}$-graded.

C.4. Strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings

In this section, we will provide sufficient conditions for the Toeplitz and Cuntz-Pimsner rings to be strongly $\mathbb{Z}$-graded. This is an algebraic analogue of recent work by Chirvasitu [7] where he gave necessary and sufficient conditions for the gauge action of a Cuntz-Pimsner C^*-algebra to be free. Unfortunately, his proofs rely on topological arguments which do not seem to generalize fully to the algebraic setting.

We begin by introducing the following new condition that is stronger than Condition (FS):

Definition C.4.1. Let R be a ring. An R-system (P, Q, ψ) is said to satisfy *Condition (FS')* if there exist some $\Theta \in \mathcal{F}_P(Q)$ and $\Phi \in \mathcal{F}_Q(P)$ such that $\Theta(q) = q$ and $\Phi(p) = p$ for every $q \in Q$ and $p \in P$.

We will later give an example (see Example C.4.5) which shows that Condition (FS) and Condition (FS') are in fact different. We omit the proof of the following proposition as it is a straightforward analogue of the corresponding statement for Condition (FS).

Proposition C.4.2. (cf. [5, Lem. 3.8]) Let R be a ring and let (P, Q, ψ) be an R-system. If (P, Q, ψ) satisfies condition (FS'), then $(P^{\otimes n}, Q^{\otimes n}, \psi_n)$ satisfies condition (FS') for every integer $n \geq 1$.

Throughout the rest of this section, we assume that R is a unital ring and that (P, Q, ψ) is a unital R-system. The following result characterizes Condition (FS'):

Proposition C.4.3. Let R be a unital ring and let (P, Q, ψ) be a unital R-system. The following assertions are equivalent:

(a) (P, Q, ψ) satisfies Condition (FS');
(b) $\mathrm{id}_Q = \Delta(1_R) \in \mathcal{F}_P(Q)$ and $\mathrm{id}_P = \Gamma(1_R) \in \mathcal{F}_Q(P)$. In this case, $\mathcal{L}_P(Q) = \mathcal{F}_P(Q)$ and $\mathcal{L}_Q(P) = \mathcal{F}_Q(P)$ are unital rings;
(c) (P, Q, ψ) satisfies Condition (FS), Q_R is finitely generated as a right R-module and ${}_RP$ is finitely generated as a left R-module.

PROOF. (a) $\Leftrightarrow$ (b): Consider the inclusions in (23). If 1_R is the multiplicative identity element of R, then $\mathrm{id}_Q = \Delta(1_R) \in \mathcal{L}_P(Q)$ is the multiplicative identity element for the ring $\mathcal{L}_P(Q)$. First assume that (P, Q, ψ) satisfies Condition (FS'). Then, $\Theta \in \mathcal{F}_P(Q)$ is a multiplicative identity element of the ring $\mathcal{L}_P(Q)$. Hence, $\Theta = \Delta(1_R) = \mathrm{id}_Q$

which implies that $\mathcal{L}_P(Q) = \mathcal{F}_P(Q)$. Similarly, $\Phi = \Gamma(1_R) = \mathrm{id}_P$ which implies that $\mathcal{L}_Q(P) = \mathcal{F}_Q(P)$. The converse statement follows by noting that $\Delta(1_R)(q) = 1_R \cdot q = q$ and $\Gamma(1_R)(p) = p \cdot 1_R = p$ for all $q \in Q$ and $p \in P$.

(b) $\Rightarrow$ (c): Assume that $\mathrm{id}_P(Q) \in \mathcal{F}_P(Q)$ and $\mathrm{id}_Q(P) \in \mathcal{F}_Q(P)$. By choosing $\Theta := \mathrm{id}_P(Q)$ and $\Phi := \mathrm{id}_Q(P)$ in Definition C.2.8, we see that (P, Q, ψ) satisfies Condition (FS). Furthermore, there are some $q_1, \ldots, q_n \in Q$ and $p_1, \ldots, p_n \in P$ such that $\mathrm{id}_P(Q) = \sum_{i=1}^n \Theta_{q_i,p_i}$. For any $q' \in Q$ we then have that,

$$q' = \mathrm{id}_P(Q)(q') = \sum_{i=1}^n \Theta_{q_i,p_i}(q') = \sum_{i=1}^n q_i \cdot \psi(p_i \otimes q') \in \mathrm{Span}_R\{q_1, \ldots, q_n\}.$$

In other words, Q is finitely generated as a right R-module by the set $\{q_1, \ldots, q_n\}$. A similar argument establishes that P is finitely generated as a left R-module.

(c) $\Rightarrow$ (a): Assume that (P, Q, ψ) satisfies Condition (FS), Q is generated as a right R-module by the set $\{q_1, \ldots, q_n\}$ and that P is generated as a left R-module by the set $\{p_1, \ldots, p_m\}$ for some non-negative integers n, m and $q_i \in Q, p_i \in P$. Let $\Theta \in \mathcal{F}_P(Q)$ and $\Phi \in \mathcal{F}_Q(P)$ be such that $\Theta(q_i) = q_i$ and $\Phi(p_j) = p_j$ for all $i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$. Take an arbitrary $q' \in Q$ and note that there are some $r_i \in R$ such that $q' = \sum_{i=1}^n q_i \cdot r_i$. But since Θ is a right R-module homomorphism, it follows that $\Theta(q') = \Theta(\sum_{i=1}^n q_i \cdot r_i) = \sum_{i=1}^n \Theta(q_i) \cdot r_i = \sum_{i=1}^n q_i \cdot r_i = q'$. A similar argument shows that $\Phi(p') = p'$ for every $p \in P$. Thus, (P, Q, ψ) satisfies Condition (FS'). $\square$

Remark C.4.4. At this point, we make two remarks regarding Proposition C.4.3.

(a) Note that Condition (FS) (cf. Definition C.2.8) and Condition (FS') (cf. Definition C.4.1) relates to each other similarly to how s-unital rings relate to unital rings. In Section C.5, we will show that Condition (FS)/Condition (FS') implies that the ideals $\mathcal{T}_i\mathcal{T}_{-i}$ are s-unital/unital for $i \geq 0$.
(b) In the C^*-setting, finite generation of the Hilbert module E is equivalent to the ring of compact operators $B(E) = K(E)$ being unital. Proposition C.4.3 is the algebraic analogue of this statement.

The following system satisfies Condition (FS) but not Condition (FS'):

Example C.4.5. Let E consist of one vertex v with countably infinitely many loops $f_1, f_2, \ldots$. This is sometimes called a rose with countably many petals.

The standard Leavitt path algebra system (P, Q, ψ) attached to the graph E satisfies Condition (FS) (see [5, Expl. 5.8]). Furthermore, it is straightforward to check that (P, Q, ψ) is a unital R-system with multiplicative identity element $1_R = \eta_v$. However, since E contains infinitely many edges it follows that P and Q are not finitely generated (see Section C.2.4 and Lemma C.4.14). By Proposition C.4.3(c), (P, Q, ψ) does not satisfy Condition (FS'). In other words, (P, Q, ψ) is an example of an R-system satisfying Condition (FS) but not Condition (FS').

To prove that the Toeplitz ring is strongly $\mathbb{Z}$-graded, we need the following definition.

Definition C.4.6. Let R be a unital ring, let (P, Q, ψ) be an R-system satisfying Condition (FS') and let (S, T, σ, B) be a covariant representation of (P, Q, ψ). Then (S, T, σ, B) is called *faithful* if $\pi_{T,S}(\Delta(1_R)) = \sigma(1_R)$.

To make sense of Definition C.4.6, note that $\Delta(1_R) \in \mathcal{F}_P(Q)$ for every R-system satisfying Condition (FS') by Proposition C.4.3(b). Hence, the condition $\pi_{T,S}(\Delta(1_R)) = \sigma(1_R)$ makes sense. It also follows from Proposition C.4.3(c) that if an R-system (P, Q, ψ) admits a faithful covariant representation, then Q is finitely generated as a right R-module and P is finitely generated as a left R-module.

Next, we will consider a graded covariant representation and derive sufficient conditions for it to be strongly $\mathbb{Z}$-graded.

Lemma C.4.7. Let R be a unital ring. Suppose that (P, Q, ψ) is an R-system that satisfies Condition (FS') and that (S, T, σ, B) is a graded, injective, surjective and faithful representation of (P, Q, ψ). Then,

$$\pi_{T,S}(\Delta(1_R)) = \sigma(1_R) = 1_B \in B_1 B_{-1}.$$

Proof. By Proposition C.4.3(b) and Condition (FS'), we have that $\Delta(1_R) \in \mathcal{F}_P(Q)$. Furthermore, by faithfulness, $\pi_{T,S}(\Delta(1_R)) = \sigma(1_R)$. By Proposition C.2.9 and the assumption that the covariant representation is injective, it follows that the map $\pi_{T,S} \colon \mathcal{F}_P(Q) \to \mathrm{Span}_R\{T(q)S(p) \mid q \in Q, p \in P\}$ is a ring isomorphism. Hence, $\pi_{T,S}(\Delta(1_R)) = \sigma(1_R) = 1_B = \sum_i T(q_i)S(p_i) \in B_1 B_{-1}$ for some $q_i \in Q, p_i \in P$. □

Lemma C.4.8. Let R be a unital ring and let (P, Q, ψ) be a unital R-system such that the map $\psi \colon P \otimes Q \to R$ is surjective. Let (S, T, σ, B) be a surjective, graded covariant representation of (P, Q, ψ). Then, $1_B \in B_{-1} B_1$.

Proof. Since ψ is surjective, there is some $p \in P$ and $q \in Q$ such that $\psi(p \otimes q) = 1_R$. This yields that $1_B = \sigma(1_R) = \sigma(\psi(p \otimes q)) = S(p)T(q) \in B_{-1} B_1$. □

Finally, recall the following proposition for general $\mathbb{Z}$-graded rings (see e.g. [17, Prop. 39]):

Proposition C.4.9. Let $S = \bigoplus_{i \in \mathbb{Z}} S_i$ be a unital $\mathbb{Z}$-graded ring. Then S is strongly $\mathbb{Z}$-graded if and only if $1_S \in S_1 S_{-1}$ and $1_S \in S_{-1} S_1$.

We have now found sufficient conditions for a representation ring to be strongly $\mathbb{Z}$-graded:

Proposition C.4.10. Let R be a unital ring and let (P, Q, ψ) be a unital R-system that satisfies Condition (FS'). Let (S, T, σ, B) be an injective, surjective and graded covariant representation of (P, Q, ψ). Furthermore, suppose that the following assertions hold:
(a) (S, T, σ, B) is a faithful representation of (P, Q, ψ);
(b) ψ is surjective.
Then B is strongly $\mathbb{Z}$-graded.

PROOF. By assumption (a), it follows from Lemma C.4.7 that $1_B \in B_1 B_{-1}$. By assumption (b) and Lemma C.4.8, it follows that $1_B \in B_{-1} B_1$. Since B is a unital $\mathbb{Z}$-graded ring, it now follows from Proposition C.4.9 that B is strongly $\mathbb{Z}$-graded. □

Note that since the Toeplitz representation $(\iota_P, \iota_Q, \iota_R, \mathcal{T}_{(P,Q,\psi)})$ is injective, surjective and graded, Proposition C.4.10 gives, in particular, sufficient conditions for the Toeplitz ring to be strongly $\mathbb{Z}$-graded.

Corollary C.4.11. Let R be a unital ring and let (P, Q, ψ) be a unital R-system that satisfies Condition (FS'). Consider the Toeplitz ring $\mathcal{T}_{(P,Q,\psi)} = \bigoplus_{i \in \mathbb{Z}} \mathcal{T}_i$. If $\pi(\Delta(1_R)) = \iota_R(1_R)$ and ψ is surjective, then $\mathcal{T}_{(P,Q,\psi)}$ is strongly $\mathbb{Z}$-graded.

The requirement of faithfulness is more easily formulated when considering the relative Cuntz-Pimsner representations.

Corollary C.4.12. Let R be a unital ring and let (P, Q, ψ) be a unital R-system that satisfies Condition (FS'). Let $J \subseteq R$ be a ψ-compatible ideal. Furthermore, suppose that the following assertions hold:

(a) $1_R \in J$;
(b) ψ is surjective.

Then the relative Cuntz-Pimsner ring $\mathcal{O}_{(P,Q,\psi)}(J)$ is strongly $\mathbb{Z}$-graded.

PROOF. Recall that the Cuntz-Pimsner representation $(\iota_P^J, \iota_Q^J, \iota_R^J, \mathcal{O}_{(P,Q,\psi)}(J))$ is injective, surjective and graded. Furthermore, note that (a) implies that the identity $\iota_R^J(1_R) = \pi_{\iota_Q^J, \iota_P^J}(\Delta(1_R))$ holds in the Cuntz-Pimsner ring. This implies that the representation $(\iota_P^J, \iota_Q^J, \iota_R^J, \mathcal{O}_{(P,Q,\psi)}(J))$ is faithful. By Proposition C.4.10 and (b), we have that $\mathcal{O}_{(P,Q,\psi)}(J)$ is strongly $\mathbb{Z}$-graded. □

For the rest of this section, we apply the above theorems to the special cases of Leavitt path algebras and corner skew Laurent polynomial rings. We begin by proving that the conditions in Corollary C.4.12 are satisfied for any Leavitt path algebra associated to a finite graph without sinks.

Remark C.4.13. The Leavitt path algebra of a graph E is the Cuntz-Pimsner ring relative to the ideal J generated by the regular vertices $\mathrm{Reg}(E) \subseteq E^0$. In other words, $L_K(E) \cong_{\mathrm{gr}} \mathcal{O}_{(P,Q,\psi)}(J)$ where (P, Q, ψ) is the standard Leavitt path algebra system associated to E (see [5, Expl. 5.8] and Section C.2.4). Suppose that E is a finite graph without any sinks. We now prove that the conditions (a) and (b) in Corollary C.4.12 are satisfied.

(a) Since a singular vertex (non-regular vertex) is either an infinite emitter or a sink, by the requirements on E, it follows that $\mathrm{Reg}(E) = E^0$. This implies that $J = R$ and hence that $1_R = \sum_{v \in E^0} \eta_v \in J$.
(b) Since E does not contain any sinks, we have that for any $v \in E^0$ there is some $f \in E^1$ such that $r(f) = v$. Thus, $\eta_v = \eta_{r(f)} = \psi(\eta_{f^*} \otimes \eta_f)$. This proves that ψ is surjective.

Compare the following lemma with Example C.4.5:

Lemma C.4.14. Let K be a unital ring and let E be a directed graph with finitely many vertices. Then the standard Leavitt path algebra system (P, Q, ψ) is a unital R-system. Furthermore, (P, Q, ψ) satisfies Condition (FS') if and only if E has finitely many edges.

Proof. Recall that the standard Leavitt path algebra system (see Section C.2.4) is defined by $P = \bigoplus_{f \in E^1} K\eta_{f^*}$ and $Q = \bigoplus_{f \in E^1} K\eta_f$. The assumption that E has finitely many vertices implies that R is a unital ring and that (P, Q, ψ) is a unital R-system. By Proposition C.4.3(c), (P, Q, ψ) satisfies Condition (FS') if and only if (P, Q, ψ) satisfies Condition (FS), (i) Q is finitely generated as a right R-module and (ii) P is finitely generated as a left R-module. However, the R-system (P, Q, ψ) always satisfies Condition (FS) (see [5, Expl. 5.8]). Moreover, it follows from the definition of P and Q that (i) and (ii) hold if and only if E has finitely many edges. ☐

We can now partially recover a result obtained by Hazrat on when a Leavitt path algebra of a finite graph is strongly $\mathbb{Z}$-graded (see [11, Thm. 3.15]).

Corollary C.4.15. Let K be a unital ring and let E be a finite graph without any sinks. Then the Leavitt path algebra $L_K(E)$ is strongly $\mathbb{Z}$-graded.

Proof. By Lemma C.4.14, Remark C.4.13 and Corollary C.4.12 it follows that $L_K(E) \cong_{\mathrm{gr}} \mathcal{O}_{(P,Q,\psi)}(J)$ is strongly $\mathbb{Z}$-graded. ☐

We will now consider corner skew Laurent polynomial rings. Recall that we need to specify a unital ring R, an idempotent $e \in R$ and a corner isomorphism $\alpha \colon R \to eRe$. Moreover, recall that an idempotent $e \in R$ is called *full* if $ReR = R$. Hazrat showed (see [10, Prop. 1.6.6]) that $R[t_+, t_-; \alpha]$ is strongly $\mathbb{Z}$-graded if and only if e is a full idempotent.

Corollary C.4.16. Let R be a unital ring and let $\alpha \colon R \to eRe$ be a ring isomorphism where e is an idempotent of R. The corner skew Laurent polynomial ring $R[t_+, t_-; \alpha]$ is strongly $\mathbb{Z}$-graded if e is a full idempotent.

Proof. Let (P, Q, ψ) denote the R-system in [5, Expl. 5.6], i.e. let,

$$P = \Big\{ \sum r_i \alpha(r_i') \mid r_i, r_i' \in R \Big\}, \quad Q = \Big\{ \sum \alpha(r_i) r_i' \mid r_i, r_i' \in R \Big\}, \quad \psi(p \otimes q) = pq,$$

where the left and right actions of R on P and Q are defined by $r \cdot r_1\alpha(r_2) = rr_1\alpha(r_2)$, $r_1\alpha(r_2) \cdot r = r_1\alpha(r_2 r)$, $r \cdot \alpha(r_1)r_2 = \alpha(rr_1)r_2$, $\alpha(r_1)r_2 \cdot r = \alpha(r_1)r_2 r$ for all $r, r_1, r_2 \in R$. By [5, Expl. 5.7], the R-system (P, Q, ψ) satisfies Condition (FS). Assume that e is a full idempotent. Then,

$$\mathrm{Im}(\psi) = PQ = (ReRe)(eReR) = ReR(ee)ReR = (ReR)e(ReR) = ReR = R.$$

Hence, ψ is surjective. Furthermore, note that ${}_RP = {}_R(ReRe) = {}_R(ReR)e = {}_RRe$ as left R-modules. It follows that P is finitely generated as a left R-module. Similarly, $Q_R = (eReR)_R = eR_R$ is finitely generated as a right R-module. By Proposition C.4.3(c), it follows that (P, Q, ψ) satisfies Condition (FS'). Recall from [5, Expl. 5.7] that $J = R$ is ψ-compatible and $R[t_+, t_-; \alpha] \cong_{\mathrm{gr}} \mathcal{O}_{(P,Q,\psi)}(J)$. By Corollary C.4.12, it follows that $\mathcal{O}_{(P,Q,\psi)}(J)$ is strongly $\mathbb{Z}$-graded. Thus, $R[t_+, t_-; \alpha]$ is strongly $\mathbb{Z}$-graded. ☐

C.5. Epsilon-strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings

We will show that Condition (FS) and Condition (FS') correspond to local unit properties of the rings $\mathcal{T}_i\mathcal{T}_{-i}$ for $i > 0$. This allows us to find sufficient conditions for certain representation rings to be nearly epsilon-strongly and epsilon-strongly $\mathbb{Z}$-graded.

Proposition C.5.1. Let R be an s-unital ring and let (P, Q, ψ) be an s-unital R-system that satisfies Condition (FS). Consider the Toeplitz ring $\mathcal{T}_{(P,Q,\psi)} = \bigoplus_{i\in\mathbb{Z}} \mathcal{T}_i$. The following assertions hold:

(a) For $i \geq 0$, $\mathcal{T}_i$ is a left s-unital $\mathcal{T}_i\mathcal{T}_{-i}$-module;
(b) For $i \geq 0$, $\mathcal{T}_{-i}$ is a right s-unital $\mathcal{T}_i\mathcal{T}_{-i}$-module;
(c) $\mathcal{T}_i\mathcal{T}_{-i}$ is an s-unital ring for $i \geq 0$;
(d) $\mathcal{T}_i = \mathcal{T}_i\mathcal{T}_{-i}\mathcal{T}_i$ for every $i \in \mathbb{Z}$.

Proof. (a): Take an arbitrary integer $i \geq 0$ and an element $s \in \mathcal{T}_i$. Then, $s = \sum_j \iota_{Q^{\otimes m_j}}(q_j)\iota_{P^{\otimes n_j}}(p_j)$ for some non-negative integers $\{m_j\}, \{n_j\}$ and elements $q_j \in Q^{\otimes m_j}, p_j \in P^{\otimes n_j}$. Note that $m_j - n_j = i$ for all indices j. Furthermore, since i is non-negative, we have that $0 \leq i \leq m_j$ for all j. We will construct an element $\epsilon(s)$ such that $\epsilon(s)s = s$.

If $i = 0$, then by the assumption that (P, Q, ψ) is an s-unital R-system and Remark C.2.1, we can find some element $r \in R$ such that $r \cdot q_j = q_j$ for all j. Put $\epsilon(s) := \iota_R(r) \in \mathcal{T}_0$. Then,

$$\begin{aligned}\epsilon(s)s = \iota_R(r)\sum_j \iota_{Q^{\otimes m_j}}(q_j)\iota_{P^{\otimes n_j}}(p_j) &= \sum_j \iota_{Q^{\otimes m_j}}(r\cdot q_j)\iota_{P^{\otimes n_j}}(p_j)\\ &= \sum_j \iota_{Q^{\otimes m_j}}(q_j)\iota_{P^{\otimes n_j}}(p_j) = s.\end{aligned}$$

If $i > 0$, then let q_j' denote the ith initial segment of q_j for every j. In other words, for every j we have that $q_j = q_j' \otimes q_j''$ where $q_j' \in Q^{\otimes i}$ and $q_j'' \in Q^{\otimes(m_j - i)}$. Since (P, Q, ψ) satisfies Condition (FS), it follows by [5, Lem. 3.8] that $(P^{\otimes i}, Q^{\otimes i}, \psi_i)$ satisfies Condition (FS). Therefore, there is some $\Theta \in \mathcal{F}_{P^{\otimes i}}(Q^{\otimes i})$ such that $\Theta(q_j') = q_j'$ for all j. Invoking Proposition C.2.9, we put $\epsilon(s) := \pi_{\iota_{Q^{\otimes i}},\iota_{P^{\otimes i}}}(\Theta)$. By Proposition C.2.9 and (22), we have that, $\pi_{\iota_{Q^{\otimes i}},\iota_{P^{\otimes i}}}(\Theta) \in \mathrm{Span}_R\{\iota_{Q^{\otimes i}}(q)\iota_{P^{\otimes i}}(p) \mid q \in Q^{\otimes i}, p \in P^{\otimes i}\} \subseteq \mathcal{T}_i\mathcal{T}_{-i}$. Furthermore, by using the left relation of (24),

$$\begin{aligned}\epsilon(s)s = \pi(\Theta)\sum_j \iota_{Q^{\otimes m_j}}(q_j)\iota_{P^{\otimes n_j}}(p_j) &= \pi(\Theta)\sum_j \iota_{Q^{\otimes i}}(q_j')\iota_{Q^{\otimes(m_j-i)}}(q_j'')\iota_{P^{\otimes n_j}}(p_j)\\ &= \sum_j (\pi(\Theta)\iota_{Q^{\otimes i}}(q_j'))\iota_{Q^{\otimes(m_j-i)}}(q_j'')\iota_{P^{\otimes n_j}}(p_j)\\ &= \sum_j (\iota_{Q^{\otimes i}}(\Theta(q_j')))\iota_{Q^{\otimes(m_j-i)}}(q_j'')\iota_{P^{\otimes n_j}}(p_j)\\ &= \sum_j (\iota_{Q^{\otimes i}}(q_j'))\iota_{Q^{\otimes(m_j-i)}}(q_j'')\iota_{P^{\otimes n_j}}(p_j) = \sum_j \iota_{Q^{\otimes m_j}}(q_j)\iota_{P^{\otimes n_j}}(p_j) = s.\end{aligned}$$

(b): Analogous to (a)

(c): Let $i \geq 0$ be an arbitrary non-negative integer. Any element of $\mathcal{T}_i\mathcal{T}_{-i}$ is a finite sum $s = \sum_j a_j b_j$ where $a_j \in \mathcal{T}_i$ and $b_j \in \mathcal{T}_{-i}$. Since $\mathcal{T}_i$ is a left s-unital $\mathcal{T}_i\mathcal{T}_{-i}$-module by (a), Remark C.2.1 implies that we can find some element $t_1 \in \mathcal{T}_i\mathcal{T}_{-i}$ such that $t_1 a_j = a_j$ for all indices j. Similarly, (b) and Remark C.2.1 implies that there is some element $t_2 \in \mathcal{T}_i\mathcal{T}_{-i}$ such that $b_j t_2 = b_j$ for all indices j. Hence, $t_1 s = s$ and $st_2 = s$. This implies that $\mathcal{T}_i\mathcal{T}_{-i}$ is a left s-unital $\mathcal{T}_i\mathcal{T}_{-i}$-module and a right s-unital $\mathcal{T}_i\mathcal{T}_{-i}$-module. Thus, $\mathcal{T}_i\mathcal{T}_{-i}$ is an s-unital ring.

(d): Take an arbitrary integer $i \in \mathbb{Z}$. From the grading, it is clear that $\mathcal{T}_i\mathcal{T}_{-i}\mathcal{T}_i \subseteq \mathcal{T}_i$. It remains to show that $\mathcal{T}_i \subseteq \mathcal{T}_i\mathcal{T}_{-i}\mathcal{T}_i$. Let $s \in \mathcal{T}_i$ be an arbitrary element. First suppose that $i \geq 0$, then by (a) there is some $\epsilon(s) \in \mathcal{T}_i\mathcal{T}_{-i}$ such that $s = \epsilon(s)s \in \mathcal{T}_i\mathcal{T}_{-i}\mathcal{T}_i$. On the other hand, if $i < 0$, then by (b) there is some $\epsilon(s) \in \mathcal{T}_{-i}\mathcal{T}_i$ such that $s = s\epsilon(s) \in \mathcal{T}_i\mathcal{T}_{-i}\mathcal{T}_i$. Thus, $\mathcal{T}_i = \mathcal{T}_i\mathcal{T}_{-i}\mathcal{T}_i$ for every $i \in \mathbb{Z}$. ☐

Recall that for idempotents e, f we define the idempotent ordering by $e \leq f \iff ef = fe = e$.

Remark C.5.2. Let A be an epsilon-strongly $\mathbb{Z}$-graded ring. Let $\epsilon_i \in A_i A_{-i}$ denote the multiplicative identity element of $A_i A_{-i}$ for $i \in \mathbb{Z}$ (see Proposition C.2.4). If the gradation on A is semi-saturated, then $\epsilon_0 \geq \epsilon_1 \geq \epsilon_2 \geq \epsilon_3 \geq \dots$ and $\epsilon_0 \geq \epsilon_{-1} \geq \epsilon_{-2} \geq \epsilon_{-3} \geq \dots$.

For the next section, let (P, Q, ψ) be a unital R-system. Suppose that (P, Q, ψ) satisfies Condition (FS'). By Proposition C.4.3(b), this implies that $\Delta(1_R) \in \mathcal{F}_P(Q)$ and $\Gamma(1_R) \in \mathcal{F}_Q(P)$. Consider the Toeplitz representation $(\iota_Q, \iota_P, \iota_R, \mathcal{T}_{(P,Q,\psi)})$. We define,

$$\epsilon_0 := \iota_R(1_R), \qquad \epsilon_i := \pi_{\iota_{Q^{\otimes i}}, \iota_{P^{\otimes i}}}(\Delta^i(1_R)) = \chi_{\iota_{Q^{\otimes i}}, \iota_{P^{\otimes i}}}(\Gamma^i(1_R)) \in \mathcal{T}_i\mathcal{T}_{-i},$$

for $i > 0$.

Lemma C.5.3. The sequence $\{\epsilon_i\}_{i \geq 0}$ consists of idempotents such that $\epsilon_0 \geq \epsilon_1 \geq \epsilon_2 \geq \epsilon_3 \geq \epsilon_4 \geq \dots$ holds in the idempotent ordering.

Proof. Fix an arbitrary integer $i \geq 0$. By Proposition C.2.9, we have that $\epsilon_i = \pi(\Delta^i(1_R)) = \sum_j \iota_{Q^{\otimes i}}(q_j)\iota_{P^{\otimes i}}(p_j)$ for some $q_j \in Q^{\otimes i}$ and $p_j \in P^{\otimes i}$. Then, by the left relation in (24),

$$\begin{aligned} \epsilon_i^2 &= \sum_j \epsilon_i \iota_{Q^{\otimes i}}(q_j)\iota_{P^{\otimes i}}(p_j) = \sum_j (\pi(\Delta^i(1_R))\iota_{Q^{\otimes i}}(q_j))\iota_{P^{\otimes i}}(p_j) \\ &= \sum_j \iota_{Q^{\otimes i}}(\Delta^i(1_R)(q_j))\iota_{P^{\otimes i}}(p_j) = \sum_j \iota_{Q^{\otimes i}}(q_j)\iota_{P^{\otimes i}}(p_j) = \epsilon_i. \end{aligned}$$

Hence, ϵ_i is an idempotent.

It is clear that $\iota_R(1_R) = \epsilon_0 \geq \epsilon_1$. Take an arbitrary integer $m > 0$. We will prove that $\epsilon_m \geq \epsilon_{m+1}$. This is equivalent to $\epsilon_{m+1} = \epsilon_{m+1}\epsilon_m = \epsilon_m\epsilon_{m+1}$. We first prove that $\epsilon_m\epsilon_{m+1} = \epsilon_m$. Let $\epsilon_{m+1} = \sum_j \iota_{Q^{\otimes m+1}}(q_j)\iota_{P^{\otimes m+1}}(p_j)$. Write $q_j = q_j' \otimes q_j''$ where

$q'_j \in Q^{\otimes m}$ and $q''_j \in Q$. Then, by the left relation in (24),

$$\begin{aligned}\epsilon_m \epsilon_{m+1} &= \sum_j \epsilon_m \iota_{Q^{\otimes m+1}}(q_j)\iota_{P^{\otimes m+1}}(p_j) = \sum_j \epsilon_m \iota_{Q^{\otimes m}}(q'_j)\iota_Q(q''_j)\iota_{P^{\otimes m+1}}(p_j) \\ &= \sum_j \iota_{Q^{\otimes m}}(\Delta^m(1_R)(q'_j))\iota_Q(q''_j)\iota_{P^{\otimes m+1}}(p_j) = \sum_j \iota_{Q^{\otimes m}}(q'_j)\iota_Q(q''_j)\iota_{P^{\otimes m+1}}(p_j) \\ &= \sum_j \iota_{Q^{\otimes m+1}}(q_j)\iota_{P^{\otimes m+1}}(p_j) = \epsilon_m.\end{aligned}$$

Again, let $\epsilon_{m+1} = \sum_j \iota_{Q^{\otimes m+1}}(q_j)\iota_{P^{\otimes m+1}}(p_j)$. This time write $p_j = p'_j \otimes p''_j$ for some $p'_j \in P$ and $p''_j \in P^{\otimes m}$. Then, by the right relation in (24),

$$\begin{aligned}\epsilon_{m+1}\epsilon_m &= \sum_j \iota_{Q^{\otimes m+1}}(q_j)\iota_{P^{\otimes m+1}}(p_j)\epsilon_m = \sum_j \iota_{Q^{\otimes m+1}}(q_j)\iota_P(p'_j)\iota_{P^{\otimes m}}(p''_j)\epsilon_m \\ &= \sum_j \iota_{Q^{\otimes m+1}}(q_j)\iota_P(p'_j)\iota_{P^{\otimes m}}(p''_j)\chi(\Gamma^m(1_R)) \\ &= \sum_j \iota_{Q^{\otimes m+1}}(q_j)\iota_P(p'_j)\iota_{P^{\otimes m}}(\Gamma^m(1_R)(p''_j)) \\ &= \sum_j \iota_{Q^{\otimes m+1}}(q_j)\iota_P(p'_j)\iota_{P^{\otimes m}}(p''_j) = \sum_j \iota_{Q^{\otimes m+1}}(q_j)\iota_{P^{\otimes m+1}}(p_j) = \epsilon_m. \qquad \square\end{aligned}$$

Proposition C.5.4. Let R be a unital ring and let (P, Q, ψ) be a unital R-system that satisfies Condition (FS'). Let ϵ_i be the idempotents defined above. The following assertions hold for every $i \geq 0$:

(a) For any $s \in \mathcal{T}_i$ we have that $\epsilon_i s = s$;
(b) For any $t \in \mathcal{T}_{-i}$ we have that $t\epsilon_i = t$.

Consequently, $\mathcal{T}_i\mathcal{T}_{-i}$ is a unital ideal with multiplicative identity element ϵ_i for every $i \geq 0$.

Proof. Note that $\mathcal{T}_0$ is a unital ring with multiplicative identity element $\epsilon_0 = \iota_R(1_R)$. The statements are clear for $i = 0$.

(a): Take an arbitrary positive integer i. Consider a monomial $\iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p)$ where m, n are non-negative integers such that $m - n = i$. Then, $0 < i \leq m$. By Lemma C.5.3, $\epsilon_m \geq \epsilon_i$. Hence,

$$\begin{aligned}\iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p) &= \iota_{Q^{\otimes m}}(\Delta^m(1_R)(q))\iota_{P^{\otimes n}}(p) = \pi(\Delta^m(1_R))\iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p) \\ &= \epsilon_m\iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p) = \epsilon_i\epsilon_m\iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p) = \epsilon_i\iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p).\end{aligned}$$

Any element $s \in \mathcal{T}_i$ is a finite sum of elements of the above form (see (22)). Hence, it follows that $\epsilon_i s = s$.

(b): Take an arbitrary positive integer i. Consider a monomial $\iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p)$ where m, n are non-negative integers such that $m - n = -i$. Then $0 < i \leq n$. By Lemma C.5.3, $\epsilon_n \geq \epsilon_i$. Hence, $\iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p) = \iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(\Gamma^n(1_R)(p)) = \iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p)\chi(\Gamma^n(1_R)) = \iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p)\epsilon_n = \iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p)\epsilon_n\epsilon_i = \iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p)\epsilon_i$. Since any element $t \in \mathcal{T}_{-i}$ is a finite sum of elements of the above form, it follows that $t\epsilon_i = t$. $\square$

We will see that restricting our attention to semi-full covariant representations (S,T,σ,B) makes life easier. This special type of graded covariant representations have the property that the image of ψ_k is enough to generate the ideal $B_{-k}B_k$ for $k \geq 0$ (see Definition C.1.3). We first prove that the property of being semi-full is invariant under isomorphism in the category of surjective covariant representations $\mathcal{C}_{(P,Q,\psi)}$.

Proposition C.5.5. Let R be a ring, let (P,Q,ψ) be an R-system and suppose that $(S,T,\sigma,B) \cong_{\mathrm{r}} (S',T',\sigma',B')$ are two isomorphic covariant representations of (P,Q,ψ). If (S,T,σ,B) is semi-full, then (S',T',σ',B') is semi-full.

Proof. Let $\phi\colon B \to B'$ be the $\mathbb{Z}$-graded isomorphism coming from Lemma C.2.6. Hence,

$$\begin{aligned} B'_{-k}B'_k &= \phi(B_{-k})\phi(B_k) = \phi(B_{-k}B_k) = \phi(I^{(k)}_{\psi,\sigma}) \\ &= (\{\phi\circ\sigma(\psi_k(p\otimes q)) \mid p \in P^{\otimes k}, q \in Q^{\otimes k}\}) = I^{(k)}_{\psi,\sigma'}. \end{aligned}$$

Thus, (S',T',σ',B') is semi-full. □

We now establish sufficient conditions for a semi-full covariant representation to be nearly epsilon-strongly $\mathbb{Z}$-graded.

Proposition C.5.6. Let R be an s-unital ring and let (P,Q,ψ) be an s-unital R-system. Suppose that (S,T,σ,B) is a semi-full covariant representation of (P,Q,ψ) and that the following assertions hold:

(a) (P,Q,ψ) satisfies Condition (FS),
(b) $I^{(k)}_{\psi,\sigma}$ is s-unital for $k \geq 0$.

Then, B is nearly epsilon-strongly $\mathbb{Z}$-graded.

Proof. Let $\mathcal{T}_{(P,Q,\psi)} = \bigoplus_{i\in\mathbb{Z}} \mathcal{T}_i$ be the Toeplitz ring associated to the R-system (P,Q,ψ). By Proposition C.5.1(c), $\mathcal{T}_i\mathcal{T}_{-i}$ is s-unital for every $i \geq 0$. By Theorem C.2.5, there is a $\mathbb{Z}$-graded ring epimorphism $\eta\colon \mathcal{T}_{(P,Q,\psi)} \to B$. Since the image of an s-unital ring under a ring homomorphism is in turn s-unital, it follows that $B_iB_{-i} = \eta(\mathcal{T}_i)\eta(\mathcal{T}_{-i}) = \eta(\mathcal{T}_i\mathcal{T}_{-i})$ is s-unital for every $i \geq 0$. Furthermore, by Proposition C.5.1(d), we have that $\mathcal{T}_i = \mathcal{T}_i\mathcal{T}_{-i}\mathcal{T}_i$ for every $i \in \mathbb{Z}$. Applying η to both sides yields, $B_i = B_iB_{-i}B_i$. Hence, B is symmetrically $\mathbb{Z}$-graded.

Next, we show that B_iB_{-i} is s-unital for $i < 0$. Since (S,T,σ,B) is semi-full, we have that $B_{-k}B_k = I^{(k)}_{\psi,\sigma}$ for $k \geq 0$. Hence, (b) implies that B_iB_{-i} is s-unital for $i < 0$. Thus, we have showed that B_iB_{-i} is s-unital for $i \in \mathbb{Z}$ and that B is symmetrically $\mathbb{Z}$-graded. By Proposition C.2.4(a), it follows that $B = \bigoplus_{i\in\mathbb{Z}} B_i$ is nearly epsilon-strongly $\mathbb{Z}$-graded. □

The proof of the following proposition is entirely analogous to the proof of Proposition C.5.6.

Proposition C.5.7. Let R be a unital ring and let (P,Q,ψ) be a unital R-system. Suppose that (S,T,σ,B) is a semi-full covariant representation of (P,Q,ψ) and that the following assertions hold:

(a) (P,Q,ψ) satisfies Condition (FS'),

(b) $I^{(k)}_{\psi,\sigma}$ is unital for $k \geq 0$.
Then, B is epsilon-strongly $\mathbb{Z}$-graded.

On the other hand, a covariant representation (S, T, σ, B) does not need to be semi-full for the ring B to be epsilon-strongly $\mathbb{Z}$-graded (see Example C.7.3).

C.6. Characterization up to graded isomorphism

In this section, we finally give characterizations of unital strongly, nearly epsilon-strongly and epsilon-strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings up to $\mathbb{Z}$-graded isomorphism.

Theorem C.6.1. Let $\mathcal{O}_{(P,Q,\psi)}$ be a Cuntz-Pimsner ring of some system (P, Q, ψ). If $\mathcal{O}_{(P,Q,\psi)}$ is nearly epsilon-strongly $\mathbb{Z}$-graded and $\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1) \cap (\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1))^{\perp} = \{0\}$, then,

$$\mathcal{O}_{(P,Q,\psi)} \cong_{\mathrm{gr}} \mathcal{O}_{(P',Q',\psi')},$$

where (P', Q', ψ') is an R'-system such that $\mathcal{O}_{(P',Q',\psi')}$ is well-defined and the following assertions hold:
(a) (P', Q', ψ') is an s-unital R'-system;
(b) $(\iota^{CP}_{P'}, \iota^{CP}_{Q'}, \iota^{CP}_{R'}, \mathcal{O}_{(P',Q',\psi')})$ is a semi-full covariant representation of (P', Q', ψ');
(c) (P', Q', ψ') satisfies Condition (FS);
(d) $I^{(k)}_{\psi',\iota^{CP}_{\mathcal{O}_0}}$ is s-unital for $k \geq 0$.

Conversely, if (P', Q', ψ') is an R'-system such that $\mathcal{O}_{(P',Q',\psi')}$ is well-defined and (a)-(d) hold, then $\mathcal{O}_{(P',Q',\psi')}$ is nearly epsilon-strongly $\mathbb{Z}$-graded.

Proof. If the Cuntz-Pimsner ring $\mathcal{O}_{(P,Q,\psi)}$ is nearly epsilon-strongly $\mathbb{Z}$-graded and the condition $\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1) \cap (\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1))^{\perp} = \{0\}$ holds, then it follows from Corollary C.3.12 that the Cuntz-Pimsner ring is graded isomorphic to $\mathcal{O}_{(\mathcal{O}_{-1},\mathcal{O}_1,\psi')}$ and that (a)-(d) are satisfied.

