

Units, zero-divisors and idempotents in rings graded by torsion-free groups

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Abstract. The three famous problems concerning units, zero-divisors and idempotents in group rings of torsion-free groups, commonly attributed to Kaplansky, have been around for more than 60 years and still remain open in characteristic zero. In this article, we introduce the corresponding problems in the considerably more general context of arbitrary rings graded by torsion-free groups. For natural reasons, we will restrict our attention to rings without non-trivial homogeneous zero-divisors with respect to the given grading. We provide a partial solution to the extended problems by solving them for rings graded by unique product groups. We also show that the extended problems exhibit the same (potential) hierarchy as the classical problems for group rings. Furthermore, a ring which is graded by an arbitrary torsion-free group is shown to be indecomposable, and to have no non-trivial central zero-divisor and no non-homogeneous central unit. We also present generalizations of the classical group ring conjectures.

1 Introduction

With a few exceptions, notably [19], the first articles on group rings of infinite groups appeared in the early 1950s. A key person in that line of research was Kaplansky, known for his many deep contributions to ring theory and operator algebra. In his famous talk, given at a conference that was held on June 6–8, 1956 at Shelter Island, Rhode Island, New York, he proposed twelve problems in the theory of rings [22, 23], one of which has become known as *the zero-divisor problem* (for group rings).

Although popularized by Kaplansky, the zero-divisor problem and its corresponding conjecture had in fact already been introduced by Higman in his 1940 thesis [20, p. 77] (see also [48, p. 112]). In [20], Higman also introduced the so-called *unit problem* and the corresponding *unit conjecture*. A third problem which is closely related to the previous two is *the idempotent problem*. For clarity, we now recall the exact formulation of the three problems.

Problem 1 (Higman/Kaplansky). Let K be a field, let G be a torsion-free group and denote by $K[G]$ the corresponding group ring.

- (a) Is every unit in $K[G]$ necessarily trivial, i.e. a scalar multiple of an element of G ?
- (b) Is $K[G]$ necessarily a domain?
- (c) Is every idempotent in $K[G]$ necessarily trivial, i.e. either 0 or 1?

Many of the problems in Kaplansky's original list [22] have been solved. The zero-divisor problem and the unit problem have been solved, in the affirmative, for several important classes of groups (see e.g. [7, 10, 13, 15, 25, 27]). Significant progress has been made on the idempotent problem using algebraic as well as analytical methods (see e.g. [8, 14] and [21, 30, 31, 45]). For a thorough account of the development on the above problems during the 1970s, we refer the reader to Passman's extensive monograph [42]. In recent years, computational approaches have been proposed as means of attacking the zero-divisor problem (see [12, 49]). In 2021, there was a major breakthrough when Gardam [16] presented a counterexample to the unit conjecture in characteristic 2 (see Remark 3.8). Soon thereafter, Murray [32] presented counterexamples in arbitrary prime characteristic. Nevertheless, for a general group G , all three problems remain open for group rings of characteristic zero.

We should point out that the problems exhibit a (potential) hierarchy. Indeed, an affirmative answer to the unit problem, for a fixed group G , implies an affirmative answer to the zero-divisor problem, which in turn implies an affirmative answer to the idempotent problem for the same group G (see [53, Remark 1.1]).

In the last two decades, the idempotent problem has regained interest, mainly due to its connection with the Baum–Connes conjecture in operator algebra (see e.g. [53]) via the so-called Kadison–Kaplansky conjecture for reduced group C^* -algebras. The idempotent problem is also connected to the Farrell–Jones conjecture (see [2]). Moreover, Lück [28] has shown that if G is a torsion-free group and K is a subfield of \mathbb{C} which satisfies the Atiyah conjecture [28, Conjecture 10.3] with coefficients in K , then the zero-divisor problem has an affirmative answer for $K[G]$. Altogether, this shows that Problem 1, in particular in characteristic zero, remains highly relevant to modern mathematics.

In this article, we will consider Problem 1 from a more general point of view, namely that of group graded rings. Let G be a group with identity element e . Recall that a ring R is said to be G -graded (or graded by G) if there is a collection $\{R_g\}_{g \in G}$ of additive subgroups of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. Furthermore, a G -graded ring R is called *strongly G -graded* (or *strongly graded by G*) if $R_g R_h = R_{gh}$ for all $g, h \in G$.

Crossed products, and more generally group graded rings, appear naturally in many branches of mathematics, e.g. in non-commutative geometry, in the repre-

sensation theory of Artin algebras, in the computation of Grothendieck groups, in the study of singularities, in orbifold theory, in the Galois theory of skew fields, in the investigation of Calabi–Yau algebras derived from superpotentials, and in the representation theory of certain preprojective algebras. For an introduction to the theory of group graded rings, we refer the reader to the excellent monographs [18, 33, 34, 43].

Note that the group ring $K[G]$ may be equipped with a canonical strong G -grading by putting $R := K[G]$ with $R_g := Kg$ for $g \in G$. With this in mind, it is natural to ask whether it would make sense to extend Problem 1 to the more general context of strongly group graded rings. It turns out that it does. In fact, we propose the following even more general set of problems which will be the main focus of this article.

Problem 2. Let G be a torsion-free group and R a unital G -graded ring equipped with a non-degenerate (see Definition 2.1) G -grading such that R_e is a domain.

- (a) Under the assumption that $\text{char}(R_e) = 0$, is every unit in R necessarily homogeneous with respect to the given G -grading?
- (b) Is R necessarily a domain?
- (c) Is every idempotent in R necessarily trivial?

