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# Pure semisimple and Köthe group rings 

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#### Abstract

In this article, we provide a complete characterization of abelian group rings which are Köthe rings. We also provide characterizations of (possibly nonabelian) group rings over division rings which are Köthe rings, both in characteristic zero and in prime characteristic, and prove a Maschke type result for pure semisimplicity of group rings. Furthermore, we illustrate our results by several examples.


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## 1. Introduction

Throughout this article, all rings are assumed to be associative. Recall that nowadays a unital ring $S$ is said to be a left (resp. right) Köthe ring if every left (resp. right) $S$-module is a direct sum of cyclic submodules. If $S$ is both a left and a right Köthe ring, then $S$ is simply called a Köthe ring. If $S$ is both a left (resp. right) artinian ring and a left (resp. right) principal ideal ring, then we say that $S$ is a left (resp. right) artinian principal ideal ring. If $S$ is both a left and a right artinian principal ideal ring, then $S$ is simple called an artinian principal ideal ring. A unital ring $S$ is said to be a left (resp. right) pure semisimple ring if every left (resp. right) $S$-module is a direct sum of finitely generated modules.

In [17] Gottfried Köthe proved the following result.
Theorem 1.1 (Köthe). Let S be a unital ring. If S is an artinian principal ideal ring, then S is a Köthe ring.
Köthe [17] also showed that if $S$ is a unital commutative artinian Köthe ring, then $S$ is necessarily a principal ideal ring. Later, Cohen and Kaplansky [5] showed that if $S$ is a unital commutative Köthe ring, then $S$ is necessarily an artinian principal ideal ring. By combining Köthe's results with those of Cohen and Kaplansky [5] one obtains the following characterization in the commutative setting.

Theorem 1.2 (Köthe, Cohen \& Kaplansky). Let S be a unital commutative ring. Then S is a Köthe ring if and only if $S$ is an artinian principal ideal ring.

Although a (possibly non-commutative) Köthe ring is necessarily artinian (see Proposition 3.1), the above characterization does not hold in the non-commutative setting. Indeed, in [20] Nakayama gave an example of a non-commutative Köthe ring which fails to be a principal ideal ring.

Nevertheless, as was shown by Behboodi et al. [3], Theorem 1.2 can in fact be generalized to certain (potentially) non-commutative rings, namely the abelian rings.

[^0]Theorem 1.3 (Behboodi et al.). Let $S$ be a unital abelian ring, i.e. a unital ring in which every idempotent is central. Then $S$ is a Köthe ring if and only if $S$ is an artinian principal ideal ring.

For further reading on the investigation of Köthe rings, we refer the reader to [3, 8-11].
Recall that given a unital ring $R$ and a group $G$, the group ring $R[G]$ is a free left $R$-module with (a copy of) $G$ as its basis and with multiplication defined as the bilinear extension of the rule $r_{1} g_{1} \cdot r_{2} g_{2}=r_{1} r_{2} g_{1} g_{2}$, for $r_{1}, r_{2} \in R$ and $g_{1}, g_{2} \in G$. For an excellent introduction to the theory of group rings, we refer the reader to Passman's extensive book [24], and the references therein. For examples of applications in coding theory, we refer the reader to [16] which examines certain codes based on commutative Köthe group rings.

In this article, we will attempt to answer the following questions.
Question 1. Given a ring $R$ and a group $G$, when is the group ring $R[G]$ a Köthe ring?
Question 2. Given a ring $R$ and a group $G$, when is the group ring $R[G]$ a pure semisimple ring?
Here is an outline of this article.
In Section 2, we gather some definitions and well-known facts from group ring theory that we need in the sequel. In Section 3, we make some useful observations on general rings. We also obtain necessary conditions for a group ring to be a Köthe ring or pure semisimple (see Proposition 3.4). In Section 4, we prepare for Section 5, by establishing a characterization of abelian Köthe group rings over local rings (see Theorem 4.7). In Section 5, we completely answer Question 1 for abelian group rings, by proving our first main result.

