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Graded von Neumann regularity of rings graded by semigroups

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Abstract
In this article, we give a complete characterization of semigroup graded rings which are graded von Neumann regular. We also demonstrate our results by applying them to several classes of examples, including matrix rings and groupoid graded rings.

Keywords Graded ring · Regular ring · von Neumann regular ring · Semigroup · Groupoid

Mathematics Subject Classification 16W50 · 16E50

1 Introduction

In 1936, von Neumann (1936) introduced the class of *regular rings* which nowadays are known as *von Neumann regular rings*. Recall that a ring R is *von Neumann regular* if $r \in rRr$ for every $r \in R$. Note that any division ring, for instance, is von Neumann regular. On the other hand, the ring of integers $\mathbb{Z}$ is not von Neumann regular. For a thorough account of the theory of von Neumann regular rings, we refer the reader to Goodearl (1991).

Let S be a semigroup. A ring R is said to be S-*graded* (or *graded by* S) if there is a collection of additive subgroups $\{R_s\}_{s \in S}$ of R such that $R = \bigoplus_{s \in S} R_s$ and $R_s R_t \subseteq R_{st}$ for all $s, t \in S$. If G is a group, then a G-graded ring R is said to be *graded von Neumann regular* if, for all $g \in G$ and $r \in R_g$, the relation $r \in rRr$ holds. This notion was introduced by Năstăsescu and van Oystaeyen who also gave a characterization of graded von Neumann regular rings in the class of unital strongly group graded rings

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(see (Năstăsescu and van Oystaeyen 1982, Cor. C.I.5.3)). In Lännström (2021), the first named author of this article gave a characterization of (non-unital) group graded rings which are graded von Neumann regular. The purpose of this article is to generalize that result to the setting of rings graded by semigroups.

Recall that a semigroup S is said to be *regular* if $Q(s) := \{x \in S \mid s = sxs\}$ is nonempty for every $s \in S$. One can show that $Q(s)$ is nonempty for every $s \in S$ if and only if $V(s) := \{x \in S \mid s = sxs \text{ and } x = xsx\}$ is nonempty for every $s \in S$. The class of regular semigroups contains two subclasses which are of great importance; the inverse semigroups and the completely regular semigroups. Now, we introduce the notion of a graded von Neumann regular ring in the context of semigroup graded rings (cf. Definition 4).

Definition 1 Let S be a semigroup. An S-graded ring R is said to be *graded von Neumann regular* if for all $s \in S$, $r \in R_s$ and $t \in V(s)$, there is some $y \in R_t$ such that $r = ryr$.

Note that if S is a group, then the above definition coincides with the one used by Năstăsescu and van Oystaeyen (1982).

Throughout this article, all rings are assumed to be associative, but not necessarily commutative nor unital. Here is an outline of this article.

In Sect. 2, we introduce and record characterizations of epsilon-strong, respectively nearly epsilon-strong, semigroup gradings. In Sect. 3, we prove the main results of this article (see Theorems 8 and 11). In Sect. 4, we apply our results to several different classes of examples. Notably, we obtain a complete characterization of groupoid graded rings which are graded von Neumann regular in the sense of Definition 4 (see Theorem 17).

2 Nearly epsilon-strong semigroup gradings

In this section, we introduce epsilon-strong, respectively nearly epsilon-strong, gradings in the context of semigroup graded rings. We also characterize such gradings (see Propositions 1 and 4). Throughout this section, we let S denote an arbitrary semigroup.

Remark 1 Let R be an S-graded ring.

(a) For each $e \in E(S) = \{s \in S \mid s^2 = s\}$, R_e is a subring of R.
(b) For all $s \in S$ and $t \in V(s)$, R_s is an R_{st}-R_{ts}-bimodule.
(c) For all $s \in S$ and $t \in V(s)$, $R_s R_t$ is an ideal of R_{st}, and $R_t R_s$ is an ideal of R_{ts}.

