# Hilbert's Basis Theorem for Non-associative and Hom-associative Ore Extensions 

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#### Abstract

We prove a hom-associative version of Hilbert's basis theorem, which includes as special cases both a non-associative version and the classical Hilbert's basis theorem for associative Ore extensions. Along the way, we develop hom-module theory. We conclude with some examples of both non-associative and hom-associative Ore extensions which are all noetherian by our theorem.


Keywords Hilbert's basis theorem • Hom-associative algebras • Hom-associative Ore extensions • Hom-modules • Non-commutative noetherian rings

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## 1 Introduction

Hom-associative algebras are not necessarily associative algebras, the associativity condition being replaced by $\alpha(a) \cdot(b \cdot c)=(a \cdot b) \cdot \alpha(c)$, where $\alpha$ is a linear map referred to as a twisting map, and $a, b, c$ arbitrary elements in the algebra. Both associative algebras and non-associative algebras can thus be seen as hom-associative algebras; in the first case, by taking $\alpha$ equal to the identity map, and in the latter case by taking $\alpha$ equal to the zero map.

Historically hom-associative algebras originate in the development of hom-Lie algebras, the latter introduced by Hartwig, Larsson and Silvestrov as generalizations of Lie algebras, the Jacobi identity now twisted by a vector space homomorphism; the "hom"

[^0]referring to this homomorphism [6]. The introduction of these generalizations of Lie algebras was mainly motivated by an attempt to study so-called $q$-deformations of the Witt and Virasoro algebras within a common framework. Makhlouf and Silvestrov then introduced hom-associative algebras as the natural counterparts to associative algebras; taking a homassociative algebra and defining the commutator as a new multiplication gives a hom-Lie algebra, just as with the classical relation between associative algebras and Lie algebras [7]. It was later discovered that there existed formally rigid associative algebras that could now be formally deformed when considered as hom-associative algebras [10], this indicating that hom-associative algebras could be useful in studying deformations as well. Since then, many papers have been written in the subject, and other algebraic structures have been discovered to have natural counterparts in the "hom-world" as well, such as e.g. hom-coalgebras, hom-bialgebras, and hom-Hopf algebras [8, 9].

A class of algebras that contains many examples of formally rigid associative algebras such as e.g. the Weyl algebras and some universal enveloping algebras of Lie algebras, is that of Ore extensions. Ore extensions were first introduced under the name non-commutative polynomial rings by Ore [14]. Non-associative Ore extensions were introduced by Nystedt, Öinert, and Richter in the unital case [12] (see also [13] for a further extension to monoid Ore extensions). The construction was later generalized to non-unital, hom-associative Ore extensions by the authors of the present article and Silvestrov [3]. They give examples, including hom-associative versions of the first Weyl algebra, the quantum plane, and a universal enveloping algebra of a Lie algebra, all of which are formal deformations of their associative counterparts [1, 2].

In this paper, we prove a hom-associative version of Hilbert's basis theorem (Theorem 1), including as special cases both a non-associative version (Corollary 3) and the classical Hilbert's basis theorem for associative Ore extensions (Remark 7). In order to prove this, we develop hom-module theory and a notion of being hom-noetherian (Section 3). Whereas the hom-module theory does not require a multiplicative identity element, the hom-associative Ore extensions in this article are all assumed to be unital. We conclude with some examples of unital, non-associative and hom-associative Ore extensions which are noetherian as a consequence of our main theorem. In more detail, the article is organized as follows:

Section 2 provides preliminaries from the theory of hom-associative algebras, and of unital, hom-associative Ore extensions as developed in [3].

Section 3 deals with hom-modules over non-unital, hom-associative rings. Whereas the notion of a hom-module was first introduced in [10], the theory developed here is new as far as we can tell.

Section 4 contains the proof of a hom-associative version of Hilbert's basis theorem, including as special cases a non-associative and the classical associative version.

Section 5 contains examples of unital, non-associative and hom-associative Ore extensions which are all noetherian by the aforementioned theorem.

## 2 Preliminaries

Throughout this paper, by non-associative algebras we mean algebras which are not necessarily associative, including in particular associative algebras. We call a non-associative algebra $A$ unital if there exists an element $1 \in A$ such that for any element $a \in A$, $a \cdot 1=1 \cdot a=a$. By non-unital algebras, we mean algebras which are not necessarily unital, including unital algebras by definition.

### 2.1 Hom-associative Algebras

This section is devoted to restating some basic definitions and general facts concerning hom-associative algebras.

Definition 1 (Hom-associative algebra) A hom-associative algebra over an associative, commutative, and unital ring $R$, is a triple $(M, \cdot, \alpha)$ consisting of an $R$-module $M$, a binary operation $: M \times M \rightarrow M$ linear over $R$ in both arguments, and an $R$-linear map $\alpha: M \rightarrow M$ satisfying, for all $a, b, c \in M, \alpha(a) \cdot(b \cdot c)=(a \cdot b) \cdot \alpha(c)$. The map $\alpha$ is referred to as the twisting map.

Remark 1 A hom-associative algebra over $R$ is in particular a non-unital, non-associative $R$-algebra, and in case $\alpha$ is the identity map, a non-unital, associative $R$-algebra. Moreover, any non-unital, non-associative $R$-algebra is a hom-associative $R$-algebra with twisting map equal to the zero map.

Remark 2 If $A$ is a unital, hom-associative algebra, then $\alpha$ is completely determined by $\alpha(1)$ since $\alpha(a)=\alpha(a)(1 \cdot 1)=a \cdot \alpha(1)$ for any $a \in A$.

Definition 2 (Morphism of hom-associative algebras) A morphism from a hom-associative $R$-algebra $A:=(M, \cdot, \alpha)$ to a hom-associative $R$-algebra $A^{\prime}:=\left(M^{\prime}, I^{\prime}, \alpha^{\prime}\right)$ is an $R$-linear map $f: M \rightarrow M^{\prime}$ such that $f \circ \alpha=\alpha^{\prime} \circ f$ and $f(a \cdot b)=f(a) \cdot^{\prime} f(b)$ for all $a, b \in M$. If $f$ is bijective, the two are isomorphic, written $A \cong A^{\prime}$.

Definition 3 (Hom-associative subalgebra) Let $A:=(M, \cdot, \alpha)$ be a hom-associative algebra and $N$ a submodule of $M$ that is closed under the multiplication • and invariant under $\alpha$. The hom-associative algebra $\left(N, \cdot,\left.\alpha\right|_{N}\right)$ is said to be a hom-subalgebra of $A$.

Definition 4 (Hom-ideal) A right (left) hom-ideal of a hom-associative $R$-algebra $A$ is an $R$-submodule $I$ of $A$ such that $\alpha(I) \subseteq I$, and for all $a \in A, i \in I, i \cdot a \in I(a \cdot i \in I)$. If $I$ is both a left and a right hom-ideal, it is simply called a hom-ideal.

Note that a hom-ideal of a hom-associative algebra $A$ is in particular a hom-subalgebra of $A$.

Remark 3 In case a hom-associative algebra has twisting map equal to the identity map or the zero map, a right (left) hom-ideal is simply a right (left) ideal.

Definition 5 (Hom-associative ring) A hom-associative ring is a hom-associative algebra over the ring of integers.

Definition 6 (Opposite hom-associative ring) Let $S:=(R, \cdot, \alpha)$ be a hom-associative ring. The opposite hom-associative ring of $S$, written $S^{\mathrm{op}}$, is the hom-associative ring ( $R,{ }^{\mathrm{op}}, \alpha$ ) where $r{ }_{\text {op }} s:=s \cdot r$ for any $r, s \in R$.