Conversely, let (P', Q', ψ') be an R'-system such that $\mathcal{O}_{(P',Q',\psi')}$ exists and (a)-(d) are satisfied. Applying Proposition C.5.6 to the covariant representation

$$(\iota^{CP}_{P'}, \iota^{CP}_{Q'}, \iota^{CP}_{R'}, \mathcal{O}_{(P',Q',\psi')}),$$

it follows that $\mathcal{O}_{(P',Q',\psi')}$ is nearly epsilon-strongly $\mathbb{Z}$-graded. □

For epsilon-strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings, we obtain the following result:

Theorem C.6.2. Let $\mathcal{O}_{(P,Q,\psi)}$ be a Cuntz-Pimsner ring of some system (P, Q, ψ). If $\mathcal{O}_{(P,Q,\psi)}$ is epsilon-strongly $\mathbb{Z}$-graded and $\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1) \cap (\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1))^{\perp} = \{0\}$, then,

$$\mathcal{O}_{(P,Q,\psi)} \cong_{\mathrm{gr}} \mathcal{O}_{(P',Q',\psi')},$$

where (P', Q', ψ') is an R'-system such that $\mathcal{O}_{(P',Q',\psi')}$ is well-defined and the following assertions hold:
(a) (P', Q', ψ') is a unital R'-system;
(b) $(\iota^{CP}_{P'}, \iota^{CP}_{Q'}, \iota^{CP}_{R'}, \mathcal{O}_{(P',Q',\psi')})$ is a semi-full covariant representation of (P', Q', ψ');
(c) (P', Q', ψ') satisfies Condition (FS');

(d) $I^{(k)}_{\psi',\iota^{CP}_{\mathcal{O}_0}}$ is unital for $k \geq 0$.

Conversely, if (P', Q', ψ') is an R'-system such that $\mathcal{O}_{(P',Q',\psi')}$ is well-defined and (a)-(d) hold, then $\mathcal{O}_{(P',Q',\psi')}$ is epsilon-strongly $\mathbb{Z}$-graded.

Proof. Assume that (P', Q', ψ') is an R'-system such that $\mathcal{O}_{(P',Q',\psi')}$ exists and the assertions in (a)-(d) hold. Then Proposition C.5.7 implies that $\mathcal{O}_{(P',Q',\psi')}$ is epsilon-strongly $\mathbb{Z}$-graded.

Conversely, assume that $\mathcal{O}_{(P,Q,\psi)}$ is epsilon-strongly $\mathbb{Z}$-graded and $\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1) \cap (\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1))^{\perp} = \{0\}$. Note that, in particular, $\mathcal{O}_{(P,Q,\psi)}$ is nearly epsilon-strongly $\mathbb{Z}$-graded. Hence, by Theorem C.6.1, $\mathcal{O}_{(P,Q,\psi)} \cong_{\mathrm{gr}} \mathcal{O}_{(\mathcal{O}_{-1},\mathcal{O}_1,\psi')}$ where $(\mathcal{O}_{-1}, \mathcal{O}_1, \psi')$ is an s-unital $\mathcal{O}_0$-system that satisfies Condition (FS) and such that (b) is satisfied. Furthermore (see Corollary C.3.12),

$$(i_{\mathcal{O}_{-1}}, i_{\mathcal{O}_1}, i_{\mathcal{O}_0}, \mathcal{O}_{(P,Q,\psi)}) \cong_{\mathrm{r}} (\iota^{CP}_{\mathcal{O}_{-1}}, \iota^{CP}_{\mathcal{O}_1}, \iota^{CP}_{\mathcal{O}_0}, \mathcal{O}_{(\mathcal{O}_{-1},\mathcal{O}_1,\psi')}). \tag{29}$$

First note that since the $\mathbb{Z}$-grading is assumed to be epsilon-strong it follows that $\mathcal{O}_i$ is a unital $\mathcal{O}_i\mathcal{O}_{-i}$–$\mathcal{O}_{-i}\mathcal{O}_i$-bimodule for each $i \in \mathbb{Z}$ (see Definition C.2.2). This implies that $(\mathcal{O}_{-1}, \mathcal{O}_1, \psi')$ is a unital $\mathcal{O}_0$-system. Hence, (a) is satisfied.

Next, we prove that the $\mathcal{O}_0$-system $(\mathcal{O}_{-1}, \mathcal{O}_1, \psi')$ satisfies Condition (FS'). Since $\mathcal{O}_{(P,Q,\psi)}$ is assumed to be epsilon-strongly $\mathbb{Z}$-graded, it follows from [18, Prop. 7(iv)] that $\mathcal{O}_i$ is a finitely generated $\mathcal{O}_0$-bimodule for every $i \in \mathbb{Z}$. In particular, $\mathcal{O}_1$ and $\mathcal{O}_{-1}$ are finitely generated $\mathcal{O}_0$-bimodules and it follows from Proposition C.4.3(c) that $(\mathcal{O}_{-1}, \mathcal{O}_1, \psi)$ satisfies Condition (FS'). In other words, (c) holds.

Moreover, it follows from Proposition C.2.4(b) that, in particular, $\mathcal{O}_{-k}\mathcal{O}_k$ is unital for $k \geq 0$. Hence, $\mathcal{O}_{-k}\mathcal{O}_k = I^{(k)}_{\psi',\iota^{CP}_{\mathcal{O}_0}}$ is unital for $k \geq 0$. This establishes (d). $\square$

For unital strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings, we obtain the following complete characterization:

Theorem C.6.3. Let $\mathcal{O}_{(P,Q,\psi)}$ be a Cuntz-Pimsner ring of some system (P, Q, ψ). Then, $\mathcal{O}_{(P,Q,\psi)}$ is unital strongly $\mathbb{Z}$-graded if and only if

$$\mathcal{O}_{(P,Q,\psi)} \cong_{\mathrm{gr}} \mathcal{O}_{(P',Q',\psi')}$$

where (P', Q', ψ') is an R'-system such that $\mathcal{O}_{(P',Q',\psi')}$ is well-defined and the following assertions hold:

(a) (P', Q', ψ') is a unital R'-system;
(b) $(\iota^{CP}_{P'}, \iota^{CP}_{Q'}, \iota^{CP}_{R'}, \mathcal{O}_{(P',Q',\psi')})$ is a semi-full and faithful covariant representation of (P', Q', ψ');
(c) ψ' is surjective.

Proof. By Proposition C.4.10, (a) and (c) are sufficient for the ring $\mathcal{O}_{(P',Q',\psi')}$ to be strongly $\mathbb{Z}$-graded.

Conversely, assume that $\mathcal{O}_{(P,Q,\psi)}$ is unital strongly $\mathbb{Z}$-graded. In particular, $\mathcal{O}_{(P,Q,\psi)}$ is epsilon-strongly $\mathbb{Z}$-graded. Moreover, $\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1) \cap (\mathrm{Ann}_{\mathcal{O}_0}(\mathcal{O}_1))^{\perp} = \{0\}$ by Lemma C.3.10(b). Then, by Theorem C.6.2, $\mathcal{O}_{(P,Q,\psi)} \cong_{\mathrm{gr}} \mathcal{O}_{(\mathcal{O}_{-1},\mathcal{O}_1,\psi')}$ where

$(\mathcal{O}_{-1}, \mathcal{O}_1, \psi')$ satisfies Condition (FS'), (b) is satisfied and $I^{(k)}_{\psi', \iota^{CP}_{\mathcal{O}_0}}$ is unital for $k \geq 0$. Since $\mathcal{O}_{(\mathcal{O}_{-1}, \mathcal{O}_1, \psi)}$ is unital strongly $\mathbb{Z}$-graded,

$$1_{\mathcal{O}_{(\mathcal{O}_{-1}, \mathcal{O}_1, \psi')}} = \iota^{CP}_{\mathcal{O}_0}(1_{\mathcal{O}_0}) \in \mathcal{O}_0 = \mathcal{O}_{-1}\mathcal{O}_1 = I^{(1)}_{\psi', \iota^{CP}_{\mathcal{O}_0}}.$$

Since $\iota^{CP}_{\mathcal{O}_0}$ is injective, we get that $1_{\mathcal{O}_0} \in \mathrm{Im}(\psi')$. Hence, ψ' is surjective.

Furthermore, since $\mathcal{O}_{(\mathcal{O}_{-1}, \mathcal{O}_1, \psi')}$ is an epsilon-strongly $\mathbb{Z}$-graded ring that is also strongly $\mathbb{Z}$-graded, we must have $\epsilon_1 = 1$ (see [18, Prop. 8]) where ϵ_1 is the multiplicative identity element of the ring $\mathcal{O}_1\mathcal{O}_{-1}$. By Condition (FS') and Proposition C.4.3(b), we have that $\Delta(1_{\mathcal{O}}) \in \mathcal{F}_P(Q)$. Then, by Proposition C.2.9, $\pi_{\iota^{CP}_{\mathcal{O}_1}, \iota^{CP}_{\mathcal{O}_{-1}}}(\Delta(1_{\mathcal{O}_0})) \in \mathcal{O}_1\mathcal{O}_{-1}$ is a multiplicative identity element of $\mathcal{O}_1\mathcal{O}_{-1}$. Thus, $\pi_{\iota^{CP}_{\mathcal{O}_1}, \iota^{CP}_{\mathcal{O}_{-1}}}(\Delta(1_{\mathcal{O}_0})) = \epsilon_1 = 1$ and therefore $(\iota^{CP}_{\mathcal{O}_{-1}}, \iota^{CP}_{\mathcal{O}_1}, \iota^{CP}_{\mathcal{O}_0}, \mathcal{O}_{(\mathcal{O}_{-1}, \mathcal{O}_1, \psi')})$ is a faithful representation of $(\mathcal{O}_{-1}, \mathcal{O}_1, \psi')$. ☐

C.7. Examples

In this section, we collect some important examples.

Example C.7.1. (Non-nearly epsilon-strongly $\mathbb{Z}$-graded Cuntz-Pimsner ring) Let R be an idempotent ring that is not s-unital (see e.g. [16, Expl. 5]). Put $P = Q = \{0\}$ and let $\psi \equiv 0$ be the zero map. Note that (P, Q, ψ) is an R-system that satisfies Condition (FS') trivially. It is not hard to see that the Toeplitz ring is given by $\mathcal{T}_0 = R$, and $\mathcal{T}_i = \{0\}$ for all $i \neq 0$. Furthermore, note that $\ker \Delta = R$. Recall that an ideal J of R is called faithful if $J \cap \ker \Delta = \{0\}$. Clearly, $J := (0)$ is the maximal faithful and ψ-compatible ideal of R. It follows that the Cuntz-Pimsner ring $\mathcal{O}_{(P,Q,\psi)}$ is well-defined and coincides with the Toeplitz ring. Since $\mathcal{T}_0 = R = R^2 = \mathcal{T}_0\mathcal{T}_0$ is not s-unital it follows by Proposition C.2.4(a) that the Cuntz-Pimsner ring $\mathcal{O}_{(P,Q,\psi)} = \mathcal{T}_{(P,Q,\psi)}$ is not nearly epsilon-strongly $\mathbb{Z}$-graded. This shows that the assumption of (P, Q, ψ) being an s-unital system in Proposition C.5.6 cannot be removed.

The following example shows that for some graphs, the standard Leavitt path algebra covariant representation is semi-full (see Section C.2.4).

Example C.7.2.

$$\bullet_{v_1} \xrightarrow{f_1} \bullet_{v_2}$$

Let K be a unital ring and let E consist of two vertices v_1, v_2 connected by a single edge f. Consider the associated standard Leavitt path algebra system (P, Q, ψ) and the standard Leavitt path algebra covariant representation $(\iota^{CP}_Q, \iota^{CP}_P, \iota^{CP}_R, \mathcal{O}_{(P,Q,\psi)})$. To save space we write $I_k = I^{(k)}_{\psi, \iota^{CP}_R}$ for $k \geq 0$. Note that $I_0 = (\{v_1, v_2\})$, $I_1 = (v_2)$ and $I_k = (0)$ for $k > 2$. Furthermore, since $f_1 f_1^* = v_1$ we see that $(L_K(E))_0 = I_0$.

Moreover, note that $(L_K(E))_1 = \mathrm{Span}_K\{f_1\}$, $(L_K(E))_{-1} = \mathrm{Span}_K\{f_1^*\}$ and hence we see that $(L_K(E))_{-1}(L_K(E))_1 = (v_2) = I_1$. Thus, $(\iota_P, \iota_Q, \iota_R, \mathcal{O}_{(P,Q,\psi)})$ is a semi-full covariant representation of (P, Q, ψ). Furthermore, (P, Q, ψ) satisfies Condition (FS') since E is finite (see Lemma C.4.14) and I_k is unital for $k \geq 0$. Thus, $L_K(E)$ is epsilon-strongly $\mathbb{Z}$-graded by Theorem C.6.2.

In general, however, it is not true that the standard Leavitt path algebra covariant representation is semi-full as the following example shows.

Example C.7.3. (cf. [17, Expl. 4.1]) Let K be a unital ring and consider the following finite directed graph E.

$$\bullet_{v_1} \xleftarrow{f_1} \bullet_{v_2} \xrightarrow{f_2} \bullet_{v_3} \xleftarrow{f_3} \bullet_{v_4} \xleftarrow{f_4} \bullet_{v_5}$$

Let (P, Q, ψ) be the standard Leavitt path algebra system associated to E and consider the standard Leavitt path algebra covariant representation,

$$(\iota_Q^{CP}, \iota_P^{CP}, \iota_R^{CP}, \mathcal{O}_{(P,Q,\psi)}). \tag{30}$$

We write $S_i = (L_K(E))_i$ and $I_i = I^{(i)}_{\psi, \iota_R^{CP}}$ to save space. Note that,

$$\begin{aligned}
&S_0 = \mathrm{Span}_K\{v_1, v_2, v_3, v_4, v_5, f_1 f_1^*, f_2 f_3^*, f_2 f_2^*, f_3 f_2^*\},\\
&S_1 = \mathrm{Span}_K\{f_1, f_2, f_3, f_4, f_4 f_3 f_2^*\}, \qquad S_{-1} = \mathrm{Span}_K\{f_1^*, f_2^*, f_3^*, f_4^*, f_2 f_3^* f_4^*\},\\
&S_2 = \mathrm{Span}_K\{f_4 f_3\}, \qquad S_{-2} = \mathrm{Span}_K\{f_3^* f_4^*\}, \quad \text{and} \quad S_n = \{0\}, \text{ for } |n| > 2.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
I_0 &= (\{v_1, v_2, v_3, v_4, v_5\}), & I_1 &= (\{v_1, v_3, v_4\}),\\
I_2 &= (\{v_3\}), & I_k &= (0), \quad k > 2.
\end{aligned}$$

In particular, we have that $S_{-1}S_1 = (\{v_1, v_3, v_4, f_2 f_2^*\}) \supsetneq I_1$ because $f_2 f_2^* \notin I_1$. Hence, the standard Leavitt path algebra covariant representation is not semi-full. In any case, however, we have that $\mathcal{O}_{(P,Q,\psi)} \cong_{\mathrm{gr}} L_K(E)$ (see Section C.2.4). On the other hand, by Proposition C.3.11, we have that $L_K(E)$ is pre-CP. Thus, by Corollary C.3.9, $L_K(E)$ is realized by the Cuntz-Pimsner representation,

$$(\iota^{CP}_{(L_K(E))_{-1}}, \iota^{CP}_{(L_K(E))_1}, \iota^{CP}_{(L_K(E))_0}, \mathcal{O}_{((L_K(E))_{-1}, (L_K(E))_1, \psi')}) \tag{31}$$

of the $(L_K(E))_0$-system $(L_K(E))_{-1}, (L_K(E))_1, \psi')$. Moreover, the corollary implies that (31) is semi-full and $\mathcal{O}_{((L_K(E))_{-1}, (L_K(E))_1, \psi')} \cong_{\mathrm{gr}} L_K(E)$. Since (30) is not semi-full and (31) is semi-full, it follows by Proposition C.5.5 that the covariant representations (30) and (31) cannot be isomorphic. Thus, $L_K(E)$ is realizable as a Cuntz-Pimsner ring in two different ways.

The following example shows that (a) is crucial in Theorem C.6.2. It also gives an example of a nearly epsilon-strongly $\mathbb{Z}$-graded ring that is not epsilon-strongly $\mathbb{Z}$-graded.

Example C.7.4. (cf. [14, Expl. 4.5]) Let K be a unital ring and consider the infinite discrete graph E consisting of countably infinitely many vertices but no edges.

$$\bullet_{v_1} \quad \bullet_{v_2} \quad \bullet_{v_3} \quad \bullet_{v_4} \quad \bullet_{v_5} \quad \bullet_{v_6} \quad \bullet_{v_7} \quad \bullet_{v_8} \quad \bullet_{v_9} \quad \bullet_{v_{10}} \quad \cdots$$

The standard Leavitt path algebra system is given by $R = \bigoplus_{v \in E^0} \eta_v$, $P = Q = \{0\}$. The R-system (P, Q, ψ) trivially satisfies Condition (FS'). However, (P, Q, ψ) is not unital as R does not have a multiplicative identity element. However, note that (P, Q, ψ) is s-unital.

We show that the standard Leavitt path algebra covariant representation of E is semi-full. Since $P = Q = \{0\}$ and $\psi = 0$ it follows that the grading is given by $\mathcal{O}_0 = R$ and $\mathcal{O}_i = \{0\}$ for $i \neq 0$ (see Example C.7.1). Furthermore, $I^{(k)}_{\psi,\iota_R^{CP}} = (0)$ for $k > 0$. Thus, the standard Leavitt path algebra covariant representation satisfies (b)-(d) in Theorem C.6.2 but not (a). Since E contains infinitely many vertices, $L_K(E)$ is not unital (see [1, Lem. 1.2.12]). By Remark C.2.3, $L_K(E)$ is not epsilon-strongly $\mathbb{Z}$-graded (cf. [14, Expl. 4.5]). Thus, (a) in Theorem C.6.1 cannot be removed. On the other hand, it follows from Theorem C.6.1 that $L_K(E)$ is nearly epsilon-strongly $\mathbb{Z}$-graded.

C.8. Noetherian and artinian corner skew Laurent polynomial rings

We end this article by characterizing noetherian and artinian corner skew Laurent polynomial rings. The following proposition can be proved in a straightforward manner using direct methods, but we show it as a special case of our results.

Proposition C.8.1. Let R be a unital ring, let $e \in R$ be an idempotent and let $\alpha\colon R \to eRe$ be a corner ring isomorphism. Then the corner skew Laurent polynomial ring $R[t_+, t_-; \alpha]$ is epsilon-strongly $\mathbb{Z}$-graded.

PROOF. Recall that $R[t_+, t_-; \alpha] = \bigoplus_{i\in\mathbb{Z}} A_i$ is $\mathbb{Z}$-graded by putting $A_0 = R$, $A_i = Rt_+^i$ for $i < 0$ and $A_i = t_-^i R$ for $i > 0$. Let $\psi'\colon A_{-1} \otimes A_1 \to A_0$ be the map defined by $\psi(a' \otimes a) = a'a$ for $a' \in A_{-1}$ and $a \in A_1$. Since $A_0 = R$ is a unital ring, the A_0-system (A_{-1}, A_1, ψ') is unital. In [19, Expl. 3.4], it is shown that $R[t_+, t_-; \alpha]$ satisfies the conditions in Theorem C.3.3. This implies that (A_{-1}, A_1, ψ') satisfies Condition (FS) and,

$$(i_{A_{-1}}, i_{A_1}, i_{A_0}, R[t_+, t_-; \alpha]) \cong_{\mathrm{r}} (\iota^{CP}_{A_{-1}}, \iota^{CP}_{A_1}, \iota^{CP}_{A_0}, \mathcal{O}_{(A_{-1}, A_1, \psi')}). \tag{32}$$

Note that $A_1 = t_- R$ is finitely generated as a right A_0-module and $A_{-1} = Rt_+$ is finitely generated as a left A_0-module. It follows from Proposition C.4.3, that (A_{-1}, A_1, ψ') satisfies Condition (FS'). Furthermore, by Proposition C.3.4, the covariant representation (32) is semi-full.

Next, we show that $I^{(k)}_{\psi',\iota_R^{CP}} = A_{-k}A_k$ is unital with multiplicative identity element $t_+^k t_{-i}^k = i(e) \in A_{-k}A_k$ for each $k > 0$. Fix a non-negative integer $k > 0$ and note that any element $x \in A_{-k}A_k = (Rt_+^k)(t_{-i}^k R)$ is a finite sum of elements of the form $rt_+^k t_-^k r' = t_+^k \alpha^{-k}(r) t_-^k r' = rt_+^k \alpha^{-k}(r') t_-^k$ where $r, r' \in R$. For any $r, r' \in R$, we get that,

$$i(e) r t_+^k t_-^k r' = (t_+^k t_-^k)(t_+^k \alpha^{-k}(r) t_-^k r') = t_+^k (t_-^k t_+^k) \alpha^{-k}(r) t_-^k r' = t_+^k (1) \alpha^{-k}(r) t_-^k r' = r t_+^k t_-^k r'.$$

It follows that $i(e)x = x$. A similar argument shows that $xi(e) = x$. By Theorem C.6.2, it now follows that $R[t_+, t_-; \alpha]$ is epsilon-strongly $\mathbb{Z}$-graded. □

We recall the following Hilbert basis theorem for epsilon-strongly $\mathbb{Z}$-graded rings.

Theorem C.8.2. ([13, Thm. 1.1, Thm. 1.2]) Let $S = \bigoplus_{i\in\mathbb{Z}} S_i$ be an epsilon-strongly $\mathbb{Z}$-graded ring. The following assertions hold:

(a) If S_0 is left (right) noetherian, then S is left (right) noetherian;

(b) If S_0 is left (right) artinian and there exists some positive integer n such that $S_i = \{0\}$ for all $|i| > n$, then S is left (right) artinian.

Applying Theorem C.8.2 to the special case of corner skew Laurent polynomial rings, we obtain the following result.

Corollary C.8.3. Let R be a unital ring and let $\alpha\colon R \to eRe$ be a ring isomorphism where e is an idempotent of R. Consider the corner skew Laurent polynomial ring $R[t_+, t_-; \alpha]$. The following assertions hold:

(a) $R[t_+, t_-; \alpha]$ is left (right) noetherian if and only if R is left (right) noetherian;

(b) $R[t_+, t_-; \alpha]$ is neither left nor right artinian.

Proof. (a): Straightforward.

(b): By Proposition C.8.1, $R[t_+, t_-; \alpha] = \bigoplus_{i \in \mathbb{Z}} A_i$ is epsilon-strongly $\mathbb{Z}$-graded where $A_0 = R$, $A_i = Rt_+^i$ for $i < 0$ and $A_i = t_-^i R$ for $i > 0$. By Theorem C.8.2(b), $R[t_+, t_-; \alpha]$ is left (right) artinian if and only if A_0 is left (right) artinian and

$$|\mathrm{Supp}(R[t_+, t_-; \alpha])| < \infty.$$

However, since $t_+^n \neq 0$ for every $n > 0$, it follows that $A_{-n} = Rt_+^n \neq \{0\}$ for every $n > 0$. Hence, $\mathrm{Supp}(R[t_+, t_-; \alpha])$ is infinite and $R[t_+, t_-; \alpha]$ is neither left nor right artinian. □

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PAPER D

A characterization of graded von Neumann regular rings with applications to Leavitt path algebras

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Daniel Lännström

We prove a new characterization of graded von Neumann regular rings involving the recently introduced class of nearly epsilon-strongly graded rings. As our main application, we generalize Hazrat's result that Leavitt path algebras over fields are graded von Neumann regular. More precisely, we show that a Leavitt path algebra $L_R(E)$ with coefficients in a unital ring R is graded von Neumann regular if and only if R is von Neumann regular. We also prove that both Leavitt path algebras and corner skew Laurent polynomial rings over von Neumann regular rings are semiprimitive and semiprime. Thereby, we generalize a result by Abrams and Aranda Pino on the semiprimitivity of Leavitt path algebras over fields.

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D.1. Introduction

An associative ring R is called *von Neumann regular* if $a \in aRa$ holds for every $a \in R$. This type of ring was first considered by von Neumann in the study of operator algebras and has since then been extensively studied (see e.g. Goodearl's monograph [14]). There are several well-known equivalent statements for a unital ring to be von Neumann regular. Notably, a unital ring R is von Neumann regular if and only if every finitely generated left (right) ideal of R is generated by an idempotent. Examples of von Neumann regular rings are plentiful. For instance, any field is von Neumann regular. On the other hand, the ring of integers $\mathbb{Z}$ (or any non-field integral domain) is not von Neumann regular.

Let G be a group with neutral element e and let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring (see Section D.2.2). The ring S is called *graded von Neumann regular* if, for every $g \in G$ and $a \in S_g$, the relation $a \in aSa$ holds. In the case of unital rings, this notation was introduced by Năstăsescu and van Oystaeyen [21] and has been further studied in [9, 15, 26, 27]. In this article, we will continue the study initiated by Hazrat [16] of non-unital graded von Neumann regular rings. There are results on the graded ideal structure of graded von Neumann regular rings (see [16, Prop. 1-2] and Proposition D.2.5) which make this class of rings interesting to us. For the special class of unital strongly group graded rings, the following result highlights a connection between von Neumann regularity and graded von Neumann regularity:

Theorem D.1.1 (Năstăsescu and van Oystaeyen [21, Cor. C.I.1.5.3])**.** Let $S = \bigoplus_{g \in G} S_g$ be a unital strongly G-graded ring. Then S is graded von Neumann regular if and only if S_e is von Neumann regular.

Theorem D.1.1 was originally proved by Năstăsescu and van Oystaeyen using Dade's theorem (see e.g. [15, Thm. 1.5.1]). An elementwise proof of Theorem D.1.1 was later given by Yahya [27, Thm. 3]. In this article, we recover Theorem D.1.1 as a special case of our characterization of general graded von Neumann regular rings (see Theorem D.1.2).

The notion of an *epsilon-strongly graded ring* (see Definition D.2.6) was introduced by Nystedt, Öinert and Pinedo [24] as a generalization of unital strongly graded rings. This class of graded rings includes: unital partial crossed products (see [24, pg. 2]), corner skew Laurent polynomial rings (see [20, Thm. 8.1]) and Leavitt path algebras of finite graphs (see [23, Thm. 1.2]). The further generalization to *nearly epsilon-strongly graded rings* (see Definition D.2.9) was recently introduced by Nystedt and Öinert [23]. In this article, we study the relation between these two recently introduced classes of graded rings and the classical notion of graded von Neumann regular rings. Our main result is the following characterization:

Theorem D.1.2. Let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. Then S is graded von Neumann regular if and only if S is nearly epsilon-strongly G-graded and S_e is von Neumann regular.

By applying Theorem D.1.2, we generalize Theorem D.1.1 to the class of epsilon-strongly graded rings (see Corollary D.3.11). This allows us to characterize when unital partial crossed products (see Corollary D.6.1), corner skew Laurent polynomial rings (see Corollary D.6.2) and Leavitt path algebras over unital rings (see Theorem D.1.4) are graded von Neumann regular.

D.1.1. Applications to Leavitt path algebras. Given a directed graph E and a field K, the *Leavitt path algebra* $L_K(E)$ is an associative $\mathbb{Z}$-graded K-algebra. These algebras were introduced by Ara, Moreno and Pardo [7] and independently by Abrams and Aranda Pino [3]. In the 15 years since their introduction, Leavitt path algebras have found applications in general ring theory and matured into a research topic of their own (see e.g. [1]). There are many results in the literature relating properties of the graph E with algebraic properties of $L_K(E)$. For instance, $L_K(E)$ is von Neumann regular if and only if E is an acyclic directed graph (see [4]). For graded von Neumann regularity, Hazrat has obtained the following result:

Theorem D.1.3 (Hazrat [16])**.** Let K be a field and let E be a directed graph. Then the $\mathbb{Z}$-graded Leavitt path algebra $L_K(E)$ is graded von Neumann regular.

Tomforde [25] introduced Leavitt path algebras over commutative unital rings and proved that many results carry over to his generalized setting. Leavitt path algebras over $\mathbb{Z}$ were considered by Johansen and Sørensen [18] in connection to the classification program of Leavitt path algebras. In this article, we follow Hazrat [17] and consider Leavitt path algebras $L_R(E)$ where R is a general, possibly non-commutative associative ring. We want to relate algebraic properties of the ring R to algebraic properties of $L_R(E)$. In that vein, we will establish the following generalization of Theorem D.1.3:

Theorem D.1.4. Let R be a unital ring and let E be a directed graph. Then the $\mathbb{Z}$-graded Leavitt path algebra $L_R(E)$ is graded von Neumann regular if and only if R is von Neumann regular.

Remark D.1.5. The statement of Theorem D.1.4 does not hold if E is the null graph, i.e. the graph without any vertices or edges (see Remark D.4.7).

Hazrat [16] outlines an approach where Theorem D.1.1 is used to prove Theorem D.1.3 for the proper subclass of strongly graded Leavitt path algebras. For the general case, however, he uses a more involved technique based on corner skew Laurent polynomial rings. In this article, we employ Theorem D.1.2 together with Hazrat's original proof idea to establish Theorem D.1.4.

The rest of this article is organized as follows:

In Section D.2, we recall some preliminaries on non-unital von Neumann regular rings (Section D.2.1), group graded rings (Section D.2.2), epsilon-strongly and nearly epsilon-strongly graded rings (Section D.2.3) and direct limits of graded rings (Section D.2.4).

In Section D.3 and Section D.4, we prove Theorem D.1.2 respectively Theorem D.1.4.

In Section D.5, we show that a Leavitt path algebra over a von Neumann regular ring is both semiprimitive and semiprime (see Corollary D.5.4). Our result generalizes a well-known result by Abrams and Aranda Pino [2] for Leavitt path algebras over fields.

In Section D.6, we apply our results to unital partial crossed products (Corollary D.6.1) and corner skew Laurent polynomial rings (Corollary D.6.2 and Corollary D.6.3).

D.2. Preliminaries

Throughout this article, all rings are assumed to be associative but not necessarily unital.

D.2.1. Non-unital von Neumann regular rings. A ring R is called *s-unital* if $x \in xR \cap Rx$ for every $x \in R$. Equivalently, a ring R is s-unital if, for every $x \in R$, there exist some $e, e' \in R$ such that $x = ex = xe'$. A ring is called *unital* if it is equipped with a non-zero multiplicative identity element. A subset E of R is called a *set of local units* for R if E consists of commuting idempotents such that for every $x \in R$ there exists some $e \in E$ such that $x = ex = xe$. Note that a ring with a set of local units is s-unital. For more details about s-unital rings and rings with local units, we refer the reader to the survey article [22].

A ring R is called *von Neumann regular* if for every $x \in R$ there is some $y \in R$ such that $x = xyx$. In fact, every von Neumann regular ring is s-unital:

Proposition D.2.1. (cf. [22, Prop. 20]) Let R be a ring. If R is von Neumann regular, then R is s-unital.

Proof. Take an arbitrary $x \in R$. Then there exists some $y \in R$ such that $x = xyx$. Letting $e := xy$ and $e' := yx$, we see that $x = xyx = ex = xe'$ and hence R is s-unital. $\square$

One of the famous classical characterizations of von Neumann regularity for unital rings generalizes to s-unital rings verbatim. We include parts of the proof for the convenience of the reader:

Proposition D.2.2. Let R be an s-unital ring. Then the following assertions are equivalent:

(a) R is von Neumann regular;
(b) every principal right (left) ideal of R is generated by an idempotent;
(c) every finitely generated right (left) ideal of R is generated by an idempotent.

Proof. $(a) \Rightarrow (b)$: Take an arbitrary $x \in R$ and let $y \in R$ such that $x = xyx$. Consider the right ideal xR. Then $xy \in R$ is an idempotent such that $xR = xyR$.

$(b) \Rightarrow (c)$: See [14, Thm. 1.1].

$(c) \Rightarrow (a)$: Take an arbitrary $x \in R$. Then $xR = fR$ for some idempotent $f \in R$. Since R is s-unital, $f \in fR = xR$ and hence $f = xy$ for some $y \in R$. Similarly, $x \in xR = fR$ and hence $x = fr$ for some $r \in R$. Then $x = fr = f^2r = f(fr) = fx = xyx$. $\square$

D.2.2. Group graded rings. Let G be a group with neutral element e. A *G-grading* of a ring S is a collection $\{S_g\}_{g\in G}$ of additive subsets of S such that $S = \bigoplus_{g\in G} S_g$ and $S_g S_h \subseteq S_{gh}$ for all $g, h \in G$. The ring S is then called *G-graded.* If the stronger condition $S_g S_h = S_{gh}$ holds for all $g, h \in G$, then the grading is called *strong* and S is called *strongly G-graded.* The subsets S_g are called the *homogeneous components* of S. The *principal component*, S_e, is a subring of S. A *homogeneous element* $s \in S$ is an element such that $s \in S_g$ for some $g \in G$. Every element of S decomposes uniquely into a sum of homogeneous elements. A left/right/two-sided ideal I of S is called a left/right/two-sided *graded ideal* of S if $I = \bigoplus_{g\in G}(I \cap S_g)$.

Recall that a G-graded ring S is *graded von Neumann regular* if and only if $a \in aSa$ for every homogeneous element $a \in S$. However, it is possible to make this condition more precise. The following result is well-known, but we have chosen to include a proof for the convenience of the reader.

Proposition D.2.3. A G-graded ring S is graded von Neumann regular if and only if, for every homogeneous $a \in S_g$ there is some homogeneous $b \in S_{g^{-1}}$ such that $a = aba$.

Proof. The 'if' direction is clear. Conversely, take an arbitrary homogeneous element $a \in S_g$. By assumption, there exists some $b \in S$ such that $a = aba$. Let $b = \sum_{h \in G} b_h$ be the decomposition of b. Note that $a = aba = \sum_{h \in G} ab_h a$. Since the decomposition is unique, it follows that $b = b_{g^{-1}} \in S_{g^{-1}}$. □

A G-graded ring that is von Neumann regular is graded von Neumann regular. On the other hand, the following is an example of a graded von Neumann regular ring which is not von Neumann regular:

Example D.2.4. Let K be a field and consider the Laurent polynomial ring $K[x, x^{-1}]$ with its canonical $\mathbb{Z}$-grading, i.e. $K[x, x^{-1}] = \bigoplus_{i \in \mathbb{Z}} Kx^i$. A routine check shows that this gives a strong $\mathbb{Z}$-grading. Since K is von Neumann regular, it follows by Theorem D.1.1 that $K[x, x^{-1}]$ is graded von Neumann regular. On the other hand, $K[x, x^{-1}]$ is an integral domain which is not a field. Therefore, $K[x, x^{-1}]$ is not von Neumann regular.

Let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring and suppose that E is a set of local units for S. If E consists of homogeneous idempotents, then E is called *a set of homogeneous local units*. Notably, every Leavitt path algebra has a set of homogeneous local units (see Section D.5). Moreover, Hazrat has established the following 'graded version' of Proposition D.2.2:

Proposition D.2.5. ([16, Prop. 1]) Let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. Suppose that S has a set of homogeneous local units. Then the following three assertions are equivalent:

(a) S is graded von Neumann regular;
(b) every principal right (left) graded ideal of S is generated by a homogeneous idempotent;
(c) every finitely generated right (left) graded ideal of S is generated by a homogeneous idempotent.

D.2.3. Nearly epsilon-strongly graded rings. Next, we recall two special types of group graded rings generalizing the classical notion of unital strongly group graded rings. Nystedt, Öinert and Pinedo [24] recently introduced the class of epsilon-strongly G-graded rings:

Definition D.2.6. ([24, Prop. 7(iii)]) Let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. Suppose that for every $g \in G$ there is an element $\epsilon_g \in S_g S_{g^{-1}}$ such that for every $s \in S_g$ the relations $\epsilon_g s = s = s\epsilon_{g^{-1}}$ hold. Then S is called *epsilon-strongly G-graded*.

Remark D.2.7. Let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. We make the following two remarks:

(a) If S is a unital strongly G-graded ring, then $1 \in S_g S_{g^{-1}}$ for every $g \in G$ (see e.g. [21, Prop. 1.1.1]). In this case, S is epsilon-strongly G-graded with $\epsilon_g := 1$ for every $g \in G$. This proves that unital strongly G-graded rings are epsilon-strongly G-graded.

(b) If S is an epsilon-strongly G-graded ring, then S is a unital ring (see [19, Prop. 3.8]). In other words, only unital rings admit epsilon-strong G-gradings.

The following is an example of a $\mathbb{Z}$-graded ring that is epsilon-strongly $\mathbb{Z}$-graded but not strongly $\mathbb{Z}$-graded:

Example D.2.8. Let R be a unital ring and consider the following $\mathbb{Z}$-grading of the full matrix ring $M_2(R)$:

$$(M_2(R))_0 := \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \quad (M_2(R))_{-1} := \begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix}, \quad (M_2(R))_1 := \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix},$$

and $(M_2(R))_i := \left\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right\}$ for $|i| > 1$. Note that,

$$(M_2(R))_1(M_2(R))_{-1} = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}, \quad (M_2(R))_{-1}(M_2(R))_1 = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}.$$

A routine check shows that $M_2(R)$ is epsilon-strongly $\mathbb{Z}$-graded with,

$$\epsilon_1 := \begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix}, \quad \epsilon_{-1} := \begin{pmatrix} 0 & 0 \\ 0 & 1_R \end{pmatrix}, \quad \epsilon_0 := \begin{pmatrix} 1_R & 0 \\ 0 & 1_R \end{pmatrix} \text{ and } \epsilon_i := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ for } |i| > 1.$$

However, since an epsilon-strong $\mathbb{Z}$-grading is strong if and only if $\epsilon_i = 1$ for every $i \in \mathbb{Z}$ (see [23, Prop. 3.2]), it follows that the above $\mathbb{Z}$-grading of $M_2(R)$ is not strong.

Crucial to our investigation, Nystedt and Öinert have shown that a Leavitt path algebra associated to a finite directed graph is epsilon-strongly $\mathbb{Z}$-graded (see [23, Thm. 1.2]). Seeking to generalize their result to include any Leavitt path algebra (i.e. possibly non-finite graphs), they introduced nearly epsilon-strongly graded rings.

Definition D.2.9. ([23, Prop. 3.3]) Let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. Suppose that for every $g \in G$ and $s \in S_g$ there are elements $\epsilon_g(s) \in S_g S_{g^{-1}}, \epsilon'_g(s) \in S_{g^{-1}} S_g$ such that the relations $\epsilon_g(s)s = s = s\epsilon'_g(s)$ hold. Then S is called *nearly epsilon-strongly G-graded*.

Every Leavitt path algebra is indeed nearly epsilon-strongly $\mathbb{Z}$-graded (see [23, Thm. 1.3]). The following is a trivial example of a nearly epsilon-strongly G-graded ring:

Example D.2.10. Let G be a group and let R be an s-unital ring that is not unital (for instance, let $R := C_c(\mathbb{R})$ with pointwise multiplication). Put $R_e := R$ and $R_g := \{0\}$ for every $g \neq e$. This gives a G-grading of R called *the trivial G-grading*. For every $x \in R$ there are some $e, e' \in R$ such that $x = ex = xe'$. Letting $\epsilon_e(x) := e \in R = R^2 = R_e R_e$ and $\epsilon'_e(x) = e' \in R = R^2 = R_e R_e$ in Definition D.2.9, we see that R is nearly epsilon-strongly G-graded. On the other hand, since R is not unital, it follows from Remark D.2.7(b) that R cannot be epsilon-strongly G-graded.

D.2.4. Direct limits in the category of graded rings. We will recall some properties of the category of group graded rings. Let $S = \bigoplus_{g \in G} S_g$ and $T = \bigoplus_{g \in G} T_g$ be two G-graded rings. A ring homomorphism $\phi \colon S \to T$ is called *graded* if $\phi(S_g) \subseteq T_g$ for every $g \in G$. If $\phi \colon S \xrightarrow{\sim} T$ is a graded ring isomorphism, then we write $S \cong_{\text{gr}} T$ and say that S and T are *graded isomorphic*. Note that two graded isomorphic rings are also isomorphic but the reverse implication does not hold in general. If $S \cong_{\mathrm{gr}} T$, then S is graded von Neumann regular if and only if T is graded von Neumann regular.

The category of G-graded rings will be denoted by G-RING. The objects of this category are pairs $(S, \{S_g\}_{g\in G})$ where S is a ring and $\{S_g\}_{g\in G}$ is a G-grading of S. The morphisms of G-RING are the G-graded ring homomorphisms. Next, we consider direct limits in G-RING. Let $\{A_i \mid i \in I\}$ be a directed system of G-graded rings. For every $i \in I$, we have that $A_i = \bigoplus_{g\in G}(A_i)_g$. Recall (see [11, II, §11.3, Rem. 3]) that $B = \varinjlim_{i\in I} A_i$ is a G-graded ring with homogeneous components $B_g = \varinjlim_{i\in I}(A_i)_g$. In other words, the category G-RING has arbitrary direct limits. The following lemma is a graded version of a well-known result (see [10, Prop. 5.2.14]). We include a proof for the convenience of the reader.