Recall that a ring R is said to be *left s-unital* (resp. *right s-unital*) if $x \in Rx$ (resp. $x \in xR$) for every $x \in R$. A ring which is both left s-unital and right s-unital is said to be *s-unital*. A ring R is said to be *left unital* (resp. *right unital*) if it has a left unity element (resp. a right unity element), i.e. if there is an element $u \in R$ such that $ur = r$ (resp. $ru = r$) for every $r \in R$.

Epsilon-strongly and *nearly epsilon-strongly* group graded rings were introduced and studied in Nystedt et al. (2018) and Nystedt and Öinert (2020), respectively. Here we introduce the corresponding notions for semigroup gradings.

Definition 2 Let R be an S-*graded* ring.

(i) If $R_s = R_s R_t R_s$ for all $s \in S$ and $t \in V(s)$, then R is said to be *symmetrically* S-*graded*.
(ii) If $R_s R_t = R_{st}$ for all $s, t \in S$, then R is said to be *strongly* S-*graded*.
(iii) If R is symmetrically S-graded, and $R_s R_t$ is unital for all $s \in S$ and $t \in V(s)$, then R is said to be *epsilon-strongly* S-*graded*.
(iv) If R is symmetrically S-graded, and $R_s R_t$ is s-unital for all $s \in S$ and $t \in V(s)$, then R is said to be *nearly epsilon-strongly* S-*graded*.

Remark 2 When specializing the above definition to the case where S is a group, one recovers the definitions from Nystedt and Öinert (2020) and Nystedt et al. (2018).

Proposition 1 *Let R be an S-graded ring. The following two assertions are equivalent:*

(i) *R is epsilon-strongly S-graded;*
(ii) *For all $s \in S$ and $t \in V(s)$, there exist $\epsilon_{s,t} \in R_s R_t$ and $\epsilon'_{t,s} \in R_t R_s$ such that the equalities $\epsilon_{s,t} r = r = r\epsilon'_{t,s}$ hold for every $r \in R_s$.*

Proof (i)⇒(ii): Suppose that R is epsilon-strongly S-graded. Take $s \in S$, $t \in V(s)$ and $r \in R_s$. By assumption, $R_s R_t$ is unital. Note that $s \in V(t)$, and hence $R_t R_s$ is also unital. Let $\epsilon_{s,t}$ and $\epsilon'_{t,s}$ denote the unity elements of $R_s R_t$ and $R_t R_s$, respectively. Using that R is symmetrically S-graded, there exist $n \in \mathbb{N}$, $a_i, c_i \in R_s$ and $b_i \in R_t$, for $i \in \{1, \ldots, n\}$, such that $r = \sum_{i=1}^{n} a_i b_i c_i$. Take $i \in \{1, \ldots, n\}$. Since $a_i b_i \in R_s R_t$ we get that $\epsilon_{s,t} a_i b_i = a_i b_i$ and thus $\epsilon_{s,t} r = \sum_{i=1}^{n} \epsilon_{s,t} a_i b_i c_i = \sum_{i=1}^{n} a_i b_i c_i = r$. Similarly, $r\epsilon'_{t,s} = r$.

(ii)⇒(i): Suppose that (ii) holds. Take $s \in S$ and $t \in V(s)$. It is clear that $R_s R_t R_s \subseteq R_{sts} = R_s$. Now, we show the reversed inclusion. Take $r \in R_s$. By assumption, $\epsilon_{s,t} \in R_s R_t$ and $\epsilon_{s,t} r = r$. In particular, it follows that $r = \epsilon_{s,t} r \in R_s R_t R_s$. This shows that R is symmetrically S-graded. Clearly, $\epsilon_{s,t}$ is a left unity element of $R_s R_t$. Note that $s \in V(t)$. Hence, there is an element $\epsilon'_{s,t} \in R_s R_t$ such that $r'\epsilon'_{s,t} = r'$ for every $r' \in R_t$. This means that $\epsilon'_{s,t}$ is a right unity element of $R_s R_t$. In fact, $\epsilon_{s,t} = \epsilon_{s,t}\epsilon'_{s,t} = \epsilon'_{s,t}$ is the unity element of $R_s R_t$. □