This article is organized as follows. In Section 2, we record the most important notation and preliminaries concerning group graded rings that we will need in the sequel. In particular, we show that the assumptions on the grading in Problem 2 make our rings especially well-behaved (see Proposition 2.6 and Proposition 2.9). In Section 3, using a result of Strojnowski, we solve Problem 2 for unique product groups (see Theorem 3.4). In particular, this solves Problem 2 in the cases where G is abelian or R is commutative (see Example 3.7 and Corollary 3.10). In Section 4, we show that if G is an arbitrary torsion-free group and R is a G -graded ring satisfying the assumptions in Problem 2, then R is a prime ring (see Theorem 4.4). In Section 5, we employ the primeness result from Section 4 to show that the unit, zero-divisor and idempotent problems for group graded rings exhibit the same (potential) hierarchy as the classical problems for group rings (see Theorem 5.2). In Section 6, we show that, for an arbitrary torsion-free group G and a G -graded ring R with a non-degenerate grading such that R_e is a domain (of arbitrary characteristic), there is no non-homogeneous central unit, no non-trivial central zero-divisor and no non-trivial central idempotent (see Theorem 6.2). In Section 7, we obtain several useful results concerning gradings by quotient groups (see e.g. Proposition 7.3 and Proposition 7.5) which are used to solve Problem 2 for G -crossed products when G belongs to a special class of solvable groups (see

Theorem 7.7). In Section 8, we formulate a conjecture (see Conjecture 8.1) which generalizes the classical unit, zero-divisor and idempotent conjectures for group rings.

2 Preliminaries on group graded rings

Throughout this section, let G be a multiplicatively written group with identity element e , and let R be a (not necessarily unital) G -graded ring.

Consider an arbitrary element $r \in R$. Note that $r \in R$ has a unique decomposition of the form $r = \sum_{g \in G} r_g$, where $r_g \in R_g$ is zero for all but finitely many $g \in G$. The *support* of r is defined as the finite set $\text{Supp}(r) := \{g \in G \mid r_g \neq 0\}$. If $r \in R_g$ for some $g \in G$ and r is nonzero, then r is said to be *homogeneous of degree g* and we write $\deg(r) = g$. Note that R_e is a subring of R . If R is unital, then $1_R \in R_e$ (see e.g. [34, Proposition 1.1.1]).

We shall now highlight two types of G -gradings which play central roles in this article.

Definition 2.1. (a) R is said to have a *non-degenerate G -grading* (cf. [11, 39]) if, for each $g \in G$ and each nonzero $r_g \in R_g$, we have $r_g R_{g^{-1}} \neq \{0\}$, $R_{g^{-1}} r_g \neq \{0\}$.

(b) R is said to have a *fully component regular G -grading* if $r_g s_h \neq 0$ for any $g, h \in G$, $r_g \in R_g \setminus \{0\}$ and $s_h \in R_h \setminus \{0\}$.

Remark 2.2. (a) Every strong G -grading on a unital ring R is non-degenerate. Indeed, take $g \in G$ and $r_g \in R_g$, and suppose that $r_g R_{g^{-1}} = \{0\}$. Then

$$r_g = r_g 1_R \in r_g R_e = r_g R_{g^{-1}} R_g = \{0\}.$$

Thus, $r_g = 0$. Similarly, $R_{g^{-1}} r_g = \{0\}$ implies $r_g = 0$.

(b) The term *fully component regular* has been chosen to capture the essence of those gradings. There is no immediate connection to the *component regular* gradings considered by Passman in e.g. [43, p. 16], which are in fact special types of non-degenerate gradings.

A grading may be both non-degenerate and fully component regular, but, as the following example shows, the two notions are quite independent.

Example 2.3. In the following two examples, the grading group is $G := (\mathbb{Z}, +)$.

(a) Consider the polynomial ring $R := \mathbb{R}[t]$ in one indeterminate. We may define a \mathbb{Z} -grading on R by putting $R_n := \mathbb{R}t^n$ for $n \geq 0$, and $R_n := \{0\}$ for $n < 0$. Clearly, this grading is fully component regular, but it is degenerate.

(b) Consider the ring of 2×2 -matrices with real entries, $R := M_2(\mathbb{R})$. We may define a \mathbb{Z} -grading on R by putting

$$R_0 := \begin{pmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{R} \end{pmatrix}, \quad R_1 := \begin{pmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{pmatrix}, \quad R_{-1} := \begin{pmatrix} 0 & 0 \\ \mathbb{R} & 0 \end{pmatrix} \quad \text{and} \quad R_n := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

whenever $|n| > 1$. This grading is neither fully component regular, nor strong, but one easily sees that it is non-degenerate.

Given a subgroup $H \subseteq G$, we may define the subset $R_H := \bigoplus_{h \in H} R_h$ of R . Note that R_H is an H -graded subring of R . If R is unital, then R_H is also unital with $1_R = 1_{R_H} \in R_e$. The corresponding *projection map* from R to R_H is defined by

$$\pi_H: R \rightarrow R_H, \quad \sum_{g \in G} r_g \mapsto \sum_{h \in H} r_h,$$

and it is clearly additive. In fact, it is an R_H -bimodule homomorphism.

Lemma 2.4. *Let H be a subgroup of G . If $a \in R$ and $b \in R_H$, then we have $\pi_H(ab) = \pi_H(a)b$ and $\pi_H(ba) = b\pi_H(a)$.*

Proof. Take $a \in R$ and $b \in R_H$. Put $a' := a - \pi_H(a)$. Clearly, $a = a' + \pi_H(a)$ and $\text{Supp}(a') \subseteq G \setminus H$. If $g \in G \setminus H$ and $h \in H$, then $gh \notin H$. Thus, we have $\text{Supp}(a'b) \subseteq G \setminus H$. Hence,

$$\begin{aligned} \pi_H(ab) &= \pi_H((a' + \pi_H(a))b) = \pi_H(a'b) + \pi_H(\pi_H(a)b) \\ &= 0 + \pi_H(a)b = \pi_H(a)b. \end{aligned}$$

Analogously, one may show that $\pi_H(ba) = b\pi_H(a)$. □

Lemma 2.5. *Suppose that the G -grading on R is non-degenerate. Let H be a subgroup of G and let $r \in R_H$. The following assertions hold:*

- (i) *r is a left (right) zero-divisor in R_H if and only if r is a left (right) zero-divisor in R ;*
- (ii) *r is left (right) invertible in R_H if and only if r is left (right) invertible in R .*

Proof. The “only if” statements are trivial. We only need to show the “if” statements. The proofs of the right-handed claims are treated analogously and are therefore omitted.