### 2.2 Unital, Non-associative Ore Extensions

In this section, we recall from [3] some basic definitions and results concerning unital, non-associative Ore extensions. We denote by $\mathbb{N}$ the set of all non-negative integers,
and by $\mathbb{N}_{>0}$ the set of all positive integers. Let $R$ be a unital, non-associative ring, $\delta: R \rightarrow R$ and $\sigma: R \rightarrow R$ additive maps such that $\sigma(1)=1$ and $\delta(1)=0$. As a set, a unital, non-associative Ore extension of $R$, written $R[X ; \sigma, \delta]$, consists of all formal sums $\sum_{i \in \mathbb{N}} a_{i} X^{i}$, called polynomials, where only finitely many $a_{i} \in R$ are nonzero. We endow $R[X ; \sigma, \delta]$ with the following addition and multiplication, holding for any $m, n \in \mathbb{N}$ and $a_{i}, b_{i} \in R$ :

$$
\sum_{i \in \mathbb{N}} a_{i} X^{i}+\sum_{i \in \mathbb{N}} b_{i} X^{i}=\sum_{i \in \mathbb{N}}\left(a_{i}+b_{i}\right) X^{i}, \quad a X^{m} \cdot b X^{n}=\sum_{i \in \mathbb{N}}\left(a \cdot \pi_{i}^{m}(b)\right) X^{i+n} .
$$

Here, $\pi_{i}^{m}$, referred to as a $\pi$ function, denotes the sum of all $\binom{m}{i}$ possible compositions of $i$ copies of $\sigma$ and $m-i$ copies of $\delta$ in arbitrary order. For instance, $\pi_{1}^{2}=\sigma \circ \delta+\delta \circ \sigma$. We also define $\pi_{0}^{0}:=\operatorname{id}_{R}$ and $\pi_{i}^{m} \equiv 0$ whenever $i<0$, or $i>m$. The identity element 1 in $R$ also becomes an identity element in $R[X ; \sigma, \delta]$ upon identification with $1 X^{0}$. We also think of $X$ as an element of $R[X ; \sigma, \delta]$ by identifying it with the monomial $1 X$. At last, defining two polynomials to be equal if and only if their corresponding coefficients are equal and imposing distributivity of the multiplication over addition make $R[X ; \sigma, \delta]$ a unital, nonassociative, and non-commutative ring. By identifying any $a \in R$ with $a X^{0} \in R[X ; \sigma, \delta]$, we see that $R$ is a subring of $R[X ; \sigma, \delta]$.

Definition 7 ( $\sigma$-derivation) Let $R$ be a unital, non-associative ring where $\sigma$ is a unital endomorphism and $\delta$ an additive map on $R$. Then $\delta$ is called a $\sigma$-derivation if $\delta(a \cdot b)=$ $\sigma(a) \cdot \delta(b)+\delta(a) \cdot b$ holds for all $a, b \in R$. If $\sigma=\operatorname{id}_{R}$, then $\delta$ is simply a derivation.

Remark 4 If $\delta$ is a $\sigma$-derivation on a unital, non-associative ring $R$, then $\delta(1)=0$.
Lemma 1 (Properties of $\pi$ functions) Let $R$ be a unital, non-associative ring, $\sigma$ a unital endomorphism and $\delta$ a $\sigma$-derivation on $R$. Then, in $R[X ; \sigma, \delta]$, the following hold for all $a, b \in R$ and $l, m, n \in \mathbb{N}$ :
(i) $\sum_{i \in \mathbb{N}} \pi_{i}^{m}\left(a \cdot \pi_{l-i}^{n}(b)\right)=\sum_{i \in \mathbb{N}} \pi_{i}^{m}(a) \cdot \pi_{l}^{i+n}(b)$.
(ii) $\pi_{l}^{m+1}=\pi_{l-1}^{m} \circ \sigma+\pi_{l}^{m} \circ \delta=\sigma \circ \pi_{l-1}^{m}+\delta \circ \pi_{l}^{m}$.

Proof A proof of (i) in the associative setting can be found in [11]. However, the proof makes no use of associativity, so we can conclude that (i) holds in the non-associative setting as well.

Regarding (ii), we first recall that $\pi_{l}^{m+1}$ consists of the sum of all $\binom{m+1}{l}$ possible compositions of $l$ copies of $\sigma$ and $m+1-l$ copies of $\delta$. Therefore, we can split the sum into a part containing $\sigma$ innermost (outermost) and a part containing $\delta$ innermost (outermost). When $l=0$, we immediately see that the result holds as $\pi_{-1}^{m}:=0$. When $l>m, \pi_{l}^{m}:=0$, and in case also $l>m+1, \pi_{l}^{m+1}=\pi_{l-1}^{m}:=0$. In case $l=m+1, \pi_{l}^{l}=\pi_{l-1}^{l-1} \circ \sigma=\sigma \circ \pi_{l-1}^{l-1}$, so we can conclude that (ii) holds when $l=0$ and when $l>m$. For the remaining case $1 \leq l \leq m$, we use the recursive formula for binomial coefficients $\binom{m+1}{l}=\binom{m}{l-1}+\binom{m}{l}$ and simply count the terms in the two parts of the sum.

For any unital, hom-associative ring $R$ with twisting map $\alpha$, we extend $\alpha$ homogeneously to an additive map on $R[X ; \sigma, \delta]$ by putting $\alpha\left(\sum_{i \in \mathbb{N}} a_{i} X^{i}\right):=\sum_{i \in \mathbb{N}} \alpha\left(a_{i}\right) X^{i}$ for any $a_{i} \in R$. The next proposition makes use of this construction.

Proposition 1 (Sufficient conditions for hom-associativity of $R[X ; \sigma, \delta]$ [3]) Let $R$ be a unital, hom-associative ring with twisting map $\alpha, \sigma$ a unital endomorphism and $\delta$ a $\sigma$-derivation that both commute with $\alpha$. Extend $\alpha$ homogeneously to $R[X ; \sigma, \delta]$. Then $R[X ; \sigma, \delta]$ is a unital, hom-associative Ore extension with twisting map $\alpha$.

## 3 Hom-Module Theory

In this section, we develop the theory of hom-modules over non-unital, hom-associative rings. The proofs have been omitted since they are nearly identical to the classical proofs of the associative case.

### 3.1 Basic Definitions and Theorems

Definition 8 (Hom-module) Let $R$ be a non-unital, hom-associative ring with twisting map $\alpha_{R}$, multiplication written with juxtaposition. Let $M$ be an additive group with a group homomorphism $\alpha_{M}: M \rightarrow M$, also called a twisting map. A right $R$-hom-module $M_{R}$ consists of $M$ and an operation $\cdot: M \times R \rightarrow M$, called scalar multiplication, such that for all $r_{1}, r_{2} \in R$ and $m_{1}, m_{2} \in M$, the following hold:

$$
\begin{array}{rlr}
\left(m_{1}+m_{2}\right) \cdot r_{1} & =m_{1} \cdot r_{1}+m_{2} \cdot r_{1} & \text { (right-distributivity), } \\
m_{1} \cdot\left(r_{1}+r_{2}\right) & =m_{1} \cdot r_{1}+m_{1} \cdot r_{2} & \text { (left-distributivity), } \\
\alpha_{M}\left(m_{1}\right) \cdot\left(r_{1} r_{2}\right) & =\left(m_{1} \cdot r_{1}\right) \cdot \alpha_{R}\left(r_{2}\right) & \text { (hom-associativity). } \tag{M3}
\end{array}
$$

A left $R$-hom-module is defined analogously and written ${ }_{R} M$.
For the sake of brevity, we also allow ourselves to write $M$ in case it does not matter whether it is a right or a left $R$-hom-module, and simply call it an $R$-hom-module. Furthermore, any two right (left) $R$-hom-modules are assumed to be equipped with the same twisting map $\alpha_{R}$ on $R$.

Remark 5 A hom-associative ring $R$ is both a right $R$-hom-module $R_{R}$ and a left $R$-hommodule ${ }_{R} R$.