Corollary 2 *If R is an epsilon-strongly S-graded ring, then R_e is unital for every $e \in E(S)$.*

Proof Suppose that R is an epsilon-strongly S-graded ring. Take $e \in E(S)$. Note that $e \in V(e)$. There is some $\epsilon_{e,e} \in R_e R_e$ and $\epsilon'_{e,e} \in R_e R_e$ such that $\epsilon_{e,e} r = r$ and $r\epsilon'_{e,e} = r$ for every $r \in R_e$. In particular, $\epsilon_{e,e} = \epsilon_{e,e}\epsilon'_{e,e} = \epsilon'_{e,e}$. □

Proposition 3 (Nystedt (2019, Prop. 2.11), Tominaga (1976)) *A ring R is left s-unital (resp. right s-unital) if and only if for any finite subset V of R there is $u \in R$ such that for every $v \in V$ the equality $uv = v$ (resp. $vu = v$) holds.*

Proposition 4 *Let R be an S-graded ring. The following two assertions are equivalent:*

(i) *R is nearly epsilon-strongly S-graded;*

(ii) *For all $s \in S$, $t \in V(s)$ and $r \in R_s$, there exist $\epsilon_{s,t}(r) \in R_s R_t$ and $\epsilon'_{t,s}(r) \in R_t R_s$ such that the equalities $\epsilon_{s,t}(r)r = r = r\epsilon'_{t,s}(r)$ hold.*

Proof (i)⇒(ii): Suppose that R is nearly epsilon-strongly S-graded. Take $s \in S$, $t \in V(s)$ and $r \in R_s$. By assumption, $R_s R_t$ is s-unital. Note that $s \in V(t)$, and hence $R_t R_s$ is also s-unital. Using that R is symmetrically S-graded, there exist $n \in \mathbb{N}$, $a_i, c_i \in R_s$ and $b_i \in R_t$, for $i \in \{1, \ldots, n\}$, such that $r = \sum_{i=1}^{n} a_i b_i c_i$. Since $a_i b_i \in R_s R_t$ and $b_i c_i \in R_t R_s$ for every $i \in \{1, \ldots, n\}$, from Proposition 3, it follows that there exist $\epsilon_{s,t}(r) \in R_s R_t$ and $\epsilon'_{t,s}(r) \in R_t R_s$ such that the equalities $\epsilon_{s,t}(r)a_i b_i = a_i b_i$ and $b_i c_i \epsilon'_{t,s}(r) = b_i c_i$ hold for every $i \in \{1, \ldots, n\}$. Thus, $\epsilon_{s,t}(r)r = r = r\epsilon'_{t,s}(r)$.

(ii)⇒(i): Suppose that (ii) holds. Symmetry of the S-grading is shown in the same way as for Proposition 1. Take $s \in S$, $t \in V(s)$ and $r \in R_s$. By assumption, there is $\epsilon_{s,t}(r) \in R_s R_t$ such that $\epsilon_{s,t}(r)r = r$. Thus, $R_s R_t$ is left s-unital. Note that $s \in V(t)$. Take $r' \in R_t$. By assumption, there is $\epsilon'_{s,t} \in R_s R_t$ such that $r'\epsilon'_{s,t} = r'$. Thus, $R_s R_t$ is right s-unital. We conclude that $R_s R_t$ is s-unital. □

3 Graded von Neumann regularity

In this section, we establish the main results of this article (see Theorems 8 and 11). Throughout this section, unless stated otherwise, S is assumed to be an arbitrary semigroup. Recall that $E(S)$ denotes the set of idempotents in S.

Lemma 5 *If R is an S-graded ring which is graded von Neumann regular, then the following two assertions hold:*

(i) *R_e is von Neumann regular for every $e \in E(S)$;*
(ii) *R is nearly epsilon-strongly S-graded.*

Proof Suppose that R is S-graded and graded von Neumann regular.