- (i) Suppose that $rs = 0$ for some nonzero $s \in R$. Then, by Lemma 2.4,

$$0 = \pi_H(rs) = r\pi_H(s).$$

Without loss of generality we may assume that $\text{Supp}(s) \cap H \neq \emptyset$, and thus we have $0 \neq \pi_H(s) \in R_H$. For otherwise, we may take some $g \in \text{Supp}(s)$ and some nonzero $x_{g^{-1}} \in R_{g^{-1}}$ such that $e \in \text{Supp}(sx_{g^{-1}})$. Note that $r(sx_{g^{-1}}) = 0$ and $\text{Supp}(sx_{g^{-1}}) \cap H \neq \emptyset$.

(ii) Suppose that $sr = 1_R$ for some $s \in R$. Then, by Lemma 2.4,

$$1_{R_H} = \pi_H(1_R) = \pi_H(sr) = \pi_H(s)r$$

in R_H . □

The following result highlights a crucial property of the rings appearing in Problem 2.

Proposition 2.6. *If the G -grading on R is non-degenerate and R_e is a domain, then the G -grading is fully component regular.*

Proof. Take $g, h \in G$, $r_g \in R_g$ and $s_h \in R_h$. Suppose that $r_g s_h = 0$. Then

$$R_{g^{-1}} r_g s_h R_{h^{-1}} = \{0\}.$$

Using the fact that R_e is a domain, we get that $R_{g^{-1}} r_g = \{0\}$ or $s_h R_{h^{-1}} = \{0\}$. By the non-degeneracy of the grading, we conclude that $r_g = 0$ or $s_h = 0$. □

Remark 2.7. There are large classes of group graded rings whose gradings are non-degenerate but not necessarily strong. For example, crystalline graded rings [35], (nearly) epsilon-strongly graded rings [37, 38], and in particular Leavitt path algebras and crossed products by unital twisted partial actions. However, it should be noted that an epsilon-strongly G -graded ring R , and in particular a partial crossed product, for which R_e is a domain, is necessarily strongly graded by a subgroup of G .

Definition 2.8. (a) The *support of the G -grading on R* is defined as the set

$$\text{Supp}(R) := \{g \in G \mid R_g \neq \{0\}\}.$$

(b) If $\text{Supp}(R) = G$, then the G -grading on R is said to be *fully supported*.

It is easy to see that any strong G -grading must be fully supported. For a general G -grading, however, $\text{Supp}(R)$ need not even be a subgroup of G . As illustrated by Example 2.3 (a), $\text{Supp}(R)$ may fail to be a subgroup of G even if the G -grading is fully component regular. As the following result shows, we are in a rather fortunate situation.

Proposition 2.9. *If the G -grading on R is non-degenerate and R_e is a domain, then $\text{Supp}(R)$ is a subgroup of G .*

Proof. Take $g, h \in \text{Supp}(R)$. Using Proposition 2.6, we conclude that

$$R_{gh} \supseteq R_g R_h \neq \{0\}.$$

Thus, $gh \in \text{Supp}(R)$. Moreover, by the non-degeneracy of the grading, it is clear that $R_{g^{-1}} \neq \{0\}$. Thus, $g^{-1} \in \text{Supp}(R)$. This shows that $\text{Supp}(R)$ is a subgroup of G . \square

The following result follows immediately from Propositions 2.9 and 2.6.

Corollary 2.10. *If the G -grading on R is non-degenerate and R_e is a domain, then R has a natural grading by the subgroup $H := \text{Supp}(R)$ of G . This H -grading is non-degenerate and $R_{e_G} = R_{e_H}$ is a domain. Moreover, the H -grading is fully supported and fully component regular.*

3 Unique product groups

In this section, we will solve Problem 2 for unique product groups (see Theorem 3.4). Unique product groups were introduced by Rudin and Schneider [47] who called them Ω -groups.

Definition 3.1. Let G be a group.

- (a) G is said to be a *unique product group* if, given any two non-empty finite subsets A and B of G , there exists at least one element $g \in G$ which has a unique representation of the form $g = ab$ with $a \in A$ and $b \in B$.
- (b) G is said to be a *two unique products group* if, given any two non-empty finite subsets A and B of G with $|A| + |B| > 2$, there exist at least two distinct elements g and h of G which have unique representations of the form $g = ab$, $h = cd$ with $a, c \in A$ and $b, d \in B$.

It is clear that every two unique products group is a unique product group. In 1980, Strojnowski showed that the two properties are in fact equivalent.

Lemma 3.2 (Strojnowski [52]). *A group G is a unique product group if and only if it is a two unique products group.*

Remark 3.3. Every unique product group is necessarily torsion-free.

We shall now state and prove the main result of this section, and thereby simultaneously generalize e.g. [19, Theorem 12], [19, Theorem 13], [47, Theorem 3.2], [4, Proposition 3.6(a)] and [41, Theorem 26.2]. The following result solves Problem 2 for unique product groups.

Theorem 3.4. *Let G be a unique product group and R a unital G -graded ring. Furthermore, suppose that the G -grading on R is non-degenerate and that R_e is a domain. The following assertions hold:*

- (i) *every unit in R is homogeneous;*
- (ii) *R is a domain;*
- (iii) *every idempotent in R is trivial.*

Proof. (i) Take $x, y \in R$ which satisfy $xy = 1_R$. Put

$$A := \text{Supp}(x) \quad \text{and} \quad B := \text{Supp}(y).$$

By assumption, $|A|$ and $|B|$ are positive. We want to show that $|A| = |B| = 1$. Seeking a contradiction, suppose that $|A| > 1$. Then $|A| + |B| > 2$. Using the fact that G is a unique product group and hence, by Lemma 3.2, a two unique products group, there are two distinct elements $g, h \in AB$ such that $g = ab$ and $h = cd$ with $a, c \in A$ and $b, d \in B$. We must have $x_a y_b = 0$ or $x_c y_d = 0$ since $|\text{Supp}(xy)| = |\text{Supp}(1_R)| = |\{e\}| = 1$. But, by Proposition 2.6, R is fully component regular, and hence neither of the two equalities can hold. This is a contradiction. We conclude that $|A| = 1$, i.e. x is homogeneous. From the equality $xy = 1_R$ and the full component regularity of the grading, we get that y is also homogeneous.