Lemma D.2.11. *Let $\{A_i \mid i \in I\}$ be a directed system of G-graded rings. Suppose that A_i is graded von Neumann regular for every $i \in I$. Then $B = \varinjlim_i A_i$ is graded von Neumann regular.*

Proof. Let $(B = \varinjlim_i A_i, \phi_i)$ be the direct limit of $\{A_i \mid i \in I\}$. Recall that the canonical functions $\phi_i \colon A_i \to B = \varinjlim_i A_i$ are graded ring homomorphisms. Take an arbitrary $g \in G$ and $b_g \in B_g = \varinjlim_{i\in I}(A_i)_g$. Then, $b_g = \phi_k(a_g)$ for some $a_g \in (A_k)_g$ and $k \in I$. Since A_k is graded von Neumann regular by assumption, it follows that there is some $s \in A_k$ such that $a_g = a_g s a_g$. Applying ϕ_k to both sides yields, $b_g = b_g\phi_k(s)b_g$ for $\phi_k(s) \in B$. Thus, B is graded von Neumann regular. □

D.3. Main result

In this section, we prove our main result: Theorem D.1.2. We first show that there are G-graded rings S such that S_e is von Neumann regular while S is not graded von Neumann regular.

Example D.3.1. Let R be a von Neumann regular ring (e.g. a field) and consider the polynomial ring $R[x] = \bigoplus_{i\geq 0} Rx^i$. By putting $S_i := Rx^i$ for $i \geq 0$ and $S_i := \{0\}$ for $i < 0$, we get a $\mathbb{Z}$-grading of $R[x]$. Note that $x^2 \notin (x^2)R[x](x^2)$. Hence, $R[x]$ is not graded von Neumann regular. This example shows that the conclusion of Theorem D.1.1 does not hold for a general group graded ring.

We now consider necessary conditions for a ring to be graded von Neumann regular. The following result is well-known and follows from Proposition D.2.3:

Lemma D.3.2. *Let $S = \bigoplus_{g\in G} S_g$ be a G-graded ring. If S is graded von Neumann regular, then S_e is von Neumann regular.*

We show that all graded von Neumann regular rings are nearly epsilon-strongly graded (cf. Proposition D.2.1).

Proposition D.3.3. *Let $S = \bigoplus_{g\in G} S_g$ be a G-graded ring. If S is graded von Neumann regular, then S is nearly epsilon-strongly G-graded.*

Proof. Take an arbitrary $g \in G$ and $s \in S_g$. To prove that S is nearly epsilon-strongly G-graded, we need to show that there exist elements $\epsilon_g(s) \in S_g S_{g^{-1}}$ and

$\epsilon'_g(s) \in S_{g^{-1}}S_g$ such that $\epsilon_g(s)s = s = s\epsilon'_g(s)$. Since S is graded von Neumann regular, it follows by Proposition D.2.3 that there is some $b \in S_{g^{-1}}$ such that $s = sbs$. Then, $\epsilon_g(s) := sb \in S_gS_{g^{-1}}$ and $\epsilon'_g(s) := bs \in S_{g^{-1}}S_g$ satisfy the requirement. Hence, S is nearly epsilon-strongly G-graded. $\square$

Remark D.3.4. Theorem D.1.3 together with Proposition D.3.3 implies that every Leavitt path algebra over a field is nearly epsilon-strongly $\mathbb{Z}$-graded. The stronger statement that every Leavitt path algebras over a general unital ring is nearly epsilon-strongly $\mathbb{Z}$-graded has been proved by Nystedt and Öinert [23, Thm. 1.3].

The following definition was introduced by Clark, Exel and Pardo [12] in the context of Steinberg algebras:

Definition D.3.5. ([12, Def. 4.5]) Let $S = \bigoplus_{g\in G} S_g$ be a G-graded ring. If $S_g = S_gS_{g^{-1}}S_g$ for every $g \in G$, then we say that S is *symmetrically G-graded*.

Moreover, Nystedt and Öinert [23, Prop. 3.3] proved that every nearly epsilon-strongly G-graded ring is symmetrically G-graded. In conclusion, the following relationship holds between the mentioned classes of group graded rings:

Remark D.3.6. The following implications hold for an arbitrary G-grading $\{S_g\}_{g\in G}$ of S:

$$\text{unital strong} \implies \text{epsilon-strong} \implies \text{nearly epsilon-strong} \implies \text{symmetrical}$$

The following corollary is a direct consequence of Proposition D.3.3 and Remark D.3.6:

Corollary D.3.7. Let $S = \bigoplus_{g\in G} S_g$ be a G-graded ring. If S is graded von Neumann regular, then S is symmetrically G-graded.

Remark D.3.8. By Corollary D.3.7, the elementwise condition of graded von Neumann regularity (cf. Proposition D.2.3) implies the componentwise condition of symmetrical gradings, that is $S_g = S_gS_{g^{-1}}S_g$ for every $g \in G$. However, the reverse implication does not hold in general (see Example D.4.2(b)).

Before proving our characterization, we need the following lemma:

Lemma D.3.9. Let $S = \bigoplus_{g\in G} S_g$ be a nearly epsilon-strongly G-graded ring and suppose that S_e is von Neumann regular. Then, for every $g \in G$ and $x \in S_g$, the left S_e-ideal $S_{g^{-1}}x$ is generated by an idempotent in S_e.

Proof. Take an arbitrary $g \in G$ and $x \in S_g$. Since S is nearly epsilon-strongly G-graded, there exists some $\epsilon_g(x) \in S_gS_{g^{-1}}$ such that $\epsilon_g(x)x = x$. We can write $\epsilon_g(x) = \sum_{i=1}^k a_ib_i$ for some elements $a_1, a_2, \ldots, a_k \in S_g$ and $b_1, b_2, \ldots, b_k \in S_{g^{-1}}$. Let $c_i := b_ix \in S_{g^{-1}}x$ for $i \in \{1, 2, \ldots, k\}$. We claim that $S_{g^{-1}}x = S_ec_1 + S_ec_2 + \cdots + S_ec_k$. Indeed, let $sx \in S_{g^{-1}}x$ be an arbitrary element. Then,

$$sx = s(\epsilon_g(x)x) = s\Big(\sum_{i=1}^k a_ib_i\Big)x = \sum_{i=1}^k (sa_i)(b_ix) = \sum_{i=1}^k (sa_i)c_i.$$

Since $sa_i \in S_{g^{-1}} S_g \subseteq S_e$, it follows that $S_{g^{-1}} x$ is finitely generated by $\{c_1, c_2, \ldots, c_k\}$ as a left S_e-ideal. Moreover, since S_e is von Neumann regular, Proposition D.2.1 implies that S_e is s-unital. By Proposition D.2.2, we have that $S_{g^{-1}} x$ is generated by an idempotent in S_e. $\square$

The following proposition generalizes Yahya's proof (see [27, Thm. 3]) of Theorem D.1.1:

Proposition D.3.10. Let $S = \bigoplus_{g \in G} S_g$ be a nearly epsilon-strongly G-graded ring. If S_e is von Neumann regular, then S is graded von Neumann regular.

Proof. Suppose that S_e is von Neumann regular. Take an arbitrary $g \in G$ and $0 \neq x \in S_g$. By Proposition D.2.3, we need to show that there exists some $r \in S_{g^{-1}}$ such that $x = xrx$. By Lemma D.3.9, there is some idempotent $y \in S_e$ such that $S_{g^{-1}} x = S_e y$. Note that $y = y^2 \in S_e y = S_{g^{-1}} x$. Hence, there is some $r \in S_{g^{-1}}$ such that $y = rx$. Also note that,

$$S_g S_{g^{-1}} x = S_g (S_{g^{-1}} x) = S_g (S_e y) = S_g S_e y \subseteq S_g y. \tag{33}$$

Since S is assumed to be nearly epsilon-strongly G-graded, there exists some $\epsilon_g(x) \in S_g S_{g^{-1}}$ such that $\epsilon_g(x) x = x$. Now, using (33), we have that $x = \epsilon_g(x) x \in S_g S_{g^{-1}} x \subseteq S_g y$, and hence there exists some $x' \in S_g$ such that $x = x'y$. But then $xy = (x'y)y = x'(yy) = x'y = x$. Thus, $x = xy = xrx$. Hence, S is graded von Neumann regular. $\square$

Now we can prove our characterization of graded von Neumann regular rings:

Proof of Theorem 1.2. Let $S = \bigoplus_{g \in G} S_g$ be a G-graded ring. Suppose that S is graded von Neumann regular. Then Proposition D.3.3 and Lemma D.3.2 establish that S is nearly epsilon-strongly G-graded and that S_e is von Neumann regular, respectively. Conversely, suppose that S is nearly epsilon-strongly G-graded and that S_e is von Neumann regular. Then Proposition D.3.10 implies that S is graded von Neumann regular. $\square$

Since epsilon-strongly G-graded rings are nearly epsilon-strongly G-graded (see Remark D.3.6), the following result is a consequence of Theorem D.1.2.

Corollary D.3.11. Let $S = \bigoplus_{g \in G} S_g$ be an epsilon-strongly G-graded ring. Then S is graded von Neumann regular if and only if S_e is von Neumann regular.

Remark D.3.12. The above result is a generalization of Theorem D.1.1. Indeed, by applying Corollary D.3.11 to unital strongly group graded rings, which by Remark D.3.6 are epsilon-strongly graded, we immediately recover Theorem D.1.1.

D.4. Proof of Theorem D.1.4

In this section, we prove that a Leavitt path algebra $L_R(E)$ is graded von Neumann regular if and only if R is von Neumann regular (see Theorem D.1.4).

Let $E = (E^0, E^1, s, r)$ be a directed graph consisting of a vertex set E^0, an edge set E^1 and maps $s \colon E^1 \to E^0$ and $r \colon E^1 \to E^0$ specifying the source vertex $s(f)$ respectively range vertex $r(f)$ for each edge $f \in E^1$. A directed graph E is called *finite* if the sets E^0 and E^1 are finite. Note that we allow the *null-graph* which has no vertices and no edges. If E is not the null-graph, i.e. $E^0 \neq \emptyset$, then we write $E \neq \emptyset$. A *sink* is a vertex $v \in E^0$ such that $s^{-1}(v) = \emptyset$. An *infinite emitter* is a vertex $v \in E^0$ such that $s^{-1}(v)$ is an infinite set. A vertex is called *regular* if it is neither a sink nor an infinite emitter. The set of regular vertices of E is denoted by $\mathrm{Reg}(E)$.

For technical reasons we will consider a generalization of Leavitt path algebras introduced by Ara and Goodearl [6]:

Definition D.4.1. Let R be a unital ring and let $E = (E^0, E^1, s, r)$ be a directed graph. Moreover, let X be any subset of $\mathrm{Reg}(E)$. The *Cohn path algebra relative to* X, denoted by $C_R^X(E)$, is the free R-algebra generated by the symbols,

$$\{v \mid v \in E^0\} \cup \{f \mid f \in E^1\} \cup \{f^* \mid f \in E^1\},$$

subject to the following relations:
(i) $vv' = \delta_{v,v'}v$ for all $v, v' \in E^0$,
(ii) $s(f)f = f = fr(f)$ for all $f \in E^1$,
(iii) $r(f)f^* = f^* = f^*s(f)$ for all $f \in E^1$,
(iv) $f^*f' = \delta_{f,f'}r(f)$ for all $f, f' \in E^1$,
(v) $v = \sum_{f \in s^{-1}(v)} ff^*$ for all $v \in X$.
We let R commute with the generators.
Taking $X = \mathrm{Reg}(E)$, we obtain the *Leavitt path algebra of E over R*. In other words, we have that $L_R(E) = C_R^{\mathrm{Reg}(E)}(E)$.

Recall that a *path* is a sequence of edges $\alpha = f_1 f_2 \dots f_n$ such that $r(f_i) = s(f_{i+1})$ for $1 \leq i \leq n-1$. The *length* of α is equal to n and we write $\mathrm{len}(\alpha) = n$. We also write $s(\alpha) = s(f_1)$ and $r(\alpha) = r(f_n)$. By convention, a vertex $v \in E^0$ is considered to be a path of length 0. Moreover, there is an anti-graded involution on $C_R^X(E)$ defined by $f \mapsto f^*$ for every $f \in E^1$ and $v \mapsto v^* = v$ for every $v \in E^0$. This involution extends to paths by putting $\alpha^* = f_n^* f_{n-1}^* \dots f_1^*$. The element $\alpha \in C_R^X(E)$ is called a *real path* and $\alpha^* \in C_R^X(E)$ is called a *ghost path*. Let $\mathrm{Path}(E)$ be the set of paths in E. In particular, $\mathrm{Path}(E)$ includes the vertices of E since they are considered zero length paths. Elements of the form $\alpha\beta^* \in C_R^X(E)$ for $\alpha, \beta \in \mathrm{Path}(E)$ are called *monomials*. It can be shown that any element of $C_R^X(E)$ can be written as a finite sum $\sum r_i \alpha_i \beta_i^*$ where $r_i \in R$ and $\alpha_i, \beta_i \in \mathrm{Path}(E)$. Furthermore, there is a natural $\mathbb{Z}$-grading of relative Cohn path algebras given by,

$$(C_R^X(E))_i = \mathrm{Span}_R\{\alpha\beta^* \mid \alpha, \beta \in \mathrm{Path}(E), \mathrm{len}(\alpha) - \mathrm{len}(\beta) = i\}, \tag{34}$$

for every $i \in \mathbb{Z}$. This $\mathbb{Z}$-grading is called the *canonical $\mathbb{Z}$-grading* of $C_R^X(E)$.

The canonical $\mathbb{Z}$-grading of Leavitt path algebras was studied by Hazrat [16]. Among other results, he proved that if E is a finite graph, then $L_R(E)$ is strongly $\mathbb{Z}$-graded if and only if E has no sinks (see [16, Thm. 3.15]). Nystedt and Öinert established that $L_R(E)$ is epsilon-strongly $\mathbb{Z}$-graded if E is finite ([23, Thm. 1.2]) and that $L_R(E)$ is nearly epsilon-strongly $\mathbb{Z}$-graded for any graph E (see [23, Thm. 1.3]). For more details about Cohn path algebras and Leavitt path algebras, we refer the reader to the monograph by Abrams, Ara and Siles Molina [1].

We now consider graded von Neumann regular Leavitt path algebras. The following example shows that graded von Neumann regularity of $L_R(E)$ is dependent on R. Recall that any ring S is trivially G-graded by any group G by putting $S_e = S$ and $S_g = \{0\}$ for every $g \neq e$.

Example D.4.2. Let R be a unital ring and consider the following directed graph:

$$A_1 : \qquad \bullet_v$$

Note that since A_1 does not contain any edges, we have that the canonical $\mathbb{Z}$-grading (cf. (34)) of $L_R(A_1)$ is given by $L_R(A_1)_i = \{0\}$ for $i \neq 0$ and $L_R(A_1)_0 = \mathrm{Span}_R\{v\}$.

(a) The $\mathbb{Z}$-graded ring $L_R(A_1)$ is graded isomorphic to the coefficient ring R equipped with the trivial $\mathbb{Z}$-grading via the map defined by $r \mapsto rv$ for every $r \in R$. With this grading, every element is homogeneous. Hence, $L_R(A_1)$ is graded von Neumann regular if and only if R is von Neumann regular.

(b) Furthermore, since A_1 is a finite graph, it follows by [23, Thm. 1.2] that $L_R(A_1)$ is epsilon-strongly $\mathbb{Z}$-graded and therefore, in particular, symmetrically $\mathbb{Z}$-graded (see Remark D.3.6). If R is not von Neumann regular, then $L_R(A_1)$ is not graded von Neumann regular by (a). However, $L_R(A_1)$ is symmetrically $\mathbb{Z}$-graded. This shows that not all symmetrically graded rings are graded von Neumann regular (cf. Corollary D.3.7).

Let us now briefly discuss our method for proving Theorem D.1.4. Let R be a unital ring and let E be a directed graph. By [23, Thm. 1.3], $L_R(E)$ is nearly epsilon-strongly $\mathbb{Z}$-graded. It follows from Theorem D.1.2 that $L_R(E)$ is graded von Neumann regular if and only if $(L_R(E))_0$ is von Neumann regular. If E is a finite graph, then we can explicitly describe $(L_R(E))_0$ which allows us to lift von Neumann regularity from R to $(L_R(E))_0$ (see Theorem D.4.3). However, if E is not finite, then this approach does not seem to work.

Instead, the proof of Theorem D.1.4 proceeds as follows: We first prove the theorem in the special case of finite graphs using Corollary D.3.11 (see Corollary D.4.6). Secondly, we reduce the general case to the finite case by writing any Leavitt path algebra as a direct limit of Leavitt path algebras of finite graphs (see Proposition D.4.12). This latter reduction step is similar to the technique used by Hazrat [16, Steps (II)-(IV)] to establish Theorem D.1.3.

D.4.1. Finite graphs. Let R be a unital ring and let E be a finite graph. The principal component $(L_R(E))_0$ is a unital subring of $L_R(E)$ with multiplicative identity element $1_{(L_R(E))_0} = \sum_{v \in E^0} v$ (see e.g. [1, Lem. 1.2.12(iv)]). We begin by characterizing when $(L_R(E))_0$ is von Neumann regular. This will follow from a more general structure theorem. For Leavitt path algebras over fields, this result was showed by Ara, Moreno and Pardo (see the proof of [7, Thm. 5.3]). However, their proof generalizes to Leavitt path algebras over unital rings in a straightforward manner. Define a filtration of $(L_R(E))_0$ as follows. For $n \geq 0$, put,

$$D_n = \mathrm{Span}_R\{\alpha\beta^* \mid \mathrm{len}(\alpha) = \mathrm{len}(\beta) \leq n\}.$$

It is straightforward to show that D_n is an R-subalgebra of $(L_R(E))_0$. For $v \in E^0$ and $n > 0$ let $P(n, v)$ denote the set of paths γ with $\mathrm{len}(\gamma) = n$ and $r(\gamma) = v$. Let $\mathrm{Sink}(E)$ denote the set of sinks in E. Moreover, recall that a matricial ring is a finite product of full matrix rings.

Theorem D.4.3. ([1, Cor. 2.1.16]) Let R be a unital ring and let E be a finite directed graph. For a non-negative integer n, let $M_n(R)$ denote the full $n\times n$-matrix ring. Then,

$$(L_R(E))_0 = \bigcup_{n\geq 0} D_n.$$

Moreover, we have that,

$$D_0 \cong \prod_{v\in E^0} R,$$

$$D_n \cong \prod_{\substack{0\leq i\leq n-1\\ v\in \operatorname{Sink}(E)}} M_{|P(i,v)|}(R) \times \prod_{v\in E^0} M_{|P(n,v)|}(R),$$

as R-algebras. In particular, $(L_R(E))_0$ is a direct limit of matricial rings over R (as an object in the category of unital rings).

We can now establish the following lemma:

Lemma D.4.4. Let R be a unital ring and let E be a finite directed graph. If R is von Neumann regular, then $(L_R(E))_0$ is von Neumann regular.

Proof. The principal component $(L_R(E))_0$ is the direct limit of matricial rings over R by Theorem D.4.3. A full matrix ring over R is von Neumann regular if and only if R is von Neumann regular. Moreover, recall that unital von Neumann regular rings are closed under direct limits (see [10, Prop. 5.2.14]). It follows that $(L_R(E))_0$ is von Neumann regular if R von Neumann is regular. $\square$

For the converse statement, we do not need the assumption that E is a finite graph, but E cannot be the null-graph (see Remark D.4.7).

Lemma D.4.5. Let R be a unital ring and let $E\neq\emptyset$ be a directed graph. If $(L_R(E))_0$ is von Neumann regular, then R is von Neumann regular.

Proof. Suppose that $(L_R(E))_0$ is von Neumann regular and fix an arbitrary vertex $v_0\in E^0$, whose existence is guaranteed by the assumption that $E\neq\emptyset$. Note that $R\hookrightarrow (L_R(E))_0$ via the map $r\mapsto rv_0$. Take an arbitrary element $0\neq t\in R$. By the assumption there is some $x\in(L_R(E))_0$ such that $tv_0=(tv_0)x(tv_0)$. Let $x=\sum_i r_i\alpha_i\beta_i^*$ for some $\alpha_i,\beta_i\in \operatorname{Path}(E)$ satisfying $\operatorname{len}(\alpha_i)=\operatorname{len}(\beta_i)$, $r(\alpha_i)=r(\beta_i)$ and $r_i\in R$ for each index i. Then,

$$tv_0=(tv_0)\Big(\sum_i r_i\alpha_i\beta_i^*\Big)(tv_0)=\sum_j tr_jt\alpha_j\beta_j^*, \tag{35}$$

where the sum goes over all indices j such that $s(\alpha_j)=s(\beta_j)=v_0$. Consider the finite set $M=\{\alpha_j\}$ of α_j's appearing in right hand sum of (35). Let α_m be a fixed path of maximal length appearing in M. Multiplying both sides of (35) with α_m^* from the left yields,

$$\alpha_m^*(tv_0)=\alpha_m^*\Big(\sum_j tr_jt\alpha_j\beta_j^*\Big)=\sum_j tr_jt\alpha_m^*\alpha_j\beta_j^*=\sum_j tr_jt(\alpha_m^*\alpha_j)\beta_j^*. \tag{36}$$

Since $s(\alpha_m) = v_0$, we have that $r(\alpha_m^*) = v_0$ and hence $\alpha_m^*(tv_0) = t\alpha_m^* \neq 0$. Recall (see [1, Lem. 1.2.12(i)]) that for any paths $\delta, \mu \in \mathrm{Path}(E)$ we have that,

$$\delta^*\mu = \begin{cases} \kappa & \text{if } \mu = \delta\kappa \text{ for some } \kappa \\ \sigma^* & \text{if } \delta = \mu\sigma \text{ for some } \sigma \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Using (37) and the assumption that α_m is of maximal length it follows that if $\alpha_m = \alpha_j \alpha_j'$ then $\alpha_m^* \alpha_j = (\alpha_j')^*$. Otherwise, i.e. if α_j is not an initial segment of α_m, we have that $\alpha_m^* \alpha_j = 0$. Hence, (36) simplifies to,

$$0 \neq t\alpha_m^* = \sum_k tr_k t(\beta_k')^*,$$

for some paths β_k'. Since ghost paths are R-linearly independent (see [25, Prop. 4.9]), it follows that $t\alpha_m^* = tr_k t\alpha_m^*$ for some index k. Hence, $t = tr_k t$ and thus R is von Neumann regular. ☐

Hazrat proved that if K is a field and E a finite directed graph, then $L_K(E)$ is a graded von Neumann regular ring (see [16, pg. 5]). His proof is based on a reduction to corner skew Laurent polynomial rings. Here, we obtain a generalization of his result:

Corollary D.4.6. Let R be a unital ring and let $E \neq \emptyset$ be a finite directed graph. Then $L_R(E)$ is graded von Neumann regular if and only if R is von Neumann regular.

Proof. By [23, Thm. 1.2], $L_R(E)$ is epsilon-strongly $\mathbb{Z}$-graded. It follows from Corollary D.3.11 that $L_R(E)$ is graded von Neumann regular if and only if $(L_R(E))_0$ is von Neumann regular. By Lemma D.4.4 and Lemma D.4.5, $(L_R(E))_0$ is von Neumann regular if and only if R is von Neumann regular. The statement now follows. ☐

Remark D.4.7. The null-graph is a degenerate case that needs to be excluded in Corollary D.4.6. Let R be a unital ring and let $\emptyset$ be the null-graph, i.e. the graph without any vertices or edges. In this case, $L_R(\emptyset)$ is the zero ring which is trivially von Neumann regular and hence also graded von Neumann regular. In other words, the Leavitt path algebra $L_R(\emptyset)$ is graded von Neumann regular for any unital ring R.

For the rest of this section, we will reduce the general case of Theorem D.1.4 to the finite case dealt with in Corollary D.4.6. For Leavitt path algebras over a field, it is known that any Leavitt path algebra is the direct limit of Leavitt path algebras associated to finite graphs (see [1, Cor. 1.6.11]). We will show that this property generalizes to Leavitt path algebras over general unital rings.

D.4.2. Cohn path algebras as Leavitt path algebras. A surprising but well-known result is that any relative Cohn path algebra with coefficients in a field is graded isomorphic to a Leavitt path algebra over the same field. We recall the construction from [1, Def. 1.5.16]. Consider a pair (E, X) where $X \subseteq \mathrm{Reg}(E)$. Define a new graph $E(X)$ in the following way. Let $Y := \mathrm{Reg}(E) \setminus X$ and add new vertices $Y' = \{v' \mid v \in Y\}$. The new graph $E(X)$ is given by:

$$(E(X))^0 = E^0 \sqcup Y' \qquad \text{and} \qquad (E(X))^1 = E^1 \sqcup \{e' \mid r(e) \in Y\},$$

where e' is a new edge going from $s(e)$ to the new vertex $r(e)'$.

Proposition D.4.8. (cf. [1, Thm. 1.5.18]) Let R be a unital ring and let E be a directed graph. Let $E(X)$ denote the directed graph defined above. Then,

$$C_R^X(E) \cong_{\text{gr}} L_R(E(X)).$$

PROOF. Let $Y = \text{Reg}(E) \setminus X$. Define a map $\phi\colon C_R^X(E) \to L_R(E(X))$. For $v \in E^0$, let $\phi(v) = v + v'$ if $v \in Y$ and $\phi(v) = v$ otherwise. For $f \in E^1$, let $\phi(f) = f + f'$ if $r(f) \in Y$ and $\phi(f) = f$ otherwise. Moreover, let $\phi(f^*) = \phi(f)^*$ for all $f \in E^1$. Using the same arguments as in [1, Thm. 1.5.18], it follows that ϕ is a well-defined ring isomorphism. Furthermore, it is clear from the definition that ϕ is $\mathbb{Z}$-graded. □

D.4.3. The Cohn path algebra functor. Let E and F be directed graphs. A *graph homomorphism* $\phi\colon E \to F$ is a pair of maps $(\phi^0\colon E^0 \to F^0, \phi^1\colon E^1 \to F^1)$ satisfying the conditions $s(\phi^1(f)) = \phi^0(s(f))$ and $r(\phi^1(f)) = \phi^0(r(f))$ for every $f \in E^1$. Recall (see [1, Def. 1.6.2]) that the category $\mathscr{G}$ consists of objects of the form (E, X) where E is a directed graph and $X \subseteq \text{Reg}(E)$ is a subset of regular vertices. If $(F, Y), (E, X) \in \text{Ob}(\mathscr{G})$, then $\psi = (\psi^0, \psi^1)$ is a morphism in $\mathscr{G}$ if the following conditions are satisfied:

(a) $\psi\colon F \to E$ is a graph homomorphism such that ψ^0 and ψ^1 are injective;
(b) $\psi^0(Y) \subseteq X$;
(c) For every $v \in Y$, the restriction $\psi^1\colon s_F^{-1}(v) \to s_E^{-1}(\psi^0(v))$ is a bijection.

The category $\mathscr{G}$ has arbitrary direct limits (see [1, Prop. 1.6.4]). We will define a functor C_R from $\mathscr{G}$ to $\mathbb{Z}$-RING for any unital ring R.

Lemma D.4.9. (cf. [1, Lem. 1.6.3]) Let $\psi\colon (F, Y) \to (E, X)$ be a morphism in $\mathscr{G}$. Then there is an induced $\mathbb{Z}$-graded ring homomorphism $\bar{\psi}\colon C_R^Y(F) \to C_R^X(E)$.

PROOF. Put $\bar{\psi}(v) = \psi^0(v)$, $\bar{\psi}(f) = \psi^1(f)$ and $\bar{\psi}(f^*) = \psi^1(f)^*$ for all $v \in F^0, f \in F^1$. We show that $\bar{\psi}$ respects the relations (i)-(v) in Definition D.4.1.

(i): Take arbitrary vertices $u, v \in F^0$ such that $u \neq v$. Then, since ψ^0 is injective by (a), it follows that $\bar{\psi}(u)\bar{\psi}(v) = \psi^0(u)\psi^0(v) = 0$. Furthermore, $\bar{\psi}(u)\bar{\psi}(u) = \psi^0(u)\psi^0(u) = \psi^0(u) = \bar{\psi}(u)$. This shows that $\bar{\psi}$ preserves (i).

(ii)-(iii): The assumption in (a) that ψ is a graph homomorphism implies that (ii) and (iii) are preserved.

(iv): Follows by injectivity of ψ^1 similarly to (i).

(v): Let $v \in F^0$ with $s_F^{-1}(v) \neq \emptyset$. By (c), ψ^1 maps $s_F^{-1}(v)$ bijectively onto $s_E^{-1}(\psi^0(v))$. Hence,

$$\bar{\psi}(v) = \psi^0(v) = \sum_{f \in s_E^{-1}(\psi^0(v))} ff^* = \sum_{f \in s_F^{-1}(v)} \psi^1(f)\psi^1(f)^* = \sum_{f \in s_F^{-1}(v)} \bar{\psi}(f)\bar{\psi}(f).$$

Thus, $\bar{\psi}$ extends to a well-defined ring homomorphism $C_R^Y(F) \to C_R^X(E)$. Furthermore, it follows directly from the definition that $\bar{\psi}$ is $\mathbb{Z}$-graded. □

The following functor has previously only been considered in the case of coefficients in a field. But in fact, the properties we need also hold true for arbitrary coefficient rings.

Definition D.4.10. Let R be a unital ring. Define the *Cohn path algebra functor* by,

$$\begin{aligned} C_R\colon \mathscr{G} &\to \mathbb{Z}\text{-RING}, \\ (E,X) &\mapsto C_R^X(E) \\ \psi &\mapsto \bar{\psi}, \end{aligned}$$

for all objects $(E,X)\in \mathrm{Ob}(\mathscr{G})$ and morphisms ψ in $\mathscr{G}$.

Lemma D.4.11. (cf. [1, Prop. 1.6.4]) The functor C_R preserves direct limits.

Proof. Let $((E_i,X_i),(\phi_{ji})_{i,j\in I, j\geq i})$ be a directed system in $\mathscr{G}$ with direct limit $((E,X),\psi_i)$. We show that $(C_R^X(E),\bar{\psi}_i)$ is the direct limit of the directed system $(C_R^{X_i}(E_i),\bar{\phi}_{ji})$. Let A be a $\mathbb{Z}$-graded ring and $\gamma_i\colon C_R^{X_i}(E_i)\to A$ be a family of compatible morphisms. We need to show that there is a $\mathbb{Z}$-graded ring homomorphism $\gamma\colon C_R^X(E)\to A$ making the following diagram commute:

$$\begin{tikzcd}
C_R^{X_i}(E_i) \arrow[rr, "\bar{\phi}_{ij}"] \arrow[dr, "\bar{\psi}_i"] \arrow[ddr, bend right, "\gamma_i"'] & & C_R^{X_j}(E_j) \arrow[dl, "\bar{\psi}_j"'] \arrow[ddl, bend left, "\gamma_j"] \\
& C_R^X(E) \arrow[d, dashed, "\gamma"] & \\
& A &
\end{tikzcd}$$

Define $\gamma\colon C_R^X(E)\to A$ by,

$$\gamma(\psi_i^s(\alpha)) = \gamma_i(\alpha), \quad \gamma(\psi_i^s(\alpha)^*) = \gamma_i(\alpha^*),$$

for all $\alpha\in E_i^s$, $i\in I$, and $s\in\{0,1\}$. It remains to show that this gives a well-defined $\mathbb{Z}$-graded ring homomorphism. We show that γ preserves the relations (i)-(v) in Definition D.4.1.

(i): Let $u\in E^0$. Then there is some $i\in I$ and $u_0\in E_i^0$ satisfying $\psi_i(u_0)=u$. Hence, by definition,

$$\gamma(u)^2 = \gamma(\psi_i^0(u_0))^2 = \gamma_i(u_0)^2 = \gamma_i(u_0^2) = \gamma_i(u_0) = \gamma(u).$$

Let $u\neq v\in E^0$ and take some $i\in I$ such that $u_0,v_0\in E_i^0$ and $\psi_i^0(u_0)=u, \psi_i^0(v_0)=v$. Since $u_0\neq v_0$, it follows that $\gamma(u)\gamma(v)=\gamma_i(u_0)\gamma_i(v_0)=\gamma_i(u_0v_0)=\gamma_i(0)=0$.

(ii): Let $f\in E^1$. Then there is some $i\in I$ such that there are $f_0\in E_i^1$ and $v_0\in E_i^0$ satisfying $\psi_i^1(f_0)=f$ and $\psi_i^0(v_0)=s(f)$. By assumption (a), ψ_i is an injective graph homomorphism, which implies that $v_0=s(f_0)$. Then,

$$\begin{aligned} \gamma(s(f))\gamma(f) &= \gamma(\psi_i^0(v_0))\gamma(\psi_i^1(f_0)) = \gamma_i(v_0)\gamma_i(f_0) = \gamma_i(v_0f_0) = \gamma_i(s(f_0)f_0) = \\ &= \gamma_i(f_0) = \gamma(\psi_i^1(f_0)) = \gamma(f). \end{aligned}$$

(iii): Analogous to (ii).

(iv): Let $f \in E^1$ and take $i \in I$ such that there is some $f_0 \in E_i^1$ and $v_0 \in E_i^0$ satisfying $\psi_i^1(f_0) = f$ and $\psi_i^0(v_0) = r(f)$. By assumption (a), ψ_i is an injective graph homomorphism, which implies that $v_0 = r(f_0)$. Then,

$$\begin{aligned}\gamma(f^*)\gamma(f) &= \gamma(\psi_i^1(f_0^*))\gamma(\psi_i^1(f_0)) = \gamma_i(f_0^*)\gamma_i(f_0) = \gamma_i(f_0^* f_0) = \gamma_i(r(f_0) = \\ &= \gamma_i(v_0) = \gamma(\psi_i^0(v_0)) = \gamma(v).\end{aligned}$$

(v): Let $v \in E^0$ and take $i \in I$ such that there is some $v_0 \in E_i^0$ satisfying $\psi_i^0(v_0) = v$. Since, ψ_i^1 maps $s_{E_i}^{-1}(v_0)$ bijectively onto $s_E^{-1}(\psi_i^0(v_0)) = s_E^{-1}(v)$, it follows that,

$$\gamma(v) = \gamma(\psi_i^0(v_0)) = \gamma_i(v_0) = \sum_{f \in s_{E_i}^{-1}(v_0)} \gamma_i(f)\gamma_i(f^*) = \sum_{f \in s_{E_i}^{-1}(v_0)} \gamma(\psi_i^1(f))\gamma(\psi_i^1(f^*)) =$$

$$= \sum_{f' \in S_E^{-1}(v)} \gamma(f')\gamma((f')^*).$$

Thus, γ is a well-defined ring homomorphism. Moreover, it follows directly from the definition that it is $\mathbb{Z}$-graded. Since γ_i and $\gamma \circ \bar{\psi}_i$ agree on the generators $E_i^0 \cup E_i^1 \cup (E_i^1)^*$ of $C_R^X(E)$, it follows that $\gamma_i = \gamma \circ \bar{\psi}_i$. Hence, the diagram commutes and we are done. □

Proposition D.4.12. (cf. [1, Cor. 1.6.11]) Let R be a unital ring and let E be a graph. Then there exists a directed system $\{(F_i, Y_i) \mid i \in I\}$ in $\mathscr{G}$ such that the following assertions hold:

(a) every F_i is a finite directed graph;
(b) $L_R(E) \cong_{\text{gr}} \varinjlim_i C_R^{Y_i}(F_i) \cong_{\text{gr}} \varinjlim_i L_R(F_i(Y_i))$.

In other words, $L_R(E)$ is graded isomorphic to the direct limit of Leavitt path algebras associated to the finite graphs $F_i(Y_i)$.

Proof. By [1, Lem. 1.6.9], we have that $(E, \text{Reg}(E))$ is the direct limit of some directed system $\{(F_i, Y_i) \mid i \in I\}$ where every F_i is a finite graph. Since the functor C_R preserves direct limits by Lemma D.4.11, we have that $L_R(E) \cong_{\text{gr}} \varinjlim_i C_R^{Y_i}(F_i)$. By Proposition D.4.8, it follows that $C_R^{Y_i}(F_i) \cong_{\text{gr}} L_R(F_i(Y_i))$ for each $i \in I$. Hence, $\varinjlim_i C_R^{Y_i}(F_i) \cong_{\text{gr}} \varinjlim_i L_R(F_i(Y_i))$. □

The following proposition establishes the difficult direction of Theorem D.1.4:

Proposition D.4.13. Let R be a unital ring and let E be a directed graph. If R is von Neumann regular, then $L_R(E)$ is graded von Neumann regular.

Proof. If $E = \emptyset$ is the null-graph, then $L_R(E)$ is trivially graded von Neumann regular (see Remark D.4.7). Next, suppose that $E \neq \emptyset$. Since R is von Neumann regular, Corollary D.4.6 implies that $L_R(F)$ is graded von Neumann regular for any finite graph F. By Proposition D.4.12, $L_R(E)$ is graded isomorphic to a direct limit of Leavitt path algebras associated to finite graphs. Thus, by Lemma D.2.11, we have that $L_R(E)$ is graded von Neumann regular. □

We are now ready to give a complete proof of Theorem D.1.4.

Proof of Theorem D.1.4. Let $E \neq \emptyset$ be a directed graph. First suppose that R is von Neumann regular. Then Proposition D.4.13 implies that $L_R(E)$ is graded von Neumann regular.

Conversely, suppose that $L_R(E)$ is graded von Neumann regular. Then, by Lemma D.3.2, it follows that $(L_R(E))_0$ is von Neumann regular. Moreover, by Lemma D.4.5, this implies that R is von Neumann regular. □

Remark D.4.14. Let E be an arbitrary graph. Since $\mathbb{C}$ is von Neumann regular it follows from Theorem D.1.4 that $L_{\mathbb{C}}(E)$ is graded von Neumann regular. On the other hand, since $\mathbb{Z}$ is not von Neumann regular, we conclude that $L_{\mathbb{Z}}(E)$ is not graded von Neumann regular. Hence, Theorem D.1.4, in particular, algebraically differentiate Leavitt path algebras with coefficients in $\mathbb{C}$ respectively $\mathbb{Z}$. The author feels that this differentiaton is especially interesting considering the strange behaviour of Leavitt path algebras with coefficients in $\mathbb{Z}$ observed by Johansen and Sørensen [18].

D.5. Semiprime and semiprimitive Leavitt path algebras

In this section, we apply our results to obtain sufficient conditions for a Leavitt path algebra over a unital ring to be semiprimitive and semiprime.

Abrams and Aranda Pino showed that if K is a field and E is a graph, then the Leavitt path algebra $L_K(E)$ is both semiprime and semiprimitive (see [2, Prop. 6.1-6.3]). However, Leavitt path algebras over non-field rings are not always semiprime nor semiprimitive. Indeed, for the graph A_1 in Example D.4.2, we have that $L_R(A_1) \cong R$ for any unital ring R. Hence, $L_R(A_1)$ is semiprime/semiprimitive if and only if R is semiprime/semiprimitive.

Let R be a ring. Recall that an ideal I of R is called *semiprime* if $aRa \subseteq I$ implies that $a \in I$ for every $a \in R$. A ring is called *semiprime* if its zero ideal is semiprime. Moreover, recall that R is called *semiprimitive* if the Jacobson radical $J(R) = 0$. Let S be a G-graded ring and recall that S is said to have *homogeneous local units* if there is a set of local units E for S consisting of homogeneous idempotents. The following lemma by Abrams and Arando Pino generalizes Bergman's famous result that if S is a unital $\mathbb{Z}$-graded ring, then $J(S)$ is a graded ideal of S (see [21, Cor. A.I.7.15]):

Lemma D.5.1. ([2, Lem. 6.2]) Let S be a $\mathbb{Z}$-graded ring. Suppose that S has a set of homogeneous local units. Then $J(S)$ is a graded ideal of S.

Remark D.5.2. Let S be a unital $\mathbb{Z}$-graded ring. Then $E = \{1_S\}$ is a set of local units for S. Moreover, note that $1_S \in S_e$ is a homogeneous element. Hence, it follows from Lemma D.5.1 that $J(S)$ is a graded ideal of S.

We have Leavitt path algebras in mind when we state the following result, but we will also need it in the next section.

Proposition D.5.3. (cf. [16, Prop. 2(4)], [2, Prop. 6.3]) Let S be a $\mathbb{Z}$-graded ring. Suppose that S is graded von Neumann regular and that S has a set of homogeneous local units. Then S is semiprimitive and semiprime.