Definition 9 (Morphism of hom-modules) A morphism from a right (left) $R$-hom-module $M$ to a right (left) $R$-hom-module $M^{\prime}$ is an additive map $f: M \rightarrow M^{\prime}$ such that $f \circ \alpha_{M}=$ $\alpha_{M^{\prime}} \circ f$ and $f(m \cdot r)=f(m) \cdot r(f(r \cdot m)=r \cdot f(m))$ hold for all $m \in M$ and $r \in R$. If $f$ is also bijective, the two are isomorphic, written $M \cong M^{\prime}$.

Definition 10 (Hom-submodule) Let $M$ be a right (left) $R$-hom-module. An $R$-homsubmodule, or just hom-submodule, is an additive subgroup $N$ of $M$ that is closed under scalar multiplication and invariant under $\alpha_{M} . N$ is then a right (left) $R$-hom-module with twisting maps $\alpha_{R}$ and $\alpha_{N}$, the latter being given by the restriction of $\alpha_{M}$ to $N$. We denote that $N$ is a hom-submodule of $M$ by $N \leq M$ or $M \geq N$, and in case $N$ is a proper subgroup of $M$, by $N<M$ or $M>N$.

Proposition 2 (Image and preimage under hom-module morphism) Let $f: M \rightarrow M^{\prime}$ be a morphism of right (left) $R$-hom-modules, $N \leq M$ and $N^{\prime} \leq M^{\prime}$. Then $f(N)$ and $f^{-1}\left(N^{\prime}\right)$ are hom-submodules of $M^{\prime}$ and $M$, respectively.

Proposition 3 (Intersection of hom-submodules) The intersection of any set of homsubmodules of a right (left) $R$-hom-module is a hom-submodule.

Definition 11 (Generating set of hom-submodule) Let $S$ be a nonempty subset of a right (left) $R$-hom-module $M$. The intersection $N$ of all hom-submodules of $M$ that contain $S$ is called the hom-submodule generated by $S$, and $S$ is called a generating set of $N$. If there is a finite generating set of $N$, then $N$ is called finitely generated.

The direct sum of hom-modules and the quotient of a hom-module by a hom-submodule are defined in the obvious way. Standard results such as the isomorphism theorems then also generalize to this setting.

### 3.2 The Hom-Noetherian Conditions

Recall that a family $\mathcal{F}$ of subsets of a set $S$ satisfies the ascending chain condition if there is no properly ascending infinite chain $S_{1} \subset S_{2} \subset \ldots$ of subsets from $\mathcal{F}$. Furthermore, an element in $\mathcal{F}$ is called a maximal element of $\mathcal{F}$ provided there is no element of $\mathcal{F}$ that properly contains that element.

Proposition 4 (The hom-noetherian conditions for hom-modules) Let $M$ be a right (left) $R$-hom-module. Then the following conditions are equivalent:
(NM1) $M$ satisfies the ascending chain condition on its hom-submodules.
(NM2) Any nonempty family of hom-submodules of $M$ has a maximal element.
(NM3) Any hom-submodule of $M$ is finitely generated.
Definition 12 (Hom-noetherian module) A right (left) $R$-hom-module is called homnoetherian if it satisfies the three equivalent conditions of Proposition 4 on its homsubmodules.

Appealing to Remark 5, all properties that hold for right (left) hom-modules necessarily also hold for hom-associative rings, replacing "hom-submodule" by "right (left) hom-ideal". Hence we have the following:

Corollary 1 (The hom-noetherian conditions for hom-associative rings) Let $R$ be a nonunital, hom-associative ring. Then the following conditions are equivalent:
(NR1) $R$ satisfies the ascending chain condition on its right (left) hom-ideals.
(NR2) Any nonempty family of right (left) hom-ideals of $R$ has a maximal element.
(NR3) Any right (left) hom-ideal of $R$ is finitely generated.

Definition 13 (Hom-noetherian ring) A non-unital, hom-associative ring $R$ is called right (left) hom-noetherian if it satisfies the three equivalent conditions of Corollary 1 on its right (left) hom-ideals. If $R$ satisfies the conditions on both its right and its left hom-ideals, it is called hom-noetherian.

Remark 6 If the twisting map is either the identity map or the zero map, a right (left) homnoetherian ring is simply a right (left) noetherian ring (cf. Remark 3). If $R$ is a unital, hom-associative ring, then all right (left) hom-ideals of $R$ are actually right (left) ideals; if
$I$ is a right hom-ideal of $R$ and $i \in I$, then $\alpha(i)=\alpha(i) \cdot(1 \cdot 1)=i \cdot \alpha(1) \in I$, and similarly for the left case. In particular, $R$ is right (left) hom-noetherian if and only if $R$ is right (left) noetherian.

Proposition 5 (Surjective hom-noetherian hom-module morphism) The hom-noetherian conditions are preserved by surjective morphisms of right (left) R-hom-modules.

Proposition 6 (Hom-noetherian condition on quotient hom-module) Let $M$ be a right (left) $R$-hom-module, and $N \leq M$. Then $M$ is hom-noetherian if and only if $M / N$ and $N$ are hom-noetherian.

Corollary 2 (Finite direct sum of hom-noetherian modules) Any finite direct sum of homnoetherian modules is hom-noetherian.

## 4 Hilbert's Basis Theorem for Hom-associative Ore Extensions

In this section, we consider unital, non-associative Ore extensions $R[X ; \sigma, \delta]$ over some unital, non-associative ring $R$. First, recall from Proposition 1 that if $R$ is hom-associative, then a sufficient condition for $R[X ; \sigma, \delta]$ to be hom-associative is that $\sigma$ is a unital endomorphism and $\delta$ a $\sigma$-derivation that both commute with the twisting map $\alpha$ of $R$, the latter extended homogeneously to $R[X ; \sigma, \delta]$. Moreover, from Remark 6, for unital, hom-associative rings, being right (left) hom-noetherian is the same as being right (left) noetherian. Also recall that the associator is the map $(\cdot, \cdot, \cdot): R \times R \times R \rightarrow R$ defined by $(r, s, t)=(r \cdot s) \cdot t-r \cdot(s \cdot t)$ for any $r, s, t \in R$. The left, middle, and right nucleus of $R$ are denoted by $N_{l}(R), N_{m}(R)$, and $N_{r}(R)$, respectively. As sets, they are defined as $N_{l}(R):=\{r \in R:(r, s, t)=0, s, t \in R\}, N_{m}(R):=\{s \in R:(r, s, t)=0, r, t \in R\}$, and $N_{r}(R):=\{t \in R:(r, s, t)=0, r, s \in R\}$. The nucleus of $R$, written $N(R)$, is defined as the set $N(R):=N_{l}(R) \cap N_{m}(R) \cap N_{r}(R)$. By the associator identity $u \cdot(r, s, t)+(u, r, s) \cdot t+(u, r \cdot s, t)=(u \cdot r, s, t)+(u, r, s \cdot t)$, holding for all $r, s, t, u \in R$, $N_{r}(R), N_{m}(R), N_{r}(R)$, and hence also $N(R)$, are all associative subrings of $R$.

Proposition 7 (Associator of $X^{k}$ ) Let $R[X ; \sigma, \delta]$ be a unital, non-associative Ore extension of a unital, non-associative ring $R$. Assume $\sigma$ is a unital endomorphism and $\delta$ a $\sigma$-derivation on $R$. Then $X^{k} \in N(R[X ; \sigma, \delta])$ for any $k \in \mathbb{N}$.