(i) Take $e \in E(S)$ and $r \in R_e$. Using that $e = e^3$, by graded von Neumann regularity, there is some $y \in R_e$ such that $r = ryr$. Thus, R_e is von Neumann regular.
(ii) Take $s \in S$, $t \in V(s)$ and $r \in R_s$. Using that S is graded von Neumann regular, there is some $y \in R_t$ such that $r = ryr$. Put $\epsilon_{s,t}(r) := ry \in R_s R_t$ and $\epsilon'_{t,s}(r) := yr \in R_t R_s$. Clearly, $\epsilon_{s,t}(r)r = ryr = r$ and $r\epsilon'_{t,s}(r) = ryr = r$. By Proposition 4 we conclude that R is nearly epsilon-strongly S-graded. □

We recall the following well-known characterization of von Neumann regularity. For a proof, see e.g. (Lännström 2021, Proposition 2.2).

Proposition 6 *Let T be an s-unital ring. The following three assertions are equivalent:*

(i) *T is von Neumann regular;*
(ii) *Every principal left (resp. right) ideal of T is generated by an idempotent;*
(iii) *Every finitely generated left (resp. right) ideal of T is generated by an idempotent.*

Lemma 7 *Let R be a nearly epsilon-strongly S-graded ring and suppose that R_e is von Neumann regular for every $e \in E(S)$. Then, for all $s \in S$, $t \in V(s)$ and $r \in R_s$, the left R_{ts}-ideal $R_t r$ is generated by an idempotent in R_{ts}.*

Proof Take $s \in S, t \in V(s)$ and $r \in R_s$. Put $I := R_t r$. Immediately from the grading, it follows that $R_t r \subseteq R_t R_s \subseteq R_{ts}$, and that $R_{ts} R_t r \subseteq R_{tst} r = R_t r$. Thus, $R_t r$ is a left ideal of R_{ts}.

Using that R is nearly epsilon-strongly S-graded, there is an element $y \in R_s R_t$ such that $yr = r$. We may write $y = \sum_{i=1}^k a_i b_i$ where $a_1, \ldots, a_k \in R_s$ and $b_1, \ldots, b_k \in R_t$. Note that $c_i := b_i r \in R_t R_s \subseteq R_{ts}$ with $ts \in E(S)$. Let $d \in R_t r$ be arbitrary. We may find some $r' \in R_t$ such that $d = r'r$. Note that $r'a_i \in R_t R_s \subseteq R_{ts}$ and $d = r'r = r'yr = \sum_{i=1}^k r'a_i b_i r = \sum_{i=1}^k (r'a_i)(b_i r) = \sum_{i=1}^k (r'a_i)c_i$. This shows that $I = \sum_{i=1}^k R_{ts} c_i$. Thus, I is finitely generated. Using that $ts \in E(S)$, R_{ts} is von Neumann regular and s-unital by assumption. By Proposition 6, I is generated by an idempotent in R_{ts}. □

Theorem 8 *Let S be a semigroup and let R be an S-graded ring. The following two assertions are equivalent:*

(i) *R is graded von Neumann regular;*
(ii) *R is nearly epsilon-strongly S-graded, and R_e is von Neumann regular for every $e \in E(S)$.*

Proof (i)⇒(ii): This follows immediately from Lemma 5.

(ii)⇒(i): Suppose that (ii) holds. Take $s \in S$, $t \in V(s)$ and $r \in R_s$. We need to show that there is some $y \in R_t$, such that $r = ryr$. By Lemma 7 there is an idempotent $u \in R_{ts}$ such that $R_t r = R_{ts} u$. Note that $u = u^2 \in R_{ts} u = R_t r$. Thus, we may find some $r' \in R_t$ such that $u = r'r$. Moreover,

$$R_s R_t r = R_s (R_t r) = R_s (R_{ts} u) \subseteq R_{sts} u = R_s u.$$

Using that R is nearly epsilon-strongly S-graded, there is some $\epsilon_{s,t}(r) \in R_s R_t$ such that $\epsilon_{s,t}(r) r = r$. Thus, from the above equation we get that $r = \epsilon_{s,t}(r) r \in R_s R_t r \subseteq R_s R_{ts} u \subseteq R_{sts} u = R_s u$. Hence, there is some $r'' \in R_s$ such that $r = r''u$. We get that $ru = (r''u)u = r''(u^2) = r''u = r$. Thus, $r = ru = r(r'r)$. This shows that R is graded von Neumann regular. □

Corollary 9 *Let S be a semigroup and let R be an epsilon-strongly S-graded ring. Then R is graded von Neumann regular if and only if R_e is von Neumann regular for every $e \in E(S)$.*

Remark 3 Let R be a strongly S-graded ring. Note that R is nearly epsilon-strongly S-graded if and only if R_e is s-unital for every $e \in E(S)$.