Proof. By Lemma D.5.1, $J(S)$ is a graded ideal. Let $x \in J(S)$ be a homogeneous element and consider the graded left ideal $Sx \subseteq J(S)$. It follows from Proposition D.2.5

that there is some idempotent f such that $Sx = Sf$. Recall that the Jacobian radical does not contain any non-zero idempotents. But $f = f^2 \in Sf = Sx \subseteq J(S)$, which implies that $f = 0$. Since S has a set of local units, it follows that $x \in Sx = Sf = 0$ and hence $x = 0$. Thus, $J(S) = 0$ and hence S is semiprimitive.

By [16, Prop. 2(2)], every graded ideal of a non-unital graded von Neumann regular ring is semiprime. Thus, the zero ideal of S is semiprime and hence S is semiprime. □

Finally, we prove that Leavitt path algebras with coefficients in a von Neumann regular ring are semiprimitive and semiprime.

Corollary D.5.4. Let R be a unital ring and let E be a directed graph. If R is von Neumann regular, then $L_R(E)$ is semiprimitive and semiprime.

Proof. Note that $E = \{v \mid v \in E^0\}$ is a set of local units for $L_R(E)$ consisting of homogeneous elements. Suppose that R is von Neumann regular. Then, by Theorem D.1.4, $L_R(E)$ is graded von Neumann regular. It follows from Proposition D.5.3 that $L_R(E)$ is semiprimitive and semiprime. □

Remark D.5.5. Since a field is von Neumann regular, it follows that Corollary D.5.4 generalizes Abrams and Aranda Pino's result that Leavitt path algebras over fields are semiprimitive and semiprime (see [2, Prop. 6.1-6.3]).

D.6. More applications

In this last section, we apply our results to unital partial crossed products and corner skew Laurent polynomial rings. Partial crossed products were introduced as a generalization of the classical crossed products (see [13]). Among these, the *unital partial crossed products* were shown to be especially well-behaved (see e.g. [8]). Let R be a unital ring and let G be a group with neutral element e. A *unital twisted partial action of G on R* (see [24, pg. 2]) is a triple $(\{\alpha_g\}_{g \in G}, \{D_g\}_{g \in G}, \{w_{g,h}\}_{(g,h) \in G \times G})$ satisfying certain technical relations. To this triple, it is possible to associate an epsilon-strongly G-graded algebra $R \star_\alpha^\omega G$ called the *unital partial crossed product*. The following result shows that unital partial crossed products behave similarly to classical crossed products with regards to graded von Neumann regularity.

Corollary D.6.1. Let G be a group, let R be a unital ring and let $R \star_\alpha^\omega G$ be a unital partial crossed product. Then the unital partial crossed product $R \star_\alpha^\omega G$ is graded von Neumann regular if and only if R is von Neumann regular.

Proof. The ring $R \star_\alpha^\omega G$ is epsilon-strongly G-graded (see [24, pg. 2]) with principal component R. The statement now follows from Corollary D.3.11. □

The general construction of fractional skew monoid rings was introduced by Ara, Gonzalez-Barroso, Goodearl and Pardo in [5]. We consider the special case of a fractional skew monoid ring by a corner isomorphism which is also called a *corner skew Laurent polynomial ring*. Let R be a unital ring and let $\alpha\colon R \to eRe$ be a corner ring isomorphism where e is an idempotent of R. The corner skew Laurent polynomial ring, denoted by $R[t_+, t_-; \alpha]$, is a unital epsilon-strongly $\mathbb{Z}$-graded ring (see [20, Prop. 8.1]).

The following result was proved by Hazrat using direct methods. We recover it as a special case of Corollary D.3.11.

Corollary D.6.2 (cf. [16, Prop. 8])**.** Let R be a unital ring, let e be an idempotent of R and let $\phi\colon R \to eRe$ be a corner isomorphism. The corner skew Laurent polynomial ring $R[t_+, t_-, \phi]$ is graded von Neumann regular if and only if R is a von Neumann regular ring.

Proof. By [20, Prop. 8.1], $R[t_+, t_-, \phi]$ is epsilon-strongly $\mathbb{Z}$-graded with principal component R. The desired conclusion now follows from Corollary D.3.11. □

We end this article by given sufficient conditions for a corner skew Laurent polynomial ring to be semiprimitive and semiprime.

Corollary D.6.3. Let R be a unital ring, let e be an idempotent of R and let $\phi\colon R \to eRe$ be a corner isomorphism. If R is a von Neumann regular ring, then the corner skew Laurent polynomial ring $R[t_+, t_-, \phi]$ is semiprimitive and semiprime.

Proof. Since $R[t_+, t_-, \phi]$ is a unital $\mathbb{Z}$-graded ring, it follows that $E = \{1_R\}$ is a set of local units (see Remark D.5.2). Suppose that R is von Neumann regular. Then, by Corollary D.6.2, it follows that $R[t_+, t_-, \phi]$ is graded von Neumann regular. The conclusion now follows by Proposition D.5.3. □

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PAPER E

Prime group graded rings with applications to partial crossed products and Leavitt path algebras

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Daniel Lännström

Patrik Lundström

Johan Öinert

Stefan Wagner

In this article we generalize a classical result by Passman on primeness of unital strongly group graded rings to the class of nearly epsilon-strongly group graded rings which are not necessarily unital. Using this result, we obtain (i) a characterization of prime s-unital strongly group graded rings, and, in particular, of infinite matrix rings and of group rings over s-unital rings, thereby generalizing a well-known result by Connell; (ii) characterizations of prime s-unital partial skew group rings and of prime unital partial crossed products; (iii) a generalization of the well-known characterizations of prime Leavitt path algebras, by Larki and by Abrams-Bell-Rangaswamy.

E.1. Introduction

Let S be a ring. By this we mean that S is associative but not necessarily unital. Unless otherwise stated, ideals of S are assumed to be two-sided. Recall that a proper ideal P of S is called *prime* if for all ideals A and B of S, $A \subseteq P$ or $B \subseteq P$ holds whenever $AB \subseteq P$. The ring S is called *prime* if $\{0\}$ is a prime ideal of S. The class of prime rings contains many well-known constructions, for instance left or right primitive rings, simple rings and matrix rings over integral domains. Prime rings also generalize integral domains to a non-commutative setting. Indeed, a commutative ring is prime if and only if it is an integral domain.

Throughout this article, G denotes a multiplicatively written group with neutral element e. Recall that S is called *G-graded*, if for each $x \in G$ there is an additive subgroup S_x of S such that $S = \bigoplus_{x\in G} S_x$, as additive groups, and for all $x, y \in G$, the inclusion $S_x S_y \subseteq S_{xy}$ holds. If in addition, $S_x S_y = S_{xy}$ holds for all $x, y \in G$, then S is said to be *strongly G-graded*. An interesting problem, studied for the past 50 years, concerns finding necessary and sufficient conditions for different classes of group graded rings to be prime, see [3, 9, 10, 29, 28, 39, 40, 41, 37, 35, 36]. In the case when S is unital and strongly G-graded, Passman has completely solved this problem by proving the following rather involved result:

Theorem E.1.1 (Passman [36, Thm. 1.3])**.** *Suppose that S is a unital and strongly G-graded ring. Then S is not prime if and only if there exist:*

(i) *subgroups $N \lhd H \subseteq G$ with N finite,*
(ii) *an H-invariant ideal I of S_e such that $I^x I = \{0\}$ for every $x \in G \setminus H$, and*
(iii) *nonzero H-invariant ideals $\tilde{A}, \tilde{B}$ of S_N such that $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A}\tilde{B} = \{0\}$.*

Let us briefly explain the notation used in the formulation of this result as well as some technical aspects of Passman's proof of it. Suppose that I is an ideal of the subring S_e. If $x \in G$, then I^x denotes the S_e-ideal $S_{x^{-1}} I S_x$. Let H, N be subgroups of G. The ideal I is called *H-invariant* if $I^x \subseteq I$ holds for every $x \in H$; S_N denotes $\bigoplus_{x\in N} S_x$, which is clearly a subring of S. In [36] Passman provided a "combinatorial" proof of Theorem E.1.1 by combining two main ideas. First, a coset counting method, also known as the "Δ-method", developed by Passman [39] and Connell [10], secondly, the "bookkeeping procedure" introduced by Passman in [37] which involves a careful study of the action of the group G on the lattice of ideals of S_e. In [36] Passman also showed that analogous criteria exist for *semiprimeness* of strongly group graded rings. In this article, however, only the concept of primeness will be studied.

In a subsequent article [35] Passman obtained an analogue of Theorem E.1.1 for the slightly larger class of unital G-graded rings which are *cancellative*, that is, rings S having the property that for all $x, y \in G$ and all homogeneous subsets $U, V \subseteq S$, the implication $US_xS_yV = \{0\} \Rightarrow US_{xy}V = \{0\}$ holds. It is clear that strongly G-graded rings are cancellative. However, not all cancellative G-graded rings are strongly graded. For instance, *the first Weyl algebra* $A_1(\mathbb{F})$, over a field $\mathbb{F}$ of characteristic zero, is a $\mathbb{Z}$-graded ring which is not strongly graded but still cancellative, since it is a *domain* (see e. g. [16, Chap. 2]).

The motivation for the present article is the observation that many important examples of group graded rings are *not* cancellative, but may still be prime. Indeed, suppose that R is a unital ring and let $S := M_n(R)$ denote the ring of $n \times n$-matrices with entries in R. Then it is easy to see that S is prime if and only if R is prime. On the other hand, one can construct group gradings on S that are not cancellative. Consider the case $n = 2$. Let e_{ij} denote the matrix with 1 in position ij and zeros elsewhere. If $G = \mathbb{Z}$ and we put $S_0 := Re_{11} + Re_{22}$, $S_1 := Re_{12}$, $S_{-1} := Re_{21}$, and $S_x := \{0\}$, for $x \in \mathbb{Z} \setminus \{0, 1, -1\}$, then this defines a G-grading on S satisfying $S_1 \cdot S_1S_{-1} \cdot S_0 = \{0\}$

but $S_1 \cdot S_0 \cdot S_0 = Re_{12} \neq \{0\}$. This shows that the grading is not cancellative. In a similar fashion, one may define non-cancellative $\mathbb{Z}$-gradings on $M_n(R)$, for every $n \geq 2$.

This phenomenon is not confined to rings of matrices. In fact, these structures can be considered as special cases of so-called *Leavitt path algebras* $L_R(E)$, over a unital ring R, defined by directed graphs E (for the details, see Section E.14). All Leavitt path algebras carry a canonical $\mathbb{Z}$-grading. One can show that with this grading, a Leavitt path algebra defined by a finite graph E, is cancellative if and only if it is strongly graded (see Proposition E.2.24 and Proposition E.14.4). However, it is easy to give examples of Leavitt path algebras which are not strongly $\mathbb{Z}$-graded. Nevertheless, the question of primeness of such structures has been completely resolved in the case when R is commutative, for any directed graph E, by Larki [25] building upon previous work by Abrams, Bell and Rangaswamy [2, Thm. 1.4]. Their results involve a certain "connectedness" property on the set E^0 of vertices of E. Namely, a directed graph E is said to satisfy *condition (MT-3)* if for all $u, v \in E^0$, there exist $w \in E^0$ and paths from u to w and from v to w.

Theorem E.1.2 (Larki [25, Prop. 4.5])**.** *Suppose that E is a directed graph and that R is a unital commutative ring. Then $L_R(E)$ is prime if and only if R is an integral domain and E satisfies condition (MT-3).*

The purpose of the present article is to prove a primeness result (see Theorem E.1.3) that holds for a class of group graded rings that contains all unital strongly group graded rings as well as many types of group graded rings that are not cancellative. The rings that we consider are the nearly epsilon-strongly group graded rings introduced by Nystedt and Öinert in [31]. Recall that a, not necessarily unital, G-graded ring S is called *nearly epsilon-strongly G-graded* if for every $x \in G$ and every $s \in S_x$, there exist $\epsilon_x(s) \in S_x S_{x^{-1}}$ and $\epsilon'_x(s) \in S_{x^{-1}} S_x$ such that the equalities $\epsilon_x(s)s = s = s\epsilon'_x(s)$ hold. Note that every nearly epsilon-strongly G-graded ring S is necessarily *non-degenerately G-graded*, i. e. for every $x \in G$ and every nonzero $s \in S_x$, we have $sS_{x^{-1}} \neq \{0\}$ and $S_{x^{-1}}s \neq \{0\}$. In loc. cit. it is shown that every Leavitt path algebra, equipped with its canonical $\mathbb{Z}$-grading, is nearly epsilon-strongly graded. In addition, s-unital partial skew group rings and unital partial crossed products are nearly epsilon-strongly graded (see Section E.13).

Here is the main result of this article:

Theorem E.1.3. *Suppose that G is a group and that S is a G-graded ring. Consider the following five assertions:*

(a) *S is not prime.*
(b) *There exist:*
  (i) *subgroups $N \lhd H \subseteq G$,*
  (ii) *an H-invariant ideal I of S_e such that $I^x I = \{0\}$ for every $x \in G \setminus H$, and*
  (iii) *nonzero ideals $\tilde{A}, \tilde{B}$ of S_N such that $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A} S_H \tilde{B} = \{0\}$.*
(c) *There exist:*
  (i) *subgroups $N \lhd H \subseteq G$ with N finite,*
  (ii) *an H-invariant ideal I of S_e such that $I^x I = \{0\}$ for every $x \in G \setminus H$, and*
  (iii) *nonzero ideals $\tilde{A}, \tilde{B}$ of S_N such that $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A} S_H \tilde{B} = \{0\}$.*

(d) *There exist:*
   (i) *subgroups* $N \lhd H \subseteq G$ *with* N *finite,*
   (ii) *an* H*-invariant ideal* I *of* S_e *such that* $I^x I = \{0\}$ *for every* $x \in G \setminus H$*, and*
   (iii) *nonzero* H*-invariant ideals* $\tilde{A}, \tilde{B}$ *of* S_N *such that* $\tilde{A}, \tilde{B} \subseteq IS_N$*,* $\tilde{A}S_H\tilde{B} = \{0\}$*.*

(e) *There exist:*
   (i) *subgroups* $N \lhd H \subseteq G$ *with* N *finite,*
   (ii) *an* H*-invariant ideal* I *of* S_e *such that* $I^x I = \{0\}$ *for every* $x \in G \setminus H$*, and*
   (iii) *nonzero* H/N*-invariant ideals* $\tilde{A}, \tilde{B}$ *of* S_N *such that* $\tilde{A}, \tilde{B} \subseteq IS_N$*,* $\tilde{A}\tilde{B} = \{0\}$*.*

The following assertions hold:

(1) *If* S *is non-degenerately* G*-graded, then* (e)$\Longrightarrow$(d)$\Longrightarrow$(c)$\Longrightarrow$(b)$\Longrightarrow$(a)*.*
(2) *If* S *is nearly epsilon-strongly* G*-graded, then* (a)$\Longleftrightarrow$(b)$\Longleftrightarrow$(c)$\Longleftrightarrow$(d)$\Longleftrightarrow$(e)*.*

Let us make four remarks on Theorem E.1.3. First of all, this result is applicable to rings which are *not necessarily unital*. Secondly, unital strongly G-graded rings (see Lemma E.2.16) and cancellatively G-graded rings (see [35, Lem. 1.2]) satisfy $r.\operatorname{Ann}_S(S_x) = \{0\}$, and $l.\operatorname{Ann}_S(S_x) = \{0\}$, for every $x \in G$. However, many important classes of group graded rings rarely satisfy such annihilator conditions, for instance Leavitt path algebras [1, 4, 2, 6, 25, 43] and partial crossed products [11, 13, 12]. Thus, Theorem E.1.3 allows us to consider classes of rings which are unreachable by the results of [35, 36]. Thirdly, we would like to motivate why assertions (b), (c) and (d) appear in Theorem E.1.3. By allowing N to be infinite, assertion (b) creates more flexibility when attempting to prove that S is non-prime. Assertion (c) is identical to the assertion in [35, Thm. 2.3], and assertion (d) is essentially identical to the assertion in [36, Thm. 1.3]. Finally, it might be possible to generalize assertion (2) of Theorem E.1.3 beyond the class of nearly epsilon-strongly graded rings (see Remark E.12.8).

Here is a detailed outline of this article.

In Section E.2, we state our conventions on groups, rings and modules. We also provide preliminary results on different types of graded rings such as epsilon-strongly graded rings, nearly epsilon-strongly graded rings and cancellatively graded rings. In Section E.3, we consider H-invariant ideals and record some of their basic properties. In Section E.4, we obtain a one-to-one correspondence between graded ideals of S and G-invariant ideals of the principal component S_e. We also give a characterization of prime nearly epsilon-strongly G-graded rings in the case when G is an ordered group. In Section E.5, we prove the implication (b)$\Rightarrow$(a) of Theorem E.1.3 for non-degenerately G-graded rings. In Section E.6, we obtain some technical results that will be necessary in Section E.7, where we provide the bulk of results needed to establish Theorem E.1.3. Our approach is very much influenced by Passman [36]. In particular, we utilize a version of the Δ-method. In Section E.8, we prove the implication (a)$\Rightarrow$(e) of Theorem E.1.3 for nearly epsilon-strongly graded rings. In Section E.9, the proof of Theorem E.1.3 is finalized. We also show that Theorem E.1.1 can be recovered from Theorem E.1.3. In Section E.10, we use Theorem E.1.3 to obtain the following generalization of a result by Passman (see [36, Cor. 4.6]):

Theorem E.1.4. *Suppose that* G *is torsion-free and that* S *is nearly epsilon-strongly* G*-graded. Then* S *is prime if and only if* S_e *is* G*-prime.*

The remaining sections are devoted to applications of our findings. In Section E.11, we obtain an s-unital analogue of Passman's Theorem E.1.1 (see Corollary E.11.1) and consider $\mathbb{Z}$-graded Morita context algebras and $\mathbb{Z}$-graded infinite matrix rings. In Section E.12, we apply Theorem E.1.3 to group rings. Notably, we obtain the following non-unital generalization of Connell's [10] classical characterization:

Theorem E.1.5. *Suppose that R is an s-unital ring and that G is a group. Then the group ring $R[G]$ is prime if and only if R is prime and G has no non-trivial finite normal subgroup.*

In Section E.13, we apply our results to s-unital partial skew group rings (see Theorem E.13.5 and Theorem E.13.7) and to unital partial crossed products (see Theorem E.13.9 and Theorem E.13.10). In Section E.14, we use Theorem E.1.3 to obtain a characterization of prime Leavitt path algebras, thereby generalizing Theorem E.1.2 by allowing the coefficient ring R to be non-commutative:

Theorem E.1.6. *Suppose that E is a directed graph and that R is a unital ring. Then the Leavitt path algebra $L_R(E)$ is prime if and only if R is prime and E satisfies condition (MT-3).*

E.2. Preliminaries

In this section, we recall some useful notions and conventions on groups, rings and modules. We also provide some preliminary results on different types of graded rings such as epsilon-strongly graded rings, nearly epsilon-strongly graded rings and cancellatively graded rings. These results will be utilized in subsequent sections.

E.2.1. Groups. For the entirety of this article, G denotes a multiplicatively written group with neutral element e. Let H be a subgroup of G. The *index* of H in G is denoted by $[G : H]$. Take $g \in G$. The *order* of g is denoted by $\mathrm{ord}(g)$. The *centralizer* of g in G is defined to be the subgroup $C_G(g) := \{x \in G \mid xg = gx\}$ of G. Recall that the *finite conjugate center* of G is the subgroup $\Delta(G) := \{g \in G \mid [G : C_G(g)] < \infty\}$ of G. The *almost centralizer of H in G* is the subgroup $D_G(H) := \{x \in G \mid [H : C_H(x)] < \infty\}$ of G. Note that $D_G(H) \cap H = \Delta(H)$. By the orbit-stabilizer theorem, $\Delta(G)$ can equivalently be described as the set of elements of G with only finitely many conjugates in G. If G is equipped with a total order relation $\leq$ such that for all $a, b, x, y \in G$ the inequality $a \leq b$ implies the inequality $xay \leq xby$, then G is called an *ordered group*.

E.2.2. Rings and modules. Throughout this article, all rings are assumed to be associative but not necessarily unital. Let R be a ring. If U and V are subsets of R, then UV denotes the set of finite sums of elements of the form uv where $u \in U$ and $v \in V$. We say that R is *unital* if it has a nonzero multiplicative identity element. In this article, we will also consider the following weaker notion of unitality. The ring R is called *s-unital* if for every $r \in R$ the inclusion $r \in rR \cap Rr$ holds. For future reference, we recall the following:

Proposition E.2.1 (Tominaga [30, Prop. 12], [44])**.** *A ring R is s-unital if and only if for any finite subset V of R there is $u \in R$ such that for every $v \in V$ the equalities $uv = vu = v$ hold.*

If M is a left R-module and U is a subset of M, then the *left annihilator of* U is defined to be the set $l.\operatorname{Ann}_R(U) := \{r \in R \mid r \cdot u = 0, \ \forall u \in U\}$. If N is a right R-module and V is a subset of N, then the right annihilator $r.\operatorname{Ann}_R(V)$ is defined analogously.

E.2.3. Group graded rings. For the rest of this article S denotes a nonzero G-graded ring. Note that the *principal component* S_e is a subring of S and every $x \in G$, the set $S_x S_{x^{-1}}$ is an ideal of S_e. The *support of* S, denoted by $\operatorname{Supp}(S)$, is the set of $x \in G$ with $S_x \neq \{0\}$. In general, $\operatorname{Supp}(S)$ need not be a subgroup of G (see [32, Rmk. 46]). Take $s \in S$. Then $s = \sum_{x \in G} s_x$, for unique $s_x \in S_x$, such that $s_x = 0$ for all but finitely many $x \in G$. The *support of* s, denoted by $\operatorname{Supp}(s)$, is the set of $x \in G$ with $s_x \neq 0$.

Proposition E.2.2. *The ring S is strongly G-graded if and only if for every $x \in G$ the equalities $S_x S_e = S_e S_x = S_x$ and $S_x S_{x^{-1}} = S_e$ hold.*

Proof. Suppose that for every $x \in G$ the equalities $S_x S_e = S_e S_x = S_x$ and $S_x S_{x^{-1}} = S_e$ hold. Take $x, y \in G$. Then $S_{xy} = S_{xy} S_e = S_{xy} S_{y^{-1}} S_y \subseteq S_{xyy^{-1}} S_y = S_x S_y \subseteq S_{xy}$. Thus $S_x S_y = S_{xy}$. The converse statement is trivial. □

Remark E.2.3. Suppose that S is unital strongly G-graded. Then, for every $x \in G$, the relations $0 \neq 1_S \in S_e = S_x S_{x^{-1}}$ hold (see e. g. [28, Prop. 1.1.1]). Therefore, $\operatorname{Supp}(S) = G$.

The following notion was first introduced by Clark, Exel and Pardo in the context of Steinberg algebras [8, Def. 4.5]:

Definition E.2.4. The ring S is said to be *symmetrically G-graded* if for every $x \in G$, the equality $S_x S_{x^{-1}} S_x = S_x$ holds.

Remark E.2.5. If S is symmetrically G-graded, then $\operatorname{Supp}(S)^{-1} = \operatorname{Supp}(S)$.

Note that strongly G-graded rings are symmetrically G-graded. As the following example shows, a grading which is not strong may fail to be symmetrical:

Example E.2.6. Let R be a unital ring and consider the standard $\mathbb{Z}$-grading on the polynomial ring $R[x] = \bigoplus_{i \in \mathbb{Z}} S_i$ where $S_i := Rx^i$ for $i \geq 0$, and $S_i := \{0\}$ for $i < 0$. Clearly, $\operatorname{Supp}(S)^{-1} \neq \operatorname{Supp}(S)$ and thus, by Remark E.2.5, it follows that the grading is not symmetrical.

Next, we will consider another special type of grading. Passman appears to have been the first to give the following definition (see also [9, 34]):

Definition E.2.7 ([38, p. 32])**.** The ring S is said to be *non-degenerately G-graded* if for every $x \in G$ and every nonzero $s \in S_x$, we have $sS_{x^{-1}} \neq \{0\}$ and $S_{x^{-1}} s \neq \{0\}$.

Clearly, every unital strongly G-graded ring is non-degenerately G-graded.

E.2.4. Epsilon-strongly graded rings. Now, we consider a generalization of unital strongly graded rings, introduced by Nystedt, Öinert and Pinedo [32, Def. 4, Prop. 7].

Definition E.2.8. The ring S is called *epsilon-strongly G-graded* if for every $x \in G$ there exists $\epsilon_x \in S_x S_{x^{-1}}$ such that for all $s \in S_x$ the equalities $\epsilon_x s = s = s\epsilon_{x^{-1}}$ hold.

Example E.2.9. Let R be a unital ring and consider the following $\mathbb{Z}$-grading on the ring $M_2(R)$ of 2×2-matrices with entries in R:

$$(M_2(R))_0 := \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \quad (M_2(R))_1 := \begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix}, \quad (M_2(R))_{-1} := \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix},$$

and $(M_2(R))_i$ zero if $|i| > 1$. Clearly, this grading is not strong, but epsilon-strong with

$$\epsilon_1 = \begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1_R \end{pmatrix}.$$

Suppose that R is prime. Then $M_2(R)$ is also prime, but $(M_2(R))_0$ is not prime. This is an example of a prime epsilon-strongly $\mathbb{Z}$-graded ring whose principal component is not prime.

Moreover, unital partial crossed products (see [32]), Leavitt path algebras of finite graphs (see [31]), and certain Cuntz-Pimsner rings (see [23]) are classes of graded rings that are epsilon-strongly graded. A further generalization was introduced by Nystedt and Öinert:

Definition E.2.10 ([31, Def. 10])**.** The ring S is called *nearly epsilon-strongly G-graded* if for every $x \in G$ and every $s \in S_x$ there exist $\epsilon_x(s) \in S_x S_{x^{-1}}$ and $\epsilon_x(s)' \in S_{x^{-1}} S_x$ such that the equalities $\epsilon_x(s)s = s = s\epsilon_x(s)'$ hold.

Notably, every Leavitt path algebra with its natural $\mathbb{Z}$-grading is nearly epsilon-strongly $\mathbb{Z}$-graded whereas only Leavitt path algebras of finite graphs are epsilon-strongly $\mathbb{Z}$-graded (cf. [31, Thm. 28, Thm. 30]).

Proposition E.2.11 ([31, Prop. 11])**.** *The ring S is nearly epsilon-strongly G-graded if and only if S is symmetrically G-graded and for every $x \in G$ the ring $S_x S_{x^{-1}}$ is s-unital.*

Remark E.2.12. The following implications hold for all G-graded rings:

unital strong $\Longrightarrow$ epsilon-strong $\Longrightarrow$ nearly epsilon-strong $\Longrightarrow$ symmetrical

Proposition E.2.13. *Suppose that S is nearly epsilon-strongly G-graded. Then, $s \in sS_e \cap S_e s$ for every $s \in S$. In particular, S is s-unital and S_e is an s-unital subring of S.*

Proof. Proposition E.2.11 yields, in particular, that (i) $S_e = S_e S_e S_e$ and (ii) $S_e S_e = S_e^2$ is an s-unital ring. But (i) gives that $S_e = S_e^3 \subseteq S_e^2 \subseteq S_e$. Thus, $S_e = S_e^2$ is s-unital.

Let $s = \sum_{y \in G} s_y \in S$ with $s_y \in S_y$. Fix $x \in \mathrm{Supp}(s)$. By Proposition E.2.11, there are finitely many elements $a_i \in S_x S_{x^{-1}} \subseteq S_e$, $b_j \in S_{x^{-1}} S_x \subseteq S_e$ and $s_i, s'_j \in S_x$ such that $s_x = \sum_i a_i s_i = \sum_j s'_j b_j$. Now, since S_e is s-unital, there is some $e_x \in S_e$ such that $e_x a_i = a_i$ and $b_j e_x = b_j$ for all i, j (see Proposition E.2.1). Then, $e_x s_x = s_x = s_x e_x$. Hence, we can find such an $e_x \in S_e$ for every $x \in \mathrm{Supp}(s)$. Since $\mathrm{Supp}(s)$ is a finite set, it follows from Proposition E.2.1 that there is some $e_s \in S_e$ such that $e_s e_x = e_x = e_x e_s$

for every $x \in \operatorname{Supp}(s)$. Then $e_s s = \sum e_s s_x = \sum e_s(e_x s_x) = \sum (e_s e_x) s_x = \sum e_x s_x = \sum s_x = s$, where the sum runs over $\operatorname{Supp}(s)$. Similarly, $s e_s = s$. $\square$

Not every symmetrically G-graded ring is nearly epsilon-strongly G-graded:

Example E.2.14. Let R be an idempotent ring that is not s-unital (see e. g. [30, Expl. 2.5]). Consider the G-graded ring S defined by $S_e := R$ and $S_x := \{0\}$ if $x \in G \setminus \{e\}$. Clearly, S is symmetrically G-graded, but by Proposition E.2.13 S is not nearly epsilon-strongly G-graded.

Proposition E.2.15 ([31, Prop. 3.4])**.** *If S is nearly epsilon-strongly G-graded, then S is non-degenerately G-graded.*

Lemma E.2.16. *If S is s-unital strongly G-graded, then the following assertions hold:*
(a) *S is nearly epsilon-strongly G-graded.*
(b) *$s \in sS_e \cap S_e s$ for every $s \in S$.*
(c) *$r.\operatorname{Ann}_S(S_x) = \{0\}$ for every $x \in G$.*

Proof. (a): Clearly, S is symmetrically G-graded. By [24, Lem. 6.8], $S_x S_{x^{-1}} = S_e$ is s-unital for every $x \in G$. The desired conclusion now follows from Proposition E.2.11.

(b): This follows from (a) and Proposition E.2.13.

(c): Take $x \in G$ and $s \in r.\operatorname{Ann}_S(S_x)$. Then $\{0\} = S_{x^{-1}} S_x s = S_e s$. Thus, we get $s = 0$ by (b). $\square$

Remark E.2.17. If S is strongly G-graded, then S is s-unital if and only if S_e is s-unital.

In the rest of this article, we will freely use the fact that nearly epsilon-strongly graded rings are symmetrically graded, non-degenerately graded, and s-unital without further comment. For additional characterizations of (nearly) epsilon-strongly graded rings, we refer to [27].

E.2.5. Induced gradings. Now, we recall two important functorial constructions. For more details, we refer the reader to [22]. The first construction assigns a subring of S with an inherited grading. Let H be a subgroup of G and put $S_H := \bigoplus_{x \in H} S_x$. Note that S_H is an H-graded ring that is also a subring of S. Consider the map $\pi_H \colon S \to S_H$ defined by

$$\pi_H \left(\sum_{x \in G} s_x \right) = \sum_{x \in H} s_x.$$

The following result is well-known (see e. g. [33, Lem. 2.4]):

Lemma E.2.18. *The map $\pi_H \colon S \to S_H$ is an S_H-bimodule homomorphism.*

We can "map down" nonzero ideals when the ring is non-degenerately G-graded:

Lemma E.2.19. *Suppose that H is a subgroup of G. If A is a left (resp. right) ideal of S, then $\pi_H(A)$ is a left (resp. right) ideal of S_H. If, in addition, S is non-degenerately G-graded and A is nonzero, then $\pi_H(A)$ is nonzero.*

Proof. The first statement immediately follows from Lemma E.2.18. For the second statement suppose that S is non-degenerately G-graded and that A is a nonzero left S-ideal. Pick a nonzero $a \in A$ and $x \in \mathrm{Supp}(a)$. Then, since S is non-degenerately G-graded, $\{0\} \neq S_{x^{-1}} a_x = \pi_{\{e\}}(S_{x^{-1}} a) \subseteq \pi_{\{e\}}(A) \subseteq \pi_H(A)$. The case when A is a right ideal is proved similarly. □

We now describe the second construction: Given a normal subgroup N of G, we define *the induced G/N-grading* on S in the following way. For every $C \in G/N$, put $S_C := \bigoplus_{x \in C} S_x$. This yields a G/N-grading on S. The following non-trivial result, proved by Lännström, will be essential later on in this article:

Proposition E.2.20 ([22, Prop. 5.8])**.** *Suppose that S is nearly epsilon-strongly G-graded and that N is a normal subgroup of G. Then the induced G/N-grading on S is nearly epsilon-strong.*

We will also need the following result:

Proposition E.2.21. *Suppose that S is non-degenerately G-graded and that N is a normal subgroup of G. Then the induced G/N-grading on S is non-degenerate.*

Proof. Take $x \in G$ and a nonzero $a \in S_{xN}$. Write $a = a_{xn_1} + a_{xn_2} + \ldots + a_{xn_k}$ where $n_1, \ldots, n_k \in N$ are all distinct and $a_{xn_i} \neq 0$ for every i. By non-degeneracy of the G-grading there is some $c_{n_1^{-1}x^{-1}} \in S_{Nx^{-1}} = S_{x^{-1}N}$ such that $c_{n_1^{-1}x^{-1}} a_{xn_1} \neq 0$. Note that

$$\pi_{\{e\}}(c_{n_1^{-1}x^{-1}} a) = c_{n_1^{-1}x^{-1}} a_{xn_1} \neq 0.$$

Hence, $c_{n_1^{-1}x^{-1}} a \neq 0$. This shows that $S_{x^{-1}N} a \neq \{0\}$. Similarly, $a S_{x^{-1}N} \neq \{0\}$. Thus, the induced G/N-grading on S is non-degenerate. □

E.2.6. Cancellatively graded rings. We now briefly discuss Passman's notion of cancellatively graded rings. We will, however, not work with this class of rings outside of this section.

In [35] Passman extended his results from [36] to the class of *cancellatively group graded rings* which generalizes the class of unital strongly group graded rings. To avoid any confusion, we wish to point out that Passman's [36] notion of *H-stability* is used interchangeably with our notion of H-invariance. Recall from the introduction that a unital G-graded ring S is called *cancellative* if for all $x, y \in G$ and all homogeneous subsets $U, V \subseteq S$, the implication $US_xS_yV = \{0\} \Rightarrow US_{xy}V = \{0\}$ holds. Clearly, all strongly graded rings are cancellative. However, e. g. canonical $\mathbb{Z}$-gradings on Leavitt path algebras (see Section E.14) need not be cancellative.

Remark E.2.22. Lännström has observed that if S is epsilon-strongly G-graded, then S must be unital (see [22, Prop. 3.8]). Moreover, ϵ_x is central in S_e for every $x \in G$ (see [32]).

Lemma E.2.23. *The following assertions hold for each $x \in G$:*

(a) *If S is symmetrically G-graded, then $r.\mathrm{Ann}_S(S_x) = r.\mathrm{Ann}_S(S_{x^{-1}}S_x)$.*
(b) *If S is epsilon-strongly G-graded, then $r.\mathrm{Ann}_S(S_x) = r.\mathrm{Ann}_S(S_{x^{-1}}S_x) = r.\mathrm{Ann}_S(\epsilon_{x^{-1}})$.*

Proof. (a): Suppose that S is symmetrically G-graded. If $s \in r.\operatorname{Ann}_S(S_x)$, then $S_{x^{-1}}S_x s = \{0\}$, which implies that $r.\operatorname{Ann}_S(S_x) \subseteq r.\operatorname{Ann}_S(S_{x^{-1}}S_x)$. If, conversely, $s \in r.\operatorname{Ann}_S(S_{x^{-1}}S_x)$, then $S_x s = S_x S_{x^{-1}} S_x s = \{0\}$. Thus, $r.\operatorname{Ann}_S(S_{x^{-1}}S_x) \subseteq r.\operatorname{Ann}_S(S_x)$.

(b): Suppose that S is epsilon-strongly G-graded. Then $S_{x^{-1}}S_x = \epsilon_{x^{-1}}S_e = S_e\epsilon_{x^{-1}}$, which entails that $r.\operatorname{Ann}_S(S_{x^{-1}}S_x) = r.\operatorname{Ann}_S(S_e\epsilon_{x^{-1}}) = r.\operatorname{Ann}_S(\epsilon_{x^{-1}})$, where the last equality follows from the fact that $1_S = 1_{S_e}$. □

Proposition E.2.24. *Suppose that S is epsilon-strongly G-graded. Then the following assertions are equivalent:*

(a) *the grading on S is strong;*
(b) *for every $x \in G$, the equality $r.\operatorname{Ann}_S(S_x) = \{0\}$ holds;*
(c) *the grading on S is cancellative.*

Proof. (a)⇒(b): Take $x \in G$. For $s \in S$, we note that

$$S_x s = \{0\} \implies S_{x^{-1}}S_x s = \{0\} \implies S_e s = \{0\} \implies 1_S \cdot s = 0.$$

Hence, $r.\operatorname{Ann}_S(S_x) = \{0\}$.

(b)⇒(c): By [35, Lem. 1.2], S is cancellative if and only if, for every $x \in G$, (i) $S_x S_{x^{-1}}$ is a so-called *middle cancellable ideal* of S_e and (ii) $r.\operatorname{Ann}_S(S_x) = \{0\}$. In the special case of epsilon-strongly graded rings, (ii) actually implies (i). Let $x \in G$ and recall that $S_x S_{x^{-1}}$ being middle cancellable means that $US_xS_{x^{-1}}V = \{0\}$ implies that $UV = \{0\}$ for all subsets $U, V \subseteq S_e$. Moreover, note that $S_x S_{x^{-1}} = \epsilon_x S_e$ for some central element $\epsilon_x \in S_e$ and

$$US_xS_{x^{-1}}V = \{0\} \iff U\epsilon_x S_e V = \{0\} \iff \epsilon_x U S_e V = \{0\} \implies \epsilon_x UV = \{0\}.$$

Now, note that, using Lemma E.2.23, we get

$$\{0\} = r.\operatorname{Ann}_S(S_{x^{-1}}) = r.\operatorname{Ann}_S(S_xS_{x^{-1}}) = r.\operatorname{Ann}_S(\epsilon_x S_e) \supseteq r.\operatorname{Ann}_S(\epsilon_x).$$

Hence, $UV = \{0\}$ whenever $US_xS_{x^{-1}}V = \{0\}$. Thus, $S_xS_{x^{-1}}$ is middle cancellable for every $x \in G$. In other words, (ii) implies (i).

(c)⇒(a): Suppose that the grading on S is not strong. There is some $x \in G$ such that $\epsilon_x \neq 1_S$. Put $U = V := \{1_S - \epsilon_x\}$ and note that $1 - \epsilon_x$ is an idempotent. Clearly, $UV = \{1_S - \epsilon_x\} \neq \{0\}$, since $\epsilon_x \neq 1_S$. However, we also have that $US_xS_{x^{-1}}V = U\epsilon_x S_e V = \{0\}$ which shows that $S_xS_{x^{-1}}$ is not a middle cancellable ideal of S_e. By [35, Lem. 1.2] (see also the above proof of (b)⇒(c)), the grading is not cancellative. □

Proposition E.2.25. *If S is unital and cancellatively G-graded, then* $\operatorname{Supp}(S) = G$.

Proof. Take $x \in G$. Since S is unital, we get that $S_e S_{xx^{-1}} S_e = S_e S_e S_e = S_e \neq \{0\}$. Thus, by cancellativity, we get $S_e S_x S_{x^{-1}} S_e \neq \{0\}$. Hence, $S_x \neq \{0\}$. □

Recall that a unital strongly G-graded ring S also satisfies $\operatorname{Supp}(S) = G$ (see Remark E.2.3). However, $\operatorname{Supp}(S) = G$ need not hold, in general, for nearly epsilon-strongly graded rings.

Remark E.2.26. Proposition E.2.24 demonstrates that epsilon-strongly graded rings which can be reached by Passman's "cancellative results" [35] are, in fact, unital strongly graded. Thus, that case has already been treated by Passman in [36].

E.3. Invariant ideals

Recall that S is a G-graded ring. If S is strongly G-graded, then there is an action of G on the lattice of ideals of S_N for any normal subgroup N of G (see [36, Sec. 5.2]). The purpose of this section is to investigate this construction for more general classes of G-graded rings.

Definition E.3.1. If I is a subset of S and $x \in G$, then we define $I^x := S_{x^{-1}} I S_x$.

Lemma E.3.2. *If $x \in G$ and I is an ideal of S_e, then I^x is an ideal of S_e.*

Proof. Clearly, I^x is an additive subgroup of S_e. Since $S_{x^{-1}}$ and S_x are S_e-bimodules, it follows that $S_e I^x = S_e S_{x^{-1}} I S_x \subseteq S_{x^{-1}} I S_x = I^x$. Similarly, we have $I^x S_e \subseteq I^x$. □

Recall that if H, K are subsets of G, then K is said to be *normalized by H* if $Kx = xK$ for every $x \in H$.