Theorem 1.4. Let $R$ be a unital ring, let $G$ be a group, and suppose that $R[G]$ is abelian. The following two assertions are equivalent:
(i) The group ring $R[G]$ is a Köthe ring;
(ii) There is a positive integer $n$ and local rings ( $R_{i}, M_{i}$ ), for $i \in\{1, \ldots, n\}$, such that $R=R_{1} \times \cdots \times R_{n}$, and $R$ is a Köthe ring. The group $G$ is finite and $p^{\prime}$-by-cyclic $p$, for every $p \in \mathbb{Z}^{+} \cap\left\{\operatorname{char}\left(R_{i} / M_{i}\right) \mid 1 \leq i \leq n\right\}$. Moreover, $|G| \cdot 1_{R_{i}} \in U\left(R_{i}\right)$ whenever $R_{i}$ is not semiprimitive.

Furthermore, in Section 6 we completely answer Question 1 both in the case where $R$ is a division ring of characteristic zero (see Theorem 6.1), and in the case where $R$ is a division ring of prime characteristic and $G$ is a finite Dedekind group (see Theorem 6.4). In Section 7, we introduce a technique involving pure projective modules as a means to tackle Question 2 and use it to prove a Maschke type result for pure semisimplicity of group rings, which is our second main result.

Theorem 1.5. Let $R$ be a unital ring and let $G$ be a finite group. Suppose that $|G| \cdot 1_{R} \in U(R)$. Then $R$ is left (resp. right) pure semisimple if and only if the group ring $R[G]$ is left (resp. right) pure semisimple.

For commutative group rings, the above result allows us to give an alternative proof of our first main result. In Section 8, we present examples of group rings which are Köthe rings resp. not Köthe rings.

## 2. Notation and preliminaries

In this section, we recall notation, important definitions and earlier results.

### 2.1. Group rings

Let $R$ be a unital ring and let $G$ be a multiplicatively written group. The group ring of $G$ over $R$ is denoted by $R[G]$. Each element $a \in R[G]$ can be uniquely written as $a=\sum_{g \in G} r_{g} g$ where $r_{g} \in R$ is zero for all but finitely many $g \in G$.

Remark 2.1. For any unital ring $S$, we let $U(S)$ denote the group of invertible elements of $S$.

### 2.2. Artinian group rings

We will invoke the following result by Connell [6].
Theorem 2.2 (Connell). Let R be a unital ring and let $G$ be a group. The group ring $R[G]$ is left (resp. right) artinian if and only if $R$ is left (resp. right) artinian and $G$ is finite.

### 2.3. Principal ideal group rings

If $\mathcal{A}$ and $\mathcal{B}$ are two classes of groups, then we say that a group $G$ is $\mathcal{A}$-by- $\mathcal{B}$ if there exists a normal subgroup $N$ of $G$, such that $N \in \mathcal{A}$ and $G / N \in \mathcal{B}$. For a finite group $G$, we shall say that $G$ is a $p$-group if the order of $G$ is a power of the prime number $p$, and that $G$ is a $p^{\prime}$-group if the order of $G$ is relatively prime to $p$.

For finite groups, the equivalence (ii) $\Longleftrightarrow$ (iii) in the theorem below follows immediately from [7, Theorem 3] (see also [23, Theorem 4.1]). Using the exact same proof technique as in [7, 23], it is also possible to establish the equivalence (i) $\Longleftrightarrow$ (iii). For related results and generalizations, we refer the reader to [15, Chapter 6].

Theorem 2.3 (Passman, Dorsey). Let $K$ be a division ring, let $G$ be a finite group, and consider the group ring $K[G]$. The following three assertions are equivalent:
(i) $K[G]$ is a left principal ideal ring;
(ii) $K[G]$ is a right principal ideal ring;
(iii) $\operatorname{char}(K)=0$, or $\operatorname{char}(K)=p>0$ and $G$ is $p^{\prime}$-by-cyclic $p$.