Corollary 10 *Let S be a semigroup and let R be a strongly S-graded ring for which R_e is s-unital for every $e \in E(S)$. Then R is graded von Neumann regular if and only if R_e is von Neumann regular for every $e \in E(S)$.*

Recall that a semigroup S is said to be an *inverse semigroup* if, for every $s \in S$, the set $V(s)$ contains exactly one element.

Theorem 11 *Let S be an inverse semigroup and let R be an S-graded ring. The following three assertions are equivalent:*

(i) *R is graded von Neumann regular;*
(ii) *For all $s \in S$ and $r \in R_s$, there exist $t \in V(s)$ and $y \in R_t$ such that $r = ryr$;*
(iii) *R is nearly epsilon-strongly S-graded, and R_e is von Neumann regular for every $e \in E(S)$.*

Proof (i)⇔(iii): This follows from Theorem 8.
(i)⇔(ii): This follows since $|V(s)| = 1$ for every $s \in S$. □

4 Applications

In this section we present several classes of examples to which our main results can be applied.

4.1 Semigroup rings

Let A be an s-unital ring and let S be a semigroup. The *semigroup ring* $A[S]$ consists of all finite sums of the form $\sum_{s \in S} a_s \delta_s$, where $a_s \in A$ and δ_s is a formal symbol for every $s \in S$. Addition is defined in the natural way and multiplication is given by extending the rule $a\delta_s a'\delta_{s'} := aa'\delta_{ss'}$ for $a, a' \in A$ and $s, s' \in S$. Note that $A[S]$ is strongly S-graded with the canonical grading defined by $(A[S])_s := A\delta_s$, for $s \in S$. By Corollary 10, we get the following result.

Proposition 12 *Let A be an s-unital ring and let S be a semigroup containing at least one idempotent. Equip the semigroup ring $A[S]$ with the canonical S-grading. Then $A[S]$ is graded von Neumann regular if and only if A is von Neumann regular.*

4.2 Matrix rings

Example 1 *(cf.* (Kelarev 2002, Sect. 3.9)*)* Let A be a unital ring and consider the full matrix ring $M_n(A)$ for some arbitrary $n > 0$. For $i, j \in \{1, \ldots, n\}$, let $e_{i,j}$ denote the standard matrix unit, i.e. the matrix with 1_A in position (i, j) and zeros everywhere else. Moreover, note that $B_n := \{0\} \cup \{e_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq n\}$ is a semigroup under matrix multiplication. By putting $(M_n(A))_0 := \{0\}$ and $(M_n(A))_s := As$ for $s \in B_n \setminus \{0\}$, we get a B_n-grading on $M_n(A)$.

Example 2 Let A be a unital ring, put $n = 3$ and consider $M_3(A)$ with its B_3-grading defined in the previous example. Note that $B_3 = \{0, e_{1,1}, e_{2,2}, e_{3,3}, e_{1,2}, e_{1,3}, e_{2,1}, e_{2,3}, e_{3,1}, e_{3,2}\}$. We note that the above B_3-grading on $M_3(A)$ is epsilon-strong. Using Corollary 9, we conclude that $M_3(A)$ is graded von Neumann regular (with respect to the B_3-grading) if and only if A is von Neumann regular.

Following Năstăsescu and van Oystaeyen (2004), we shall say that a semigroup grading on a matrix ring $M_n(A)$ is *good* if every standard matrix unit $e_{i,j}$ is homogeneous, i.e. if for all $i, j \in \{1, \dots, n\}$ there is some $s \in S$ such that $e_{i,j} \in (M_n(A))_s$.