(ii) Take two nonzero elements $x, y \in R$. Seeking a contradiction, suppose that $xy = 0$. Using the fact that G is a unique product group, there is some $a \in \text{Supp}(x)$ and some $b \in \text{Supp}(y)$ such that $x_a y_b = 0$. By Proposition 2.6, this is a contradiction.

(iii) This follows from (ii) since $u^2 = u \Leftrightarrow u(u - 1_R) = 0$. □

Remark 3.5. Note that Theorem 3.4 (ii) also holds if R is non-unital. We want to point out that Malman has already proved Theorem 3.4 (ii) in [29, Lemma 3.13].

Remark 3.6. The proof of Theorem 3.4 (i) yields a seemingly stronger conclusion than the one we aim to prove. But in fact, note that, for a G -graded ring R with a fully component regular grading, the following three assertions are equivalent:

- (L) every left invertible element in R is homogeneous;
- (R) every right invertible element in R is homogeneous;
- (U) every unit in R is homogeneous.

There is an abundance of classes of groups to which Theorem 3.4 can be applied.

Example 3.7. Typical examples of unique product groups are the diffuse groups (see [6, 24]) and in particular the right (or left) orderable groups, including e.g. all free groups, all torsion-free nilpotent groups, and hence all torsion-free abelian groups.

Remark 3.8. For many years, it was not known whether every torsion-free group necessarily had the unique product property. However, in 1987, Rips and Segev [46] presented an example of a torsion-free group without the unique product property. Since then, a growing number of examples of torsion-free non-unique product groups have surfaced (see e.g. [1, 9, 17, 44, 50]). Notably, Passman [42, p. 606] showed that the so-called (*Passman*) *fours group*

$$P := \langle a, b \mid b^{-1}a^2b = a^{-2}, a^{-1}b^2a = b^{-2} \rangle$$

is torsion-free and non-right-orderable. Promislow [44] later showed that P is not a unique product group. And more than 30 years later, Gardam [16] showed that the group ring $K[P]$ has non-trivial units, where K is the field of two elements. It is interesting to note that P is a solvable group, and hence $K[P]$ is in fact a domain (see [25, Theorem 1.4]).

For an arbitrary torsion-free group G , we have that $Z(G)$ is a torsion-free abelian group, and thus a unique product group. Theorem 3.4 now yields the following result.

Corollary 3.9. *Let G be a torsion-free group and R a unital G -graded ring whose G -grading is non-degenerate. If R_e is a domain, then every unit in $R_{Z(G)}$ is homogeneous, $R_{Z(G)}$ is a domain and every idempotent in $R_{Z(G)}$ is trivial.*

Corollary 3.10. *Let G be a torsion-free group and R a unital commutative G -graded ring whose G -grading is non-degenerate. If R_e is a domain, then every unit in R is homogeneous, R is an integral domain and every idempotent in R is trivial.*

Proof. By Corollary 2.10, there is a torsion-free group H such that R may be equipped with an H -grading which is fully supported and fully component regular. Take $g, h \in H$. There are nonzero homogeneous elements $r_g \in R_g$ and $r_h \in R_h$ such that $r_g r_h = r_h r_g \neq 0$. Thus, $R_{gh} \cap R_{hg} \neq \emptyset$, which yields $gh = hg$. This shows that H is a torsion-free abelian group. The result now follows from Theorem 3.4. \square

Theorem 3.4, Corollary 3.9 and Corollary 3.10 may be applied to G -crossed products, and in particular to group rings, but more generally to strongly group

graded rings. We shall now apply the aforementioned theorem to a few examples of group graded rings whose gradings are (typically) not strong.

Example 3.11. (a) Let \mathbb{F} be a field and consider *the first Weyl algebra*

$$R := \mathbb{F}\langle x, y \rangle / (yx - xy - 1).$$

It is an easy exercise to show that R is a domain, but this elementary fact is also an immediate consequence of Theorem 3.4. Indeed, note that, by assigning suitable degrees to the generators, $\deg(x) := 1$ and $\deg(y) := -1$, R becomes graded by the unique product group $(\mathbb{Z}, +)$. Moreover, $R_0 = \mathbb{F}[xy]$ is a domain and the \mathbb{Z} -grading is non-degenerate. Thus, the first Weyl algebra R is a domain.

(b) More generally, let D be a ring, let $\sigma := (\sigma_1, \dots, \sigma_n)$ be a set of commuting automorphisms of D , and let $a := (a_1, \dots, a_n)$ be an n -tuple with nonzero entries from $Z(D)$ satisfying $\sigma_i(a_j) = a_j$ for $i \neq j$. Given this data, it is possible to define the corresponding *generalized Weyl algebra* $R := D(\sigma, a)$ (see [3] or [35]). One may show that R is \mathbb{Z}^n -graded with $R_e = D$. If D is a domain, then the \mathbb{Z}^n -grading is non-degenerate and Theorem 3.4 yields that R is a domain. Thus, we have recovered [3, Proposition 1.3 (2)].

(c) Any *crystalline graded ring* $R := A \diamond_{\sigma}^{\alpha} G$ (see [35]) is equipped with a non-degenerate G -grading with $R_e = A$. If A is a domain and G is a unique product group, then R is a domain by Theorem 3.4.

4 Primeness

As a preparation for Section 5, in this section, we will give a sufficient condition for a ring R graded by a torsion-free group G to be prime (see Theorem 4.4).

Recall that a group G is said to be an *FC-group* if each $g \in G$ has only a finite number of conjugates in G . Equivalently, G is an FC-group if $[G : C_G(g)] < \infty$ for each $g \in G$. Given a group G , we define the subset

$$\Delta(G) := \{g \in G \mid g \text{ has only finitely many conjugates in } G\}.$$

It is not difficult to see that $\Delta(G)$ is a subgroup of G .