Proof By identifying $X^{0}$ with $1 \in R, X^{0} \in N(R[X ; \sigma, \delta])$. We now wish to show that $X \in N(R[X ; \sigma, \delta])$. In order to do so, we must show that $X$ associates with all polynomials in $R[X ; \sigma, \delta]$. Due to distributivity, it is however sufficient to prove that $X$ associates with arbitrary monomials $a X^{m}$ and $b X^{n}$ in $R[X ; \sigma, \delta]$. To this end, first note that $a X^{m} \cdot X=$ $\sum_{i \in \mathbb{N}}\left(a \cdot \pi_{i}^{m}(1)\right) X^{i+1}=a X^{m+1}$ since $\sigma$ is unital by assumption, and $\delta(1)=0$ by Remark 4. Then,

$$
\begin{aligned}
\left(a X^{m} \cdot b X^{n}\right) \cdot X & =\left(\sum_{i \in \mathbb{N}}\left(a \cdot \pi_{i}^{m}(b)\right) X^{i+n}\right) \cdot X=\sum_{i \in \mathbb{N}}\left(\left(a \cdot \pi_{i}^{m}(b)\right) X^{i+n}\right) \cdot X \\
& =\sum_{i \in \mathbb{N}}\left(a \cdot \pi_{i}^{m}(b)\right) X^{i+n+1}=a X^{m} \cdot b X^{n+1}=a X^{m} \cdot\left(b X^{n} \cdot X\right),
\end{aligned}
$$

so $X \in N_{r}(R[X ; \sigma, \delta])$. Also, by using (ii) in Lemma 1,

$$
\begin{aligned}
\left(a X^{m} \cdot X\right) \cdot b X^{n} & =a X^{m+1} \cdot b X^{n}=\sum_{i \in \mathbb{N}}\left(a \cdot \pi_{i}^{m+1}(b)\right) X^{i+n} \\
& =\sum_{i \in \mathbb{N}}\left(a \cdot\left(\pi_{i-1}^{m} \circ \sigma(b)+\pi_{i}^{m} \circ \delta(b)\right)\right) X^{i+n} \\
& =\sum_{j \in \mathbb{N}}\left(a \cdot \pi_{j}^{m}(\sigma(b))\right) X^{j+n+1}+\sum_{i \in \mathbb{N}}\left(a \cdot \pi_{i}^{m}(\delta(b))\right) X^{i+n} \\
& =a X^{m} \cdot \sigma(b) X^{n+1}+a X^{m} \cdot \delta(b) X^{n}=a X^{m} \cdot\left(\sigma(b) X^{n+1}+\delta(b) X^{n}\right) \\
& =a X^{m} \cdot \sum_{i \in \mathbb{N}}\left(1 \cdot \pi_{i}^{1}(b)\right) X^{n+i}=a X^{m} \cdot\left(X \cdot b X^{n}\right)
\end{aligned}
$$

so $X \in N_{m}(R[X ; \sigma, \delta])$. By a similar calculation, $X \in N_{l}(R[X ; \sigma, \delta])$, so $X \in$ $N(R[X ; \sigma, \delta])$. Since $N(R[X ; \sigma, \delta])$ is a ring it also contains all powers of $X$, so $X^{k} \in$ $N(R[X ; \sigma, \delta])$ for any $k \in \mathbb{N}$.

Proposition 8 (Hom-modules of $R[X ; \sigma, \delta]$ ) Let $R$ be a unital, noetherian, homassociative ring with twisting map $\alpha, \sigma$ a unital endomorphism and $\delta$ a $\sigma$-derivation that both commute with $\alpha$. Extend $\alpha$ homogeneously to $R[X ; \sigma, \delta]$. Then, for any $m \in \mathbb{N}$, $\sum_{i=0}^{m} X^{i} R\left(\sum_{i=0}^{m} R X^{i}\right)$ is a hom-noetherian right (left) $R$-hom-module.

Proof Let us prove the right case; the left case is similar, but slightly simpler. Put $M=$ $\sum_{i=0}^{m} X^{i} R$. First note that $M$ really is a subset of $R[X ; \sigma, \delta]$, where the elements are of the form $\sum_{i=0}^{m} 1 X^{i} \cdot r_{i} X^{0}$ with $r_{i} \in R$. When identifying $1 X^{i}$ with $X^{i}$ and $r_{i}$ with $r_{i} X^{0}$, this gives us elements of the form $\sum_{i=0}^{m} X^{i} \cdot r_{i}$. Using this identification also allows us to write the multiplication in $R$, which in Definition 8 is done by juxtaposition, by "." instead. The purpose of this is do be consistent with our previous notation.

Since distributivity follows from that in $R[X ; \sigma, \delta]$, it suffices to show that the multiplication in $R[X ; \sigma, \delta]$ is a scalar multiplication, and that we have twisting maps $\alpha_{M}$ and $\alpha_{R}$ that give us hom-associativity. To this end, for any $r \in R$ and any element in $M$ (which is of the form described above), by using Proposition 7,

$$
\begin{equation*}
\left(\sum_{i=0}^{m} X^{i} \cdot r_{i}\right) \cdot r=\sum_{i=0}^{m}\left(X^{i} \cdot r_{i}\right) \cdot r=\sum_{i=0}^{m} X^{i} \cdot\left(r_{i} \cdot r\right) \tag{1}
\end{equation*}
$$

and the latter is clearly an element of $M$. Now, we claim that $M$ is invariant under the homogeneously extended twisting map on $R[X ; \sigma, \delta]$. To follow the notation in Definition 8 , let us denote this map when restricted to $M$ by $\alpha_{M}$, and that of $R$ by $\alpha_{R}$. Then, by using the additivity of $\alpha_{M}$ and $\alpha_{R}$, as well as the fact that the latter commutes with $\delta$ and $\sigma$, we get

$$
\begin{align*}
\alpha_{M}\left(\sum_{i=0}^{m} X^{i} \cdot r_{i}\right) & =\alpha_{M}\left(\sum_{i=0}^{m} \sum_{j \in \mathbb{N}} \pi_{j}^{i}\left(r_{i}\right) X^{j}\right)=\sum_{i=0}^{m} \sum_{j \in \mathbb{N}} \alpha_{M}\left(\pi_{j}^{i}\left(r_{i}\right) X^{j}\right) \\
& =\sum_{i=0}^{m} \sum_{j \in \mathbb{N}} \alpha_{R}\left(\pi_{j}^{i}\left(r_{i}\right)\right) X^{j}=\sum_{i=0}^{m} \sum_{j \in \mathbb{N}} \pi_{j}^{i}\left(\alpha_{R}\left(r_{i}\right)\right) X^{j}=\sum_{i=0}^{m} X^{i} \cdot \alpha_{R}\left(r_{i}\right) \tag{2}
\end{align*}
$$

which again is an element of $M$. At last, let $r, s \in R$ be arbitrary. Then,

$$
\begin{aligned}
\alpha_{M}\left(\sum_{i=0}^{m} X^{i} \cdot r_{i}\right) \cdot(r \cdot s) & \stackrel{(2)}{=}\left(\sum_{i=0}^{m} X^{i} \cdot \alpha_{R}\left(r_{i}\right)\right) \cdot(r \cdot s) \stackrel{(1)}{=} \sum_{i=0}^{m} X^{i} \cdot\left(\alpha_{R}\left(r_{i}\right) \cdot(r \cdot s)\right) \\
& =\sum_{i=0}^{m} X^{i} \cdot\left(\left(r_{i} \cdot r\right) \cdot \alpha_{R}(s)\right) \stackrel{(1)}{=}\left(\sum_{i=0}^{m} X^{i} \cdot\left(r_{i} \cdot r\right)\right) \cdot \alpha_{R}(s) \\
& \stackrel{(1)}{=}\left(\left(\sum_{i=0}^{m} X^{i} \cdot r_{i}\right) \cdot r\right) \cdot \alpha_{R}(s),
\end{aligned}
$$

which proves hom-associativity. What is left to prove is that $M$ is hom-noetherian. Now, let us define $f: \bigoplus_{i=0}^{m} R \rightarrow M$ by $\left(r_{0}, r_{1}, \ldots, r_{m}\right) \mapsto \sum_{i=0}^{m} X^{i} \cdot r_{i}$ for any $\left(r_{0}, r_{1}, \ldots, r_{m}\right) \in$ $\bigoplus_{i=0}^{m} R$. We see that $f$ is additive, and for any $r \in R$, we have $f\left(\left(r_{0}, r_{1}, \ldots, r_{m}\right) \bullet\right.$ $r)=f\left(\left(r_{0}, r_{1}, \ldots, r_{m}\right)\right) \cdot r$. A similar argument gives $f\left(\alpha_{\oplus_{i=0}^{m} R}\left(\left(r_{0}, r_{1}, \ldots, r_{m}\right)\right)\right)=$ $\alpha_{M}\left(f\left(\left(r_{0}, r_{1}, \ldots, r_{m}\right)\right)\right)$, which shows that $f$ is a morphism of two right $R$-hom-modules. Moreover, $f$ is surjective, and so by Proposition 5, $M$ is hom-noetherian.