Lemma 13 *Let A be a unital ring, let n be a positive integer and let S be an inverse semigroup defining a good grading on $M_n(A)$. If $\deg(e_{i,j}) = s$, then $\deg(e_{j,i}) = s^{-1}$.*

Proof Suppose that $\deg(e_{i,j}) = s$. Using that the grading is good, there is some $t \in S$ such that $\deg(e_{j,i}) = t$. By inspecting the equations $e_{i,j}e_{j,i}e_{i,j} = e_{i,j}$ and $e_{j,i} = e_{j,i}e_{i,j}e_{j,i}$ we conclude that $sts = s$ and $t = tst$, i.e. $t = s^{-1}$. □

Proposition 14 *Let A be a unital ring, let n be a positive integer and let S be an inverse semigroup defining a good grading on $M_n(A)$. Furthermore, suppose that for every $e \in E(S)$, we have either $(M_n(A))_e = \{0\}$ or that there are integers $i_1, i_2, \dots, i_k \in \{1, \dots, n\}$ such that $(M_n(A))_e = Ae_{i_1,i_1} + Ae_{i_2,i_2} + \cdots + Ae_{i_k,i_k}$. Then $M_n(A)$ is graded von Neumann regular if and only if A is von Neumann regular.*

Proof We begin by showing that $R := M_n(A)$ is epsilon-strongly S-graded using Proposition 1. Take $s \in S$. If $R_s = \{0\}$, then there is nothing to check. Otherwise, using that the grading is good, we may write $R_s = Ae_{i_1,j_1} + Ae_{i_2,j_2} + \cdots + Ae_{i_k,j_k}$ for some $i_1, \dots, i_k, j_1, \dots, j_k$. Supported by Lemma 13, we may put $\epsilon_{s,s^{-1}} := e_{i_1,i_1} + e_{i_2,i_2} + \cdots + e_{i_k,i_k} \in R_s R_{s^{-1}}$ and note that $\epsilon_{s,s^{-1}} r = r$ for every $r \in R_s$. One can show that $R_s R_{s^{-1}} R_s = R_s$ in the same way as in the proof of Proposition 1. Also note that $r' \epsilon_{s,s^{-1}} = r'$ for every $r' \in R_{s^{-1}}$.

The "only if" statement follows from Corollary 9 and the fact that $Ae_{1,1}$ is of idempotent degree. Indeed, if we suppose that $\deg(e_{1,1}) = s$ and inspect the equation $e_{1,1}e_{1,1} = e_{1,1}$, then we conclude that $R_{s^2} \cap R_s \neq \{0\}$, which implies $s^2 = s$.

Now, we show the "if" statement. Suppose that A is von Neumann regular. Take $e \in E(S)$. By assumption, $R_e = \{0\}$or$R_e = Ae_{i_1,i_1} + Ae_{i_2,i_2} + \cdots + Ae_{i_k,i_k}$ and it is readily verified that R_e is von Neumann regular. The desired conclusion now follows from Corollary 9. □

4.3 Groupoid graded rings

Throughout this section, we let G denote an arbitrary groupoid, i.e. a small category in which every morphism is invertible. The family of objects and morphisms of G will be denoted by $\mathrm{ob}(G)$ and $\mathrm{mor}(G)$, respectively. As usual one identifies an object e with the identity morphism Id_e, and hence $\mathrm{ob}(G) \subseteq \mathrm{mor}(G)$. To ease notation, we will write G instead of $\mathrm{mor}(G)$. If $g \in G$, then the domain and codomain of g will be denoted by $d(g)$ and $c(g)$, respectively. We let $G^{(2)}$ denote the set of all pairs $(g, h) \in G \times G$ that are composable, i.e. such that $d(g) = c(h)$.

Recall that a ring R is said to be *G-graded* if there are additive subgroups $\{R_g\}_{g \in G}$ of R such that $R = \oplus_{g \in G} R_g$ and, for $g, h \in G$, we have $R_g R_h \subseteq R_{gh}$ if $(g, h) \in G^{(2)}$, and $R_g R_h = \{0\}$ otherwise.