Definition E.3.3 (cf. [35, p. 406])**.** Suppose that H is a subgroup of G and that I is a subset of S. Then I is called *H-invariant* if $I^x \subseteq I$ for every $x \in H$. Furthermore, if K is a subset of G which is normalized by H, then we say that I is *H/K-invariant* if $S_{x^{-1}K} I S_{xK} \subseteq I$ for every $x \in H$.

In the special case of s-unital (and in particular unital) strongly G-graded rings, our definition coincides with Passman's notion of invariance used in [36]:

Lemma E.3.4. *Suppose that H is a subgroup of G and that S is s-unital strongly G-graded. Then a subset I of S is H-invariant if and only if $I^x = I$ for every $x \in H$.*

Proof. Suppose that I is H-invariant. Take $x \in H$. By Lemma E.2.16(b) we have

$$I \subseteq S_e I S_e = (S_{x^{-1}} S_x) I (S_{x^{-1}} S_x) = S_{x^{-1}} (S_x I S_{x^{-1}}) S_x \subseteq S_{x^{-1}} I S_x = I^x \subseteq I.$$

This shows that $I^x = I$. The converse statement is trivial. □

Example E.3.5. Let us again look at Example E.2.9. Let J, J' be nonzero R-ideals and consider the following ideals of $(M_2(R))_0$:

$$I = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \qquad \text{and} \qquad I' = \begin{pmatrix} J & 0 \\ 0 & J' \end{pmatrix}.$$

It is easily checked that I is $\mathbb{Z}$-invariant but $I^x = I$ does not hold for every $x \in \mathbb{Z}$. Moreover, if $J \not\subseteq J'$, then a quick verification shows that I' is not $\mathbb{Z}$-invariant. However, I' is invariant with respect to any proper non-trivial subgroup of $\mathbb{Z}$.

More examples of invariant ideals may be found in Example E.13.11 and Example E.14.3. The following result is essential and will often be used implicitly in the rest of this article:

Lemma E.3.6 (cf. [36, Lem. 5.7])**.** *Suppose that I and J are subsets of S. Then the following assertions hold for all $x, y \in G$:*

(a) $(I^x)^y \subseteq I^{xy}$

(b) $I^x J^x \subseteq (IJ)^x$ *if I or J is an ideal of S_e.*

(c) *If $I \subseteq J$, then $I^x \subseteq J^x$.*

PROOF. (a): $(I^x)^y = S_{y^{-1}}(S_{x^{-1}}IS_x)S_y = (S_{y^{-1}}S_{x^{-1}})I(S_xS_y) \subseteq S_{(xy)^{-1}}IS_{xy} = I^{xy}$.

(b): $I^xJ^x = S_{x^{-1}}I(S_xS_{x^{-1}})JS_x \subseteq S_{x^{-1}}IS_eJS_x \subseteq S_{x^{-1}}IJS_x = (IJ)^x$.

(c): $I^x = S_{x^{-1}}IS_x \subseteq S_{x^{-1}}JS_x = J^x$. □

For unital strongly G-graded rings, the inclusions in (a) and (b) of Lemma E.3.6 are, in fact, equalities (see [36, Lem. 5.7]). However, Example E.3.15 below shows that the inclusion in Lemma E.3.6(a) can be strict for some nearly epsilon-strongly graded rings. We now prove that the inclusion in Lemma E.3.6(b) is actually an equality for nearly epsilon-strongly graded rings.

Definition E.3.7. If I is a subset of S, then we say that I is *ϵ-invariant* if for every $x \in G$, the equality $S_xS_{x^{-1}}I = IS_xS_{x^{-1}}$ holds.

Remark E.3.8. If S is epsilon-strongly G-graded, H is a subgroup of G and I is an ideal of S_H, then the statement

$$S_xS_{x^{-1}}I = IS_xS_{x^{-1}}, \qquad \forall x \in G \tag{38}$$

is equivalent to the statement

$$\epsilon_x I = I\epsilon_x, \qquad \forall x \in G. \tag{39}$$

Note that if $H = \{e\}$, then (39) (and hence also (38)) is true since the elements ϵ_x, for $x \in G$, are central idempotents in S_e (see Remark E.2.22). This justifies our usage of the term "ϵ-invariant".

Lemma E.3.9. *If S is nearly epsilon-strongly G-graded, then every ideal of S_e is ϵ-invariant.*

PROOF. Take $x \in G$ and let I be an ideal of S_e. We prove that $S_xS_{x^{-1}}I \subseteq IS_xS_{x^{-1}}$. The reversed inclusion can be shown in an analogous fashion and is therefore left to the reader. Take $s_x \in S_x$, $s_{x^{-1}} \in S_{x^{-1}}$ and $a \in I$. Since $s_{x^{-1}}a \in S_{x^{-1}}$, there is $\epsilon'_{x^{-1}}(s_{x^{-1}}a) \in S_xS_{x^{-1}}$ such that $s_{x^{-1}}a = s_{x^{-1}}a \cdot \epsilon'_{x^{-1}}(s_{x^{-1}}a)$. Using that $s_xs_{x^{-1}}a \subseteq I$, it follows that $s_xs_{x^{-1}}a = (s_xs_{x^{-1}}a) \cdot \epsilon'_{x^{-1}}(s_{x^{-1}}a) \in IS_xS_{x^{-1}}$. □

Proposition E.3.10. *Suppose that S is symmetrically G-graded, N is a normal subgroup of G, and that I, J are ideals of S_N. If I or J is ϵ-invariant, then $(IJ)^x = I^xJ^x$ for every $x \in G$.*

PROOF. Suppose that I is ϵ-invariant. Then, since S is symmetrically G-graded, we get

$$I^xJ^x = S_{x^{-1}}IS_xS_{x^{-1}}JS_x = S_{x^{-1}}S_xS_{x^{-1}}IJS_x = S_{x^{-1}}IJS_x = (IJ)^x$$

for every $x \in G$. The case when J is ϵ-invariant can be treated analogously. □

Combining Lemma E.3.9 and Proposition E.3.10 we obtain the following result:

Corollary E.3.11. *If S is nearly epsilon-strongly G-graded and I, J are ideals of S_e, then $(IJ)^x = I^xJ^x$ for every $x \in G$.*

Proposition E.3.12. *If S is nearly epsilon-strongly G-graded and I is an ideal of S_e, then the following assertions hold:*
(a) *$IS_y = S_y I^y$ and $S_{y^{-1}} I = I^y S_{y^{-1}}$ for every $y \in G$.*
(b) *If H is a subgroup of G and I is H-invariant, then $IS_y = S_y I$ for every $y \in H$.*

Proof. (a): Take $y \in G$. Since S is symmetrically G-graded and I is ϵ-invariant by Lemma E.3.9, we have $IS_y = I(S_y S_{y^{-1}} S_y) = I(S_y S_{y^{-1}})S_y = (S_y S_{y^{-1}})IS_y = S_y(S_{y^{-1}} I S_y) = S_y I^y$. Similarly, we get that $S_{y^{-1}} I = (S_{y^{-1}} S_y S_{y^{-1}})I = S_{y^{-1}}(S_y S_{y^{-1}})I = S_{y^{-1}} I(S_y S_{y^{-1}}) = (S_{y^{-1}} I S_y)S_{y^{-1}} = I^y S_{y^{-1}}$.

(b): Take $y \in H$. By (a), we get $IS_y = S_y I^y \subseteq S_y I = I^{y^{-1}} S_y \subseteq IS_y$. Thus, $S_y I = IS_y$. □

In the following lemma we use the induced quotient grading described in Section E.2.5.

Lemma E.3.13. *Suppose that N is a normal subgroup of G. If I is a G/N-invariant subset of S_N, then I is G-invariant.*

Proof. Suppose that I is G/N-invariant. Take $x \in G$. Then, we have that $S_{x^{-1}} I S_x \subseteq S_{x^{-1}N} I S_{xN} \subseteq I$. □

Remark E.3.14. In Passman's original setting of unital strongly G-graded rings an important property that is repeatedly used is that, for $y \in G$, $S_y I = IS_y$ if and only if $I^y = I$ for any ideal I of S_H, where H is a subgroup of G. In our generalized setting, we will have to make do with the result in Proposition E.3.12 which only holds for ideals of the principal component.

The identity $(I^x)^y = I^{xy}$, for all $x, y \in G$, does not hold in general when working with nearly epsilon-strongly G-graded rings. Before giving an example for which this identity fails, note that if $x \notin \mathrm{Supp}(S)$, then $I^x = \{0\}$ for every ideal I of S_e.

Example E.3.15. Let R be an s-unital ring and let G be a non-trivial group. Consider the nearly epsilon-strong G-graded ring S defined by $S_e := R$ and $S_x := \{0\}$ for $x \in G \setminus \{e\}$. Now, consider the nonzero ideal R of R and let $x \in G \setminus \{e\}$. Then $\{0\} \neq R = R^{xx^{-1}} \neq (R^x)^{x^{-1}} = \{0\}$, because $x \notin \mathrm{Supp}(S)$.

Lemma E.3.16. *Suppose that S is nearly epsilon-strongly G-graded, K is a subgroup of G and that I and J are ideals of S_e. Then the following assertions hold:*
(a) *If I, J are K-invariant, then IJ is K-invariant.*
(b) *If I is K-invariant, then $r.\mathrm{Ann}_{S_e}(I)$ is K-invariant.*

Proof. (a): This follows from Corollary E.3.11.

(b): Take $x \in G$. From Proposition E.3.12, it follows that

$$I \cdot S_{x^{-1}}(r.\mathrm{Ann}_{S_e}(I))S_x \subseteq S_{x^{-1}} I(r.\mathrm{Ann}_{S_e}(I))S_x = S_{x^{-1}}(I \cdot r.\mathrm{Ann}_{S_e}(I))S_x = \{0\}. \quad \square$$

Lemma E.3.17. *If $x \in G$ and F is a family of subsets of S, then $(\sum_{I \in F} I)^x = \sum_{I \in F} I^x$.*

Proof. $(\sum_{I \in F} I)^x = S_{x^{-1}}(\sum_{I \in F} I)S_x = \sum_{I \in F} S_{x^{-1}} I S_x = \sum_{I \in F} I^x$. □

Definition E.3.18. For $H \subseteq G$ and $M \subseteq S$ we define $M^H := \sum_{h \in H} S_{h^{-1}} M S_h$.

Lemma E.3.19. *With the above notation the following assertions hold:*

(a) *If H is a subgroup of G and $M \subseteq S$, then M^H is an H-invariant subset of S.*
(b) *If S_e is s-unital and I is an ideal of S_e, then I^G is the smallest G-invariant ideal of S_e containing I.*

Proof. (a): Take $x \in H$. Combining Lemma E.3.6(a) and Lemma E.3.17, we deduce that $(M^H)^x = \left(\sum_{y \in H} M^y\right)^x = \sum_{y \in H} (M^y)^x \subseteq \sum_{y \in H} M^{yx} = M^H$.

(b): From (a), it follows that I^G is G-invariant. Clearly, I^G is an ideal of S_e and $I = I^e \subseteq I^G$ by s-unitality of S_e. Suppose now that J is a G-invariant S_e-ideal such that $I \subseteq J$. Then, by Lemma E.3.6(c), $I^x \subseteq J^x \subseteq J$ for every $x \in G$ and hence we get $I^G = \sum_{x \in G} I^x \subseteq J$. □

Lemma E.3.20. *The following assertions hold:*

(a) *Suppose that S is non-degenerately G-graded. Let I be a subset of S_e and let $x \in \mathrm{Supp}(S)$ be such that $I(S_x S_{x^{-1}}) = I$ or $(S_x S_{x^{-1}})I = I$. If $I \neq \{0\}$, then $I^x \neq \{0\}$.*
(b) *Suppose that S is symmetrically G-graded. Then for every S_e-ideal I and every $x \in G$, we have $I^x(S_{x^{-1}} S_x) = I^x$.*

Proof. (a): Suppose that $I^x = \{0\}$. Since S is non-degenerately G-graded, we have $IS_x = \{0\}$ or $S_{x^{-1}} I = \{0\}$. Hence, $\{0\} = IS_x S_{x^{-1}} = I$ or $\{0\} = S_x S_{x^{-1}} I = I$.

(b): For every $x \in G$, we get $I^x(S_{x^{-1}} S_x) = S_{x^{-1}} I S_x (S_{x^{-1}} S_x) = S_{x^{-1}} I S_x = I^x$. □

Later on, we need to consider ideals I satisfying $I^x I = \{0\}$ for every $x \in G \setminus H$ for some subgroup H of G. The following result will allow us to replace I with I^H.

Proposition E.3.21 (cf. [36, Lem. 5.5])**.** *Suppose that S is nearly epsilon-strongly G-graded and that H is a subgroup of G. Let I be an ideal of S_e such that $I^x I = \{0\}$ for every $x \in G \setminus H$. Then $(I^H)^x (I^H) = \{0\}$ for every $x \in G \setminus H$.*

Proof. Take $x \in G$ such that $(I^H)^x I^H \neq \{0\}$. There exist $h_1, h_2 \in H$ such that $\{0\} \neq (I^{h_1})^x I^{h_2} \subseteq I^{h_1 x} I^{h_2}$, by Lemma E.3.6(a). By Lemma E.3.20(b), we have $I^{h_1 x} \cdot (I^{h_2}(S_{h_2^{-1}} S_{h_2})) = I^{h_1 x} \cdot (I^{h_2}) = I^{h_1 x} I^{h_2}$. Hence, Lemma E.3.20(a) applies to the S_e-ideal $I^{h_1 x} I^{h_2}$. Thus, $\{0\} \neq (I^{h_1 x} I^{h_2})^{h_2{}^{-1}} \subseteq I^{h_1 x h_2{}^{-1}} I$. By assumption, $h_1 x h_2{}^{-1} \in H$ and hence $x \in H$. □

Lemma E.3.22. *Suppose that S is nearly epsilon-strongly G-graded and that S_e is G-semiprime. Furthermore, let H be a subgroup of G and let I be an H-invariant ideal of S_e such that $I^x I = \{0\}$ for every $x \in G \setminus H$. Then the following assertions hold:*

(a) *The ideal I does not contain any nonzero nilpotent H-invariant ideal.*
(b) *Let W be a subgroup of H of finite index. Then I does not contain any nonzero nilpotent W-invariant ideal.*

Proof. (a): Seeking a contradiction, suppose that J is a nonzero H-invariant ideal of S_e such that $J^2 = \{0\}$ and $J \subseteq I$. First we show that $J^x J = \{0\}$ for every $x \in G$. Indeed, for $x \in H$ we have $J^x J \subseteq J^2 = \{0\}$ while for $x \in G \setminus H$ we have $J^x J \subseteq I^x I = \{0\}$ by Lemma E.3.6(c). Next, note that $J^G = \sum_{x \in G} J^x$ is a nonzero G-invariant ideal of S_e by Lemma E.3.19. We claim that $J^G J^G = \{0\}$. If we assume that the claim holds, then we get the desired contradiction, since S_e is assumed to be G-semiprime. Now, we prove the claim. Seeking a contradiction, suppose that $J^G J^G \neq \{0\}$. Then $J^G J^G = \left(\sum_{x\in G} J^x\right)\left(\sum_{y\in G} J^y\right) = \sum_{x,y\in G} J^x J^y \neq \{0\}$. Hence there are $x, y \in G$ such that $J^x J^y \neq \{0\}$. By Lemma E.3.20(b), we have $J^x J^y(S_{y^{-1}}S_y) = J^x J^y$, and therefore Lemma E.3.20(a) implies that $\{0\} \neq (J^x J^y)^{y^{-1}}$. Moreover, by Corollary E.3.11, we have $\{0\} \neq (J^x J^y)^{y^{-1}} = (J^x)^{y^{-1}}(J^y)^{y^{-1}} \subseteq J^{xy^{-1}} J = \{0\}$, which is a contradiction.

(b): Seeking a contradiction, suppose that $J \subseteq I$ is a nonzero W-invariant ideal of S_e such that $J^2 = \{0\}$. Let $Wx_1, Wx_2, \ldots, Wx_n$ be a set of representatives of the right cosets of W in H and, for every $i \in \{1,\ldots,n\}$, let $J^{Wx_i} := \sum_{y\in W} J^{yx_i}$. We wish to prove that $J' := J^{Wx_1} + J^{Wx_2} + \ldots + J^{Wx_n}$ is a nonzero H-invariant nilpotent ideal contained in I.

To begin with, note that for all $y_1, y_2 \in W$ and $i \in \{1,\ldots,n\}$ we have

$$J^{y_1x_i}J^{y_2x_i} = S_{(y_1x_i)^{-1}}JS_{y_1x_i}S_{(y_2x_i)^{-1}}JS_{y_2x_i} \subseteq S_{(y_1x_i)^{-1}}JS_{y_1y_2^{-1}}JS_{y_2x_i}.$$

Using that $y_1y_2^{-1} \in W$ and that J is W-invariant, Proposition E.3.12(b) yields $JS_{y_1y_2^{-1}}J = S_{y_1y_2^{-1}}JJ = \{0\}$. Hence, $J^{y_1x_i}J^{y_2x_i} = \{0\}$, and therefore it follows that

$$\left(J^{Wx_i}\right)^2 = \left(\sum_{y_1\in W} J^{y_1x_i}\right)\left(\sum_{y_2\in W} J^{y_2x_i}\right) = \sum_{y_1,y_2\in W} J^{y_1x_i}J^{y_2x_i} = \{0\}.$$

In other words, J^{Wx_i} is a nilpotent ideal for every $i \in \{1,\ldots,n\}$. Since J' is a finite sum of nilpotent ideals, we conclude that J' is also a nilpotent ideal.

Next, we prove that J' is H-invariant. For this we repeatedly use Lemma E.3.6. Note that for all $i \in \{1,\ldots,n\}$ and $y \in H$, we have $(J^{Wx_i})^y \subseteq J^{Wx_iy} = J^{Wx_j}$ for some $j \in \{1,\ldots,n\}$ with $Wx_iy = Wx_j$. Now, by Lemma E.3.17, $(J')^y = (J^{Wx_1})^y + \ldots + (J^{Wx_n})^y \subseteq J'$ and hence J' is H-invariant. Finally, we show that $J' \subseteq I$. Note that $J \subseteq I$ implies $J^{yx_i} \subseteq I^{yx_i}$ for every $y \in W$. In addition, we have $I^{yx_i} \subseteq I$, since I is H-invariant. It follows that $J^{Wx_i} \subseteq I$ for every $i \in \{1,\ldots,n\}$, which gives the inclusion $J' \subseteq I$.

Summarizing, we have established that J' is indeed an H-invariant nilpotent ideal contained in I, but by virtue of (a) we must have $J' = \{0\}$. However, writing Wx_j for the right coset containing e, we get $\{0\} \neq J = J^e \subseteq J^{Wx_j} \subseteq J'$. This contradiction proves the assertion. □

E.4. Graded prime ideals

Recall that S is a G-graded ring. In this section, we obtain a correspondence between graded prime ideals of S and G-prime ideals of S_e, in the case when S is nearly epsilon-strongly G-graded. Using that correspondence, we establish a primeness result in the case when G is ordered (see Corollary E.4.14). That result will be generalized in Section E.10, using more elaborate methods. We wish to emphasize that the rest of this article does not depend on the results of this section.

Definition E.4.1. An ideal I of S is called *graded* if $I = \bigoplus_{x\in G}(I \cap S_x)$.

Example E.4.2. This example illustrates that a graded ring may have infinitely many ideals but only trivial graded ideals. Indeed, consider the complex Laurent polynomial ring equipped with the standard $\mathbb{Z}$-grading, that is, $\mathbb{C}[t,t^{-1}] = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}t^i$. This is clearly a strong $\mathbb{Z}$-grading and hence also nearly epsilon-strong. Every point of the circle gives rise to a maximal ideal of $\mathbb{C}[t,t^{-1}]$. On the other hand, the only graded ideals are $\{0\}$ and $\mathbb{C}[t,t^{-1}]$.

Let I be an ideal of S. Then $I_e := I \cap S_e$ is an S_e-ideal. Conversely, if J is an S_e-ideal, then SJS is a graded ideal of S. For strongly graded rings we have the following bijection:

Proposition E.4.3 ([28, Prop. 2.11.7])**.** *If S is unital strongly G-graded, then the map $I \mapsto I_e$ is a bijection between the set of graded ideals of S and the set of G-invariant ideals of S_e.*

We now generalize Proposition E.4.3 to nearly epsilon-strongly graded rings (see Theorem E.4.7). To this end, we need three lemmas.

Lemma E.4.4 (cf. [36, Expl. 2.7.3])**.** *If S_e is s-unital and I is an ideal of S_e, then I is G-invariant if and only if $(SIS)_e = I$.*

Proof. Suppose that $I^x = S_{x^{-1}} I S_x \subseteq I$ for every $x \in G$. Then $(SIS)_e = SIS \cap S_e \subseteq I$. The reversed inclusion follows since S_e is s-unital. Conversely, suppose that $(SIS)_e = I$. Then $S_{x^{-1}} I S_x \subseteq SIS \cap S_e = (SIS)_e = I$ for every $x \in G$. Thus, I is G-invariant. $\square$

Lemma E.4.5. *If I is a graded ideal of S, then I_e is a G-invariant ideal of S_e.*

Proof. Take $x \in G$. Then $S_{x^{-1}} I_e S_x \subseteq (S_{x^{-1}} I S_x) \cap (S_{x^{-1}} S_e S_x) \subseteq I \cap S_e = I_e$. $\square$

Lemma E.4.6 (cf. [18, Prop. 1.1.34])**.** *Suppose that S is nearly epsilon-strongly G-graded. If I is a graded ideal of S, then $SI_eS = SI_e = I_eS = I$.*

Proof. Using that S is s-unital, we get $I_e \subseteq I_eS$ and $I_e \subseteq SI_e$. Hence, $SI_e \subseteq SI_eS \subseteq I$ and similarly $I_eS \subseteq SI_eS \subseteq I$. Next, we prove that $I \subseteq SI_e$. Since I is graded, it is enough to show that $a_x \in SI_e$ for every homogeneous $a_x \in I \cap S_x$. Indeed, since S is nearly epsilon-strongly G-graded, we have $a_x = \epsilon_x(a_x) \cdot a_x$ for some $\epsilon_x(a_x) \in S_x S_{x^{-1}}$. Write $\epsilon_x(a_x) = \sum_i c_i b_i$ for finitely many $c_i \in S_x$ and $b_i \in S_{x^{-1}}$. Then $a_x = \sum_i c_i b_i a_x$. Note that for any i we have $b_i a_x \in S_{x^{-1}} S_x \subseteq S_e$ and $b_i a_x \in I$ thus yielding $a_x \in I \cap S_e = I_e$. Hence, $a_x = \sum_i c_i b_i a_x \in S_x I_e \subseteq SI_e$. By an analogous argument the inclusion $I \subseteq I_eS$ follows. We conclude that $I = SI_e = I_eS$. Consequently, $I = SI_e \subseteq SI_eS \subseteq I$. $\square$

Theorem E.4.7. *Suppose that S is nearly epsilon-strongly G-graded. The map $I \mapsto I_e$ is a bijection between the sets $\{$graded ideals of $S\}$ and $\{G$-invariant ideals of $S_e\}$. The inverse map is given by $J \mapsto SJS$.*

Proof. Let I be a graded ideal. By Lemma E.4.5, I_e is a G-invariant ideal of S_e. In other words, the map $I \mapsto I_e$ is well-defined. Furthermore, by Lemma E.4.6, we have $SI_eS = I$ establishing that $I \mapsto I_e$ is injective. Next, suppose that J is a G-invariant ideal of S_e. By Lemma E.4.4, $(SJS)_e = J$ proving that $I \mapsto I_e$ is surjective. $\square$

Later on we will apply Theorem E.4.7 to Leavitt path algebras (see Section E.14).

Definition E.4.8. A proper graded ideal P of S is called *graded prime* if for all graded ideals A, B of S, we have $A \subseteq P$ or $B \subseteq P$ whenever $AB \subseteq P$. A proper G-invariant ideal Q of S_e is called *G-prime* if for all G-invariant ideals A, B of S_e, we have $A \subseteq Q$ or $B \subseteq Q$ whenever $AB \subseteq Q$. The ring S_e is called *G-prime* if $\{0\}$ is a G-prime ideal of S_e.

For unital strongly G-graded rings, the bijection $I \mapsto I_e$ from Theorem E.4.7 restricts to a bijection between graded prime ideals of S and G-prime ideals of S_e (see [28, Prop. 2.11.7]). We proceed to show that the same holds for nearly epsilon-strongly G-graded rings.

Lemma E.4.9. *Suppose that S is nearly epsilon-strongly G-graded. If I is a graded ideal of S such that I_e is a G-prime ideal of S_e, then I is graded prime.*

PROOF. Suppose that A, B are graded ideals of S such that $AB \subseteq I$. Then $A_e B_e \subseteq AB \cap S_e \subseteq I \cap S_e = I_e$. By Theorem E.4.7, the S_e-ideals A_e, B_e are G-invariant. Since I_e is G-prime, we have $A_e \subseteq I_e$ or $B_e \subseteq I_e$. Assume w.l.o.g. that $A_e \subseteq I_e$. Then $A = SA_eS \subseteq SI_eS = I$ by Theorem E.4.7. Thus, I is a graded prime ideal of S. □

Lemma E.4.10. *Suppose that S is nearly epsilon-strongly G-graded. If I is a graded prime ideal of S, then I_e is a G-prime ideal of S_e.*

PROOF. Clearly, I_e is an ideal of S_e. Suppose that A, B are G-invariant ideals of S_e such that $AB \subseteq I_e$. We need to show that $A \subseteq I_e$ or $B \subseteq I_e$. By Theorem E.4.7, $A = (SAS)_e$, $B = (SBS)_e$ and $S(AB)S \subseteq SI_eS = I$. Clearly, SAS and SBS are graded ideals of S. By Lemma E.4.6, $SASSBS = S(AB)S \subseteq I$. Since I is graded prime, we have $SAS \subseteq I$ or $SBS \subseteq I$. Assume w.l.o.g. that $SAS \subseteq I$. Then $A \subseteq I_e$ and thus I_e is G-prime. □

By combining Lemma E.4.9 and Lemma E.4.10 we get the desired bijection:

Theorem E.4.11. *Suppose that S is nearly epsilon-strongly G-graded. The map $I \mapsto I_e$ restricts to a bijection between the sets $\{$graded prime ideals of $S\}$ and $\{G$-prime ideals of $S_e\}$.*

We now generalize a well-known result by Năstăsescu and Van Oystaeyen to the setting of s-unital group graded rings:

Proposition E.4.12 (cf. [29, Prop. II.1.4]). *Suppose that G is an ordered group and that S is s-unital. If I is a graded ideal of S, then I is graded prime if and only if I is prime.*

PROOF. Suppose that I is graded prime. For every $k \geq 0$, let $P(k)$ be the following statement:

$$a, b \in S \text{ satisfy } aSb \subseteq I \text{ and } |\operatorname{Supp}(a)| + |\operatorname{Supp}(b)| \leq k \Longrightarrow a \in I \text{ or } b \in I.$$

We proceed by induction to show that $P(k)$ holds for every $k \geq 0$.

Base case: $k = 0$. If $|\operatorname{Supp}(a)| + |\operatorname{Supp}(b)| = 0$, then $a = b = 0 \in I$.

Inductive step: Take $k \geq 0$ such that $P(k)$ holds. Suppose that $aSb \subseteq I$ and $|\operatorname{Supp}(a)| + |\operatorname{Supp}(b)| = k+1$. Put $m := |\operatorname{Supp}(a)|$ and $n := |\operatorname{Supp}(b)|$. Then we can write $a = \sum_{i=1}^{m} a_{x_i}$ and $b = \sum_{j=1}^{n} b_{y_j}$ where $x_1, \dots, x_m \in G$ and $y_1, \dots, y_n \in G$ satisfy $x_1 < \cdots < x_m$ and $y_1 < \cdots < y_n$. Take $z \in G$. For any $s_z \in S_z$ we have $as_z b \in I$. Using that G is an ordered group and that I is a graded ideal, we get $a_{x_m} s_z b_{y_n} \in I$. This shows that $a_{x_m} S b_{y_n} \subseteq I$. By graded primeness of I, and s-unitality of S, we get $a_{x_m} \in S a_{x_m} S \subseteq I$ or $b_{y_n} \in S b_{y_n} S \subseteq I$.

Case 1: $a_{x_m} \in I$. Put $a' = a - a_{x_m}$. Then $a'Sb = aSb - a_{x_m}Sb \subseteq I - I = I$. Since $|\operatorname{Supp}(a')| + |\operatorname{Supp}(b)| < k+1$, the induction hypothesis yields that $a' \in I$ or $b \in I$, and hence that $a = a' + a_{x_m} \in I$ or $b \in I$.

Case 2: $b_{y_n} \in I$. Put $b' = b - b_{y_n}$. Then $aSb' = aSb - aSb_{y_n} \subseteq I - I = I$. Since $|\operatorname{Supp}(a)| + |\operatorname{Supp}(b')| < k+1$, the induction hypothesis yields that $a \in I$ or $b' \in I$, and hence that $a \in I$ or $b = b' + b_{y_n} \in I$.

Therefore, $P(k+1)$ holds.

Now, let A, B be nonzero ideals of S with $AB \subseteq I$. Seeking a contradiction, suppose that there are $a \in A \setminus I$ and $b \in B \setminus I$. Since A and B are ideals, it follows that $aSb \subseteq AB \subseteq I$. Since $P(k)$ holds for every $k \geq 0$, we get that $a \in I$ or $b \in I$, which is a contradiction.

The converse statement is trivial. □

By Proposition E.4.12, we immediately obtain the following partial generalization of a result by Abrams and Haefner [3, Thm. 3.2]:

Corollary E.4.13. *Suppose that G is an ordered group and that S is s-unital. Then S is graded prime if and only if S is prime.*

Combining the above result with Theorem E.4.11, we immediately get the following:

Corollary E.4.14. *Suppose that G is an ordered group and that S is nearly epsilon-strongly G-graded. Then S is prime if and only if S_e is G-prime.*

Example E.4.15. Let R be a unital ring.

(a) Consider the Laurent polynomial ring $R[t, t^{-1}] = \bigoplus_{i \in \mathbb{Z}} Rt^i$ equipped with its canonical strong $\mathbb{Z}$-grading. Since t is central in $R[t, t^{-1}]$, any ideal I of R satisfies $t^{-n} I t^n = I$. Thus, every ideal of R is $\mathbb{Z}$-invariant. Hence, R being $\mathbb{Z}$-prime is equivalent to R being prime. Therefore, Corollary E.4.14 implies that $R[t, t^{-1}]$ is prime if and only if R is prime.

(b) More generally, let G be an ordered group and consider the group ring $R[G]$. Note that for any ideal I of $R = (R[G])_e$ we have $\delta_{x^{-1}} I \delta_x = \delta_{x^{-1}} \delta_x I = I$ for every $x \in G$. Thus, every ideal of R is G-invariant. By Corollary E.4.14, it follows that $R[G]$ is prime if and only if R is prime (see e. g. [21, Thm. 6.29]).

E.5. The "easy" direction

Recall that S is a G-graded ring. In this section, we prove the implication (b)⇒(a) of Theorem E.1.3 for non-degenerately G-graded rings (see Proposition E.5.3).

Lemma E.5.1 (cf. [36, Lem. 1.4]). *Suppose that S is non-degenerately G-graded, that H is a subgroup of G, and that I is an ideal of S_e which satisfies $I^x I = \{0\}$ for every $x \in G \setminus H$. Then the following two assertions hold:*

(a) $IS_xI = \{0\}$ *for every* $x \in G \setminus H$.
(b) $ISI \subseteq IS_H \subseteq S_H$.

PROOF. (a): Take $x \in G \setminus H$ and $s \in IS_xI$. By assumption, $S_{x^{-1}}IS_xI = I^xI = \{0\}$ and hence $S_{x^{-1}}s = \{0\}$. Using that S is non-degenerately G-graded, we get that $s = 0$.

(b): Employing part (a), we get $ISI = \bigoplus_{x\in G} IS_xI = \bigoplus_{x\in H} IS_xI \subseteq S_H$. ☐

Lemma E.5.2. *Suppose that S is non-degenerately G-graded and that N is a subgroup of G. If $\tilde{A}$ is a nonzero subset of S_N, then $S\tilde{A}S$ is a nonzero ideal of S.*

PROOF. Clearly, $S\tilde{A}S$ is an ideal of S. Choose a nonzero $a \in \tilde{A}$. Let $n \in \mathrm{Supp}(a) \subseteq N$. By non-degeneracy of the G-grading, there is some $s_{n^{-1}} \in S_{n^{-1}}$ and some $t_e \in S_e$ such that $t_e a_n s_{n^{-1}} \neq 0$. Therefore, $t_e a s_{n^{-1}} \in S\tilde{A}S \setminus \{0\}$. This shows that $S\tilde{A}S$ is nonzero. ☐

Proposition E.5.3 (cf. [36, Thm. 1.3]). *Suppose that S is non-degenerately G-graded and that there exist*

(i) *subgroups $N \lhd H \subseteq G$,*
(ii) *an H-invariant ideal I of S_e such that $I^xI = \{0\}$ for every $x \in G \setminus H$, and*
(iii) *nonzero ideals $\tilde{A}, \tilde{B}$ of S_N such that $\tilde{A}, \tilde{B} \subseteq IS_N$, and $\tilde{A}S_H\tilde{B} = \{0\}$.*

Then S is not prime.

PROOF. If $x \in H$, then the second condition in (iii) implies that $\tilde{A}S_x\tilde{B} = \{0\}$.

If $x \in G \setminus H$, then the first condition in (iii) implies that $\tilde{A}S_x\tilde{B} \subseteq (IS_N)S_x(IS_N)$. Since $S_NS_x = \bigoplus_{n\in N} S_nS_x \subseteq \bigoplus_{n\in N} S_{nx}$ and $nx \in G \setminus H$, it follows from Lemma E.5.1 that

$$IS_NS_xI \subseteq \bigoplus_{n\in N} IS_{nx}I = \{0\}.$$

Hence, $\tilde{A}S_x\tilde{B} = \{0\}$ for every $x \in G$, and thus $\tilde{A}S\tilde{B} = \{0\}$. Now, by (iii) and Lemma E.5.2 it follows that $A := S\tilde{A}S$ and $B := S\tilde{B}S$ are nonzero ideals of S satisfying $AB = (S\tilde{A}S)(S\tilde{B}S) \subseteq S(\tilde{A}S\tilde{B})S = \{0\}$. This shows that S is not prime. ☐

Remark E.5.4. Note that N is not required to be finite in Proposition E.5.3.

In an attempt to ease the technical notation, we now introduce the following notion.

Definition E.5.5 (NP-datum). Let S be a G-graded ring. An *NP-datum* for S is a quintuple $(H, N, I, \tilde{A}, \tilde{B})$ with the following three properties:

(NP1) H is a subgroup of G, and N is a finite normal subgroup of H,
(NP2) I is a nonzero H-invariant ideal of S_e such that $I^xI = \{0\}$ for every $x \in G \setminus H$, and
(NP3) $\tilde{A}, \tilde{B}$ are nonzero ideals of S_N such that $\tilde{A}, \tilde{B} \subseteq IS_N$, and $\tilde{A}\tilde{B} = \{0\}$.

An NP-datum $(H, N, I, \tilde{A}, \tilde{B})$ is said to be *balanced* if it satisfies the following property:

(NP4) $\tilde{A}, \tilde{B}$ are nonzero ideals of S_N such that $\tilde{A}, \tilde{B} \subseteq IS_N$, and $\tilde{A}S_H\tilde{B} = \{0\}$.

Remark E.5.6. (a) If S is nearly epsilon-strongly G-graded, then (NP4) implies (NP3).

(b) Suppose that S is s-unital strongly G-graded. An NP-datum $(H, N, I, \tilde{A}, \tilde{B})$ for S is necessarily balanced whenever $\tilde{A}$ or $\tilde{B}$ is H-invariant. Indeed, suppose that $\tilde{A}$ is H-invariant. For any $h \in H$, we get that $\tilde{A}S_h\tilde{B} = S_e\tilde{A}S_h\tilde{B} = S_hS_{h^{-1}}\tilde{A}S_h\tilde{B} \subseteq S_h\tilde{A}\tilde{B} = \{0\}$ by Lemma E.2.16. The proof of the case when $\tilde{B}$ is H-invariant is analogous.

Corollary E.5.7. *Suppose that S is non-degenerately G-graded. If S_e is not G-prime, then S has a balanced NP-datum $(H, N, I, \tilde{A}, \tilde{B})$ for which $\tilde{A}, \tilde{B}$ are H/N-invariant.*

Proof. If S_e is not G-prime, then there are nonzero G-invariant ideals $\tilde{A}, \tilde{B}$ of S_e such that $\tilde{A}\tilde{B} = \{0\}$. We claim that $(G, \{e\}, S_e, \tilde{A}, \tilde{B})$ is a balanced NP-datum. Conditions (NP1), (NP2) and (NP3) are trivially satisfied. We now check condition (NP4). Take $x \in G$. Seeking a contradiction, suppose that $\tilde{A}S_x\tilde{B} \neq \{0\}$. Note that $\tilde{A}S_x\tilde{B} \subseteq S_x$. By non-degeneracy of the G-grading, $S_{x^{-1}} \cdot \tilde{A}S_x\tilde{B} \neq \{0\}$. Since $\tilde{A}$ is G-invariant, we get that $S_{x^{-1}}\tilde{A}S_x\tilde{B} \subseteq \tilde{A}\tilde{B} = \{0\}$, which is a contradiction. Note that, trivially, $\tilde{A}, \tilde{B}$ are both $G/\{e\}$-invariant. $\square$

By combining the above results we get the following.

Corollary E.5.8. *Suppose that S is non-degenerately G-graded. If S is prime, then S_e is G-prime.*

E.6. Passman pairs and the Passman replacement argument

In this section, we generalize a technical result by Passman [36]. Recall that S is a G-graded ring. We are interested in pairs (J, M) where J is a nonzero ideal of S_e and $M \subseteq G$ is a subset such that $J^xJ = \{0\}$ for every $x \in G \setminus M$. Given such a pair (J, M), where M is of a certain type, we will find another pair (K, L) where $K \subseteq J$ is a nonzero ideal of S_e and L is a subgroup of G. Crucially, the new pair (K, L) satisfies $K^xK = \{0\}$ for $x \in G \setminus L$. Passman's original proof relies on S being unital and strongly G-graded, and provides a construction of the ideal K. As we will see, his main argument generalizes to our extended setting, although we do not get an explicit description of the ideals.

Definition E.6.1. If I is a nonzero ideal of S_e and $M \subseteq G$ is such that $I^xI = \{0\}$ for every $x \in G \setminus M$, then we call (I, M) a *Passman pair.*

Proposition E.6.2 (cf. [36, Lem. 2.1])**.** *Suppose that S is nearly epsilon-strongly G-graded and that (J, M) is a Passman pair where $M = \bigcup_{k=1}^n g_kG_k$ for some subgroups $G_1, \ldots, G_n$ of G and $g_1, \ldots, g_n \in G$. Then there exist a nonzero ideal $K \subseteq J$ of S_e and a subgroup L of G such that (K, L) is a Passman pair. In addition, $[L : L \cap G_k] < \infty$ for some $k \in \{1, \ldots, n\}$.*

We now fix a group G and a finite family $\{G_1, \ldots, G_n\}$ of subgroups of G. To establish Proposition E.6.2, we need the following:

Lemma E.6.3. *Suppose that S is nearly epsilon-strongly G-graded. Let $B = \{A_1, \ldots, A_l\}$ be a family of subgroups of G such that for all $i, j \in \{1, \ldots, l\}$ there is some $k \in \{1, \ldots, n\}$ such that $A_j \subseteq G_k$, and $A_i \cap A_j \in B$. Let (J, M) be a Passmain pair. Suppose that $M = \bigcup_{k=1}^t g_kA_{n_k}$ where $n_1, \ldots, n_t \in \{1, \ldots, l\}$ and $g_1, \ldots, g_t \in G$. Then there* *exist a nonzero ideal* $K \subseteq J$ *of* S_e *and a subgroup* L *of* G *such that* (K, L) *is a Passman pair. In addition, if* B *is non-empty then* $[L : L \cap A_j] < \infty$ *for some* $j \in \{1, \dots, l\}$.

PROOF. The proof proceeds by induction over $|B|$. If $|B| = 0$, then the assumption that $(J, \emptyset)$ is a Passman pair implies that $(J, \{e\})$ is a Passman pair. Next, suppose that $|B| \geq 1$. Let A be a maximal element of B ordered by inclusion and note that $B' = B \setminus \{A\}$ is closed under intersections. We consider the following set of Passman pairs:

$$P := \{(K, N) \mid \{0\} \neq K \subseteq J,\ N = \bigcup_{j=1}^{s} g_j A_{k_j} \text{ for } k_1, \dots, k_s \in \{1, \dots, l\},\ g_1, \dots, g_s \in G\}$$

Note that P is non-empty since $(J, M) \in P$. For $(K, N) \in P$ with $N = \cup_{j=1}^{s} g_j A_{k_j}$, we let $\mathrm{Supp}(K, N)$ be the subset $\{A_{k_1}, \dots, A_{k_s}\} \subseteq B$. Let $\deg(K, N)$ be the number of times that $A = A_{k_j}$ in the expression of N.