Lemma 2 (Properties of $R[X ; \sigma, \delta]^{\mathrm{op}}$ ) Let $R$ be a unital, hom-associative ring with twisting map $\alpha, \sigma$ an automorphism and $\delta$ a $\sigma$-derivation that both commute with $\alpha$. Extend $\alpha$ homogeneously to $R[X ; \sigma, \delta]$. Then the following hold:
(i) $\sigma^{-1}$ is an automorphism on $R^{o p}$ that commutes with $\alpha$.
(ii) $-\delta \circ \sigma^{-1}$ is a $\sigma^{-1}$-derivation on $R^{o p}$ that commutes with $\alpha$.
(iii) $R[X ; \sigma, \delta]^{o p} \cong R^{o p}\left[X ; \sigma^{-1},-\delta \circ \sigma^{-1}\right]$.

Proof That $\sigma^{-1}$ is an automorphism and $-\delta \circ \sigma^{-1}$ a $\sigma^{-1}$-derivation on $R^{o p}$ is an exercise in [5] that can be solved without any use of associativity. Now, since $\alpha$ commutes with $\delta$ and $\sigma$, for any $r \in R^{\mathrm{op}}, \sigma\left(\alpha\left(\sigma^{-1}(r)\right)\right)=\alpha\left(\sigma\left(\sigma^{-1}(r)\right)\right)=\alpha(r)$, so by applying $\sigma^{-1}$ to both sides, $\alpha\left(\sigma^{-1}(r)\right)=\sigma^{-1}(\alpha(r))$. From this, it follows that $-\delta\left(\sigma^{-1}(\alpha(r))\right)=-\delta\left(\alpha\left(\sigma^{-1}(r)\right)\right)=$ $\alpha\left(-\delta\left(\sigma^{-1}(r)\right)\right)$, which proves the first and second statement.

For the third statement, let us start by putting $S:=R^{\circ \mathrm{op}}\left[X ; \sigma^{-1},-\delta \circ \sigma^{-1}\right]$ and $S^{\prime}:=$ $R[X ; \sigma, \delta]^{\mathrm{op}}$, and then define a map $f: S \rightarrow S^{\prime}$ by $\sum_{i=0}^{n} r_{i} X^{i} \mapsto \sum_{i=0}^{n} r_{i} \cdot$ op $X^{i}$ for $n \in \mathbb{N}$. We claim that $f$ is an isomorphism of hom-associative rings. First, note that an arbitrary element of $S^{\prime}$ by definition is of the form $p:=\sum_{i=0}^{m} a_{i} X^{i}$ for some $m \in \mathbb{N}$ and $a_{i} \in R^{\mathrm{op}}$. Then,

$$
\begin{aligned}
p & =\underbrace{X^{m} \cdot \sigma^{-m}\left(a_{m}\right)+b_{m-1} X^{m-1}+\cdots+b_{0}}_{=a_{m} X^{m}}+\cdots+\underbrace{X \cdot \sigma^{-1}\left(a_{1}\right)+\delta\left(\sigma^{-1}\left(a_{1}\right)\right)}_{=a_{1} X}+a_{0} \\
& =X^{m} \cdot \sigma^{-m}\left(a_{m}\right)+X^{m-1} \cdot a_{m-1}^{\prime}+\cdots+X \cdot a_{1}^{\prime}+a_{0}^{\prime} \\
& =\sigma^{-m}\left(a_{m}\right) \cdot{ }_{\text {op }} X^{m}+a_{m-1}^{\prime} \cdot \mathrm{op} X^{m-1}+\cdots+\cdot a_{1}^{\prime} \cdot \text { op } X+a_{0}^{\prime} \in \operatorname{im} f,
\end{aligned}
$$

for some $a_{m-1}^{\prime}, b_{m-1}, \ldots, a_{0}^{\prime}, b_{0} \in R^{\mathrm{op}}$, so $f$ is surjective. The second last step also shows that $\sum_{i=0}^{m} R X^{i} \subseteq \sum_{i=0}^{m} X^{i} R$ as sets, and a similar calculation shows that $\sum_{i=0}^{m} X^{i} R \subseteq$
$\sum_{i=0}^{m} R X^{i}$, so that as sets, $\sum_{i=0}^{m} R X^{i}=\sum_{i=0}^{m} X^{i} R$. Hence, if $\sum_{i=0}^{m} r_{i} \cdot{ }^{\circ}$ op $X^{i}=$ $\sum_{j=0}^{m^{\prime}} r_{j}^{\prime} \cdot$ op $X^{j}$ for some $r_{i}, r_{j}^{\prime} \in R^{\mathrm{op}}$ and $m, m^{\prime} \in \mathbb{N}$, then $m=m^{\prime}$ and so

$$
\begin{align*}
0 & =\sum_{i=0}^{m}\left(r_{i}-r_{i}^{\prime}\right) \cdot{ }_{\mathrm{op}} X^{i}=\sum_{i=0}^{m} X^{i} \cdot\left(r_{i}-r_{i}^{\prime}\right)=\sum_{i=0}^{m} \sum_{j \in \mathbb{N}} \pi_{j}^{i}\left(r_{i}-r_{i}^{\prime}\right) X^{j} \\
& =\sum_{j=0}^{m} \sum_{i=0}^{m} \pi_{j}^{i}\left(r_{i}-r_{i}^{\prime}\right) X^{j} \Longrightarrow 0=\sum_{i=0}^{m} \pi_{j}^{i}\left(r_{i}-r_{i}^{\prime}\right) X^{j} \quad \text { for } 0 \leq j \leq m, \tag{3}
\end{align*}
$$

where the implication comes from comparing coefficients with the left-hand side, which is equal to zero. Let us prove by induction that $r_{j}=r_{j}^{\prime}$ for $0 \leq j \leq m$. Put $k=m-j$, where $m$ is fixed, and consider the statement $\mathrm{P}(k): r_{m-k}=r_{m-k}^{\prime}$ for $0 \leq k \leq m$.

Base case $(\mathrm{P}(0)): k=0 \Longleftrightarrow j=m$, so using that $\sigma$ is an automorphism,

$$
0 \stackrel{(3)}{=} \sum_{i=0}^{m} \pi_{m}^{i}\left(r_{i}-r_{i}^{\prime}\right) X^{m}=\sigma^{m}\left(r_{m}-r_{m}^{\prime}\right) X^{m} \Longrightarrow 0=r_{m}-r_{m}^{\prime} \text {. }
$$

Induction step (For $0 \leq k \leq m:(\mathrm{P}(k) \rightarrow \mathrm{P}(k+1))$ ): By putting $j=m-(k+1)$ and then using the induction hypothesis,

$$
0 \stackrel{(3)}{=} \sum_{i=0}^{m} \pi_{m-(k+1)}^{i}\left(r_{i}-r_{i}^{\prime}\right) X^{m-(k+1)}=\sigma^{m-(k+1)}\left(r_{m-(k+1)}-r_{m-(k+1)}^{\prime}\right),
$$

which implies $0=r_{m-(k+1)}=r_{m-(k+1)}^{\prime}$. Hence $r_{j}=r_{j}^{\prime}$ for $0 \leq j \leq m$, so $\sum_{i=0}^{m} r_{i} \cdot \mathrm{op} X^{i}=\sum_{j=0}^{m^{\prime}} r_{j}^{\prime} \cdot$ op $X^{j} \Longrightarrow \sum_{i=0}^{m} r_{i} X^{i}=\sum_{j=0}^{m^{\prime}} r_{j}^{\prime} X^{j}$, proving that $f$ is injective. Additivity of $f$ follows immediately from the definition by using distributivity. Using additivity also makes it sufficient to consider only two arbitrary monomials $a X^{m}$ and $b X^{n}$ in $S$ when proving that $f$ is multiplicative. To this end, let us use the following notation for multiplication in $S: a X^{m} \bullet b X^{n}:=\sum_{i \in \mathbb{N}}\left(a \cdot{ }_{\text {op }} \bar{\pi}_{i}^{m}(b)\right) X^{i+n}$, and then use induction over $n$ and $m$;