Definition 3 Let R be a *G-graded* ring.

(i) If $R_g = R_g R_{g^{-1}} R_g$ for every $g \in G$, then R is said to be *symmetrically G-graded*.

(ii) If R is symmetrically G-graded, and $R_g R_{g^{-1}}$ is unital for every $g \in G$, then R is said to be *epsilon-strongly G-graded*.
(iii) If R is symmetrically G-graded, and $R_g R_{g^{-1}}$ is s-unital for every $g \in G$, then R is said to be *nearly epsilon-strongly G-graded*.

Remark 4 It is not difficult to see that (ii) above is equivalent (cf. (Nystedt et al. 2020, Proposition 37)) to the definition of an epsilon-strongly groupoid graded ring introduced in (Nystedt et al. 2020, Definition 34).

For groupoid graded rings it is suitable to make the following definition (cf. (Năstăsescu and van Oystaeyen 1982) and Definition 1).

Definition 4 Let G be a groupoid. A G-graded ring R is said to be *graded von Neumann regular* if for all $g \in G$ and $r \in R_g$, the relation $r \in rRr$ holds.

Remark 5 Note that a G-graded ring R is graded von Neumann regular if and only if for all $g \in G$ and $r \in R_g$, there is some $y \in R_{g^{-1}}$ such that $r = ryr$.

The proof of the following result follows that of Proposition 4 closely and is therefore omitted.

Proposition 15 *Let G be a groupoid and let R be a G-graded ring. The following two assertions are equivalent:*

(i) *R is nearly epsilon-strongly G-graded;*
(ii) *For all $g \in G$ and $r \in R_g$, there exist $\epsilon_{g,g^{-1}}(r) \in R_g R_{g^{-1}}$ and $\epsilon'_{g^{-1},g}(r) \in R_{g^{-1}} R_g$ such that the equalities $\epsilon_{g,g^{-1}}(r)r = r = r\epsilon'_{g^{-1},g}(r)$ hold.*

We will now associate a certain inverse semigroup $S(G)$ with the groupoid G. Indeed, put $S(G) := G \cup \{0_S\}$ and define a binary operation $*_{S(G)}$ on $S(G)$ as follows. For all $g, h \in G$ we define:

- $g *_{S(G)} h := gh$ if $(g, h) \in G^{(2)}$,
- $g *_{S(G)} h := 0_S$ if $(g, h) \notin G^{(2)}$,
- $g *_{S(G)} 0_S := 0_S$, $0_S *_{S(G)} h := 0_S$, and $0_S *_{S(G)} 0_S := 0_S$.

It is readily verified that $S(G)$ is an inverse semigroup. If R is a G-graded ring, then we may endow R with an $S(G)$-grading by keeping R_s for every $s \in G$ and simply adding the component $R_{0_S} := \{0\}$.

The proof of the following result is straightforward.

Proposition 16 *Let G be a groupoid and let R be a G-graded ring. Simultaneously consider R as an $S(G)$-graded ring, where $S(G)$ is the semigroup associated with G. Then R is epsilon-strongly G-graded (resp. nearly epsilon-strongly G-graded) if and only if R is epsilon-strongly $S(G)$-graded (resp. nearly epsilon-strongly $S(G)$-graded).*

The following result generalizes (Lännström 2021, Theorem 1.2) and (Bagio et al. 2021, Proposition 3.5).

Theorem 17 *Let G be a groupoid and let R be a G-graded ring. The following three assertions are equivalent:*

(i) *R is graded von Neumann regular;*
(ii) *For all $g \in G$ and $r \in R_g$, there exists $y \in R_{g^{-1}}$ such that $r = ryr$;*
(iii) *R is nearly epsilon-strongly G-graded, and R_e is von Neumann regular for every $e \in \mathrm{ob}(G)$.*

Proof The proof follows by combining Remark 5, Proposition 16 and Theorem 11. □

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