The following useful lemma can be shown in various ways (see e.g. [36]).

Lemma 4.1 (Neumann [36]). *Every torsion-free FC-group is abelian.*

The next lemma is used in the work of Passman and can be shown by induction on the number of subgroups. We omit the proof and instead refer the reader to [41, Lemma 1.2].

Lemma 4.2 (Passman [41]). *Let L be a group and let H_1, H_2, \dots, H_n be a finite number of subgroups of L . Suppose that there exists a finite collection of elements $s_{i,j} \in L$ for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ such that $L = \bigcup_{i,j} H_i s_{i,j}$. Then for some $k \in \{1, \dots, n\}$, we have $[L : H_k] < \infty$.*

Now we use Passman's lemma to prove the next lemma which is crucial to this section.

Lemma 4.3. *Let G be a group and consider the subgroup $H := \Delta(G)$. Suppose that F is a non-empty finite subset of H and that A, B are non-empty finite subsets of $G \setminus H$. Let $f \in F$ and $h \in H$ be arbitrary. There exists some $g \in C_G(F)$ such that $fh \notin g^{-1}AgB$.*

Proof. Put $L := C_G(F)$ and suppose that $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_m\}$. Let $f \in F$ and $h \in H$ be arbitrary. Seeking a contradiction, suppose that

$$fh \in g^{-1}AgB \quad \text{for all } g \in L.$$

For each $i \in \{1, \dots, n\}$, we define the subgroup $H_i := L \cap C_G(a_i)$.

For $(i, j) \in \{1, \dots, n\} \times \{1, \dots, m\}$, if a_i is conjugate to fhb_j^{-1} by an element of L , then choose $s_{i,j} \in L$ such that $s_{i,j}^{-1}a_i s_{i,j} = fhb_j^{-1}$. Otherwise, choose $s_{i,j} = e$. Note that $L \supseteq \bigcup_{i,j} H_i s_{i,j}$.

Now, let $g \in L$ be arbitrary. By our assumption, there exist i and j such that $fh = g^{-1}a_i g b_j$. That is, $fhb_j^{-1} = g^{-1}a_i g = s_{i,j}^{-1}a_i s_{i,j}$. From the last equality, we get that $gs_{i,j}^{-1} \in L \cap C_G(a_i) = H_i$. Thus, $g \in H_i s_{i,j}$. Since g was arbitrarily chosen, it is clear that $L \subseteq \bigcup_{i,j} H_i s_{i,j}$. This shows that $L = \bigcup_{i,j} H_i s_{i,j}$.

By Lemma 4.2, there is some $k \in \{1, \dots, n\}$ such that $[L : H_k] < \infty$. Consider the chains of subgroups $G \supseteq L \supseteq H_k$ and $G \supseteq C_G(a_k) \supseteq H_k$. Recall that $[G : C_G(r)] < \infty$ for each $r \in F$. Thus, $[G : L] < \infty$, and hence $[G : H_k] < \infty$. This shows that $[G : C_G(a_k)] < \infty$. But this is a contradiction since $a_k \in G \setminus H$. \square

Theorem 4.4. *Let G be a torsion-free group and R a G -graded ring. If the G -grading on R is non-degenerate and R_e is a domain, then R is a prime ring.*

Proof. By Corollary 2.10, $G' := \text{Supp}(R)$ is a torsion-free subgroup of G . Moreover, R can be equipped with a non-degenerate and fully supported G' -grading. Thus, we will without loss of generality assume that the G -grading on R is fully supported.

Put $H := \Delta(G)$. Note that H is a torsion-free FC-group. Thus, by Lemma 4.1, H is torsion-free abelian. Theorem 3.4 now yields that R_H is a domain.

Seeking a contradiction, suppose that there are nonzero ideals I and J of R such that $I \cdot J = \{0\}$. By Proposition 2.6, the G -grading on R is fully component regular. Using this and Lemma 2.4, it is clear that $\pi_H(I)$ and $\pi_H(J)$ are nonzero ideals of R_H .

Choose some $x \in I$ and $y \in J$ such that $\pi_H(x) \neq 0$ and $\pi_H(y) \neq 0$. Put

$$x' := \pi_H(x) \quad \text{and} \quad x'' := x - x'.$$

Note that $F := \text{Supp}(x') \subseteq H$ and $A := \text{Supp}(x'') \subseteq G \setminus H$. Similarly, put

$$y' := \pi_H(y) \quad \text{and} \quad y'' := y - y',$$

and note that $\text{Supp}(y') \subseteq H$ and $B := \text{Supp}(y'') \subseteq G \setminus H$.

Choose some $f \in F$ and $h \in \text{Supp}(y')$. Put $L := C_G(F)$ and let $g \in L$ be arbitrary. Choose some nonzero elements $r_g \in R_g$ and $r_{g^{-1}} \in R_{g^{-1}}$. Then we have $r_{g^{-1}}xr_g \subseteq I$, and thus

$$\begin{aligned} 0 &= r_{g^{-1}}xr_g \cdot y = (r_{g^{-1}}x'r_g + r_{g^{-1}}x''r_g)(y' + y'') \\ &= r_{g^{-1}}x'r_gy' + r_{g^{-1}}x''r_gy' + r_{g^{-1}}x'r_gy'' + r_{g^{-1}}x''r_gy''. \end{aligned}$$

Now, by combining the facts that the G -grading on R is fully supported and fully component regular and that R_H is a domain, it is not difficult to see that $r_{g^{-1}}x'r_gy' \neq 0$. In fact, $fh \in \text{Supp}(r_{g^{-1}}x'r_gy') = \text{Supp}(x'y') \subseteq H$. Using the fact that H is a subgroup of G which is closed under conjugation, we note that $\text{Supp}(r_{g^{-1}}x''r_gy') \cap H = \emptyset$ and $\text{Supp}(r_{g^{-1}}x'r_gy'') \cap H = \emptyset$, and hence we must have $fh \in \text{Supp}(r_{g^{-1}}x''r_gy'') = g^{-1}AgB$. But $g \in L$ may be chosen arbitrarily, and thus Lemma 4.3 yields a contradiction. This shows that R is prime. \square

If R is a unital ring and $x, y \in Z(R)$ are nonzero elements satisfying $xy = 0$, then $I := xR$ and $J := yR$ are nonzero ideals of R such that $I \cdot J = \{0\}$. Thus, we obtain the following corollary which generalizes a conclusion which, using results of Burns [8], is already well known for group rings.