Remark 2.4. (a) Let $I$ be an ideal of a ring $S$ and let $s+I$ be an idempotent element of $S / I$. Following [1, p. 301] we say that $s+I$ can be lifted (to $u$ ) modulo $I$ in case there is an idempotent $u \in S$ such that $s+I=u+I$. We say that idempotents lift modulo $I$ in case every idempotent in $S / I$ can be lifted to an idempotent in $S$.
(b) A ring $S$ is said to be local if $S$ has a unique maximal left ideal, or equivalently, if $S$ has a unique maximal right ideal. If $S$ is a local ring, then its unique maximal left ideal coincides with its unique maximal right ideal and with the Jacobson radical $J(S)$. We will let ( $S, M$ ) denote a local ring $S$ together with its maximal ideal $M$.
(c) Recall from [7, p. 397] that, given a unital local artinian principal ideal ring $R$, a finite group $G$ is said to be $R$-admissible if $|G| \cdot 1_{R / J(R)} \in U(R / J(R))$ and each centrally primitive idempotent in $(R / J(R))[G]$ can be lifted to a centrally primitive idempotent in $\left(R / J(R)^{2}\right)[G]$.

For finite groups, the following result follows immediately from [7, Theorem 4].
Theorem 2.5 (Dorsey). Let $(R, M)$ be a unital local artinian principal ideal ring, let $G$ be a finite group, and consider the group ring $R[G]$. The following two assertions are equivalent:
(i) $R[G]$ is a principal ideal ring;
(ii) $\operatorname{char}(R / M)=0$ : If $R$ is not a division ring, then $G$ is an $R$-admissible group.
$\operatorname{char}(R / M)=p>0: G$ is $p^{\prime}$-by-cyclic $p$. If $R$ is not a division ring, then $G$ is an $R$-admissible group.

### 2.4. Pure projectivity and pure semisimplicity

Let $S$ be a unital ring. A short exact sequence

of left (resp. right) $S$-modules is said to be pure exact if $D \otimes_{S} \mathcal{E}$ (resp. $\mathcal{E} \otimes_{S} D$ ) is an exact sequence (of abelian groups) for any right (resp. left) $S$-module $D$.

A submodule $P$ of a left $S$-module $M$ is said to be a pure submodule of $M$ if and only if the following holds: For any $m$-by- $n$ matrix $A=\left(a_{i j}\right)$ with entries in $S$, and any set $y_{1}, \ldots, y_{m}$ of elements of $P$, if there exist elements $x_{1}, \ldots, x_{n}$ in $M$ such that $\sum_{j=1}^{n} a_{i j} x_{j}=y_{i}$ for $i \in\{1, \ldots, m\}$ then there also exist elements $x_{1}^{\prime}, \ldots, x_{n}^{\prime}$ in $P$ such that $\sum_{j=1}^{n} a_{i j} x_{j}^{\prime}=y_{i}$ for $i \in\{1, \ldots, m\}$. Pure submodules of right $S$-modules are defined analogously.
Remark 2.6. $\mathcal{E}$ is pure exact if and only if $f(A)$ is a pure submodule of $B$.
A module is said to be pure projective if it is projective with respect to pure exact sequences. We recall the following characterization of pure semisimplicity from [26, Section 53.6] and [13, Theorem 13].

Proposition 2.7. Let $S$ be a unital ring. The following three assertions are equivalent:
(i) S is left (resp. right) pure semisimple;
(ii) Every left (resp. right) S-module is pure projective;
(iii) Every left (resp. right) S-module is a direct sum of indecomposable modules.

By the definitions it is clear that any left (resp. right) Köthe ring is necessarily left (resp. right) pure semisimple. As it turns out, for commutative rings the converse also holds. Indeed, we recall the following result from [10, Theorem 3].

Theorem 2.8 (Girvan). Let $S$ be a unital commutative ring. Then $S$ is a Köthe ring if and only if every $S$-module is pure projective.

## 3. General observations

In this section, we record some observations on general rings and group rings. We begin by recalling the following result from [26, Section 53.6]

Proposition 3.1. Let $S$ be a unital ring. If S is a left (resp. right) pure semisimple ring, then $S$ is a left (resp. right) artinian ring.