Now, choose $(K, N) \in P$ of minimal degree. We consider two mutually exclusive cases:

Case 1: $\deg(K, N) = 0$. In this case $\mathrm{Supp}(K, N) \subseteq B'$. Hence, the induction hypothesis applies and we conclude that there exists some Passman pair (I, L) such that $\{0\} \neq I \subseteq K \subseteq J$ and L is a subgroup of G.

Case 2: $\deg(K, N) = m > 0$. Let $N = z_1 A \cup z_2 A \cup \ldots \cup z_m A \cup T$ where T is a finite union of cosets of groups in B'. Put

$$L := \Big\{ g \in G \mid g\Big(\bigcup_{i=1}^{m} z_i A\Big) = \bigcup_{i=1}^{m} z_i A \Big\}.$$

Our goal is to prove that (K, L) is a Passman pair. Note that L is the stabiliser of $\bigcup_{i=1}^{m} z_i A$. Thus, L is in fact a subgroup of G. Take $x \in G$ such that $K^x K \neq \{0\}$. We will show that $x \in L$. Indeed, if $h = x^{-1} h'$ for some $h' \in G \setminus N$, then

$$\begin{aligned}(K^x K)^h (K^x K) &= ((K^x)^h K^h)(K^x K) \subseteq \\ &\subseteq K^{xh} K^h K^x K \subseteq K^{xh} K = K^{x x^{-1} h'} K = K^{h'} K = \{0\}\end{aligned}$$

where we have used Lemma E.3.2, Lemma E.3.6 and Corollary E.3.11. Similarly, if $h \in G \setminus N$, then $(K^x K)^h (K^x K) \subseteq (K^{xh} K^h)(K^x K) \subseteq K^h K = \{0\}$. In other words, $(K^x K)^h (K^x K) = \{0\}$ for every $h \in (G \setminus N) \cup x^{-1}(G \setminus N) = G \setminus (N \cap x^{-1} N)$. Thus, $(K^x K, N \cap x^{-1} N)$ is a Passman pair. Since $K^x K \subseteq K \subseteq J$ and $N \cap x^{-1} N$ is a finite union of cosets in B, it follows that $(K^x K, N \cap x^{-1} N) \in P$. Let $m' := \deg(K^x K, N \cap x^{-1} N)$. By minimality of m, we have $m' \geq m > 0$. Note that $x^{-1} N = x^{-1} z_1 A \cup x^{-1} z_2 A \cup \ldots \cup x^{-1} z_m A \cup x^{-1} T$. Since A is maximal, the m' cosets of A in $N \cap x^{-1} N$ must come from $(\cup_{i=1}^{m} z_i A) \cap (\cup_{i=1}^{m} x^{-1} z_i A)$. Moreover, cosets are either equal or disjoint, and hence $m' \leq m$. This shows that $m' = m$ and $\cup_{i=1}^{m} z_i A = x^{-1}(\cup_{i=1}^{m} z_i A)$ which in turn shows that $x \in L$. Summarizing, we have established that $K^x K = \{0\}$ for every $x \in G \setminus L$, i.e. (K, L) is a Passman pair.

Now, suppose that B is non-empty. It remains to show that $[L : L \cap A_j] < \infty$ for some $j \in \{1, \dots, l\}$. Consider G acting from the left on the left cosets of A, i.e. $G \curvearrowright \{gA \mid g \in G\}$ by $g_1 \cdot g_2 A = g_1 g_2 A$ for all $g_1, g_2 \in G$. Note that L acts on the finite set of cosets $D = \{z_1A, z_2A, \ldots, z_mA\}$. Let $i \in \{1, \ldots, m\}$ be arbitrary. A short computation shows that $\mathrm{Stab}_G(z_iA) = z_iAz_i^{-1}$. Thus, $\mathrm{Stab}_L(z_iA) = z_iAz_i^{-1} \cap L$. Hence, by the orbit-stabilizer theorem we have $|L \cdot z_iA| = [L : L \cap z_iAz_i^{-1}]$. Using that D is a finite set, we conclude that the orbit of z_iA is finite, i. e. $|L \cdot z_iA| < \infty$. Thus, we have $[L : L \cap z_iAz_i^{-1}] < \infty$ for every $i \in \{1, \ldots, m\}$.

We consider two mutually exclusive cases.

Case A: $L \cap z_iA = \emptyset$ for every i. Note that $K^xK = \{0\}$ for every $x \in G \setminus N \cup G \setminus L = G \setminus (N \cap L)$. By the case assumption, we have $N \cap L = T \cap L$. We see that $T \cap L$ is a finite union of cosets from the set $B'' = \{A' \cap T \mid A' \in B'\}$.

Note that $|B''| \leq |B'| < |B|$. By the induction hypothesis, it follows that there is a Passman pair (I, L') satisfying the required properties.

Case B: $L \cap z_iA \neq \emptyset$ for some i. Let $a \in A$ be such that $z_ia \in L \cap z_iA$. Since $(z_ia)A = z_iA$, we may assume that $z_i \in L$ by choosing another representative of the coset. It follows that $L \cap A \cong L \cap z_iAz_i^{-1}$ via the map defined by $a \mapsto z_iaz_i^{-1}$ for every $a \in A \cap L$. As noted above we have $[L : L \cap z_iAz_i^{-1}] < \infty$ and hence $[L : L \cap A] < \infty$. Consequently, $[L : L \cap A_j] < \infty$ with $A_j := A$ as required. ☐

We are now ready to give a proof of Proposition E.6.2:

Proof of Proposition E.6.2. Let (J, M) be a Passman pair such that $M = \bigcup_{k=1}^{n} g_kG_k$ for some subgroups $G_1, G_2, \ldots, G_n$ of G. Furthermore, let B denote the closure of $\{G_1, G_2, \ldots, G_n\}$ with respect to intersections. Then M is a finite union of left cosets of subgroups of B, and we may apply Lemma E.6.3. Hence, there is a Passman pair (K, L) where L is a subgroup of G and $K \subseteq J$ is a nonzero ideal of S_e. In addition, using that B is non-empty, we have $[L : L \cap A_i] < \infty$ for some $A_i \in B$. Since $A_i \subseteq G_k$ for some k, it follows that $L \cap A_i \subseteq L \cap G_k$. Consequently, $[L : L \cap G_k] \leq [L : L \cap A_i] < \infty$. ☐

The following result is a stronger version of Proposition E.6.2:

Proposition E.6.4 (cf. [36, Lem. 2.2]). *Suppose that S is nearly epsilon-strongly G-graded and that W is a subgroup of G of finite index. Let J be a nonzero ideal of S_e such that*

$$J^xJ = \{0\}, \qquad \forall x \in W \setminus \bigcup_{k=1}^{n} w_kH_k$$

where $H_1, \ldots, H_n$ are subgroups of W and $w_1, \ldots, w_n \in W$. Then there is a subgroup L of G and a nonzero ideal $I \subseteq J$ of S_e such that (I, L) is a Passman pair of S. In other words, $I^xI = \{0\}$ for every $x \in G \setminus L$. In addition, $[L : L \cap H_k] < \infty$ for some $k \in \{1, \ldots, n\}$.

Proof. For each positive integer m we let A_m be the set consisting of all tuples $(h_1, h_2, \ldots, h_m) \in G^m$ such that

- $J^{h_1}J^{h_2} \cdots J^{h_m} \neq \{0\}$,
- $e = h_i$ for some $i \in \{1, \ldots, m\}$, and
- $Wh_j = Wh_i$ if and only if $i = j$.

By Proposition E.2.13, S_e is s-unital and hence $J = J^e \neq \{0\}$. This shows that $e \in A_1$. Now, by assumption $[G : W] < \infty$, and hence there is a greatest integer s such that A_s is non-empty. Pick $\alpha = (h_1, h_2, \ldots, h_s) \in A_s$ and put $K := J^{h_1} J^{h_2} \cdots J^{h_s}$. Using that $\alpha \in A_s$ and that J^e is an ideal of S_e, we get that $K \subseteq J^e = J$. We will construct a set $M \subseteq G$ such that (K, M) is a Passman pair of S where M has the required form for Proposition E.6.2.

Take $x \in G$ such that $K^x K \neq \{0\}$. We begin by showing that $\{h_1 x, h_2 x, \ldots, h_s x\}$ represents the same set of right cosets of W as $\{h_1, h_2, \ldots, h_s\}$. Seeking a contradiction, suppose that there is some $i \in \{1, \ldots, s\}$ such that $W h_i x \neq W h_j$ for each $j \in \{1, \ldots, s\}$. By Corollary E.3.11 and Lemma E.3.6(a), we get that

$$\{0\} \neq K^x K \subseteq (J^{h_1 x} J^{h_2 x} \cdots J^{h_s x})(J^{h_1} J^{h_2} \cdots J^{h_s}) \subseteq J^{h_i x} J^{h_1} \cdots J^{h_s}. \tag{40}$$

Hence, $(h_i x, h_1, h_2, \ldots, h_s) \in A_{s+1}$ which contradicts the assumption on s. Thus, $\{W h_1, W h_2, \ldots, W h_s\} = \{W h_1 x, W h_2 x, \ldots, W h_s x\}$. In particular, $h_i x \in W$ for some $i \in \{1, \ldots, s\}$. By a computation similar to that in (40), we get that $\{0\} \neq K^x K \subseteq J^{h_i x} J$. Hence, by assumption we have $h_i x \in \bigcup_{k=1}^n w_k H_k$. We have thus proved that

$$K^x K = \{0\}, \qquad \forall x \in G \setminus \left(\bigcup_{i=1}^n \bigcup_{k=1}^n h_i^{-1} w_k H_k \right).$$

By Proposition E.6.2, there is a nonzero ideal $I \subseteq K \subseteq J$ of S_e and a subgroup L of G such that (I, L) is a Passman pair. Moreover, $[L : L \cap H_k] < \infty$ for some $k \in \{1, \ldots, n\}$. ☐

Remark E.6.5. Let S be nearly epsilon-strongly G-graded and let W be a subgroup of G. Then S_W is a nearly epsilon-strongly W-graded ring and $(J, \bigcup_{k=1}^n w_k H_k)$ is a Passman pair of S_W. By Proposition E.6.2, there is a subgroup L of W and a nonzero ideal $K \subseteq J$ of S_e such that (K, L) is a Passman pair of S_W. In other words, $K^x K = \{0\}$ for every $x \in W \setminus L$. In contrast, note that Proposition E.6.4 gives a Passman pair (K, L) of the larger ring S, i. e. we have $K^x K = \{0\}$ for every $x \in G \setminus L$.

E.7. Passman forms and the Δ-method

Let S be a G-graded ring. For nonzero graded ideals A, B of S, we have that $AB = \{0\}$ implies $\pi_N(A)\pi_N(B) \subseteq AB = \{0\}$ for every normal subgroup N of G. Moreover, if S is non-degenerately G-graded, then $\pi_N(A) \neq \{0\}$ and $\pi_N(B) \neq \{0\}$ by Lemma E.2.19. In this section, we consider nonzero ideals A, B of S such that $AB = \{0\}$ and show that there exist a normal subgroup N of G and nonzero ideals $\tilde{A}, \tilde{B}$ of S_N such that $\tilde{A}\tilde{B} = \{0\}$.

Recall that a ring is called *semiprime* if it contains no nonzero nilpotent ideal. Analogously, we make the following definition:

Definition E.7.1. If for every G-invariant ideal I of S_e, $I^2 = \{0\}$ implies $I = \{0\}$, then the ring S_e is called *G-semiprime*.

Remark E.7.2. (a) S_e is G-semiprime if and only if S_e contains no nonzero nilpotent G-invariant ideal.
(b) If S_e is G-prime, then S_e is G-semiprime.

We record the following result which follows directly from Remark E.7.2(b) and Corollary E.5.7:

Corollary E.7.3. *Suppose that S is nearly epsilon-strongly G-graded. If S_e is not G-semiprime, then S has a balanced NP-datum.*

Our main task for the remainder of this section is to establish Proposition E.7.4 below. Recall that, for a given group H, $\Delta(H) := \{h \in H \mid [H : C_H(h)] < \infty\}$ denotes its finite conjugate center (cf. Section E.2).

Proposition E.7.4 (cf. [36, Prop. 3.1])**.** *Suppose that S is nearly epsilon-strongly G-graded and that S_e is G-semiprime. Let A, B be nonzero ideals of S such that $AB = \{0\}$. Then there exist a subgroup H of G, a nonzero H-invariant ideal I of S_e and an element $\beta \in B$ such that the following assertions hold:*
(a) *$I^x I = \{0\}$ for every $x \in G \setminus H$;*
(b) *$I\pi_{\Delta(H)}(A) \neq \{0\}$, $I\pi_{\Delta(H)}(\beta) \neq \{0\}$;*
(c) *$I\pi_{\Delta(H)}(A) \cdot I\beta = \{0\}$.*

Using Connell's result (cf. [38, Lem. 5.2]), we show that Proposition E.7.4 holds for the special case of group rings in the following example.

Example E.7.5. Let R be a unital semiprime ring, and consider the group ring $R[G] = \bigoplus_{x \in G} R\delta_x$ with its natural strong G-grading. Let $\Delta := \Delta(G)$ and let $a, b \in R[G]$. The Δ-argument was used by Connell to prove that if $a\delta_x b = 0$ for every $x \in G$, then $\pi_\Delta(a)b = 0$. We show that Proposition E.7.4 holds in this special case:

Let A, B be nonzero ideals of $R[G]$ such that $AB = \{0\}$. Put $H := G$ and $I := R$. Since $R[G]$ is non-degenerately G-graded, we can choose $\beta \in B$ such that $\beta_e \neq 0$. Now, note that (a) is trivially satisfied. Moreover, (b) follows from Lemma E.2.19 and the fact that $\beta_e \neq 0$. Next, note that (c) asserts that $R\pi_\Delta(A) \cdot R\beta = \{0\}$. Also note that $R\pi_\Delta(A)R\beta = R\pi_\Delta(AR)\beta = R\pi_\Delta(A)\beta$. Now, let $\alpha \in A$ and let $x \in G$. Then $\alpha\delta_x\beta \subseteq ASB = AB = \{0\}$. Applying Connell's Δ-result, we have $\pi_\Delta(\alpha)\beta = 0$, and since α is arbitrary it follows that $\pi_\Delta(A)\beta = \{0\}$. Thus, $R\pi_\Delta(A)\beta = \{0\}$ which shows that (c) is satisfied.

Before proving Proposition E.7.4 we show that it also holds in the following special case:

Example E.7.6. Suppose that G is an FC-group and that S is nearly epsilon-strongly G-graded. Let A, B be nonzero ideals of S such that $AB = \{0\}$. Put $H := G$, $I := S_e$ and choose a nonzero $\beta \in B$. Since G is an FC-group, it follows that $\Delta := \Delta(G) = G$. Note that (a) is trivially satisfied. Moreover, $I\pi_\Delta(A) = S_e A = A \neq \{0\}$ and $I\pi_\Delta(\beta) = S_e\beta \ni \beta \neq 0$. Thus, (b) holds. Finally, $I\pi_\Delta(A) \cdot I\beta = S_e A \cdot S_e\beta \subseteq AS_e B \subseteq AB = \{0\}$. Hence, (c) is satisfied.

The key bookkeeping device used by Passman [36] is the notion of a *form*. We extend his definition to our generalized setting:

Definition E.7.7. Let S be a G-graded ring. Suppose that A, B are nonzero ideals of S such that $AB = \{0\}$. We say that the quadruple (H, D, I, β) is a *Passman form* for (A, B) if the following conditions are satisfied:

(a) H is a subgroup of G and $D = D_G(H) = \{x \in G \mid [H : C_H(x)] < \infty\}$;
(b) I is an H-invariant ideal of S_e such that $I^x I = \{0\}$ for every $x \in G \setminus H$;
(c) $0 \neq \beta \in B$, $I\beta \neq \{0\}$, and $IA \neq \{0\}$.

The *size* of a Passman form (H, D, I, β) is defined to be the number of right D-cosets in G meeting $\mathrm{Supp}(\beta)$.

Remark E.7.8. Passman (see [36, Prop. 7.1]) only considers forms coming from unital strongly G-graded rings. For that class of rings our definition coincides with his original definition.

Example E.7.9. Here are two examples of Passman forms:

(a) In Example E.7.5, $(G, \Delta(G), R, \beta)$ is a Passman form. Let $g_1, g_2, \ldots, g_n \in G$ be such that $\mathrm{Supp}(\beta) \subseteq \bigcup_{i=1}^n g_i\Delta(G)$ is a minimal cover (meaning that it is not possible to choose elements $h_1, \ldots, h_m \in G$ such that $\mathrm{Supp}(\beta) \subseteq \bigcup_{i=1}^m h_i\Delta(G)$, for any $m < n$). The size of the Passman form $(G, \Delta(G), R, \beta)$ is n.

(b) In Example E.7.6, (G, G, S_e, β) is a Passman form of size 1.

Later in this section we will consider Passman forms of minimal size, whose existence is guaranteed by the following:

Proposition E.7.10 (cf. [36, Lem. 7.2])**.** *Suppose that S is nearly epsilon-strongly G-graded. If A, B are nonzero ideals of S such that $AB = \{0\}$, then (A, B) has a Passman form.*

Proof. Put $H := G, D := \Delta(G)$, and $I := S_e$. Note that I is G-invariant. Furthermore, $IA = S_eA = A \neq \{0\}$. Now, let $\beta \in B \setminus \{0\}$. It remains to show that $I\beta \neq \{0\}$. To this end, write $\beta = \sum_{x\in G} \beta_x$. Since S is nearly epsilon-strongly G-graded, for every $x \in \mathrm{Supp}(\beta)$, there exists some $\epsilon_x(\beta_x) \in S_xS_{x^{-1}} \subseteq S_e = I$ such that $\epsilon_x(\beta_x)\beta_x = \beta_x$. Moreover, there is some $s \in S_e = I$ such that $s\epsilon_x(\beta_x) = \epsilon_x(\beta_x)$ for every $x \in \mathrm{Supp}(\beta)$ (see Proposition E.2.13 and Proposition E.2.1). Thus,

$$I\beta \ni s\beta = s\sum \beta_x = \sum s(\epsilon_x(\beta_x)\beta_x) = \sum (s\epsilon_x(\beta_x))\beta_x = \sum \epsilon_x(\beta_x)\beta_x = \beta \neq 0,$$

where all sums run over $\mathrm{Supp}(\beta)$. This shows that $(G, \Delta(G), S_e, \beta)$ is a Passman form. □

Proposition E.7.11 (cf. [36, Lem. 3.3(ii)])**.** *Suppose that S is non-degenerately G-graded and that A, B are nonzero ideals of S such that $AB = \{0\}$. Let (H, D, I, β) be a Passman form for (A, B). Then the following assertions hold:*

(a) $I\pi_{\Delta(H)}(A) \neq \{0\}$
(b) *There exists a Passman form (H, D, I, β') for (A, B) such that $I\pi_{\Delta(H)}(\beta') \neq \{0\}$, and hence $I\pi_D(\beta') \neq \{0\}$. Moreover, the size of (H, D, I, β') is not greater than the size of (H, D, I, β).*

Proof. (a): Note that $IA \neq \{0\}$ is a right S-ideal. By Lemma E.2.18 and Lemma E.2.19, we have $\{0\} \neq \pi_{\Delta(H)}(IA) = I\pi_{\Delta(H)}(A)$.

(b): We construct a Passman form with the required properties. Write $\beta = \sum_{x\in G} \beta_x$. By assumption, $I\beta \neq \{0\}$. Hence there is some $r \in I \subseteq S_e$ and $x \in G$ such that $r\beta_x \neq 0$. By non-degeneracy of the G-grading, we have $(r\beta_x)S_{x^{-1}} \neq \{0\}$,

i. e. there is some $\sigma_{x^{-1}} \in S_{x^{-1}}$ such that $r\beta_x\sigma_{x^{-1}} \neq 0$. Thus, $I\beta_x\sigma_{x^{-1}} \neq \{0\}$. Hence, (H, D, I, β') with $\beta' := \beta\sigma_{x^{-1}}$ is a Passman form for (A, B) such that $I\pi_{\Delta(H)}(\beta') \neq \{0\}$. We now show that the size of (H, D, I, β') is less than or equal to the size of (H, D, I, β). Suppose that (H, D, I, β) has size m and that $Dg_1, \ldots, Dg_m$ form a minimal set of right D-cosets covering $\operatorname{Supp}(\beta)$. Then

$$\operatorname{Supp}(\beta\sigma_{x^{-1}}) \subseteq \operatorname{Supp}(\beta)x^{-1} \subseteq Dg_1x^{-1} \cup Dg_2x^{-1} \cup \ldots \cup Dg_mx^{-1}$$

and hence the m right D-cosets $\{Dg_ix^{-1}\}_{i=1}^m$ cover $\operatorname{Supp}(\beta\sigma_{x^{-1}})$. Thus, the size of (H, D, I, β') is less than or equal to m. Finally, since $\Delta(H) \subseteq D$, we get $\{0\} \neq I\pi_{\Delta(H)}(\beta') \subseteq I\pi_D(\beta')$. □

Lemma E.7.12. *Suppose that S is nearly epsilon-strongly G-graded and that S_e is G-semiprime. For any G-invariant ideal I of S_e the following assertions hold:*
(a) $r.\operatorname{Ann}_{S_e}(I) = r.\operatorname{Ann}_{S_e}(I^2)$.
(b) $r.\operatorname{Ann}_S(I) = r.\operatorname{Ann}_S(I^2)$.

Proof. (a): Put $J := r.\operatorname{Ann}_{S_e}(I^2)$. Clearly, $r.\operatorname{Ann}_{S_e}(I) \subseteq J$. By Corollary E.3.11, $(IJ)^x = I^xJ^x$ for every $x \in G$, and hence IJ is a G-invariant ideal of S_e by Lemma E.3.16. Moreover, by definition $I^2J = \{0\}$, and hence $(IJ)^2 = (IJ)(IJ) \subseteq I(IJ) = I^2J = \{0\}$. Since S_e is G-semiprime, it follows that $IJ = \{0\}$. Thus, J annihilates I, i. e. $J \subseteq r.\operatorname{Ann}_{S_e}(I)$.

(b): Similarly, the inclusion $r.\operatorname{Ann}_S(I) \subseteq r.\operatorname{Ann}_S(I^2)$ is immediate. We now show the reversed inclusion. Take $\gamma = \sum_{x\in G}\gamma_x \in r.\operatorname{Ann}_S(I^2)$. Since $I^2 \subseteq S_e$, we have $I^2\gamma_x = \{0\}$ for every $x \in G$. Next, let $x \in G$. Using (a), we obtain that $\gamma_xS_{x^{-1}} \subseteq r.\operatorname{Ann}_{S_e}(I)$. In other words, $I\gamma_xS_{x^{-1}} = \{0\}$ which, by non-degeneracy of the G-grading, yields $I\gamma_x = \{0\}$, and hence $\gamma_x \in r.\operatorname{Ann}_S(I)$. Since $x \in G$ is arbitrary, it follows that $\gamma \in r.\operatorname{Ann}_S(I)$. □

Lemma E.7.13 (cf. [36, Lem. 3.3(iii)])**.** *Suppose that S is nearly epsilon-strongly G-graded and that S_e is G-semiprime. Furthermore, let A, B be nonzero ideals of S such that $AB = \{0\}$. If (H, D, I, β) is a Passman form for (A, B) of minimal size with $I\pi_D(\beta) \neq \{0\}$, then for every $\gamma \in S_D$ we have $I\gamma\beta = \{0\}$ if and only if $I\gamma\pi_D(\beta) = \{0\}$.*

Proof. Suppose that $I\gamma\beta = \{0\}$. Since π_D is an S_D-bimodule homomorphism by Lemma E.2.18, it follows that $\{0\} = \pi_D(I\gamma\beta) = I\gamma\pi_D(\beta)$.

Conversely, suppose that $I\gamma\pi_D(\beta) = \{0\}$. Take $s \in I$ and note that $s\gamma\beta \in IS_DB \subseteq B$. Seeking a contradiction, suppose that $(H, D, I, s\gamma\beta)$ is a Passman form for the pair (A, B). We show that $(H, D, I, s\gamma\beta)$ has less than minimal size. Indeed, suppose that $n \in \mathbb{N}$ is the size of (H, D, I, β), i. e. the minimal number such that $\operatorname{Supp}(\beta) \subseteq \bigcup_{i=1}^n Dg_i$ for some $g_1, \ldots, g_n \in G$. Since $I\pi_D(\beta) \neq \{0\}$, we have $\pi_D(\beta) \neq 0$. Hence, we may w.l.o.g. assume that $g_1 = e$. Moreover, it is immediate that

$$\operatorname{Supp}(s\gamma\beta) \subseteq \operatorname{Supp}(s\gamma)\operatorname{Supp}(\beta) \subseteq D\left(\bigcup_{i=1}^n Dg_i\right) \subseteq \left(\bigcup_{i=1}^n Dg_i\right).$$

By assumption, however, $0 = s\gamma\pi_D(\beta) = \pi_D(s\gamma\beta)$ which entails that $\operatorname{Supp}(s\gamma\beta) \subseteq \bigcup_{i=2}^n Dg_i$. This is a contradiction since (H, D, I, β) is assumed to be minimal. Thus, $(H, D, I, s\gamma\beta)$ is not a Passman form, and hence $Is\gamma\beta = \{0\}$ (cf. Definition E.7.7).

As this holds for every $s \in I$, we have $I^2\gamma\beta = \{0\}$. By Lemma E.7.12(b), this yields $I\gamma\beta = \{0\}$. □

E.7.1. Properties of a Passman form of minimal size. In what follows, we fix a nearly epsilon-strongly G-graded ring S such that S_e is G-semiprime, nonzero ideals A, B of S with $AB = \{0\}$, and a Passman form (H, D, I, β) for (A, B) of minimal size. Throughout this section we assume that $I\pi_{\Delta(H)}(A) \cdot I\beta \neq \{0\}$.

Lemma E.7.14 (cf. [36, Lem. 3.4])**.** *The following assertions hold:*

(a) *There exists $\alpha \in A \cap S_H$ such that $I\pi_D(\alpha)\beta \neq \{0\}$.*
(b) *For every $\alpha \in A$ there is a subgroup W of H of finite index that centralizes $\mathrm{Supp}(\pi_D(\alpha))$ and $\mathrm{Supp}(\pi_D(\beta))$.*

Proof. (a): By assumption, we have $I\pi_{\Delta(H)}(A) \cdot I\beta \neq \{0\}$. In other words, $\pi_{\Delta(H)}(A) \cdot I\beta$ is not contained in $r.\mathrm{Ann}_S(I)$. Furthermore, by Lemma E.7.12(b), we have $r.\mathrm{Ann}_S(I) = r.\mathrm{Ann}_S(I^2)$. Applying Lemma E.2.18, we get $I\pi_{\Delta(H)}(IAI)\beta = I^2\pi_{\Delta(H)}(A) \cdot I\beta \neq \{0\}$. Hence, there exists some $\alpha \in IAI \subseteq A$ such that $I\pi_{\Delta(H)}(\alpha)\beta \neq \{0\}$. Additionally, we have $\alpha \in ISI \subseteq S_H$ by Lemma E.5.1(b). Since $D \cap H = \Delta(H)$, we get $\pi_D(\alpha) = \pi_{\Delta(H)}(\alpha)$ and thus $I\pi_D(\alpha)\beta \neq \{0\}$.

(b): Note that $P := \mathrm{Supp}(\pi_D(\alpha)) \cup \mathrm{Supp}(\pi_D(\beta))$ is a finite subset of $D = D_G(H)$ and consider $W := \bigcap_{x \in P} C_H(x)$. Since $P \subseteq D$ and $[H : C_H(x)] < \infty$ for every $x \in D$, we get that $[H : W] < \infty$. □

Lemma E.7.15 (cf. [36, Lem. 3.4])**.** *Suppose that $\alpha \in A \cap S_H$ is such that $I\pi_D(\alpha)\beta \neq \{0\}$. Let W be given by Lemma E.7.14. Then there are $d_0 \in \mathrm{Supp}(\pi_D(\alpha))$ and $u \in W$ such that $I(S_e\alpha_{d_0}S_{d_0^{-1}})^u\pi_D(\alpha)\beta \neq \{0\}$.*

Proof. First put $\gamma := \pi_D(\alpha)\beta$ and write $\alpha = \sum_{x \in G} \alpha_x$, $\beta = \sum_{x \in G} \beta_x$, and $\gamma = \sum_{x \in G} \gamma_x$. Furthermore, let $J := \sum_{d \in D}(S_e\alpha_d S_{d^{-1}})^W \subseteq S_e$. Note that J is a W-invariant ideal of S_e. Using that S is nearly-espilon strongly G-graded, note that for all $d \in D, y \in G$, we have

$$\alpha_d\beta_y S_{y^{-1}d^{-1}} \subseteq S_e\alpha_d(S_{d^{-1}}S_d)\beta_y S_{y^{-1}d^{-1}} = S_e\alpha_d S_{d^{-1}} \cdot S_d\beta_y S_{y^{-1}d^{-1}} \subseteq J \cdot S_e \subseteq J. \quad (41)$$

Take $x \in G$. Then by (41), $\gamma_x S_{x^{-1}} \subseteq J$. Seeking a contradiction, suppose that $IJ\gamma = \{0\}$. Then $IJ\gamma_x S_{x^{-1}} = \{0\}$. Hence $\gamma_x S_{x^{-1}} \subseteq r.\mathrm{Ann}_{S_e}(IJ)$. Using that $IJI \subseteq IJ$, we get

$$I\gamma_x S_{x^{-1}} \subseteq IJ \cap r.\mathrm{Ann}_{S_e}(IJ). \quad (42)$$

By Lemma E.3.16, we know that $IJ \cap r.\mathrm{Ann}_{S_e}(IJ)$ is a W-invariant nilpotent ideal of S_e contained in I, and hence $IJ \cap r.\mathrm{Ann}_{S_e}(IJ) = \{0\}$ by Lemma E.3.22(b). By non-degeneracy of the G-grading, (42) implies that $I\gamma_x = \{0\}$. Since $x \in G$ is arbitrary, this yields $I\gamma = \{0\}$, i.e. $I\pi_D(\alpha)\beta = \{0\}$. This contradicts the properties of α. Consequently, $IJ\pi_D(\alpha)\beta \neq \{0\}$, i.e. there exist some $d_0 \in D$ and some $u \in W$ such that $I(S_e\alpha_{d_0}S_{d_0^{-1}})^u\pi_D(\alpha)\beta \neq \{0\}$. □

For the remainder of this section, we fix $\alpha \in A \cap S_H$ such that $|\mathrm{Supp}(\pi_D(\alpha))|$ is minimal subject to $I\pi_D(\alpha)\beta \neq \{0\}$. We also fix W given by Lemma E.7.14.

Lemma E.7.16 (cf. [36, Lem. 3.4])**.** *For every $y \in W$ and every $d \in D$, we have*

$$IS_{y^{-1}}\alpha_d S_{d^{-1}y}\pi_D(\alpha)\pi_D(\beta) = IS_{y^{-1}}\pi_D(\alpha)S_{d^{-1}y}\alpha_d\pi_D(\beta).$$

Proof. Take $y \in W$, $d \in D$, $a_{y^{-1}} \in S_{y^{-1}}$, and $b_{d^{-1}y} \in S_{d^{-1}y}$. Note that if $d \notin \mathrm{Supp}(\alpha)$, then the claim trivially holds. Therefore, we now suppose that $d \in D \cap \mathrm{Supp}(\alpha)$. Define

$$\gamma := a_{y^{-1}}\alpha_d b_{d^{-1}y}\alpha - a_{y^{-1}}\alpha b_{d^{-1}y}\alpha_d.$$

A short computation, using Lemma E.7.14(b), shows that $\gamma \in A \cap S_H$. Moreover, since π_D is an S_e-bimodule homomorphism by Lemma E.2.18, we get

$$\pi_D(\gamma) = a_{y^{-1}}\alpha_d b_{d^{-1}y}\pi_D(\alpha) - a_{y^{-1}}\pi_D(\alpha)b_{d^{-1}y}\alpha_d.$$

From this we get that $\mathrm{Supp}(\pi_D(\gamma)) \subseteq \mathrm{Supp}(\pi_D(\alpha))$. We claim that the minimality assumption on α implies that $I\pi_D(\gamma)\pi_D(\beta) = \{0\}$. If the claim holds, then we get that

$$Ia_{y^{-1}}(\alpha_d b_{d^{-1}y}\pi_D(\alpha) - \pi_D(\alpha)b_{d^{-1}y}\alpha_d)\pi_D(\beta) = \{0\}$$

and hence that

$$IS_{y^{-1}}\alpha_d S_{d^{-1}y}\pi_D(\alpha)\pi_D(\beta) = IS_{y^{-1}}\pi_D(\alpha)S_{d^{-1}y}\alpha_d\pi_D(\beta).$$

Now we show the claim. Write $\gamma = \sum_{x\in G}\gamma_x$. By considering the cases when $x \in \mathrm{Supp}(\alpha)$ and $x \notin \mathrm{Supp}(\alpha)$ separately, for each $x \in G$ we get that

$$\gamma_x = a_{y^{-1}}\alpha_d b_{d^{-1}y}\alpha_x - a_{y^{-1}}\alpha_x b_{d^{-1}y}\alpha_d. \tag{43}$$

Now, recall that $\pi_D(\gamma) = \sum_{x\in D}\gamma_x$. However, due to (43), $\gamma_d = 0$, and thus we get that $|\,\mathrm{Supp}(\pi_D(\gamma))| < |\,\mathrm{Supp}(\pi_D(\alpha))|$, since $\alpha_d \neq 0$. The minimality assumption on α therefore implies that $I\pi_D(\gamma)\beta = \{0\}$. Applying the map π_D to the former equation yields $I\pi_D(\gamma)\pi_D(\beta) = \{0\}$. □

Lemma E.7.17. *There are elements $x_1, \dots, x_n \in W$ and $g_1, \dots, g_n \in \mathrm{Supp}(\beta) \setminus D$ such that if $S_{y^{-1}}I\pi_D(\alpha)S_y\pi_D(\beta) \neq \{0\}$, then $y \in \bigcup_{k=1}^n x_k H_k$ whenever $y \in W$. Here, $H_k := C_W(g_k)$.*

Proof. Let $\tilde{\alpha} := \alpha - \pi_D(\alpha)$ and let $\tilde{\beta} := \beta - \pi_D(\beta)$. Then

$$S_{y^{-1}}I(\pi_D(\alpha) + \tilde{\alpha})S_y(\pi_D(\beta) + \tilde{\beta}) = S_{y^{-1}}I\alpha S_y\beta \subseteq S_{y^{-1}}IAS_yB \subseteq AB = \{0\}.$$

Note that $S_{y^{-1}}I\pi_D(\alpha)S_y\tilde{\beta}$ and $S_{y^{-1}}I\tilde{\alpha}S_y\pi_D(\beta)$ have support disjoint from D. On the other hand, $\{0\} \neq S_{y^{-1}}I\pi_D(\alpha)S_y\pi_D(\beta) \subseteq S_D$. Hence, $S_{y^{-1}}I\pi_D(\alpha)S_y\pi_D(\beta)$ must be additively cancelled out by $S_{y^{-1}}I\tilde{\alpha}S_y\tilde{\beta}$. In particular, these two expressions must have a support element in common, i. e. there exist $a \in \mathrm{Supp}(\tilde{\alpha}), b \in \mathrm{Supp}(\tilde{\beta}), g \in \mathrm{Supp}(\pi_D(\alpha))$, and $f \in \mathrm{Supp}(\pi_D(\beta))$ such that $y^{-1}ayb = y^{-1}gyf$. Multiplying with y from the left and with y^{-1} from the right gives $ayby^{-1} = gyfy^{-1} = gf$, where we have used the fact that $y \in W$ commutes with both $\mathrm{Supp}(\pi_D(\alpha))$ and $\mathrm{Supp}(\pi_D(\beta))$. Consequently, $yby^{-1} = a^{-1}gf$, and hence $y \in xC_W(b)$ for some fixed x depending on a, b, g, f. Since there are only a finite number of choices for the parameters a, b, g, f and $b \in \mathrm{Supp}(\tilde{\beta}) = \mathrm{Supp}(\beta) \setminus D$, the desired conclusion follows. □

Next, we will construct an ideal J of S_e that allows us to apply the Passman replacement argument (see Section E.6). In the following two lemmas we make use of the notation introduced in Lemma E.7.17.

Lemma E.7.18 (cf. [36, Lem. 3.5])**.** *For every $d \in D$,*

$$I(S_e\alpha_d S_{d^{-1}})^y \cdot \pi_D(\alpha)\beta = \{0\}, \qquad \forall y \in W \setminus \bigcup_{k=1}^{n} x_k H_k.$$

Proof. Take $y \in W$ such that $I(S_e\alpha_d S_{d^{-1}})^y \cdot \pi_D(\alpha)\beta \neq \{0\}$. Expanding this expression, we get $IS_{y^{-1}}\alpha_d S_{d^{-1}} S_y \pi_D(\alpha)\beta \neq \{0\}$. Since $S_{y^{-1}}\alpha_d S_{d^{-1}} S_y \pi_D(\alpha) \subseteq S_D$, Lemma E.7.13 implies that $IS_{y^{-1}}\alpha_d S_{d^{-1}} S_y \pi_D(\alpha)\pi_D(\beta) \neq \{0\}$. As a consequence, $IS_{y^{-1}}\alpha_d S_{d^{-1}y}\pi_D(\alpha)\pi_D(\beta) \neq \{0\}$, since $S_{d^{-1}}S_y \subseteq S_{d^{-1}y}$. By Lemma E.7.16, we get that $IS_{y^{-1}}\pi_D(\alpha)S_{d^{-1}y}\alpha_d\pi_D(\beta) \neq \{0\}$ and, due to $d^{-1}yd = y$, we even have $IS_{y^{-1}}\pi_D(\alpha)S_y\pi_D(\beta) \neq \{0\}$. Next, note that I is also a W-invariant ideal, and thus $IS_{y^{-1}} = S_{y^{-1}}I$ by Proposition E.3.12(b). It follows that

$$\{0\} \neq IS_{y^{-1}}\pi_D(\alpha)S_y\pi_D(\beta) = S_{y^{-1}}I\pi_D(\alpha)S_y\pi_D(\beta)$$

which, combined with Lemma E.7.17, yields the desired conclusion. □

Lemma E.7.19. *There exists an ideal J of S_e such that $J^yJ = \{0\}$ for every $y \in W \setminus \bigcup_{k=1}^n u^{-1}x_k H_k$.*

Proof. Set $\gamma := \pi_D(\alpha)\beta$ and write $\gamma = \sum_{x\in G}\gamma_x$. By Lemma E.7.15 there exist $d_0 \in D$ and $u \in W$ such that $I(S_e\alpha_{d_0}S_{d_0^{-1}})^u\gamma \neq \{0\}$. Hence, there exists $x \in G$ such that $I(S_e\alpha_{d_0}S_{d_0^{-1}})^u\gamma_x \neq \{0\}$. By non-degeneracy of the G-grading, we have that $J := I(S_e\alpha_{d_0}S_{d_0^{-1}})^u\gamma_x S_{x^{-1}} \neq \{0\}$ is an ideal of S_e contained in I. Recall that I is W-invariant, since W is a subgroup of H. Now, combining the fact that $J \subseteq I(S_e\alpha_{d_0}S_{d_0^{-1}})^u S_e = I(S_e\alpha_{d_0}S_{d_0^{-1}})^u$ with Lemma E.3.6, for every $y \in W$ we get

$$J^{u^{-1}y} \subseteq I^{u^{-1}y}((S_e\alpha_{d_0}S_{d_0^{-1}})^u)^{u^{-1}y} \subseteq I^{u^{-1}y}(S_e\alpha_{d_0}S_{d_0^{-1}})^y \subseteq I(S_e\alpha_{d_0}S_{d_0^{-1}})^y.$$

By Lemma E.7.18, it follows that $J^{u^{-1}y}\pi_D(\alpha)\beta = \{0\}$ for every $y \in W \setminus \bigcup_{k=1}^n x_k H_k$ or, equivalently, that $J^y\gamma = \{0\}$ for every $y \in W \setminus \bigcup_{k=1}^n u^{-1}x_k H_k$. In particular, we have $J^y\gamma_x = \{0\}$, and hence, $J^y(S_e\gamma_x S_{x^{-1}}) = \{0\}$. This shows that $J^yJ = \{0\}$ for every $y \in W \setminus \bigcup_{k=1}^n u^{-1}x_k H_k$. □

E.7.2. Establishing Proposition E.7.4. We still assume that $I\pi_{\Delta(H)}(A)\cdot I\beta \neq \{0\}$. Combining that assumption with the following lemma, we will establish Proposition E.7.4.