Corollary 4.5. *Let G be a torsion-free group and R a unital G -graded ring. If the G -grading on R is non-degenerate and R_e is a domain, then $Z(R)$ is an integral domain. In particular, every central idempotent in R is trivial.*

5 A potential hierarchy between the three problems

For group rings (cf. Problem 1), it is well known that an affirmative answer to the unit conjecture, for a fixed field K , would yield an affirmative answer to the

zero-divisor conjecture, which in turn would yield an affirmative answer to the idempotent conjecture (see e.g. [53, p. 12]). However, it is not known whether two (or all) of the three conjectures are equivalent.

In this section, we will use the main result from Section 4 to show that the corresponding problems for group graded rings (see Problem 2) exhibit the same potential hierarchy. In fact, the hierarchy is not only manifested for the family of all torsion-free groups, but already at the level of every fixed torsion-free group (see Theorem 5.2). Recall that a ring is called *reduced* if it has no nonzero nilpotent element. We begin by showing the following generalization of [42, Lemma 13.1.2].

Proposition 5.1. *Let G be a torsion-free group and R a G -graded ring whose G -grading is non-degenerate. Then R is a domain if and only if R is reduced and R_e is a domain.*

Proof. The “only if” statement is trivial. We proceed by showing the “if” statement. To this end, suppose that R_e is a domain and that R is not a domain. We need to show that R is not reduced. By Theorem 4.4, we conclude that R is a prime ring. Choose some nonzero elements $x, y \in R$ which satisfy $xy = 0$. By primeness of R , we have $yRx \neq \{0\}$. Note that $(yRx)^2 \subseteq yRxyRx = \{0\}$. Thus, there is some nonzero $z \in yRx$ which satisfies $z^2 = 0$. This shows that R is not reduced. \square

We are now ready to state and prove the main result of this section. It generalizes e.g. [42, Lemma 13.1.2]. See also [53, Remark 1.1].

Theorem 5.2. *Let G be a torsion-free group and R a unital G -graded ring. Furthermore, suppose that the G -grading on R is non-degenerate and that R_e is a domain. Consider the following assertions:*

- (i) *every unit in R is homogeneous;*
- (ii) *R is reduced;*
- (iii) *R is a domain;*
- (iv) *every idempotent in R is trivial.*

Then (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv). Moreover, (iii) \Rightarrow (ii).

Proof. (i) \Rightarrow (ii) Suppose that every unit in R is homogeneous. Let $x \in R$ be an element which satisfies $x^2 = 0$. Note that

$$(1_R + x)(1_R - x) = (1_R - x)(1_R + x) = 1_R.$$

This shows that $1_R - x$ is a unit in R , and hence, by assumption, $1_R - x \in R_g$ for some $g \in G$. Put $r_g := 1_R - x$ and $H := \langle g \rangle$, the subgroup of G generated by g . Consider the subring R_H whose H -grading is non-degenerate. Using the fact that $1_R \in R_e$, we note that $x = 1_R - r_g \in R_H$. We claim that R_H is a domain. If we assume that the claim holds, then $x^2 = 0$ implies $x = 0$ and we are done. Now we show the claim.

Case 1 ($g = e$): By assumption, $R_H = R_e$ is a domain.

Case 2 ($g \neq e$): H is an infinite cyclic group which can be ordered. The desired conclusion follows from Theorem 3.4(ii).

(ii) \Leftrightarrow (iii) This follows from Proposition 5.1.

(iii) \Rightarrow (iv) This is trivial. □

In Section 6, we will record an alternative proof of (ii) \Rightarrow (iv) in the above theorem (see Corollary 6.5).

Remark 5.3. If $u = u^2 \in R$ is an idempotent, then

$$(1_R - 2u)^2 = 1_R - 4u + 4u = 1_R.$$

Thus, if 2 is invertible in R_e , then one can directly, without invoking a primeness argument, show that (i) \Rightarrow (iv) in Theorem 5.2 by proceeding as in the proof of (i) \Rightarrow (ii).

For a G -graded ring R to be a domain, it is obviously also necessary for R_e to be a domain. However, as the following example shows, it is possible for R to have only homogeneous units without R_e being a domain.

Example 5.4. If G is a unique product group and A is a unital commutative ring, then the group ring $A[G]$ has only trivial units if and only if A is reduced and indecomposable (see [51, Proposition 2.1]).

Let G be a unique product group and let $A := C^\infty(\mathbb{R})$ be the algebra of all smooth functions $\mathbb{R} \rightarrow \mathbb{R}$ with pointwise addition and multiplication. Note that A is not a domain. However, A is reduced and indecomposable. Thus, $A[G]$ has only trivial units.

Remark 5.5. (a) While torsion-freeness of G is clearly a necessary condition for a group ring $K[G]$ to be a domain, this is not the case for strongly G -graded rings in general. Indeed, consider for instance the real quaternion algebra \mathbb{H} which is a division ring, and which is strongly graded by the finite group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(b) Example 5.4 shows that Theorem 5.2 fails to hold if the assumption on R_e is dropped.

(c) Non-degeneracy of the grading is not a necessary condition for a group graded ring to be a domain. To see this, consider e.g. Example 2.3 (a).

6 Central elements

The aim of this section is to obtain a strengthening of Corollary 4.5 by completely solving Problem 2 for central elements (see Theorem 6.2).