Lemma 3.2. Let $S$ and $T$ be unital rings, and suppose that $\varphi: S \rightarrow T$ is a surjective ring homomorphism. The following three assertions hold:
(a) If S is a left (resp. right) Köthe ring, then $T$ is also a left (resp. right) Köthe ring.
(b) If S is a left (resp. right) principal ideal ring, then $T$ is also a left (resp. right) principal ideal ring.
(c) If $S$ is a left (resp. right) pure semisimple ring, then $T$ is also a left (resp. right) pure semisimple ring.

Proof. (a) Suppose that $S$ is a left Köthe ring. We may identify $T$ with $S / \operatorname{ker}(\varphi)$. Let $M$ be an arbitrary left $S / \operatorname{ker}(\varphi)$-module. By restriction of scalars, with respect to the canonical ring homomorphism $\psi$ : $S \rightarrow S / \operatorname{ker}(\varphi), M$ may be viewed as a left $S$-module. By assumption, $M$ decomposes into a direct sum of
cyclic submodules. It is not difficult to see that $M$, viewed as a left $S / \operatorname{ker}(\varphi)$-module, also decomposes into a direct sum of cyclic submodules. This shows that $S / \operatorname{ker}(\varphi)$, and hence $T$, is a left Köthe ring. The right-handed case is treated analogously.
(b) Suppose that $S$ is a left principal ideal ring. Let $I$ be an arbitrary left ideal of $T$. The set $\varphi^{-1}(I)$ is a left ideal of $S$. By assumption, there is some $x \in S$ such that $\varphi^{-1}(I)=S x$, and by surjectivity of $\varphi$ we get that $I=\varphi(S x)=\varphi(S) \varphi(x)=T \varphi(x)$. This shows that $T$ is a left principal ideal ring. The right-handed case is treated analogously.
(c) Suppose that $S$ is a left pure semisimple ring. Let $M$ be an arbitrary left $T$-module. Following (a), we may view $M$ as a left $S / \operatorname{ker}(\varphi)$-module. By assumption (cf. Proposition 2.7), there is some index set $I$ such that $M=\oplus_{i \in I} M_{i}$, where $M_{i}$ is a finitely generated left $S$-module for every $i \in I$. Now, using that $M_{i}$ is a finitely generated $S$-module, one easily verifies that $M_{i}$ is also a finitely generated $S / \operatorname{ker}(\varphi)$-module, for every $i \in I$. This shows that $S / \operatorname{ker}(\varphi)$, and hence $T$, is a left pure semisimple ring. The right-handed case is treated analogously.

Lemma 3.3. The following two assertions hold:
(a) Let $S$ and $T$ be unital rings, and suppose that $\varphi: S \rightarrow T$ is an injective ring homomorphism. If $T$ is abelian, then $S$ is also abelian. In particular, any subring of an abelian ring is abelian in itself.
(b) Let $S=\Pi_{i=1}^{n} S_{i}$ be a direct product of unital rings $S_{1}, \ldots, S_{n}$. Then $S$ is abelian if and only if $S_{i}$ is abelian for every $i \in\{1, \ldots, n\}$.

Proof. (a) Suppose that $T$ is abelian and consider the subring $\varphi(S)$ of $T$. Take an idempotent $u \in \varphi(S) \subseteq$ $T$. By assumption, we get that $u \in \varphi(S) \cap Z(T) \subseteq Z(\varphi(S))$. Thus, $\varphi(S)$ is abelian. Using that $S \cong \varphi(S)$ we conclude that $S$ is abelian.
(b) The "only if" statement follows immediately from (a) after considering the natural embedding $\iota_{i}: S_{i} \rightarrow \Pi_{i=1}^{n} S_{i}$, for each $i \in\{1, \ldots, n\}$. Now, suppose that $S_{i}$ is abelian for every $i \in\{1, \ldots, n\}$. Take an idempotent $u \in S$. Then $u=\left(u_{1}, \ldots, u_{n}\right)$ where, by assumption, $u_{i}$ is a central idempotent in $S_{i}$, for every $i \in\{1, \ldots, n\}$. Clearly, $u \in Z(S)$. This concludes the proof of the "if" statement.