Base case $(\mathrm{P}(0,0)): f(a \bullet b)=f\left(a \cdot_{\text {op }} b\right)=a \cdot{ }_{\text {op }} b=f(a) \cdot{ }_{\text {op }} f(b)$.
Induction step over $n(\forall(m, n) \in \mathbb{N} \times \mathbb{N}(\mathrm{P}(m, n) \rightarrow \mathrm{P}(m, n+1)))$ : We know that $X \in N\left(S^{\prime}\right)$ by Proposition 7, and so by a straightforward calculation we have $f\left(a X^{m} \bullet b X^{n+1}\right)=f\left(a X^{m}\right) \cdot{ }_{\text {op }} f\left(b X^{n+1}\right)$.

Induction step over $m(\forall(m, n) \in \mathbb{N} \times \mathbb{N}(\mathrm{P}(m, n) \rightarrow \mathrm{P}(m+1, n)))$ : We know that $X \in N\left(S^{\prime o p}\right) \cap N(S)$ by Proposition 7 , and so by a straightforward calculation, $f\left(a X^{m+1} \bullet b X^{n}\right)=f\left(a X^{m+1}\right) \cdot$ op $f\left(b X^{n}\right)$. Now, according to Definition 2 with $R[X ; \sigma, \delta]$ considered as a hom-associative algebra over the integers, we are done if we can prove that $f \circ \alpha=\alpha \circ f$ for the homogeneously extended map $\alpha$. Since both $\alpha$ and $f$ are additive, it again suffices to prove that $f\left(\left(\alpha\left(a X^{m}\right)\right)=\alpha\left(f\left(a X^{m}\right)\right)\right.$ for some arbitrary monomial $a X^{m}$ in $R[X ; \sigma, \delta]$. This is verified by a simple computation.

Theorem 1 (Hilbert's basis theorem for hom-associative Ore extensions) Let $R$ be a unital, hom-associative ring with twisting map $\alpha, \sigma$ an automorphism and $\delta$ a $\sigma$-derivation that both commute with $\alpha$. Extend $\alpha$ homogeneously to $R[X ; \sigma, \delta]$. If $R$ is right (left) noetherian, then so is $R[X ; \sigma, \delta]$.

Proof This proof is an adaptation of a proof in [5] to the hom-associative setting. Let us begin with the right case, and therefore assume that $R$ is right noetherian. We wish to show
that any right ideal of $R[X ; \sigma, \delta]$ is finitely generated. Since the zero ideal is finitely generated, it is sufficient to show that any nonzero right ideal $I$ of $R[X ; \sigma, \delta]$ is finitely generated. Let $J:=\left\{r \in R: r X^{d}+r_{d-1} X^{d-1}+\cdots+r_{1} X+r_{0} \in I, r_{d-1}, \ldots, r_{0} \in R\right\}$, i.e. $J$ consists of the zero element and all leading coefficients of polynomials in $I$. We claim that $J$ is a right ideal of $R$. First, one readily verifies that $J$ is an additive subgroup of $R$. Now, let $r \in J$ and $a \in R$ be arbitrary. Then there is some polynomial $p=r X^{d}+[$ lower order terms] in $I$. Moreover, $p \cdot \sigma^{-d}(a)=r X^{d} \cdot \sigma^{-d}(a)+[$ lower order terms $]=\left(r \cdot \sigma^{d}\left(\sigma^{-d}(a)\right)\right) X^{d}+$ [lower order terms] $=(r \cdot a) X^{d}+[$ lower order terms $]$, which is an element of $I$ since $p$ is. Therefore, $r \cdot a \in J$, so $J$ is a right ideal of $R$.

Since $R$ is right noetherian and $J$ is a right ideal of $R, J$ is finitely generated, say by $\left\{r_{1}, \ldots, r_{k}\right\} \subseteq J$. All the elements $r_{1}, \ldots, r_{k}$ are assumed to be nonzero, and moreover, each of them is a leading coefficient of some polynomial $p_{i} \in I$ of degree $n_{i}$. Put $n=$ $\max \left(n_{1}, \ldots, n_{k}\right)$. Then each $r_{i}$ is the leading coefficient of $p_{i} \cdot X^{n-n_{i}}=r_{i} X^{n_{i}} \cdot X^{n-n_{i}}+$ [lower order terms] $=r_{i} X^{n}+$ [lower order terms], which is an element of $I$ of degree $n$.

Let $N:=\sum_{i=0}^{n-1} R X^{i}$. Then calculations similar to those in the proof of the third statement of Lemma 2 show that as sets, $N=\sum_{i=0}^{n-1} R X^{i}=\sum_{i=0}^{n-1} X^{i} R$. By Proposition 8 , $N$ is then a hom-noetherian right $R$-hom-module. Now, since $I$ is a right ideal of the ring $R[X ; \sigma, \delta]$ which contains $R$, in particular, it is also a right $R$-hom-module. By Proposition $3, I \cap N$ is then a hom-submodule of $N$, and since $N$ is a hom-noetherian right $R$-hom-module, $I \cap N$ is finitely generated, say by the set $\left\{q_{1}, q_{2}, \ldots, q_{t}\right\}$.

Let $I_{0}$ be the right ideal of $R[X ; \sigma, \delta]$ generated by

$$
\left\{p_{1} \cdot X^{n-n_{1}}, p_{2} \cdot X^{n-n_{2}}, \ldots, p_{k} \cdot X^{n-n_{k}}, q_{1}, q_{2}, \ldots, q_{t}\right\} .
$$

Since all the elements in this set belong to $I$, we have that $I_{0} \subseteq I$. We claim that $I \subseteq I_{0}$. In order to prove this, pick any element $p^{\prime} \in I$.

Base case $(\mathrm{P}(n))$ : If $\operatorname{deg} p^{\prime}<n, p^{\prime} \in N=\sum_{i=0}^{n-1} R X^{i}$, so $p^{\prime} \in I \cap N$. On the other hand, the generating set of $I \cap N$ is a subset of the generating set of $I_{0}$, so $I \cap N \subseteq I_{0}$, and therefore $p^{\prime} \in I_{0}$.

Induction step $(\forall m \geq n(\mathrm{P}(m) \rightarrow \mathrm{P}(m+1)))$ : Assume $\operatorname{deg} p^{\prime}=m \geq n$ and that $I_{0}$ contains all elements of $I$ with deg $<m$. Does $I_{0}$ contain all elements of $I$ with deg $<m+1$ as well? Let $r^{\prime}$ be the leading coefficient of $p^{\prime}$, so that we have $p^{\prime}=r^{\prime} X^{m}+$ [lower order terms]. Since $p^{\prime} \in I$ by assumption, $r^{\prime} \in J$. We then claim that $r^{\prime}=\sum_{i=1}^{k} \sum_{j=1}^{k^{\prime}}\left(\cdots\left(\left(r_{i} \cdot a_{i j 1}\right) \cdot a_{i j 2}\right) \cdot \cdots\right) \cdot a_{i j k^{\prime \prime}}$ for some $k^{\prime}, k^{\prime \prime} \in \mathbb{N}_{>0}$ and some $a_{i j 1}, a_{i j 2}, \ldots, a_{i j k^{\prime \prime}} \in R$. First, we note that since $J$ is generated by $\left\{r_{1}, r_{2}, \ldots, r_{k}\right\}$, it is necessary that $J$ contains all elements of that form. Secondly, we see that subtracting any two such elements or multiplying any such element from the right with one from $R$ again yields such an element, and hence the set of all elements of this form is not only a right ideal containing $\left\{r_{1}, r_{2}, \ldots, r_{k}\right\}$, but also the smallest such to do so.