Proposition 6.1. *Let G be a group and R a unital G -graded ring whose G -grading is non-degenerate. Furthermore, suppose that R_e is a domain and that the G -grading is fully supported. If x is a central element in R , then the subgroup of G generated by $\text{Supp}(x)$ is an FC-group.*

Proof. Let $x = \sum_{g \in G} x_g$ be a central element in R . Take $s \in G$. Choose some nonzero $r_s \in R_s$ and, using Proposition 2.6, note that

$$s \text{ Supp}(x) = \text{Supp}(r_s x) = \text{Supp}(x r_s) = \text{Supp}(x) s.$$

This shows that $\text{Supp}(x)$ is closed under conjugation by elements of G . Thus, by finiteness of $\text{Supp}(x)$, we get that $\text{Supp}(x) \subseteq \Delta(G)$. Let H be the subgroup of G generated by $\text{Supp}(x)$. Using the fact that H is finitely generated, we conclude that H is an FC-group. \square

We now state and prove the main result of this section.

Theorem 6.2. *Let G be a torsion-free group and R a unital G -graded ring. If the G -grading on R is non-degenerate and R_e is a domain, then the following assertions hold:*

- (i) *every central unit in R is homogeneous;*
- (ii) *R has no non-trivial central zero-divisor;*
- (iii) *R is indecomposable, i.e. every central idempotent in R is trivial.*

Proof. By Corollary 2.10, $G' := \text{Supp}(R)$ is a torsion-free subgroup of G . Moreover, R can be equipped with a non-degenerate and fully supported G' -grading. Thus, we will without loss of generality assume that the G -grading on R is fully supported.

(i) Let $x \in Z(R)$ be a unit in R . Denote by H the subgroup of G generated by $\text{Supp}(x)$. Note that, by Lemma 2.5 (ii), x is a unit in R_H . By Proposition 6.1, H

is a torsion-free FC-group. Thus, using Lemma 4.1 and Theorem 3.4, we conclude that x is homogeneous.

(ii) Let $x \in Z(R)$ be nonzero. Suppose that $xy = 0$ for some $y \in R$. Denote by H the subgroup of G generated by $\text{Supp}(x)$. By Proposition 6.1, H is a torsion-free FC-group. Using Lemma 4.1 and Theorem 3.4, we conclude that R_H is a domain. Thus, x is not a zero-divisor in R_H , and by Lemma 2.5 (i), we conclude that $y = 0$.

(iii) This follows from (ii) or from Corollary 4.5. \square

Remark 6.3. (a) Malman has essentially proved Theorem 6.2 (ii) in [29, Proposition 3.14]. Thanks to Neumann's lemma (Lemma 4.1), our proof is shorter.

(b) Note that we can immediately recover Corollary 3.10 from Theorem 6.2.

We record the following well-known lemma.

Lemma 6.4. *If R is a reduced ring, then every idempotent in R is central in R .*

Proof. Let $u \in R$ be an idempotent. Take any $r \in R$. Note that $(ur - uru)^2 = 0$ and $(ru - uru)^2 = 0$. Using the fact that R is reduced, we conclude that

$$ur - uru = 0 \quad \text{and} \quad ru - uru = 0.$$

Hence, $ur = ru$. This shows that $u \in Z(R)$. \square

By combining Lemma 6.4 and Theorem 6.2, we get the following result.

Corollary 6.5. *Let G be a torsion-free group and R a unital G -graded ring. Furthermore, suppose that the G -grading on R is non-degenerate and that R_e is a domain. If R is reduced, then every idempotent in R is trivial.*

Note that the above corollary allows us to establish the implication (ii) \Rightarrow (iv) in Theorem 5.2 without relying on the primeness argument from Section 4.

7 Gradings by quotient groups

In this section, we will show that Problem 2 can be approached by considering gradings by quotient groups (see Proposition 7.3). For G -crossed products, we obtain a more explicit connection (see Proposition 7.5), and as an application, we generalize a result of Bovdi for a special class of solvable groups (see Theorem 7.7).

Remark 7.1. Let G be a group and R a G -graded ring. If N is a normal subgroup of G , then R may be viewed as a G/N -graded ring. Indeed, by writing

$$R = \bigoplus_{g \in G} R_g = \bigoplus_{C \in G/N} \left(\bigoplus_{h \in C} R_h \right),$$

it is easy to see that this yields a G/N -grading.

For a proof of the next lemma, we refer the reader to [26, Proposition 2.21].

Lemma 7.2. *Let G be a group and R a G -graded ring whose G -grading is non-degenerate. If N is a normal subgroup of G , then the canonical G/N -grading on R is non-degenerate.*

The following result generalizes [42, Lemma 13.1.9 (i)] and [40, Corollary 3.6].

Proposition 7.3. *Let G be a group and R a unital G -graded ring whose G -grading is non-degenerate. If N is a normal subgroup of G and R_e is a domain, then the following assertions hold:*

- (i) *If $R_N := \bigoplus_{n \in N} R_n$ is a domain and G/N is a unique product group, then R is a domain.*
- (ii) *Suppose that N is torsion-free, that G/N is a unique product group and that every unit in R which is contained in R_{gN} for some $g \in G$ must be homogeneous with respect to the G -grading. Then every unit in R is homogeneous with respect to the G -grading.*

Proof. (i) We will view

$$R = \bigoplus_{g \in G} R_g = \bigoplus_{C \in G/N} \left(\bigoplus_{h \in C} R_h \right)$$

as a G/N -graded ring. By Lemma 7.2, the G/N -grading is non-degenerate, and by assumption, R_N is a domain. The desired conclusion now follows immediately from Theorem 3.4.

(ii) We begin by noting that, by assumption, every unit in R_N must be homogeneous. Thus, by Theorem 5.2, R_N is a domain. Take $x, y \in R$ which satisfy $xy = 1_R$. Let $\text{Supp}(x)$ and $\text{Supp}(y)$ denote the support of x and y , respectively, with respect to the G -grading. Let $\phi: G \rightarrow G/N$ denote the quotient homomorphism. Define a and b to be the cardinalities of $\phi(\text{Supp}(x))$ and $\phi(\text{Supp}(y))$, respectively. If $a + b > 2$, then by the unique product property of G/N (and Lemma 3.2), we will reach a contradiction in the same way as in the proof of

Theorem 3.4, by instead considering the G/N -grading. Thus, $a = b = 1$. This means that there is some $g \in G$ such that $x \in R_{gN}$ and $y \in R_{g^{-1}N}$. Now, by assumption, both x and y must be homogeneous with respect to the G -grading. \square

Remark 7.4. By taking $N = \{e\}$, note that, from Proposition 7.3 (i), we recover Theorem 3.4 (ii), and from Proposition 7.3 (ii), we recover Theorem 3.4 (i).