Proposition 3.4. Let $R$ be a unital ring and let $G$ be a group. The following two assertions hold:
(a) If the group ring $R[G]$ is a left (resp. right) pure semisimple ring, then $(R / I)[G / N]$ is a left (resp. right) pure semisimple ring for every proper ideal $I$ of $R$ and every normal subgroup $N$ of $G$. Furthermore, $(R / I)[G / N]$ is a left (resp. right) artinian ring. In particular, $R / I, R$ and $R[G]$ are left (resp. right) pure semisimple rings and left (resp. right) artinian rings, and $G$ is a finite group.
(b) If the group ring $R[G]$ is a left (resp. right) Köthe ring, then $(R / I)[G / N]$ is a left (resp. right) Köthe ring for every proper ideal I of $R$ and every normal subgroup $N$ of $G$. Furthermore, $(R / I)[G / N]$ is a left (resp. right) artinian ring. In particular, $R / I, R$ and $R[G]$ are left (resp. right) Köthe rings and left (resp. right) artinian rings, and $G$ is a finite group.

Proof. Consider the natural quotient maps $R \rightarrow R / I, r \mapsto \bar{r}$ and $G \rightarrow G / N, g \mapsto g N$. Define a map $\varphi: R[G] \rightarrow(R / I)[G / N]$ by naturally extending the rule $\varphi(r g)=\bar{r} g N$, for $r \in R$ and $g \in G$. Clearly, $\varphi$ is a surjective ring homomorphism.

Most of (a) and (b) now follow immediately from Lemma 3.2. The claims about artinianity follow from Proposition 3.1, and the finiteness of $G$ follows from Theorem 2.2.

## 4. Abelian group rings over local rings

In this section, we provide a characterization of abelian Köthe group rings over local rings (see Theorem 4.7). That result will be an essential ingredient in the proof of our first main result in Section 5.

Lemma 4.1. Let $R$ be a unital ring and let $n>1$ be an integer. The following two assertions are equivalent:
(i) $n \cdot 1_{R} \notin U(R)$;
(ii) There is a prime divisor $q$ of $n$, and a proper ideal $I$ of $R$ such that $\operatorname{char}(R / I)=q$.

Proof. (ii) $\Rightarrow$ (i) Suppose that $n=q \cdot n^{\prime}$ for a prime number $q$, and that $I$ is a proper ideal of $R$ such that $\operatorname{char}(R / I)=q$. Seeking a contradiction, suppose that $n \cdot 1_{R} \in U(R)$, i.e. there is some $r \in R$ such that $r \cdot n \cdot 1_{R}=1_{R}$. Consider the natural map $\varphi: R \rightarrow R / I$. Clearly, $1_{R}+I=\varphi\left(1_{R}\right)=\varphi\left(r \cdot n \cdot 1_{R}\right)=$ $\varphi\left(r \cdot q \cdot n^{\prime} \cdot 1_{R}\right)=\varphi(r q) \cdot \varphi\left(n^{\prime} \cdot 1_{R}\right)=\varphi(r) \cdot q \cdot \varphi\left(n^{\prime} \cdot 1_{R}\right)=\varphi(r) \cdot I \subseteq I$. This is a contradiction since $I$ is proper.
(i) $\Rightarrow$ (ii) Suppose that $n \cdot 1_{R} \notin U(R)$. There must exist a prime number $q$, such that $n=q \cdot n^{\prime}$ and such that $q \cdot 1_{R} \notin U(R)$. Consider the ideal $I:=R \cdot q \cdot 1_{R}$ of $R$. Clearly, $I$ must be proper since $q \cdot 1_{R} \notin U(R)$. Using that $q$ is prime, it is easy to see that $\operatorname{char}(R / I)=q$.

Lemma 4.2. Let $(R, M)$ be a unital local ring and let $n>1$ be an integer. The following two assertions hold:
(a) Suppose that $\operatorname{char}(R / M)=0$. Then $n \cdot 1_{R} \in U(R)$.
(b) Suppose that $\operatorname{char}(R / M)=p>0$. Then $n \cdot 1_{R} \notin U(R)$ if and only if $p$ divides $n$.