Lemma E.7.20. *There is a Passman form for (A, B) of size smaller than the size of (H, D, I, β).*

Proof. By Lemma E.7.14, we have $[H : W] < \infty$. Let J be the ideal of S_e from Lemma E.7.19. By Proposition E.6.4 there exists a subgroup L of H and a nonzero ideal $K \subseteq J$ of S_e such that $K^yK = \{0\}$ for every $y \in H \setminus L$. Furthermore, we have $[L\colon L \cap H_k] < \infty$ for some subgroup H_k of W. We claim that $(L, D_G(L), K^L, \pi_D(\alpha)\beta)$

is a Passman form of size smaller than the size of (H, D, I, β). We first check that it satisfies the conditions in Definition E.7.7.

Note that condition (a) is trivially satisfied. Moreover, it follows from Lemma E.3.19 that K^L is an L-invariant ideal of S_e. Since $K \subseteq I$, we have $K^x K = \{0\}$ for every $x \in G \setminus H$. This shows that $K^x K = \{0\}$ for every $x \in G \setminus L$. Thus, by Proposition E.3.21, $(K^L)^x (K^L) = \{0\}$ for every $x \in G \setminus L$. Hence, condition (b) is satisfied. Next, note that $\gamma := \pi_D(\alpha)\beta \in B$. It remains to show that $K^L \gamma \neq \{0\}$ and $K^L A \neq \{0\}$. Seeking a contradiction, suppose that $K^L \gamma = \{0\}$. Then $K^L \gamma_x = \{0\}$, and hence $K^L (S_e \gamma_x S_{x^{-1}}) = \{0\}$. We get that $K^L J = \{0\}$. This implies that $J \subseteq r.\operatorname{Ann}_{S_e}(K^L)$. But since $r.\operatorname{Ann}_{S_e}(K^L)$ is an L-invariant ideal by Lemma E.3.16, we deduce from Lemma E.3.19 that $J \subseteq J^L \subseteq r.\operatorname{Ann}_{S_e}(K^L)$ and hence that $K^L J^L = \{0\}$. As $K \subseteq J$, this yields $(K^L)^2 = \{0\}$, which is a contradiction by Lemma E.3.22(a). Therefore, $K^L \pi_D(\alpha)\beta \neq \{0\}$. It follows that $K^L \pi_D(\alpha) \neq \{0\}$, and hence $K^L A \neq \{0\}$, by Lemma E.2.18. Summarizing, we have shown that $(L, D_G(L), K^L, \pi_D(\alpha)\beta)$ is a Passman form.

To proceed, let n be the size of the Passman form (H, D, I, β), i.e. the number of cosets of D in H meeting $\operatorname{Supp}(\beta)$. Furthermore, let m denote the size of $(L, D_G(L), K^L, \pi_D(\alpha)\beta)$. We claim that $m < n$. To show this, first note that $D = D_G(H) \subseteq D_G(L)$ and that $\operatorname{Supp}(\pi_D(\alpha)\beta) \subseteq D \cdot \operatorname{Supp}(\beta)$. Hence, $m \leq n$. Combining the facts that $[L : L \cap H_k] < \infty$ for some $H_k = C_W(g)$ with $g \in \operatorname{Supp}(\beta) \setminus D$ and $L \cap H_k = L \cap C_W(g) = C_L(g)$, we infer that $[L : C_L(g)] < \infty$ and hence that $g \in D_G(L)$. This means that the two distinct D-cosets Dg and D are contained in $D_G(L)$. Consequently, $m < n$, as claimed. □

We are now fully prepared to prove the following:

Proof of Proposition E.7.4. Let S be nearly epsilon-strongly G-graded such that S_e is G-semiprime. Furthermore, let A, B be nonzero ideals of S such that $AB = \{0\}$. We now show that conditions (a)-(c) in Proposition E.7.4 are satisfied. By Proposition E.7.10, S admits a minimal Passman form for (A, B), say (H, D, I, β). Moreover, by Propositon E.7.11, we may assume that $I\pi_{\Delta(H)}(A) \neq \{0\}$ and that $I\pi_{\Delta(H)}(\beta) \neq \{0\}$. Hence, conditions (a) and (b) hold. Seeking a contradiction, suppose that $I\pi_{\Delta(H)}(A) \cdot I\beta \neq \{0\}$. Then the previous results, in particular Lemma E.7.20, yields a Passman form of size smaller than that of (H, D, I, β), which is the desired contradiction. Hence, $I\pi_{\Delta(H)}(A) \cdot I\beta = \{0\}$ which shows that condition (c) holds. □

E.8. The "hard" direction

Recall that S is a G-graded ring. In this section, we prove the implication (a)⇒(e) of Theorem E.1.3 for nearly epsilon-strongly G-graded rings (see Proposition E.8.9). We remind the reader that if H, K are subgroups of G, then H *normalizes* K if $Kx = xK$ for every $x \in H$. In that case it follows that $H \subseteq N_G(K)$, where $N_G(K) := \{x \in G \mid xK = Kx\}$ denotes the normalizer of K in G, and we allow ourselves to speak of H/K*-invariance* in the sense of Definition E.3.3.

Lemma E.8.1. *Suppose that H, K are subgroups of G such that H normalizes K. If $x \in H$, $k_1, k_2 \in K$, $r \in S_{xk_1}$, $\alpha \in S$ and $s \in S_{k_2 x^{-1}}$, then $\pi_K(r\alpha s) = r\pi_K(\alpha)s$.*

PROOF. Write $\alpha = \sum_{y\in G}\alpha_y$, where $\alpha_y \in S_y$ for $y \in G$. Take $y \in G$. Note that $xk_1 \cdot y \cdot k_2x^{-1} \in K$ if and only if $y \in k_1^{-1}x^{-1}Kxk_2^{-1} = k_1^{-1}Kk_2^{-1} = K$. Thus, $\pi_K(r\alpha s) = \sum_{y\in G}\pi_K(r\alpha_y s) = \sum_{y\in K}\pi_K(r\alpha_y s) = \sum_{y\in K} r\alpha_y s = r\pi_K(\alpha)s$. $\square$

By Lemma E.8.1, with $k_1 = k_2 = e$, we get the following result (cf. [36, p. 721]).

Corollary E.8.2. *Suppose that H, K are subgroups of G such that H normalizes K. For every $\alpha \in S$ and $x \in H$, we have $S_{x^{-1}}\pi_K(\alpha)S_x = \pi_K(S_{x^{-1}}\alpha S_x)$.*

Given ideals A, B of S such that $AB = \{0\}$, we will find new ideals A_1, A_2, B_1, B_2 of subrings of S satisfying $A_1B_1 = \{0\}$ and $A_2B_2 = \{0\}$.

Lemma E.8.3. *Suppose that S is nearly epsilon-strongly G-graded and that S_e is G-semiprime. If S is not prime, then there exists a subgroup H of G such that $S_{\Delta(H)}$ is not prime. In fact, there exist nonzero $H/\Delta(H)$-invariant ideals A_1, B_1 of $S_{\Delta(H)}$ such that $A_1, B_1 \subseteq IS_{\Delta(H)}$ and $A_1B_1 = \{0\}$.*

PROOF. Let A, B be nonzero ideals of S such that $AB = \{0\}$. By Proposition E.7.4, there are a subgroup H of G, a nonzero H-invariant ideal I of S_e, and $\beta \in B$ such that:

(a) $I\pi_{\Delta(H)}(A) \neq \{0\}$;
(b) $I\pi_{\Delta(H)}(\beta) \neq \{0\}$;
(c) $I\pi_{\Delta(H)}(A) \cdot I\beta = \{0\}$.

Consider the set $A_1 := I\pi_{\Delta(H)}(A) \subseteq IS_{\Delta(H)}$. Clearly, A_1 is nonzero by (a). Take $h \in H$. Then Proposition E.3.12, the fact that $\Delta(H) \subseteq H$, and Lemma E.8.1 yield

$$\begin{aligned} S_{h^{-1}\Delta(H)}A_1S_{h\Delta(H)} &= S_{h^{-1}\Delta(H)}I\pi_{\Delta(H)}(A)S_{h\Delta(H)} = IS_{h^{-1}\Delta(H)}\pi_{\Delta(H)}(A)S_{h\Delta(H)} \\ &= I\pi_{\Delta(H)}(S_{h^{-1}\Delta(H)}AS_{h\Delta(H)}) \subseteq I\pi_{\Delta(H)}(A) = A_1. \end{aligned}$$

By taking $h = e$, the above computation yields $S_{\Delta(H)}A_1S_{\Delta(H)} \subseteq A_1$. Thus, A_1 is an $H/\Delta(H)$-invariant ideal of $S_{\Delta(H)}$. Next, we define $B_1 := \sum_{h\in H} IS_{h^{-1}\Delta(H)}\pi_{\Delta(H)}(\beta)S_{h\Delta(H)}$. Clearly, $B_1 \subseteq IS_{\Delta(H)}$ and B_1 is nonzero. Take $h_1 \in H$. Using that $\Delta(H)$ is a normal subgroup of H and that I is H-invariant, we get

$$\begin{aligned} S_{h_1^{-1}\Delta(H)}B_1S_{h_1\Delta(H)} &= \sum_{h\in H} S_{h_1^{-1}\Delta(H)} \cdot IS_{h^{-1}\Delta(H)}\pi_{\Delta(H)}(\beta)S_{h\Delta(H)} \cdot S_{h_1\Delta(H)} \\ &= \sum_{h\in H} IS_{h_1^{-1}\Delta(H)}S_{h^{-1}\Delta(H)} \cdot \pi_{\Delta(H)}(\beta) \cdot S_{h\Delta(H)}S_{h_1\Delta(H)} \\ &\subseteq \sum_{h\in H} IS_{h_1^{-1}h^{-1}\Delta(H)} \cdot \pi_{\Delta(H)}(\beta) \cdot S_{hh_1\Delta(H)} = B_1. \end{aligned}$$

By taking $h_1 = e$, the above computation yields $S_{\Delta(H)}B_1S_{\Delta(H)} \subseteq B_1$. Thus, B_1 is an $H/\Delta(H)$-invariant ideal of $S_{\Delta(H)}$. By Proposition E.2.20, the induced $H/\Delta(H)$-grading on S_H is nearly epsilon-strong and hence $S_{h^{-1}\Delta(H)}S_{h\Delta(H)} \cdot \pi_{\Delta(H)}(A) = \pi_{\Delta(H)}(A)$. Using that I is H-invariant, it follows from Lemma E.2.18, Lemma E.8.1, and (c), that

$$\begin{aligned}
A_1B_1 &= I\pi_{\Delta(H)}(A) \cdot \sum_{h\in H} IS_{h^{-1}\Delta(H)}\pi_{\Delta(H)}(\beta)S_{h\Delta(H)} \\
&= \sum_{h\in H} I\pi_{\Delta(H)}(A)S_{h^{-1}\Delta(H)}\pi_{\Delta(H)}(I\beta)S_{h\Delta(H)} \\
&= \sum_{h\in H} I\cdot S_{h^{-1}\Delta(H)}S_{h\Delta(H)}\pi_{\Delta(H)}(A)\cdot S_{h^{-1}\Delta(H)}\pi_{\Delta(H)}(I\beta)S_{h\Delta(H)} \\
&= \sum_{h\in H} IS_{h^{-1}\Delta(H)}\cdot \pi_{\Delta(H)}(S_{h\Delta(H)}AS_{h^{-1}\Delta(H)})\cdot \pi_{\Delta(H)}(I\beta)S_{h\Delta(H)} \\
&\subseteq \sum_{h\in H} IS_{h^{-1}\Delta(H)}\cdot \pi_{\Delta(H)}(A)\cdot \pi_{\Delta(H)}(I\beta)S_{h\Delta(H)} \\
&= \sum_{h\in H} S_{h^{-1}\Delta(H)}\cdot I\pi_{\Delta(H)}(A)\cdot \pi_{\Delta(H)}(I\beta)S_{h\Delta(H)} \\
&= \sum_{h\in H} S_{h^{-1}\Delta(H)}\cdot \pi_{\Delta(H)}\big(I\pi_{\Delta(H)}(A)I\beta\big)\cdot S_{h\Delta(H)} = \{0\}.
\end{aligned}$$

As a result, $S_{\Delta(H)}$ is not prime. □

Lemma E.8.4. *Suppose that we are in the setting of Lemma E.8.3. Then there exists a finitely generated normal subgroup W of H such that $W \subseteq \Delta(H)$. Moreover, there exist nonzero H/W-invariant ideals A_2, B_2 of S_W such that $A_2, B_2 \subseteq IS_W$ and $A_2B_2 = \{0\}$.*

Proof. Let $A_1, B_1 \subseteq IS_{\Delta(H)}$ be as in Lemma E.8.3. Then there exist nonzero elements $a_1 \in A_1$ and $b_1 \in B_1$. Putting $P := \mathrm{Supp}(a_1) \cup \mathrm{Supp}(b_1)$ and using that A_1, B_1 are ideals of $S_{\Delta(H)}$, we see that $P \subseteq \Delta(H)$. Moreover, let W be the normal closure of P in H. Then W is clearly a finitely generated normal subgroup of H with $W \subseteq \Delta(H)$. Now, consider $A_2 := A_1 \cap S_W$ and $B_2 := B_1 \cap S_W$. Using that A_1 (resp. B_1) is $H/\Delta(H)$-invariant, we get that A_1 (resp. B_1) is H/W-invariant. Clearly, S_W is H/W-invariant. Thus, A_2 and B_2 are are nonzero H/W-invariant ideals of S_W such that $A_2, B_2 \subseteq IS_W$. Furthermore, we have $A_2B_2 \subseteq A_1B_1 = \{0\}$, which completes the proof. □

We recall the following general result regarding the finite conjugate center $\Delta(H)$ of an arbitrary group H and include parts of the proof for the convenience of the reader.

Proposition E.8.5 ([41, Lem. II.4.1.5(iii)])**.** *Suppose that H is a group and that W is a finitely generated subgroup of $\Delta(H)$. Then there exists a finite characteristic subgroup $N \lhd W$ such that W/N is torsion-free abelian.*

Proof. Put $N := \{w \in W \mid \mathrm{ord}(w) < \infty\}$. It can be shown that the commutator subgroup $W' = [W, W]$ is finite (see [41, Lem. II.4.1.5(ii)]). Thus $W' \subseteq N$. Moreover, note that W/W' is a finitely generated abelian group. By the fundamental theorem of finitely generated abelian groups, W/W' has a finite maximal torsion subgroup K, i. e. $W/W' \cong \mathbb{Z}^n \oplus K$ for some $n \geq 0$. By restricting to torsion elements, we see that $N/W' \cong K$. Thus, N is a finite subgroup of W. Since every automorphism of W

preserves element order, it follows that N is a characteristic subgroup of W. We also get that W/N is torsion-free abelian, because $W' \subseteq N$. $\square$

Definition E.8.6 (cf. [36, p. 14]). Suppose that A is a nonzero ideal of a nearly epsilon-strongly W-graded ring S_W and that $N \lhd W$. For any nonzero $a \in A$ we define $\mathrm{meet}_N(a)$ to be the number of cosets of N in W that meet $\mathrm{Supp}(a)$. Define $m := \min\{\mathrm{meet}_N(b) \mid b \in A \setminus \{0\}\}$. Let $\min_N(A)$ denote the additive span of all nonzero elements $a \in A$ such that $\mathrm{meet}_N(a) = m$.

Lemma E.8.7 (cf. [36, Lem. 4.1]). *Suppose that S is nearly epsilon-strongly G-graded and that H is a subgroup of G. Furthermore, suppose that $N \lhd W$ are subgroups of G that are normalized by H and that A is a nonzero H/W-invariant ideal of S_W. Then the following assertions hold:*

(a) *$min_N(A)$ is a nonzero H/W-invariant ideal of S_W.*
(b) *$\pi_N(A)$ is a nonzero H/W-invariant ideal of S_N.*

Proof. (a): Note that $\min_N(A)$ is nonzero by definition. We show that $\min_N(A)$ is an ideal of S_W. Let $\alpha \neq 0$ be a generator of $\min_N(A)$ and take $w \in W$. It is enough to show that $S_w\alpha$ and αS_w are contained in $\min_N(A)$. To this end, note that $\mathrm{Supp}(\alpha S_w) \subseteq (\mathrm{Supp}(\alpha))w$ and that $\mathrm{Supp}(S_w\alpha) \subseteq w(\mathrm{Supp}(\alpha))$. Since $N \lhd W$, right and left cosets of N in W coincide. Let $\{w_1N, w_2N, \ldots, w_mN\}$ be a minimal set of cosets of N that covers $\mathrm{Supp}(\alpha)$. That is, $\mathrm{Supp}(\alpha) \subseteq w_1N \cup \ldots \cup w_mN = Nw_1 \cup \ldots \cup Nw_m$ with m minimal among such covers. Hence, $\mathrm{Supp}(\alpha S_w) \subseteq Nw_1w \cup \ldots \cup Nw_mw$ and $\mathrm{Supp}(S_w\alpha) \subseteq ww_1N \cup \ldots \cup ww_mN$, and consequently, αS_w and $S_w\alpha$ meet less than or exactly m cosets of N. It follows that $\alpha S_w, S_w\alpha \in \min_N(A)$ and therefore $\min_N(A)$ is an ideal of S_W.

Next, let $\alpha \in A$ be a generator of $\min_N(A)$ and take $h \in H$. To show that $\min_N(A)$ is H/W-invariant, it is enough to show that $S_{h^{-1}W}\alpha S_{hW} \subseteq \min_N(A)$. Take $k_1, k_2 \in W$. We will show that $S_{h^{-1}k_1}\alpha S_{hk_2} \subseteq \min_N(A)$

Using that A is assumed to be H/W-invariant, we have $S_{h^{-1}k_1}\alpha S_{hk_2} \subseteq A$. Hence, it only remains to show that $S_{h^{-1}k_1}\alpha S_{hk_2}$ meets a minimal number of cosets of N. As before, let $w_1N \cup \ldots \cup w_mN$ be a minimal cover of $\mathrm{Supp}(\alpha)$. Then

$$\begin{aligned}\mathrm{Supp}(S_{h^{-1}k_1}\alpha S_{hk_2}) &\subseteq h^{-1}k_1(\mathrm{Supp}(\alpha))hk_2 \\ &\subseteq h^{-1}k_1(w_1N)hk_2 \cup h^{-1}k_1(w_2N)hk_2 \cup \ldots \cup h^{-1}k_1(w_mN)hk_2.\end{aligned}$$

Since both H and W normalize N, we get that $h^{-1}k_1(w_iN)hk_2 = (h^{-1}k_1w_ihk_2)N$. Moreover, since H normalizes W, and $k_1w_i \in W$, we have $h^{-1}(k_1w_i)h \in W$. Thus, $h^{-1}k_1w_ih \cdot k_2 \in W$. Hence, $\mathrm{Supp}(S_{h^{-1}k_1}\alpha S_{hk_2})$ meets less than or exactly m cosets of N in W. Thus, $\min_N(A)$ is H/W-invariant.

(b): By Lemma E.2.19 and Proposition E.2.15, it follows that $\pi_N(A)$ is a nonzero ideal of S_N. Take $\alpha \in A$ and $h \in H$. Since H normalizes N, Lemma E.8.1 yields $S_{h^{-1}W}\pi_N(\alpha)S_{hW} = \pi_N(S_{h^{-1}W}\alpha S_{hW}) \subseteq \pi_N(S_{h^{-1}W}AS_{hW}) \subseteq \pi_N(A)$, where the last inclusion follows by the H/W-invariance of A. This shows that $\pi_N(A)$ is H/W-invariant. $\square$

Lemma E.8.8 (cf. [36, Lem. 4.2])**.** *Suppose that S is nearly epsilon-strongly G-graded and that H is a subgroup of G. Let $N \lhd W$ be subgroups of G such that N, W are normalized by H and W/N is a unique product group. Furthermore, let A, B be nonzero ideals of S_W such that $AB = \{0\}$. Then there exist nonzero ideals A', B' of S_N such that $A'B' = \{0\}$. Moreover, the following assertions hold:*

(a) *If A (resp. B) is H/W-invariant, then A' (resp. B') is H/W-invariant.*
(b) *If $A, B \subseteq IS_W$ for some ideal $I \subseteq S_e$, then $A', B' \subseteq IS_N$.*

Proof. Put $A' := \pi_N(\min_N(A))$ and $B' := \pi_N(\min_N(B))$, and note that they are both ideals of S_N by Lemma E.8.7. Let $\alpha = \sum_{x \in G} \alpha_x \in A$ and $\beta = \sum_{x \in G} \beta_x \in B$ be generators of $\min_N(A)$ and $\min_N(B)$, respectively.

Consider the induced W/N-grading on S_W (see Section E.2.5). With this grading, S_W has principal component S_N. Moreover, it follows from Proposition E.2.20 that S_W is a nearly epsilon-strongly W/N-graded ring. Thus, we may w.l.o.g. assume that $N = \{e\}$.

Now, using the fact that W is a unique product group, we write $x_0 y_0$ for the unique product of $(\mathrm{Supp}(\alpha))(\mathrm{Supp}(\beta))$ and deduce from $\alpha\beta \subseteq AB = \{0\}$ that $\alpha_{x_0}\beta_{y_0} = 0$, since no cancelling can occur. But then $\alpha\beta_{y_0} = \sum_{x \in G} \alpha_x \beta_{y_0}$ has smaller support size than that of α. Since α meets a minimal number of cosets of N, it follows that $\alpha\beta_{y_0} = 0$. Hence, $\alpha_x \beta_{y_0} = 0$ for every $x \in W$, which in turn implies that $\alpha_x \beta$ has smaller support size than that of β. As a result, we must have $\alpha_x \beta = 0$. In consequence, we have $\alpha_x \beta_y = 0$ for all $x, y \in W$, and hence $\pi_N(\alpha)\pi_N(\beta) = \alpha_e \beta_e = 0$. Thus, $A'B' = \{0\}$.

Finally, we prove (a) and (b). If A is H/W-invariant, then it follows by Lemma E.8.7 that A' is H/W-invariant. Next, suppose that $A \subseteq IS_W$. Then, $\min_N(A) \subseteq A \subseteq IS_W$. Hence, by Lemma E.2.18, $A' = \pi_N(\min_N(A)) \subseteq \pi_N(IS_W) \subseteq IS_N$. The proof of the corresponding statements for B and B' is completely analogous. □

Proposition E.8.9. *Suppose that S is nearly epsilon-strongly G-graded. If S is not prime, then it has an NP-datum $(H, N, I, \tilde{A}, \tilde{B})$ for which $\tilde{A}, \tilde{B}$ are H/N-invariant.*

Proof. If S_e is not G-semiprime, then the desired conclusion follows from Corollary E.7.3. Now, suppose that S_e is G-semiprime. Then Proposition E.7.4 provides us with a subgroup H of G and an H-invariant ideal I of S_e such that $I^x I = \{0\}$ for every $x \in G \setminus H$. In particular, condition (NP2) holds.

To proceed, we apply Lemma E.8.3, which yields nonzero $H/\Delta(H)$-invariant ideals A_1, B_1 of $S_{\Delta(H)}$ such that $A_1 B_1 = \{0\}$. Moreover, by Lemma E.8.4 there exists a finitely generated normal subgroup W of H with $W \subseteq \Delta(H)$ and nonzero H/W-invariant ideals A_2, B_2 of S_W such that $A_2 B_2 = \{0\}$.

Next, by Proposition E.8.5 there is a finite characteristic subgroup $N \lhd W$ such that W/N is torsion-free abelian. Since N is a characteristic subgroup, we get that $N \lhd W \lhd H$. This establishes condition (NP1). Moreover, by a well-known result by Levi [26], W/N is an ordered group, and hence a unique product group. Note that H normalizes N and W. This means that Lemma E.8.8 is at our disposal, i. e. there are nonzero H/W-invariant, and in particular H/N-invariant, ideals $\tilde{A}, \tilde{B}$ of S_N such that $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A}\tilde{B} = \{0\}$. Hence, condition (NP3) holds. This shows that $(H, N, I, \tilde{A}, \tilde{B})$ is an NP-datum for S. □

E.9. Proof of the main theorem

In this section, we finish the proof of Theorem E.1.3 and show that Passman's result (see Theorem E.1.1) can be recovered from it.

Proof of Theorem E.1.3. (1) Suppose that S is non-degenerately G-graded.

(e)⇒(d): Suppose that (e) holds. By Lemma E.3.13, $\tilde{A}, \tilde{B}$ are H-invariant. It only remains to show that $\tilde{A}S_H\tilde{B} = \{0\}$. Take $x \in H$. Seeking a contradiction, suppose that $\tilde{A}S_{xN}\tilde{B} \neq \{0\}$. Note that $\tilde{A}S_{xN}\tilde{B} \subseteq S_{xN}$. By non-degeneracy of the G-grading on S, it follows that S_H is non-degenerately H-graded. Hence, by Proposition E.2.21, the H/N-grading on S_H is also non-degenerate. Consequently, $S_{x^{-1}N}\tilde{A}S_{xN}\tilde{B} \neq \{0\}$. By the H/N-invariance of $\tilde{A}$ we get that $\{0\} \neq S_{x^{-1}N}\tilde{A}S_{xN}\tilde{B} \subseteq \tilde{A}\tilde{B} = \{0\}$ which is a contradiction. We conclude that $\tilde{A}S_x\tilde{B} \subseteq \tilde{A}S_{xN}\tilde{B} = \{0\}$. Thus, $\tilde{A}S_H\tilde{B} = \{0\}$.

(d)⇒(c)⇒(b): This is trivial.

(b)⇒(a): This follows from Proposition E.5.3.

(2) Suppose that S is nearly epsilon-strongly G-graded. By Proposition E.2.15, S is non-degenerately G-graded. Hence, by (1) we get that (e)⇒(d)⇒(c)⇒(b)⇒(a). The remaining implication, (a)⇒(e), follows from Proposition E.8.9. □

Proof of Theorem E.1.1. Let S be a unital strongly G-graded ring. The claim of Theorem E.1.1 follows immediately from Remark E.5.6 and the equivalence (a)⇔(d) in Theorem E.1.3. □

E.10. Applications for torsion-free grading groups

Recall that S is a G-graded ring. In this section, we pay special attention to the case when G is torsion-free. The following result generalizes Corollary E.4.14 and establishes Theorem E.1.4:

Theorem E.10.1 (cf. [36, Cor. 4.6]). *Suppose that G is torsion-free and that S is nearly epsilon-strongly G-graded. Then S is prime if and only if S_e is G-prime.*

Proof. Suppose that S is not prime. By Theorem E.1.3, there is a balanced NP-datum
$(H, N, I, \tilde{A}, \tilde{B})$ for S. Using that G is torsion-free, we conclude that $N = \{e\}$. In consequence, $S_N = S_e$ and $I, \tilde{A}, \tilde{B}$ are all ideals of S_e. Consider the sets $\tilde{A}^G$ and $\tilde{B}^G$. By Proposition E.3.19 they are nonzero G-invariant ideals of S_e. Note that $\tilde{A}S_x\tilde{B} = \{0\}$ for every $x \in G$ by the same argument as in the proof of Proposition E.5.3. Using this, we get that $\tilde{A}^G\tilde{B}^G = \{0\}$ and hence S_e is not G-prime.

Now suppose that S is prime. By Corollary E.5.8, it follows that S_e is G-prime. □

Remark E.10.2. Note that a strongly G-graded ring with local units is necessarily nearly epsilon-strongly G-graded (see Lemma E.2.16). Hence, [3, Thm. 3.1] by Abrams and Haefner follows from Theorem E.10.1.

The following corollary is similar to a result by Öinert [33, Thm. 4.4]:

Corollary E.10.3. *Suppose that G is torsion-free and that S is nearly epsilon-strongly G-graded. If S_e is prime, then S is prime.*

Example E.10.4. Let R be a unital ring, let u be an idempotent of R, and let $\alpha : R \to uRu$ be a corner ring isomorphism. In this example we consider the corner skew Laurent polynomial ring $R[t_+, t_-, \alpha]$ which was introduced by Ara, Gonzalez-Barroso, Goodearl and Pardo in [5]. For the convenience of the reader we now briefly recall its definition: $R[t_+, t_-, \alpha]$ is the universal unital ring satisfying the following two conditions:
(a) there is a unital ring homomorphism $i : R \to R[t_+, t_-, \alpha]$;
(b) $R[t_+, t_-, \alpha]$ is the R-algebra satisfying the following equations for every $r \in R$:

$$t_-t_+ = 1, \qquad t_+t_- = i(u), \qquad rt_- = t_-\alpha(r), \qquad t_+r = \alpha(r)t_+.$$

Assigning degrees -1 to t_- and 1 to t_+ turns $R[t_+, t_-, \alpha]$ into a $\mathbb{Z}$-graded ring with principal component R. By [23, Prop. 8.1], $R[t_+, t_-, \alpha]$ is nearly epsilon-strongly $\mathbb{Z}$-graded. Hence, if R is prime, then it follows from Corollary E.10.3 that $R[t_+, t_-, \alpha]$ is also prime. Of course, when $u = 1$ and α is the identity map, then $R[t_+, t_-, \alpha]$ is the familiar ring $R[t, t^{-1}]$.

E.11. Applications to s-unital strongly graded rings

In this section, we apply our results to s-unital strongly G-graded rings. Recall that, by Lemma E.2.16, every s-unital strongly G-graded ring is nearly epsilon-strongly G-graded. Thus, by Theorem E.1.3, we obtain the following s-unital generalization of Passman's Theorem E.1.1:

Corollary E.11.1. *Suppose that S is an s-unital strongly G-graded ring. Then S is not prime if and only if it has an NP-datum $(H, N, I, \tilde{A}, \tilde{B})$ for which $\tilde{A}, \tilde{B}$ are both H-invariant.*

E.11.1. Morita context algebras. Let S be an s-unital strongly G-graded ring. For every $x \in G$ the canonical multiplication map $m_x : S_x \otimes_{S_e} S_{x^{-1}} \to S_e$, $a \otimes b \mapsto ab$ is an isomorphism of S_e-bimodules. Indeed, m_x is well-defined and surjective, using that S is strongly G-graded. Moreover, the injectivity is a consequence of the s-unitality. Noteworthily, by associativity of the multiplication, for every $x \in G$ we also have

$$\begin{aligned} m_x \otimes \mathrm{id} = \mathrm{id} \otimes m_{x^{-1}} &: S_x \otimes_{S_e} S_{x^{-1}} \otimes_{S_e} S_x \to S_x \\ m_{x^{-1}} \otimes \mathrm{id} = \mathrm{id} \otimes m_x \quad &: S_{x^{-1}} \otimes_{S_e} S_x \otimes_{S_e} S_{x^{-1}} \to S_{x^{-1}}. \end{aligned}$$

Thus, for every $x \in G$ we get a quintupel $(S_e, S_x, S_{x^{-1}}, m_x, m_{x^{-1}})$ which is usually referred to as a *strict Morita context.*

Next, let us consider an s-unital ring R and a strict Morita context $(R, M, N, \mu_1, \mu_{-1})$, i.e. we have R-bimodules M, N and R-bimodule isomorphisms

$$\mu_1 : M \otimes_R N \to R, \qquad \mu_{-1} : N \otimes_R M \to R$$

satisfying the mixed associativity conditions $\mu_1 \otimes \mathrm{id} = \mathrm{id} \otimes \mu_{-1}$ and $\mu_{-1} \otimes \mathrm{id} = \mathrm{id} \otimes \mu_1$. Furthermore, we assume that $RM = MR = M$ and $RN = NR = N$. We form a $\mathbb{Z}$-graded module S by putting

$$S_n := \begin{cases} R & n = 0 \\ M^{\otimes_R^n} & n > 0 \\ N^{\otimes_R^{-n}} & n < 0. \end{cases}$$

We wish to turn S into a $\mathbb{Z}$-graded ring. The product of two positively graded elements is just the usual tensor product $\otimes_R$ of tensor products of M's, and similarly the product of two negatively graded elements is just the usual tensor product of N's. To deal with products of mixed elements, we repeatedly make use of the maps μ_1 and μ_{-1}. By the mixed associativity conditions, this multiplication becomes associative, and hence S is a $\mathbb{Z}$-graded ring as desired. In addition, as the maps μ_1 and μ_{-1} are surjective, we may infer that S is strongly $\mathbb{Z}$-graded. Clearly, S is s-unital. Finally, Theorem E.10.1 implies that if R is $\mathbb{Z}$-prime, then S is prime.

E.11.2. s-unital strongly graded matrix rings. In what follows, let R be an s-unital ring. Let $M_{\mathbb{Z}}(R)$ denote the ring of infinite $\mathbb{Z} \times \mathbb{Z}$-matrices with only finitely many nonzero entries in R. For $r \in R$ and $i, j \in \mathbb{Z}$ we write $re_{i,j}$ for the matrix in $M_{\mathbb{Z}}(R)$ with r in the ijth position and zeros elsewhere. We regard $M_{\mathbb{Z}}(R)$ as a $\mathbb{Z}$-graded ring with respect to

$$\deg(re_{i,j}) := i - j \quad \text{for all } i, j \in \mathbb{Z} \text{ and all nonzero } r \in R. \tag{44}$$

The corresponding homogeneous components of the $\mathbb{Z}$-grading are given by

$$(M_{\mathbb{Z}}(R))_k = \bigoplus_{i \in \mathbb{Z}} Re_{i+k,i}, \qquad k \in \mathbb{Z}.$$

In particular, $(M_{\mathbb{Z}}(R))_0 = \bigoplus_{i \in \mathbb{Z}} Re_{i,i}$ is the main diagonal.

Lemma E.11.2. *The ring $M_{\mathbb{Z}}(R)$ is s-unital and strongly $\mathbb{Z}$-graded with respect to the grading defined by* (44).

Proof. Put $S := M_{\mathbb{Z}}(R)$. By Proposition E.2.1 and s-unitality of R, it follows that S is s-unital and that $S_0 S_n = S_n S_0 = S_n$, for every $n \in \mathbb{Z}$. Take $k \in \mathbb{Z}$. Since R is s-unital, and hence idempotent, we get that $S_k S_{-k} = (\sum_{i \in \mathbb{Z}} Re_{i+k,i})(\sum_{j \in \mathbb{Z}} Re_{j-k,j}) = \sum_{i \in \mathbb{Z}} R^2 e_{i+k,i} e_{i,i+k} = \sum_{i \in \mathbb{Z}} Re_{i+k,i+k} = S_0$. The claim now follows from Proposition E.2.2. □

Corollary E.11.3. *The ring $M_{\mathbb{Z}}(R)$ is prime if and only if R is prime.*

Proof. Suppose that R is not prime, i. e. there are nonzero ideals A, B of R such that $AB = \{0\}$. Then $M_{\mathbb{Z}}(A) \cdot M_{\mathbb{Z}}(B) = \{0\}$ which shows that $M_{\mathbb{Z}}(R)$ is not prime. Conversely suppose that R is prime. Note that any ideal I of $(M_{\mathbb{Z}}(R))_0$ is of the form $I = \bigoplus_{i \in \mathbb{Z}} I_i e_{i,i}$ for some family of R-ideals I_i, $i \in \mathbb{Z}$, and it is $\mathbb{Z}$-invariant if and only if $I_i = I_0$ for every $i \in \mathbb{Z}$. Next, let A, B be $\mathbb{Z}$-invariant ideals of $(M_{\mathbb{Z}}(R))_0$ such that $AB = \{0\}$. There are R-ideals A_0, B_0 such that $A = \bigoplus_{i \in \mathbb{Z}} A_0 e_{i,i}$ and $B = \bigoplus_{i \in \mathbb{Z}} B_0 e_{i,i}$. Since $AB = \{0\}$, we see that $A_0 B_0 = \{0\}$ and thus $A_0 = \{0\}$ or $B_0 = \{0\}$ due to the primeness of R. Hence, $A = \{0\}$ or $B = \{0\}$. Consequently, $(M_{\mathbb{Z}}(R))_0$ is $\mathbb{Z}$-prime and hence $M_{\mathbb{Z}}(R)$ is prime by Theorem E.10.1. □

Remark E.11.4. The above result shows that primeness of the principal component is not a necessary condition for primeness of a strongly graded ring. Nevertheless, by Corollary E.5.8, G-primeness of S_e is a necessary condition.

Now we fix $n \in \mathbb{N}$ and consider $M_n(R)$, the ring of $n \times n$-matrices with entries in R. The ring $M_n(R)$ comes equipped with a natural $\mathbb{Z}$-grading defined by

$$\deg(re_{i,j}) := i - j \quad \text{for all } i, j \in \{1, \ldots, n\} \text{ and all nonzero } r \in R. \tag{45}$$

Lemma E.11.5. *The ring $M_n(R)$ is nearly epsilon-strongly $\mathbb{Z}$-graded with respect to the grading defined by* (45).

Proof. Put $S := M_n(R)$. Take $k \in \mathbb{Z}$ and $r \in R$. Note that for i, j such that $i - j = k$, and $a, b, c \in R$ such that $abc = r$, we have $ae_{i,j}, ce_{i,j} \in S_k$, $be_{j,i} \in S_{-k}$ and $ae_{i,j}be_{j,i}ce_{i,j} = re_{i,j}$. Take $s \in S_k$. Then $s = \sum_{i-j=k} r_{i,j}e_{i,j} \in S_k$ for some $r_{i,j} \in R$. By Proposition E.2.1 and s-unitality of R, there is $u \in R$ such that $ur_{i,j} = r_{i,j}u = r_{i,j}$ for all i, j. Put $v := \sum_{i-j=k} ue_{i,j}ue_{j,i} \in S_kS_{-k}$ and $w := \sum_{i-j=k} ue_{j,i}ue_{i,j} \in S_{-k}S_k$. Then $vs = s$ and $sw = s$. This shows that S is nearly epsilon-strongly $\mathbb{Z}$-graded. $\square$

Note that if R is prime, then $(M_n(R))_0$ is $\mathbb{Z}$-prime. Hence, by Corollary E.4.14 and Lemma E.11.5, we obtain the following s-unital generalization of a well-known result:

Corollary E.11.6 (cf. [21, Prop. 10.20])**.** *The ring $M_n(R)$ is prime if and only if R is prime.*

The $\mathbb{Z}$-grading on $M_n(R)$ defined above induces a $\mathbb{Z}/n\mathbb{Z}$-grading on $M_n(R)$ (see Section E.2.5). By Lemma E.11.5 and Proposition E.2.20, this turns $M_n(R)$ into a nearly epsilon-strongly $\mathbb{Z}/n\mathbb{Z}$-graded ring. By using an argument similar to the one in the proof of Lemma E.11.2, it is not difficult to see that this grading is, in fact, strong. Hence, Corollary E.11.1 is applicable but presently it is not clear to the authors how to use it to prove Corollary E.11.6.

E.12. Applications to s-unital skew group rings

Connell [10] famously gave a characterization of when a unital group ring $R[G]$ is prime. In this section, we generalize and recover his result from our main theorem. More precisely, we describe when an s-unital group ring $R[G]$ is prime.

Let R be a (possibly non-unital) ring and let $\alpha : G \to \mathrm{Aut}(R)$ be a group homomorphism. We define the *skew group ring* $R \star_\alpha G$ as the set of all formal sums $\sum_{x \in G} r_x\delta_x$ where δ_x is a symbol for each $x \in G$ and $r_x \in R$ is zero for all but finitely many $x \in G$. Addition on $R \star_\alpha G$ is defined in the natural way and multiplication is defined by linearly extending the rules $r\delta_x r'\delta_y = r\alpha_x(r')\delta_{xy}$, for all $r, r' \in R$ and $x, y \in G$. This yields an associative ring structure on $R \star_\alpha G$. Moreover, $S = R \star_\alpha G$ is canonically G-graded by putting $S_x := R\delta_x$ for every $x \in G$. If $\alpha_x = \mathrm{id}_R$ for every $x \in G$, then we simply write $R[G]$ for $R \star_\alpha G$ and call it a *group ring*. Note that $R \star_\alpha G$ is a so-called *partial skew group ring* (see Section E.13).

Proposition E.12.1. *Suppose that R is a ring and that $\alpha : G \to \mathrm{Aut}(R)$ is a group homomorphism. The following assertions are equivalent:*

(a) *R is idempotent;*
(b) *$R \star_\alpha G$ is strongly G-graded;*
(c) *$R \star_\alpha G$ is symmetrically G-graded.*

PROOF. (a)⇒(b): Suppose that R is idempotent, i.e. $R^2 = R$. Then for all $x, y \in G$ we have $(R\delta_x)(R\delta_y) = R\alpha_x(R)\delta_{xy} = R\delta_{xy}$. In other words, $R \star_\alpha G$ is strongly G-graded.