Recall that a unital G -graded ring R is said to be a G -crossed product if, for each $g \in G$, the homogeneous component R_g contains an element which is invertible in R (see [34, Chapter 1]). Every G -crossed product is necessarily strongly G -graded (see e.g. [34, Remark 1.1.2]), and in particular, its G -grading is non-degenerate (see Remark 2.2 (a)).

The following result generalizes [42, Lemma 13.1.9 (ii)].

Proposition 7.5. *Let G be a group and R a G -crossed product. Suppose that N is a torsion-free normal subgroup of G , that G/N is a unique product group and that R_e is a domain. The following two assertions are equivalent:*

- (i) *every unit in $R = \bigoplus_{g \in G} R_g$ is homogeneous with respect to the G -grading;*
- (ii) *every unit in $R_N := \bigoplus_{n \in N} R_n$ is homogeneous with respect to the N -grading.*

Proof. (i) \Rightarrow (ii) This is trivial.

(ii) \Rightarrow (i) The G -grading on R is non-degenerate. Thus, the first part of the proof may be carried out in the same way as the proof of Proposition 7.3 (ii). Indeed, for elements $x, y \in R$ which satisfy $xy = yx = 1_R$, we get that $x \in R_{gN}$ and $y \in R_{g^{-1}N}$ for some $g \in G$. Using the fact that R is a G -crossed product, we may choose homogeneous units x' and y' of degree g^{-1} and g , respectively, such that $x'y' = y'x' = 1_R$. Note that

$$1_R = x'y' = x'(xy)y' = (x'x)(yy') = (yy')(x'x),$$

where $x'x \in R_N$ and $yy' \in R_N$. By assumption, $x'x$ and yy' are homogeneous with respect to the N -grading on R_N . Using the fact that x' and y' are homogeneous, we conclude that x and y must be homogeneous with respect to the G -grading. \square

By invoking Theorem 3.4, Example 3.7 and Theorem 5.2, we obtain the following result.

Corollary 7.6. *Let G be a torsion-free group and R a G -crossed product for which R_e is a domain. If N is an abelian normal subgroup of G such that G/N is a unique product group, then every unit in R is homogeneous with respect to the G -grading. Moreover, R is a domain and every idempotent in R is trivial.*

As an application of the above results, we will solve Problem 2 for G -crossed products by a special class of solvable groups and thereby generalize [5, Theorem 1].

Theorem 7.7. *Let G be a group and suppose that G has a finite subnormal series*

$$\langle e \rangle = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_k = G$$

with quotients G_{i+1}/G_i all of which are torsion-free abelian. If R is a G -crossed product with R_e a domain, then every unit in R is homogeneous with respect to the G -grading, and R is a domain. In particular, every idempotent in R is trivial.

Proof. Using that G_0 is a torsion-free normal subgroup of G_1 , that $R_{G_0} = R_e$ is a domain and that G_1/G_0 is a unique product group, we get by Proposition 7.5 that every unit in R_{G_1} is homogeneous. More generally, if G_i is torsion-free and every unit in R_{G_i} is homogeneous, then, by Theorem 5.2, R_{G_i} is a domain. Using the fact that G_i is normal in G_{i+1} and that G_{i+1}/G_i is a unique product group, Proposition 7.5 yields that every unit in $R_{G_{i+1}}$ is homogeneous. Furthermore, since both G_{i+1}/G_i and G_i are torsion-free, we notice that G_{i+1} is also torsion-free. By induction over i , we conclude that every unit in R is homogeneous with respect to the G -grading. Clearly, G is torsion-free. Thus, Theorem 5.2 yields that R is a domain and that every idempotent in R is trivial. \square

Remark 7.8. It is also possible to obtain Theorem 7.7 directly from Theorem 3.4. Indeed, one can show that the group G in Theorem 7.7 is right-ordered (see [42, Lemma 13.1.6]), and hence a unique product group.

8 A conjecture

Recall that, up until now, we have been able to solve Problem 2 in the affirmative in the following important cases:

- when G is a unique product group, including e.g. all torsion-free abelian groups (see Theorem 3.4 and Example 3.7);
- when R is commutative (see Corollary 3.10);
- for central elements (see Theorem 6.2).

Despite the fact that the list of torsion-free non-unique product groups is growing (see Remark 3.8), we dare, in view of our findings, present the following generalizations of the Higman–Kaplansky conjectures for group rings.

Conjecture 8.1. Let G be a torsion-free group and R a unital G -graded ring whose G -grading is non-degenerate. If R_e is a domain with $\text{char}(R_e) = 0$, then the following assertions hold:

- (a) every unit in R is homogeneous;
- (b) R is a domain;
- (c) every idempotent in R is trivial.

Remark 8.2. Using Lemma 2.5, it is not difficult to see that, in order to resolve Conjecture 8.1, it is enough to consider the case where G is finitely generated.

Remark 8.3. In their work, Dykema, Heister and Juschenko [12], and independently Schweitzer [49], identified certain classes of finitely presented torsion-free groups. Amongst other results, they showed that, in order to prove the zero-divisor conjecture for group rings over the field of two elements, it is sufficient to prove the conjecture for groups coming from the aforementioned classes of finitely presented groups. Further investigation is needed to determine whether a similar reduction is possible for group rings in general, and even more generally for the group graded rings appearing in Problem 2.

As mentioned in Section 1, the idempotent conjecture for group rings is related to the Kadison–Kaplansky conjecture for group C^* -algebras, to the Baum–Connes conjecture and to the Farrell–Jones conjecture. Moreover, the zero-divisor conjecture for group rings is related to the Atiyah conjecture [28, Conjecture 10.3]. Further investigations are required to determine which relationship (if any) Conjecture 8.1 may have to other well-known conjectures.

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