Proof. We begin with a general observation. If $n \cdot 1_{R} \notin U(R)$, then by Lemma 4.1 there is a prime divisor $q$ of $n$, and a proper ideal $I$ of $R$ such that $\operatorname{char}(R / I)=q>0$. Notice that $I \subseteq M \subseteq R$, since $M$ is maximal. From the third isomorphism theorem we get that $R / M \cong(R / I) /(M / I)$. Recall that char $(R / M)$ must divide char $(R / I)$.
(a) Seeking a contradiction, suppose that $n \cdot 1_{R} \notin U(R)$. Using that $\operatorname{char}(R / M)=0$ and $\operatorname{char}(R / I)=$ $q>0$, we get a contradiction. Thus, $n \cdot 1_{R} \in U(R)$.
(b) The "if" statement follows immediately from Lemma 4.1. Now we show the "only if" statement. Suppose that $n \cdot 1_{R} \notin U(R)$. We use Lemma 4.1 to find $q$ and $I$. Using that $\operatorname{char}(R / M)=p$ and $\operatorname{char}(R / I)=q$ we get $p=q$, and this shows that $p$ divides $n$.

The following result follows immediately from [7, Lemma 5].
Lemma 4.3. Let $(R, M)$ be a unital local artinian principal ideal ring with $\operatorname{char}(R / M)=p>0$, and let $G$ be a finite group. If $R[G]$ is a principal ideal ring and $R$ is not a division ring, then $G$ is not a p-group.

In the following remark, we record a number of facts that will be useful in the proof of the subsequent lemma.

Remark 4.4. (a) Recall from [1, Proposition 27.1] that if $I$ is a nil ideal of a ring $S$, then idempotents lift modulo $I$.
(b) If $S$ is a left artinian ring, then $J(S)$ is nilpotent (see e.g. [1, Theorem 15.20]). In particular, $J(S)$ is nil and hence by (a) idempotents in $S$ lift modulo $J(S)$.
(c) If $I$ is an ideal of a ring $S$ such that $S / I$ is semiprimitive, i.e. $J(S / I)=\{0\}$, then $J(S) \subseteq I$ (see e.g. [18, Ex. 4.11]).
(d) If $S$ is a left artinian ring, then by (b), $J(S)$ is a nilpotent ideal and hence by [18, Theorem 22.9] there is a bijective correspondence between centrally primitive idempotents of $S$ and centrally primitive idempotents of $S /\left(J(S)^{2}\right)$.

The proof of the following lemma is inspired by [7, Corollary 9].
Lemma 4.5. Let $(R, M)$ be a unital local artinian principal ideal ring, let $G$ be a finite group, and suppose that $R[G]$ is abelian. If $|G| \cdot 1_{R} \in U(R)$, then $G$ is $R$-admissible.

Proof. Given an ideal $I$ of $R$ we will let $I G$ denote the set $I \cdot R[G]$. Notice that $I G$ is an ideal of the group ring $R[G]$. It is easy to see that we get two natural ring isomorphisms

$$
\begin{equation*}
\frac{R[G]}{I G} \cong\left(\frac{R}{I}\right)[G] \quad \text { and } \quad \frac{R[G]}{I^{2} G} \cong\left(\frac{R}{I^{2}}\right)[G] . \tag{1}
\end{equation*}
$$

We will now direct our attention to the case $I=J(R)$.
Using that $R$ is artinian, [6, Proposition $9(25)]$ yields that $J(R) \subseteq J(R[G])$. From this we get that $J(R) G \subseteq J(R[G])$. Notice that $R / J(R)$ is a division ring, since $R$ is local. Moreover, $|G| \cdot 1_{R} \in U(R)$ yields $|G| \cdot 1_{R / J(R)} \in U(R / J(R))$, and hence by Maschke's theorem we conclude that $(R / J(R))[G]$ is semisimple. In particular, by (1), $(R / J(R))[G]$ and $R[G] /(J(R) G)$ are semiprimitive. By Remark 4.4(c) we get that $J(R[G]) \subseteq J(R) G$. By combining this with the previous inclusion we get that $J(R[G])=J(R) G$. From this we also get that $J(R[G])^{2}=J(R)^{2} G$. Thus, by invoking (1), we get that

$$
\begin{equation*}
(R / J(R))[G] \cong R[G] /(J(R) G)=R[G] /(J(R[G])) \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
R /\left(J(R)^{2}\right)[G] \cong R[G] /\left(J(R)^{2} G\right)=R[G] /\left(J(R[G])^{2}\right) \tag{3}
\end{equation*}
$$