Recalling that $p_{i} \cdot X^{n-n_{i}}=r_{i} X^{n}+[$ lower order terms $],\left(p_{i} \cdot X^{n-n_{i}}\right) \cdot \sigma^{-n}\left(a_{i j 1}\right)$ $=\left(r_{i} \cdot a_{i j 1}\right) X^{n}+[$ lower order terms $]$, and by iterating this multiplication from the right, we set $c_{i j}:=\left(\cdots\left(\left(\left(p_{i} \cdot X^{n-n_{i}}\right) \cdot \sigma^{-n}\left(a_{i j 1}\right)\right) \cdot \sigma^{-n}\left(a_{i j 2}\right)\right) \cdots\right) \cdot \sigma^{-n}\left(a_{i j k^{\prime \prime}}\right)$. Since $p_{i} \cdot X^{n-n_{i}}$ is a generator of $I_{0}, c_{i j}$ is an element of $I_{0}$ as well, and therefore also $q:=\sum_{i=1}^{k} \sum_{j=1}^{k^{\prime}} c_{i j}$. $X^{m-n}=r^{\prime} X^{m}+[$ lower order terms $]$. However, as $I_{0} \subseteq I$, we also have that $q \in I$, and since $p^{\prime} \in I,\left(p^{\prime}-q\right) \in I$. Now, $p^{\prime}=r^{\prime} X^{m}+[$ lower order terms $]$, so $\operatorname{deg}\left(p^{\prime}-q\right)<m$, and therefore $\left(p^{\prime}-q\right) \in I_{0}$. This shows that $p^{\prime}=\left(p^{\prime}-q\right)+q$ is an element of $I_{0}$ as well, and thus $I=I_{0}$, which is finitely generated.

For the left case, first note that any hom-associative ring $S$ is right (left) noetherian if and only if $S^{\mathrm{op}}$ is left (right) noetherian, due to the fact that any right (left) ideal of
$S$ is a left (right) ideal of $S^{\text {op }}$, and vice versa. Now, assume that $R$ is left noetherian. Then, $R^{\text {op }}$ is right noetherian, and using (i) and (ii) in Lemma $2, \sigma^{-1}$ is an automorphism and $-\delta \circ \sigma^{-1}$ a $\sigma^{-1}$-derivation on $R^{\text {op }}$ that both commute with $\alpha$. Hence, by the previously proved right case, $R^{\mathrm{op}}\left[X ; \sigma^{-1},-\delta \circ \sigma^{-1}\right]$ is right noetherian. By (iii) in Lemma 2, $R^{\mathrm{op}}\left[X ; \sigma^{-1},-\delta \circ \sigma^{-1}\right] \cong R[X ; \sigma, \delta]^{\text {op }}$. One verifies that surjective morphisms between hom-associative rings preserve the noetherian conditions (NR1), (NR2), and (NR3) in (NR1) by examining the proof of Proposition 5, changing the module morphism to that between rings instead, and "submodule" to "ideal". Therefore, $R[X ; \sigma, \delta]^{\mathrm{op}}$ is right noetherian, so $R[X ; \sigma, \delta]$ is left noetherian.

Remark 7 By putting $\alpha=\operatorname{id}_{R}$ in Theorem 1, we recover the classical Hilbert's basis theorem for associative Ore extensions.

Corollary 3 (Hilbert's basis theorem for non-associative Ore extensions) Let $R$ be a unital, non-associative ring, $\sigma$ an automorphism and $\delta$ a $\sigma$-derivation on $R$. If $R$ is right (left) noetherian, then so is $R[X ; \sigma, \delta]$.

Proof Put $\alpha \equiv 0$ in Theorem 1 .

## 5 Examples

Here we provide some examples of unital, non-associative and hom-associative Ore extensions which are all noetherian by the above theorem. First, recall that there are, up to isomorphism, only four normed, unital division algebras over the real numbers: the real numbers themselves, the complex numbers, the quaternions $(\mathbb{H})$, and the octonions $(\mathbb{O})$ [16]. The largest of the four are the octonions, and while sharing the property of not being commutative with the quaternions, the octonions are the only ones that are not associative. All of the four algebras above are noetherian, and hence also all iterated Ore extensions of them: let $D$ be any unital division algebra, and $I$ any nonzero right ideal of $D$. If $a \in D$ is an arbitrary nonzero element, then $1=a \cdot a^{-1} \in I$, so $I=D$, and analogously for the left case. As an ideal of itself, $D$ is finitely generated (by 1 , for instance), as is the zero ideal.

The derivations on any normed division algebra $D$ is a linear combination of derivations $\delta_{a, b}$ where $a, b \in D$, defined by $\delta_{a, b}(x):=[[a, b], x]-3(a, b, x)$ for all $x \in D[15]$. These derivations are called inner, and in particular, all derivations on $\mathbb{H}$ are of the form [ $a, \cdot]$ for some $a \in \mathbb{H}$.

Given a unital and associative algebra $A$ with product - over a field of characteristic different from two, one may define a unital and non-associative algebra $A^{+}$by using the Jordan product $\{\cdot, \cdot\}: A^{+} \rightarrow A^{+}$. This is given by $\{a, b\}:=\frac{1}{2}(a \cdot b+b \cdot a)$ for any $a, b \in$ A. $A^{+}$is then a Jordan algebra, i.e. a commutative algebra where any two elements $a$ and $b$ satisfy the Jordan identity, $\{\{a, b\}\{a, a\}\}=\{a,\{b,\{a, a\}\}\}$. Since inverses on $A$ extend to inverses on $A^{+}$, one may infer that if $A=\mathbb{H}$, then $A^{+}$is also noetherian. Using the standard notation $i, j, k$ for the quaternion units in $\mathbb{H}$ with defining relation $i^{2}=j^{2}=k^{2}=i j k=$ -1 , one can see that $\mathbb{H}^{+}$is not associative as e.g. $(i, i, j)_{\mathbb{H}^{+}}:=\{\{i, i\}, j\}-\{i,\{i, j\}\}=-j$.

Example 1 Let $\sigma$ be the automorphism on $\mathbb{H}$ defined by $\sigma(i)=-i, \sigma(j)=k$, and $\sigma(k)=$ $j$. Any automorphism on $\mathbb{H}$ is also an automorphism on $\mathbb{H}^{+}$, and hence $\mathbb{H}^{+}\left[X ; \sigma, 0_{\mathbb{H}}\right]$ is a unital, non-associative Ore extension. $\mathbb{H}^{+}\left[X ; \sigma, 0_{\mathbb{H}}\right]$ is then noetherian by Corollary 3 .

Example 2 Let $[j, \cdot]_{\mathbb{H}}$ be the inner derivation on $\mathbb{H}$ induced by $j$. Any derivation on $\mathbb{H}$ is also a derivation on $\mathbb{H}^{+}$, and so we may form the unital, non-associative Ore extension $\mathbb{H}^{+}\left[X ; \mathrm{id}_{\mathbb{H}},[j, \cdot]_{\mathbb{H}}\right]$ which is noetherian due to Corollary 3.