(b)⇒(c): This holds in general for strongly G-graded rings (see [24, Prop. 4.45]).

(c)⇒(a): This holds in general for symmetrically G-graded rings (see [24, Prop. 4.47]). □

Proposition E.12.2. *Suppose that R is a ring and that $\alpha : G \to \mathrm{Aut}(R)$ is a group homomorphism. The following assertions are equivalent:*

(a) *R is s-unital;*
(b) *$R \star_\alpha G$ is s-unital strongly G-graded;*
(c) *$R \star_\alpha G$ is nearly epsilon-strongly G-graded.*

PROOF. (a)⇒(b): Suppose that R is s-unital. In particular, R is idempotent. Hence, $R \star_\alpha G$ is strongly G-graded by Proposition E.12.1. It is easy to see that $R \star_\alpha G$ is s-unital.

(b)⇒(c): This follows from Lemma E.2.16.

(c)⇒(a): This holds for any nearly epsilon-strongly graded ring (see Proposition E.2.13). □

Example E.12.3. In this example we consider the s-unital ring $M_{\mathbb{N}}(\mathbb{R})$ of $\mathbb{N} \times \mathbb{N}$-matrices with only finitely many nonzero entries in $\mathbb{R}$. Recall that the group $\mathrm{SO}_3(\mathbb{R})$ of rotations in $\mathbb{R}^3$ contains a subgroup F isomorphic to free group of rank 2 (see e.g. [17, 42]). For every $x \in F \subseteq \mathrm{SO}_3(\mathbb{R})$ we may define a diagonal matrix $\mathrm{diag}(x, x, x, \ldots)$ which is row-finite and column-finite but does not belong to $M_{\mathbb{N}}(\mathbb{R})$. We thus obtain a group homomorphism $\alpha : F \to \mathrm{Aut}\big(M_{\mathbb{N}}(\mathbb{R})\big)$ by putting

$$\alpha_x(a) := \mathrm{diag}(x, x, x, \ldots)\, a\, \mathrm{diag}(x^{-1}, x^{-1}, x^{-1}, \ldots)$$

for $x \in F$ and $a \in M_{\mathbb{N}}(\mathbb{R})$. Since $M_{\mathbb{N}}(\mathbb{R})$ is simple and F is torsion-free, it follows from Corollary E.10.3 that the s-unital skew group ring $M_{\mathbb{N}}(\mathbb{R}) \star_\alpha F$ is prime.

We proceed to prove Theorem E.1.5 by using our main theorem:

Theorem E.12.4. *Suppose that R is an s-unital ring. Then the group ring $R[G]$ is prime if and only if R is prime and G has no non-trivial finite normal subgroup.*

PROOF. We prove the converse statement: $R[G]$ is not prime if and only if R is not prime or G has a non-trivial finite normal subgroup. By Proposition E.12.2, (a)⇔(c) in Theorem E.1.3 holds for $S = R[G]$. In other words, the group ring $R[G]$ is not prime if and only if it has a balanced NP-datum. We prove that the G-graded ring $R[G]$ has a balanced NP-datum if and only if R is not prime or G has a non-trivial finite normal subgroup. First note that for any ideal I of R we have $I^x = R\delta_{x^{-1}} I R\delta_x = RIR\delta_e = I\delta_e$ for every $x \in G$. In particular, every ideal of R is G-invariant.

Suppose that $(H, N, I, \tilde{A}, \tilde{B})$ is a balanced NP-datum for $R[G]$.

<u>Case 1: $H = G$.</u> Note that $N \lhd H = G$ is a finite normal subgroup of G. Condition (NP4) proves that $R[N]$ is not prime. Then either $N = \{e\}$ and R is not prime or there exists a non-trivial finite normal subgroup N of G.

Case 2: $H \subsetneq G$. Note that condition (NP2) implies that there is a nonzero ideal I of R such that $I^2 = \{0\}$. Thus R is not prime.

Now we prove the converse statement.

Case I: R is not prime. There are nonzero ideals $\tilde{A}, \tilde{B}$ of R such that $\tilde{A}\tilde{B} = \{0\}$. This implies that $\tilde{A}R\delta_x\tilde{B} = R\delta_x\tilde{A}\tilde{B} = \{0\}$ for every $x \in G$. Therefore, $\tilde{A} \cdot R[G] \cdot \tilde{B} = \{0\}$. We note that $(G, \{e\}, R, \tilde{A}, \tilde{B})$ is a balanced NP-datum.

Case II: there exists a non-trivial finite normal subgroup N of G.

Consider $H := G$ and $I := R$. Pick a nonzero $a \in R$. Let $\tilde{A}$ be the ideal of S_N generated by the element $\sum_{n \in N} a\delta_n$ and let $\tilde{B}$ be the ideal of S_N generated by the set $\{r\delta_n - r\delta_e \mid n \in N, r \in R\}$. Since N is non-trivial, it follows that $\tilde{A}$ and $\tilde{B}$ are nonzero ideals of S_N. Next, let $t \in R$, $x \in G$ and $n_1 \in N$. Then, since N is finite and normal in G,

$$\begin{aligned}\left(\sum_{n \in N} a\delta_n\right) t\delta_x(r\delta_{n_1} - r\delta_e) &= \left(\sum_{n \in N} at\delta_{nx}\right)(r\delta_{n_1} - r\delta_e) = \\ &= \left(\sum_{n \in N} at\delta_{xn}\right)(r\delta_{n_1} - r\delta_e) = \\ &= a\delta_x\left(\sum_{n \in N} t\delta_n\right)(r\delta_{n_1} - r\delta_e) = \\ &= a\delta_x\left(\sum_{n \in N} tr\delta_{nn_1} - \sum_{n \in N} tr\delta_{ne}\right) = \\ &= a\delta_x\left(\sum_{n \in N} tr\delta_n - \sum_{n \in N} tr\delta_n\right) = \\ &= 0.\end{aligned}$$

This shows that $\tilde{A} \cdot R[G] \cdot \tilde{B} = \{0\}$. Hence, $(H, N, I, \tilde{A}, \tilde{B})$ is a balanced NP-datum. □

Remark E.12.5. Note that Theorem E.12.4 applies to s-unital group rings $R[G]$ which are not necessarily unital. Hence, this application shows that our results indeed reach farther than Passman's results [37, 35, 36] which are only concerned with unital rings.

Remark E.12.6. The above result can not be generalized to s-unital (unital) skew group rings. Indeed, neither primeness of R nor the non-existence of non-trivial finite normal subroups of G are necessary conditions for primeness of an s-unital skew group ring $R \star_\alpha G$. To see this, consider the matrix algebra $M_4(\mathbb{R}) \cong \mathbb{R}^4 \star_\alpha \mathbb{Z}/4\mathbb{Z}$ as a unital skew group ring. It is well-known that $M_4(\mathbb{R})$ is prime, but $\mathbb{R}^4$ is not prime and $\mathbb{Z}/4\mathbb{Z}$ contains a non-trivial finite normal subgroup. Note, however, that in this case $\mathbb{R}^4$ is actually $\mathbb{Z}/4\mathbb{Z}$-prime.

Example E.12.7. Suppose that G is torsion-free and let $F(G, \mathbb{C})$ be the algebra of all complex-valued functions on G with finite support, under pointwise addition and multiplication. Note that $F(G, \mathbb{C})$ is s-unital. We define a map $\alpha : G \to \operatorname{Aut}\big(F(G, \mathbb{C})\big)$

by putting $\alpha_x(f)(y) := f(x^{-1}y)$ for all $x, y \in G$ and $f \in F(G, \mathbb{C})$. Clearly, $F(G, \mathbb{C})$ is G-prime. Using Theorem E.10.1, we get that $F(G, \mathbb{C}) \star_\alpha G$ is prime.

Remark E.12.8. Consider the non-unital group ring $R[G]$ where $R := 2\mathbb{Z}$ and $G := \mathbb{Z}$. Note that R is not s-unital and hence $R[G]$ is not nearly epsilon-strongly G-graded (see Proposition E.2.13). It is, however, non-degenerately G-graded. In fact, it is not difficult to see that $R[G]$ is a domain and hence prime. From this and the fact that G is torsion-free, we easily see that the equivalences (a)⇔(b)⇔(c)⇔(d)⇔(e) in Theorem E.1.3 hold for $R[G]$. This example suggests that it might be possible to generalize Theorem E.1.3.

E.13. Applications to crossed products defined by partial actions

A significant development in the study of C^*-algebras was the introduction of the notion of a *partial action* by Exel [14]. Various algebraic analogues of this notion were developed and studied during the last two decades (see e. g. [7, 13, 12]).

In this section, we apply our main theorem to obtain results on primeness of s-unital partial skew group rings (see Section E.13.1) and of unital partial crossed products (see Section E.13.2). We also apply our results to some particular examples of partial skew group rings associated with partial dynamical systems (see Section E.13.3).

E.13.1. Partial skew group rings. Recall that a *partial action of G on an s-unital ring R* (see [13, p. 1932]) is a pair $(\{\alpha_g\}_{g\in G}, \{D_g\}_{g\in G})$, where for all $g, h \in G$, D_g is a (possibly zero) s-unital ideal of R, $\alpha_g \colon D_{g^{-1}} \to D_g$ is a ring isomorphism. We require that the following conditions hold for all $g, h \in G$:

(P1) $\alpha_e = \mathrm{id}_R$;
(P2) $\alpha_g(D_{g^{-1}} D_h) = D_g D_{gh}$;
(P3) if $r \in D_{h^{-1}} D_{(gh)^{-1}}$, then $\alpha_g(\alpha_h(r)) = \alpha_{gh}(r)$.

Given a partial action of G on R, we can form the *s-unital partial skew group ring* $R \star_\alpha G := \bigoplus_{g\in G} D_g \delta_g$ where the δ_g's are formal symbols. For $g, h \in G, r \in D_g$ and $r' \in D_h$ the multiplication is defined by the rule:

$$(r\delta_g)(r'\delta_h) = \alpha_g(\alpha_{g^{-1}}(r)r')\delta_{gh}$$

It can be shown that $R \star_\alpha G$ is an associative ring (see e. g. [13, Cor. 3.2]). Moreover, $S := R \star_\alpha G$ is canonically G-graded by putting $S_g := D_g\delta_g$ for every $g \in G$.

Proposition E.13.1. *The canonical G-grading on $R \star_\alpha G$ is nearly epsilon-strong.*

Proof. Take $g \in G$. Note that

$$\begin{aligned} S_g S_{g^{-1}} = D_g \delta_g D_{g^{-1}} \delta_{g^{-1}} &= \alpha_g(\alpha_{g^{-1}}(D_g) D_{g^{-1}})\delta_e = \\ &= \alpha_g(D_{g^{-1}} D_{g^{-1}})\delta_e = \alpha_g(D_{g^{-1}})\delta_e = D_g\delta_e \end{aligned}$$

and hence

$$S_g S_{g^{-1}} S_g = (S_g S_{g^{-1}}) S_g = D_g \delta_e D_g \delta_g = D_g^2 \delta_g = D_g \delta_g = S_g.$$

This shows that the G-grading is symmetrical and that $S_g S_{g^{-1}}$ is s-unital for every $g \in G$. By Proposition E.2.11 the desired conclusion follows. □

Remark E.13.2. We will identify R with $R\delta_e$ via the canonical isomorphism.

Definition E.13.3. Let H be a subgroup of G. An ideal I of R is called *H-invariant* if $\alpha_h(ID_{h^{-1}}) \subseteq I$ for every $h \in H$. The ring R is called *G-prime* if for all G-invariant ideals I, J of R, we have $I = \{0\}$ or $J = \{0\}$, whenever $IJ = \{0\}$.

Remark E.13.4. Consider $S := R \star_\alpha G$ with its canonical G-grading.
(a) Let H be a subgroup of G. Note that, for $h \in H$, we have

$$\begin{aligned} I^h \subseteq I &\iff D_{h^{-1}}\delta_{h^{-1}} \cdot I \cdot D_h\delta_h \subseteq I\delta_e \iff \alpha_{h^{-1}}(\alpha_h(D_{h^{-1}})ID_h)\delta_e \subseteq I\delta_e \\ &\iff \alpha_{h^{-1}}(D_hID_h)\delta_e \subseteq I\delta_e \iff \alpha_{h^{-1}}(D_hID_h) \subseteq I \iff \alpha_{h^{-1}}(ID_h) \subseteq I. \end{aligned}$$

This shows that G-invariance in the sense of Definition E.13.3 is equivalent to G-invariance defined by the G-grading (see Definition E.3.3).
(b) By (a) we note that R is G-prime if and only if S_e is G-prime.

Theorem E.13.5. *Suppose that G is torsion-free and that $R \star_\alpha G$ is an s-unital partial skew group ring. Then $R \star_\alpha G$ is prime if and only if R is G-prime.*

Proof. This follows from Proposition E.13.1, Theorem E.10.1 and Remark E.13.4(b). $\square$

We proceed to characterize prime s-unital partial skew group rings for general groups.

Lemma E.13.6. *Suppose that $(\{\alpha_g\}_{g\in G}, \{D_g\}_{g\in G})$ is a partial action of G on R, and that I is an ideal of R. For any subgroup H of G, the following holds:*

$$\alpha_h(ID_{h^{-1}}) \subseteq I, \quad \forall h \in H \quad \iff \quad \alpha_h(ID_{h^{-1}}) = ID_h, \quad \forall h \in H$$

Proof. Take $h \in G$.
($\Leftarrow$): Clear, since $ID_h \subseteq I$.
($\Rightarrow$): Note that $I \cap D_h = I \cdot D_h$, by s-unitality of D_h. Thus, $\alpha_h(ID_{h^{-1}}) \subseteq I$ implies $\alpha_h(ID_{h^{-1}}) \subseteq I \cap D_h = ID_h$. By applying $\alpha_{h^{-1}}$ to both sides, and using that h is arbitrary, we get $ID_{h^{-1}} \subseteq \alpha_{h^{-1}}(ID_h) \subseteq ID_{h^{-1}}$. Hence, $\alpha_{h^{-1}}(ID_h) = ID_{h^{-1}}$. $\square$

Theorem E.13.7. *The s-unital partial skew group ring $R \star_\alpha G$ is not prime if and only if there are:*

(i) *subgroups $N \lhd H \subseteq G$ with N finite,*
(ii) *an ideal I of R such that*
   - *$\alpha_h(ID_{h^{-1}}) = ID_h$ for every $h \in H$,*
   - *$ID_g \cdot \alpha_g(ID_{g^{-1}}) = \{0\}$ for every $g \in G \setminus H$, and*

(iii) *nonzero ideals $\tilde{A}, \tilde{B}$ of $R \star_\alpha N$ such that $\tilde{A}, \tilde{B} \subseteq I\delta_e(R \star_\alpha N)$ and $\tilde{A} \cdot D_h\delta_h \cdot \tilde{B} = \{0\}$ for every $h \in H$.*

Proof. By Proposition E.13.1, we may apply Theorem E.1.3 to $S := R \star_\alpha G$. For $g \in G$, we get

$$\begin{aligned} I^g \cdot I = \{0\} &\iff D_{g^{-1}}\delta_{g^{-1}} \cdot I \cdot D_g\delta_g \cdot I\delta_e = \{0\} \iff \alpha_{g^{-1}}(\alpha_g(D_{g^{-1}}) \cdot ID_g)\delta_e \cdot I\delta_e = \{0\} \\ &\iff (\alpha_{g^{-1}}(D_g \cdot ID_g) \cdot I)\delta_e = \{0\} \iff \alpha_{g^{-1}}(D_gID_g) \cdot I = \{0\} \\ &\iff \alpha_{g^{-1}}(D_gID_g) \cdot ID_{g^{-1}} = \{0\} \iff D_gID_g \cdot \alpha_g(ID_{g^{-1}}) = \{0\}. \end{aligned}$$

Using that I, D_g are ideals of R and that D_g is s-unital, we get that $D_g I D_g \subseteq I D_g \subseteq D_g(I D_g)$. Hence, $D_g I D_g = I D_g$. We conclude that $I^g \cdot I = \{0\}$ if and only if $I D_g \cdot \alpha_g(I D_{g^{-1}}) = \{0\}$. The desired conclusion now follows by Remark E.13.4(a) and Lemma E.13.6. □

E.13.2. Unital partial crossed products. Recall that a *unital twisted partial action of G on a unital ring R* (see [32, p. 2]) is a triple

$$(\{\alpha_g\}_{g\in G}, \{D_g\}_{g\in G}, \{w_{g,h}\}_{(g,h)\in G\times G}),$$

where for all $g, h \in G$, D_g is a unital ideal of R, $\alpha_g \colon D_{g^{-1}} \to D_g$ is a ring isomorphism and $w_{g,h}$ is an invertible element in $D_g D_{gh}$. Let $1_g \in Z(R)$ denote the (not necessarily nonzero) multiplicative identity element of the ideal D_g. We require that the following conditions hold for all $g, h \in G$:

(UP1) $\alpha_e = \mathrm{id}_R$;
(UP2) $\alpha_g(D_{g^{-1}} D_h) = D_g D_{gh}$;
(UP3) if $r \in D_{h^{-1}} D_{(gh)^{-1}}$, then $\alpha_g(\alpha_h(r)) = w_{g,h}\alpha_{gh}(r) w_{g,h}^{-1}$;
(UP4) $w_{e,g} = w_{g,e} = 1_g$;
(UP5) if $r \in D_{g^{-1}} D_h D_{hl}$, then $\alpha_g(r w_{h,l}) w_{g,hl} = \alpha_g(r) w_{g,h} w_{gh,l}$.

Given a unital twisted partial action of G on R, we can form the *unital partial crossed product* $R \star_\alpha^w G := \bigoplus_{g\in G} D_g \delta_g$ where the δ_g's are formal symbols. For $g, h \in G, r \in D_g$ and $r' \in D_h$ the multiplication is defined by the rule:

$$(r\delta_g)(r'\delta_h) = r\alpha_g(r' 1_{g^{-1}}) w_{g,h} \delta_{gh}$$

It can be shown that $R \star_\alpha^w G$ is an associative ring (see e. g. [12, Thm. 2.4]). Moreover, Nystedt, Öinert and Pinedo established in [32, Thm. 35] that its natural G-grading is epsilon-strong, and in particular nearly epsilon-strong. Thus, Theorem E.1.3 is applicable.

Remark E.13.8. (a) Let H be a subgroup of G. Note that an ideal I of R is H-invariant (in the sense of Definition E.13.3) if and only if $\alpha_h(I 1_{h^{-1}}) \subseteq I$ for every $h \in H$.
(b) We also define G-primeness of R according to Definition E.13.3. By a computation, similar to the one in Remark E.13.4, we note that G-primeness of R is equivalent to G-primeness of S_e.

The next result partially generalizes Theorem E.13.5.

Theorem E.13.9. *Suppose that G is torsion-free and that $R \star_\alpha^w G$ is a unital partial crossed product. Then $R \star_\alpha^w G$ is prime if and only if R is G-prime.*

Proof. Using the fact that unital partial crossed products are epsilon-strongly graded (see [32, Thm. 35]), the desired conclusion follows from Theorem E.10.1 and Remark E.13.8(b). □

The proof of the following result is similar to the proof of Theorem E.13.5 and is therefore omitted.

Theorem E.13.10. *The unital partial crossed product $R \star_\alpha^w G$ is not prime if and only if there are:*

(i) *subgroups* $N \lhd H \subseteq G$ *with* N *finite,*

(ii) *an ideal* I *of* R *such that*
  - $\alpha_h(I1_{h^{-1}}) = I1_h$ *for every* $h \in H$,
  - $I \cdot \alpha_g(I1_{g^{-1}}) = \{0\}$ *for every* $g \in G \setminus H$, *and*

(iii) *nonzero ideals* $\tilde{A}, \tilde{B}$ *of* $R \star_\alpha^w N$ *such that* $\tilde{A}, \tilde{B} \subseteq I \cdot (R \star_\alpha^w N)$ *and* $\tilde{A} \cdot 1_h \delta_h \cdot \tilde{B} = \{0\}$ *for every* $h \in H$.

E.13.3. Partial dynamical systems. In this section we consider several examples of partial skew group rings coming from a particular type of partial dynamical system (cf. [15]).

Let X be a topological space and let A_1, A_2, B_1, B_2 be subspaces of X. Furthermore, let $h_1 : A_1 \to B_1$ and $h_2 : A_2 \to B_2$ be homeomorphisms. For the remainder of this section, G denotes the free group $\mathbb{F}_2 = \langle g_1, g_2 \rangle$. For $g \in G$ we define,

$$\theta_g = \begin{cases} h_j & \text{if } g = g_j \\ h_j^{-1} & \text{if } g = g_j^{-1} \\ \theta_{g_{k_1}}^{\pm 1} \circ \cdots \circ \theta_{g_{k_m}}^{\pm 1} & \text{if } g = g_{k_1}^{\pm} \cdots g_{k_m}^{\pm} \text{ is in reduced form,} \end{cases}$$

where $\circ$ denotes partial function composition. Moreover, we let X_g denote the domain of the function $\theta_{g^{-1}}$. We thus obtain a partial action of G on the space X which we denote by $(\{\theta_g\}_{g \in G}, \{X_g\}_{g \in G})$. This induces a partial action of G on the s-unital ring $R := C_c(X)$, of continuous compactly supported complex-valued functions on X, by putting $D_g := C_c(X_g)$ and defining $\alpha_g : D_{g^{-1}} \to D_g$ by $\alpha_g(f) := f \circ \theta_{g^{-1}}$ for every $g \in G$. Therefore, we may define the s-unital partial skew group ring $S := R \star_\alpha G$.

Example E.13.11. (a) First, we consider $X = \mathbb{R}$ with
- $h_1 : [0, \infty) \to (-\infty, 0]$, $\quad t \mapsto -t$, and
- $h_2 : \mathbb{R} \to \mathbb{R}$, $\quad t \mapsto t + 1$.

It is not difficult to see that $R = C_c(\mathbb{R})$ is not G-prime. Hence, by Theorem E.13.10, $C_c(\mathbb{R}) \star_\alpha G$ is not prime.

(b) Now we consider $X = \mathbb{R}$ with
- $h_1 : [0, \infty) \to [0, \infty)$, $\quad t \mapsto 2t$, and
- $h_2 : \mathbb{R} \to \mathbb{R}$, $\quad t \mapsto t + 1$.

It is not difficult to see that $R = C_c(\mathbb{R})$ is G-prime. Hence, by Theorem E.13.10, $C_c(\mathbb{R}) \star_\alpha G$ is prime.

Example E.13.12. Now we consider X with its discrete topology.

(a) Consider $X = \{x_1, x_2, x_3, x_4\}$ with
- $h_1 : \{x_1, x_2\} \to \{x_3, x_4\}$ given by $h_1(x_1) = x_3$ and $h_1(x_2) = x_4$, and
- $h_2 : \{x_1, x_3\} \to \{x_2, x_4\}$ given by $h_2(x_1) = x_2$ and $h_2(x_3) = x_4$.

Note that the ideals of $C_c(X) \cong \mathbb{C}^4$ correspond bijectively to the 2^4 subsets of X. For arbitrary elements $x, y \in X$ there is $g \in G$ such that $\theta_g(x) = y$. From this we conclude that $R = C_c(X)$ is G-prime. Hence, by Theorem E.13.10, $C_c(X) \star_\alpha G$ is prime.

(b) Consider $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ with
- $h_1 : \{x_1, x_2\} \to \{x_3, x_4\}$ given by $h_1(x_1) = x_3$ and $h_1(x_2) = x_4$, and
- $h_2 : \{x_1, x_3\} \to \{x_2, x_4\}$ given by $h_2(x_1) = x_2$ and $h_2(x_3) = x_4$.

The nonzero ideals $J_1 := C_c(\{x_5\})$ and $J_2 := C_c(\{x_6\})$ of $C_c(X)$ are G-invariant. Clearly, $J_1 J_2 = \{0\}$ and hence $R = C_c(X)$ is not G-prime. By Theorem E.13.10, $C_c(X) \star_\alpha G$ is not prime.

E.14. Applications to Leavitt path algebras

In this section, we use our main theorem to obtain a characterization of prime Leavitt path algebras with coefficients in an arbitrary, possibly non-commutative, unital ring (see Theorem E.14.12). Our result generalizes previous results by Abrams, Bell and Rangaswamy [2, Thm. 1.4], and Larki [25, Prop. 4.5].

The *Leavitt path algebra* $L_K(E)$ over a field K associated with a directed graph E was introduced by Ara, Moreno and Pardo in [6] and independently by Abrams and Aranda Pino in [4]. These algebras are algebraic analogues of graph C^*-algebras. For a thorough account of the history and theory of Leavitt path algebras, we refer the reader to the excellent monograph [1]. Recall that a *directed graph* E is a tuple (E^0, E^1, s, r) where E^0 is the set of vertices, E^1 is the set of edges and $s\colon E^1 \to E^0$ and $r\colon E^1 \to E^0$ are maps specifying the *source* respectively *range* of each edge. For an arbitrary $v \in E^0$, the set $s^{-1}(v) = \{e \in E^1 \mid s(e) = v\}$ is the set of edges emitted from v. If $s^{-1}(v) = \emptyset$, then v is called a *sink*. If $s^{-1}(v)$ is an infinite set, then v is called an *infinite emitter*. A vertex that is neither a sink nor an infinite emitter is called *regular*. A *path* in E is a series of edges $\alpha := f_1 f_2 \dots f_n$ such that $r(f_i) = s(f_{i+1})$ for $i \in \{1, \dots, n-1\}$, and such a path has *length* n which we denote by $|\alpha|$. By convention, we consider a vertex to be a path of length zero. The set of all paths in E is denoted by E^*.

Leavitt path algebras with coefficients in a commutative unital ring was introduced by Tomforde [43] and further studied in [20]. A further generalization was studied by Hazrat [19], and Nystedt and Öinert [31]. Following their lead, we consider Leavitt path algebras with coefficients in a general (possibly non-commutative) unital ring:

Definition E.14.1. Let E be a directed graph and let R be a unital ring. The *Leavitt path algebra of the graph E with coefficients in R*, denoted by $L_R(E)$, is the free associative R-algebra generated by the symbols $\{v \mid v \in E^0\} \cup \{f \mid f \in E^1\} \cup \{f^* \mid f \in E^1\}$ subject to the following relations:

(a) $vw = \delta_{v,w} v$ for all $v, w \in E^0$;
(b) $s(f)f = f r(f) = f$ for every $f \in E^1$;
(c) $r(f)f^* = f^* s(f) = f^*$ for every $f \in E^1$;
(d) $f^* f' = \delta_{f,f'} r(f)$ for all $f, f' \in E^1$;
(e) $\sum_{f \in E^1, s(f)=v} f f^* = v$ for every $v \in E^0$ for which $0 < |s^{-1}(v)| < \infty$.

We let every element of R commute with the generators.

Remark E.14.2. By (a), $\{v \mid v \in E^0\}$ is a set of pairwise orthogonal idempotents in $L_R(E)$.

In the following example, we use Theorem E.4.7 to describe the $\mathbb{Z}$-invariant ideals of $L_K(E)$.

Example E.14.3. Let K be a field and let E be a row-finite directed graph. Recall that there exists a bijection between graded ideals of the Leavitt path algebra $L_K(E)$ and hereditary subsets of E^0 (see [1, Thm. 2.5.9]). Furthermore, since $L_K(E)$ is naturally nearly epsilon-strongly $\mathbb{Z}$-graded (see [31, Thm. 30]), we can apply Theorem E.4.7 to infer that there is a bijection between hereditary subsets of E^0 and $\mathbb{Z}$-invariant ideals of $(L_K(E))_0$. More precisely, we obtain the following explicit description of the $\mathbb{Z}$-invariant ideals of $(L_K(E))_0$:

$$\{I(H) \mid H \subseteq E^0 \text{ hereditary vertex set}\},$$

where $I(H)$ denotes the ideal of $(L_K(E))_0$ generated by the elements $v \in H$.

Recall that $L_R(E)$ comes equipped with a canonical $\mathbb{Z}$-grading defined by $\deg(f) := 1$ and $\deg(f^*) := -1$ for every $f \in E^1$, and $\deg(v) := 0$ for every $v \in E^0$ (cf. [1, Cor. 2.1.5]). In the sequel, the following result will become useful:

Proposition E.14.4 (Nystedt and Öinert [31]). *Suppose that E is a directed graph and that R is a unital ring. Consider $L_R(E)$ with its canonical $\mathbb{Z}$-grading.*

(a) *$L_R(E)$ is nearly epsilon-strongly $\mathbb{Z}$-graded.*
(b) *If E is finite, then $L_R(E)$ is epsilon-strongly $\mathbb{Z}$-graded.*

In order to begin understanding when a Leavitt path algebra is prime, we consider a few examples.

Example E.14.5. Let R be a unital ring and let E_1 be the directed graph below:

$$E_1 : \qquad \bullet_v$$

In this case, $L_R(E_1) = vR \cong R$ is prime if and only if R is prime.

Example E.14.6. Let R be a unital ring and let E_2 be the directed graph below:

$$E_2 : \qquad \bullet_{v_1} \qquad\qquad \bullet_{v_2}$$

We have $L_R(E_2) = v_1R + v_2R \cong R \oplus R$. Note that v_1R, v_2R are nonzero ideals of $L_R(E_2)$ such that $(v_1R)(v_2R) = \{0\}$. Thus, $L_R(E_2)$ is never prime, for any ring R.

From the above examples, it is clear that a criterion for primeness of $L_R(E)$ must depend on properties of both the coefficient ring R and the graph E. To describe such a criterion, we need to introduce a preorder $\geq$ on the set of vertices E^0 of the directed graph E in the following way: We write $u \geq v$ if there is a path (possibly of length zero) from u to v. Note that $v \geq v$ for every $v \in E^0$, i. e. the preorder is reflexive. Transitivity of the preorder follows by concatenating the paths.

Definition E.14.7. A directed graph E is said to satisfy *condition (MT-3)* if the above defined preorder $\geq$ is *downward directed*, i. e. if for every pair of vertices $u, v \in E^0$, there is some $w \in E^0$ such that $u \geq w$ and $v \geq w$.

The graph E_2 in Example E.14.6 does not satisfy condition (MT-3) since there is no vertex u such that $v_1 \geq u$ and $v_2 \geq u$. The next example shows a graph satisfying condition (MT-3):

Example E.14.8. Let R be a unital ring and let E_3 be the directed graph below:

$$E_3 : \qquad \bullet_{v_1} \longrightarrow \bullet_{v_2}$$

E_3 satisfies condition (MT-3). Indeed for v_1, v_2 we have $v_1 \geq v_2$ and $v_2 \geq v_2$. A computation yields $L_R(E_3) \cong M_2(R)$ (see [23, Expl. 2.6]). By Corollary E.11.6, it follows that $L_R(E_3)$ is prime if and only if R is prime.

In the case when K is a field, Abrams, Bell and Rangaswamy have shown that $L_K(E)$ is prime if and only if E satisfies condition (MT-3) (see [2, Thm. 1.4]). For Leavitt path algebras with coefficients in a commutative unital ring, the following generalization was proved by Larki [25, Prop. 4.5]:

Proposition E.14.9. *Suppose that E is a directed graph and that R is a unital commutative ring. Then $L_R(E)$ is prime if and only if R is an integral domain and E satisfies condition (MT-3).*

We aim to generalize Proposition E.14.9 to Leavitt path algebras with coefficients in a general (possibly non-commutative) unital ring. Since Leavitt path algebras are nearly epsilon-strongly $\mathbb{Z}$-graded (see Proposition E.14.4), we will be able to obtain this generalization as a corollary to Theorem E.1.3. We begin with the following result:

Proposition E.14.10. *Suppose that E is a directed graph and that R is a unital ring. Consider the Leavitt path algebra $S = L_R(E)$. The following assertions hold:*

(a) *There exist $v, w \in E^0$ such that $SvSwS = \{0\}$ if and only if E does not satisfy condition (MT-3).*
(b) *If R is prime and there exist nonzero $r, s \in R$ and $v, w \in E^0$ such that $SrvSswS = \{0\}$, then E does not satisfy condition (MT-3).*

PROOF. (a): Suppose that E does not satisfy condition (MT-3). There exist $v, w \in E^0$ such that for every $y \in E^0$ we have $v \not\geq y$ or $w \not\geq y$. Take a monomial $r\alpha\beta^*$ in $L_R(E)$. From the properties of v and w, it follows that $vr\alpha\beta^* w = 0$. Therefore $SvSwS = \{0\}$.

Now suppose that E satisfies condition (MT-3). Take $v, w \in E^0$. There exist $y \in E^0$ and paths α, β from v to y and from w to y, respectively. Then $SvSwS \ni v \cdot v \cdot \alpha\beta^* \cdot w \cdot w = \alpha\beta^* \neq 0$.

(b): Suppose that R is prime and that there exist nonzero $r, s \in R$ and $v, w \in E^0$ such that $SrvSswS = \{0\}$. Let $P = RrR$ and $Q = RsR$. Then P and Q are nonzero ideals of R. Hence, from primeness of R it follows that PQ is a nonzero ideal of R. Take $p_i \in P$ and $q_i \in Q$, for $i \in \{1, \ldots, n\}$, such that $\sum_{i=1}^n p_i q_i \neq 0$. Seeking a contradiction, suppose that E satisfies condition (MT-3). There exist $y \in E^0$ and paths α, β from v to y and from w to y, respectively. We get $\{0\} = SrvSswS \supseteq S \cdot (RrR)v \cdot S \cdot (RsR)w \cdot S \ni \sum_{i=1}^n v \cdot p_i v \cdot \alpha\beta^* \cdot q_i w \cdot w = \sum_{i=1}^n p_i q_i \alpha\beta^* \neq 0$, which is a contradiction. □

The following result is a special case of [1, Thm. 2.2.11]. Tomforde [43] established this result for Leavitt path algebras with coefficients in a commutative unital ring. His proof generalizes verbatim to Leavitt path algebras with coefficients in a general unital ring. For the convenience of the reader, we include a full proof:

Proposition E.14.11 (cf. [43, Lem. 5.2]). *Suppose that E is a directed graph and that R is a unital ring. If $a \in (L_R(E))_0$ is nonzero, then there exist $\alpha, \beta \in E^*$, $v \in E^0$ and a nonzero $t \in R$ such that $\alpha^* a \beta = tv$.*

Proof. If we for every $N \in \mathbb{N}$ put $\mathcal{G}_N := \mathrm{Span}_R\{\alpha\beta^* \mid \alpha, \beta \in E^*, |\alpha| = |\beta| \leq N\}$, then $(L_R(E))_0 = \bigcup_{N=0}^{\infty} \mathcal{G}_N$. The proof proceeds by induction over N. Base case: $N = 0$. Take a nonzero $a \in \mathcal{G}_0$. Then $0 \neq a = \sum_{i=1}^{n} r_i v_i$ for some nonzero $r_i \in R$ and distinct vertices $v_i \in E^0$. If we put $\alpha = \beta := v_1$, then $\alpha^* a \beta = r_1 v_1$.

Inductive step: suppose that $N > 0$ and that the statement of the proposition holds for every nonzero element in $\mathcal{G}_{N-1}$. Take a nonzero $a \in \mathcal{G}_N$. Then $a = \sum_{i=1}^{M} r_i \alpha_i \beta_i^* + \sum_{j=1}^{M'} s_j v_j$, where $\alpha_i, \beta_i \in E^*$ with $|\alpha_i| = |\beta_i| \geq 1$ and $v_j \neq v_{j'}$ for all $j \neq j'$. We consider two mutually exclusive cases.

Case 1: some v_j is not regular. If v_j is an infinite emitter, then there is some edge $f \in E^1$ with $s(f) = v_j$ such that f is not included in any path α_i, β_i. Put $\alpha = \beta := f$. Then $\alpha^* a \beta = 0 + f^* s_j v_j f = s_j v_j$. If v_j is a sink, then put $\alpha = \beta := v_j$ and note that $\alpha^* a \beta = s_j v_j$.

Case 2: every v_j is regular. Then $v_j = \sum_{s(f)=v_j} f f^*$ for every j. Hence, we may write $a = \sum_{i=1}^{M''} r_i \gamma_i \delta_i^*$ where $\gamma_i, \delta_i \in E^*$ with $|\gamma_i| = |\delta_i| \geq 1$. By regrouping the elements of the sum, we may rewrite it as $a = \sum_{i=1}^{P} \sum_{j=1}^{Q} e_i x_{i,j} f_j^*$, where

- $e_i, f_i \in E^1$ with $e_i \neq e_{i'}$ for $i \neq i'$ and $f_j \neq f_{j'}$ for $j \neq j'$, and
- $x_{i,j} \in \mathcal{G}_{N-1}$ with $e_i x_{i,j} f_j^* \neq 0$ for all i, j.

Note that $e_1 x_{1,1} f_1^* \neq 0$ implies $r(e_1) x_{1,1} r(f_1) \neq 0$. By the induction hypothesis, there are $\alpha', \beta' \in E^*$ such that $(\alpha')^* r(e_1) x_{1,1} r(f_1) \beta' = tv$ for some $v \in E^0$ and $t \in R$. Put $\alpha := e_1 \alpha'$ and $\beta := f_1 \beta'$. Then $\alpha^* a \beta = (\alpha')^* e_1^* a f_1 \beta' = (\alpha')^* e_1^* e_1 x_{1,1} f_1^* f_1 \beta' = (\alpha')^* r(e_1) x_{1,1} r(f_1) \beta' = tv$. □

We can now establish Theorem E.1.6:

Theorem E.14.12. *Suppose that E is a directed graph and that R is a unital ring. The Leavitt path algebra $L_R(E)$ is prime if and only if R is prime and E satisfies condition (MT-3).*

Proof. Put $S := L_R(E)$. Suppose that R is not prime. There exist nonzero ideals I, J of R such that $IJ = \{0\}$. Let A and B be the nonzero ideals in S consisting of sums of monomials with coefficients in I and J, respectively. Then $AB = \{0\}$ which implies that S is not prime. Suppose now that E^0 does not satisfy condition (MT-3). By Proposition E.14.10(a), there exist $v, w \in E^0$ such that $SvSwS = \{0\}$. Consider the nonzero ideals $C := SvS$ and $D := SwS$ of S. Then $CD = SvSSwS \subseteq SvSwS = \{0\}$ which shows that S is not prime.

Suppose that S is not prime and that R is prime. By Proposition E.14.4, S is nearly epsilon-strongly $\mathbb{Z}$-graded. Theorem E.1.3 implies that there exist nonzero ideals $\tilde{A}, \tilde{B}$ of S_0 such that $\tilde{A} S \tilde{B} = \{0\}$. Take $a \in \tilde{A} \setminus \{0\}$ and $b \in \tilde{B} \setminus \{0\}$. By Propostion E.14.11, there exist $v, w \in E^0$, $r, s \in R \setminus \{0\}$ and $\alpha, \beta, \gamma, \delta \in E^*$ such that $\alpha^* a \beta = rv$ and $\gamma^* b \delta = sw$. Now, $SrvSswS \subseteq S\tilde{A}S\tilde{B}S = \{0\}$ and hence $SrvSswS = \{0\}$. Employing Proposition E.14.10(b), we conclude that E does not satisfy condition (MT-3). □

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ABSTRACT

The development of a general theory of strongly group graded rings was initiated by Dade, Năstăsescu and Van Oystaeyen in the 1980s, and since then numerous structural results have been established.

In this thesis we develop a general theory of so-called (nearly) epsilon-strongly group graded rings which were recently introduced by Nystedt, Öinert and Pinedo and which generalize strongly group graded rings. Moreover, we obtain applications to Leavitt path algebras, unital partial crossed products and algebraic Cuntz-Pimsner rings.

This thesis is based on five scientific papers (A, B, C, D, E).

Papers A and B are concerned with structural properties of epsilon-strongly graded rings. In Paper A, we consider an important construction called the induced quotient group grading. In Paper B, using results from Paper A, we obtain a Hilbert Basis Theorem for epsilon-strongly graded rings.

In Paper C, we study the graded structure of algebraic Cuntz-Pimsner rings. In particular, we obtain a partial characterization of unital strongly graded, epsilon-strongly graded and nearly epsilon-strongly graded algebraic Cuntz-Pimsner rings up to graded isomorphism.

In Paper D, we give a complete characterization of group graded rings that are graded von Neumann regular.

Finally, in Paper E, written in collaboration with Lundström, Öinert and Wagner, we consider prime nearly epsilon-strongly graded rings. Generalizing Passman's work from the 1980s, we give necessary and sufficient conditions for a nearly epsilon-strongly graded ring to be prime.

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