Example 3 From the Jordan identity one may infer that a map $\delta_{a, b}: J \rightarrow J$ defined by $\delta_{a, b}(x):=(a, x, b)_{J}$ for any $a, b, x \in J$ where $J$ is a Jordan algebra, is a derivation, called an inner derivation. On $\mathbb{H}^{+}$one could for instance take $a=i$ and $b=j$, resulting in $\delta_{i, j}(x)=\{\{i, x\}, j\}-\{i,\{x, j\}\}$ for any $x \in \mathbb{H}^{+}$. Then $\mathbb{H}^{+}\left[X ; \mathrm{id}_{\mathbb{H}}, \delta_{i, j}\right]$ is a unital, non-associative Ore extension which is noetherian by Corollary 3.

Example 4 Take any derivation on $\mathbb{O}$, e.g. $\delta_{i, j}$ defined by $\delta_{i, j}(x):=[[i, j], x]-3(i, j, x)$ for any $x \in \mathbb{O}$. Then $\mathbb{O}\left[X ; \mathrm{id}_{\mathbb{O}}, \delta_{i, j}\right]$ is a unital, non-associative Ore extension which is noetherian by Corollary 3 .

Example 5 For any $q \in \mathbb{R} \backslash\{0,1\}$, one may define an octonionic $q$-Weyl algebra $A_{q}(\mathbb{D})$ as the tensor product of the usual $q$-Weyl algebra $A_{q}(\mathbb{R})$ over $\mathbb{R}$, and $\mathbb{O}$. In particular, $A_{q}(\mathbb{O})$ is a free module of finite rank over $A_{q}(\mathbb{R})$, and hence it is noetherian. Alternatively, one may see that $A_{q}(\mathbb{O})$ is noetherian by noting that it is the iterated Ore extension $\mathbb{O}[Y][X ; \sigma, \delta]$ where $\sigma$ is the automorphism on $\mathbb{O}[Y]$ defined by $\sigma(Y)=q Y$, and $\delta$ the Jackson $q$-derivative, i.e. the $\sigma$-derivation defined on an arbitrary polynomial $p(Y) \in \mathbb{O}[Y]$ by

$$
\delta(p(Y)):=\frac{p(q Y)-p(Y)}{q Y-Y}=\frac{\sigma(p(Y))-p(Y)}{\sigma(Y)-Y} .
$$

Example 6 This example is a slight generalization of Example 1.1 in [4]. Let $R$ and $S$ be associative, commutative, and unital rings, and $f: R \rightarrow S$ a homomorphism. Further assume that $R$ is noetherian. Let $A$ be a non-associative, non-unital, noetherian $S$-algebra, and define a multiplication on $U:=R \times A$ by $\left(r_{1}, a_{1}\right) \cdot\left(r_{2}, a_{2}\right):=\left(r_{1} r_{2}, f\left(r_{1}\right) a_{2}+\right.$ $\left.f\left(r_{2}\right) a_{1}+a_{1} a_{2}\right)$ for any $r_{1}, r_{2} \in R$ and $a_{1}, a_{2} \in A . U$ is then unital with identity element $(1,0)$, and by defining a twisting map $\alpha$ on $U$ by $\alpha(r, a):=(p r, 0)$ for any $r \in R, a \in A$, and $p \in \operatorname{ker} f, U$ is hom-associative. Moreover, $U$ is noetherian, and if $A$ is not associative, then $U$ is not associative. Now, let $\sigma_{A}$ be an automorphism on $A$. Then $\sigma$ defined by $\sigma(r, a):=\left(r, \sigma_{A}(a)\right)$ is an automorphism on $U$. Moreover, if $\delta_{A}$ is a $\sigma_{A}$-derivation on $A$, then $\delta$ defined by $\delta(r, a):=\left(0, \delta_{A}(a)\right)$ is a $\sigma$-derivation on $U$, and both $\delta$ and $\sigma$ commute with $\alpha$. Hence, by Theorem $1, U[X ; \sigma, \delta]$ is noetherian. Here, one could e.g. take $R=\mathbb{R}[Y]$, $S=\mathbb{R}, f: \mathbb{R}[Y] \rightarrow \mathbb{R}$ the evaluation homomorphism at zero, $p \in \mathbb{R}[Y]$ any polynomial without a constant term, and $A, \sigma_{A}$, and $\delta_{A}$ any $\mathbb{R}$-algebra, $\sigma$, and $\delta$, respectively, from the previous examples.

We here include a proof that $U$ is noetherian. Suppose we have an ascending chain of right (left) ideals, $I_{1} \subseteq I_{2} \subseteq \ldots$, in $U$. Define $J_{j}=\left\{r \in R \mid \exists a \in A:(r, a) \in I_{j}\right\}$. This is an ideal in $R$. Also define $H_{j}=\left\{a \in A \mid(0, a) \in I_{j}\right\}$. This is a right (left) ideal in $A$. We thus have two ascending chains, $J_{1} \subseteq J_{2} \subseteq \ldots$ and $H_{1} \subseteq H_{2} \subseteq \ldots$, in $R$ and $A$, respectively. Since $R$ and $A$ are noetherian there is some integer $n$ such that if $k>n$ then $J_{k}=J_{n}$ and $H_{k}=H_{n}$. We claim that in fact also $I_{k}=I_{n}$. Let $(r, a) \in I_{k}$. Then $r \in J_{k}=J_{n}$ so there is $a^{\prime} \in A$ such that $\left(r, a^{\prime}\right) \in I_{n}$. It follows that $a-a^{\prime} \in H_{k}=H_{n}$, which implies $\left(0, a-a^{\prime}\right) \in I_{n}$. Hence $(r, a)=\left(r, a^{\prime}\right)+\left(0, a-a^{\prime}\right)$ is a sum of two elements in $I_{n}$ and therefore belongs to $I_{n}$.

Example 7 Let $R$ be a unital, non-associative, noetherian ring, and denote by $I$ the ideal of $R$ generated by all expressions of the form $r(s t)-(r s) t$ where $r, s, t \in R$. Define $S:=R / I$ and let $\pi: R \rightarrow S$ be the natural homomorphism. Set $U:=R \times S$ and define a multiplication $\cdot$ on $U$ by $\left(r_{1}, s_{1}\right) \cdot\left(r_{2}, s_{2}\right):=\left(r_{1} r_{2}, \pi\left(r_{1}\right) s_{2}+s_{1} \pi\left(r_{2}\right)+s_{1} s_{2}\right)$ for all $r_{1}, r_{2} \in R$ and $s_{1}, s_{2} \in R . U$ is unital with identity element ( 1,0 ), and the map $\alpha$ defined by $\alpha(r, s):=(0, \pi(r)+s)$ for all $r \in R$ and $s \in S$ is a well-defined twisting map that makes $U$ hom-associative. Since $R$ is noetherian, so is $S$, and by the same argument as in Example $6, U$ is noetherian. Moreover, if $R$ is not associative, then $U$ is not associative. Now, let $\sigma_{R}$ be an endomorphism on $R$. Then $\sigma_{R}(I) \subseteq I$, which guarantees the naturally extended endomorphism $\sigma_{S}$ on $S$ to be well-defined. By defining $\sigma(r, s):=\left(\sigma_{R}(r), \sigma_{S}(s)\right)$, we get an endomorphism $\sigma$ on $U$. Similarly, any $\sigma_{R}$-derivation $\delta_{R}$ satisfies $\delta_{R}(I) \subseteq I$, and hence the naturally extended $\sigma_{S}$-derivation $\delta_{S}$ is well-defined, and in turn gives rise to a $\sigma$-derivation $\delta$ on $U$ defined by $\delta(r, s):=\left(\delta_{R}(r), \delta_{S}(s)\right)$. Now, assume that $\sigma_{R}$ is an automorphism. Then it is clear that $\sigma_{S}$ is surjective. Moreover, $\sigma_{R}^{-1}(I) \subseteq I$, which implies that $\sigma_{S}$ is injective. Hence $\sigma$ is an automorphism. Moreover, $\alpha$ commutes with both $\delta$ and $\sigma$, and therefore $U[X ; \sigma, \delta]$ is noetherian. Here, one could e.g. take $R$ to be any base ring together with $\sigma$ and $\delta$ from the previous examples.

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