Using that $G$ is finite and that $R$ is artinian, by Theorem 2.2, $R[G]$ is artinian. Thus, by Remark 4.4(b), (2) and the fact that $R[G]$ is abelian, every centrally primitive idempotent in $(R / J(R))[G]$ can be lifted to a centrally primitive idempotent in $R[G]$.

Furthermore, by Remark 4.4(d) and (3) there is a bijective correspondence between the centrally primitive idempotents of $R[G]$ and the centrally primitive idempotents of $\left(R / J(R)^{2}\right)[G]$. In conclusion, every centrally primitive idempotent in $(R / J(R))[G]$ can be lifted to a centrally primitive idempotent in $\left(R / J(R)^{2}\right)[G]$. This shows that $G$ is $R$-admissible.

Remark 4.6. By the third isomorphism theorem for rings, we get that $R[G] / J(R[G]) \cong(R[G] /$ $\left.J(R[G])^{2}\right) /\left(J(R[G]) / J(R[G])^{2}\right)$. If we combine this with (2) and (3), then we see that $(R / J(R))[G]$ may be viewed as a quotient of $\left(R / J(R)^{2}\right)[G]$. Hence, in this situation it makes sense to speak of lifting idempotents from $(R / J(R))[G]$ to $\left(R / J(R)^{2}\right)[G]$.

Theorem 4.7. Let $(R, M)$ be a unital local ring, let $G$ be a group, and suppose that $R[G]$ is abelian. The following two assertions are equivalent:
(i) The group ring $R[G]$ is a Köthe ring;
(ii) $\operatorname{char}(R / M)=0$ : The ring $R$ is a Köthe ring and the group $G$ is finite. If $R$ is not a division ring, then $|G| \cdot 1_{R} \in U(R)$.
$\operatorname{char}(R / M)=p>0$ : The ring $R$ is a Köthe ring and the group $G$ is finite and $p^{\prime}$-by-cyclic $p$. If $R$ is not a division ring, then $|G| \cdot 1_{R} \in U(R)$.

Proof. (i) $\Rightarrow$ (ii) By Proposition 3.4, $R$ is a Köthe ring and $G$ is a finite group. If $G$ is the trivial group, then clearly $|G| \cdot 1_{R}=1_{R} \in U(R)$. Therefore, suppose that $G$ is non-trivial. Notice that $R / M$ is a division ring whose characteristic is either zero or a prime.

Case 1: $\operatorname{char}(R / M)=0$. By Lemma 4.2(a) we conclude that $|G| \cdot 1_{R} \in U(R)$.
Case 2: $\operatorname{char}(R / M)=p$. By Theorem 1.3, $R[G]$ is a principal ideal ring, and by Lemma 3.2(b) the same conclusion holds for $(R / M)[G]$. We know that $R / M$ is a division ring, and hence Theorem 2.3 yields that $G$ is $p^{\prime}$-by-cyclic $p$. Suppose that $R$ is not a division ring. Seeking a contradiction, suppose that $|G| \cdot 1_{R} \notin U(R)$. Using that $G$ is $p^{\prime}$-by-cyclic $p$, there is a normal subgroup $N$ of $G$ such that $N$ is a $p^{\prime}$ group and $G / N$ is a cyclic $p$-group. Notice that, since $R[G]$ is a principal ideal ring, Lemma 3.2(b) yields that $R[G / N]$ and $R$ are also principal ideal rings. By Lemma 4.3 we conclude that $G / N$ is not a $p$-group. Thus, $G / N$ is trivial, i.e. $N=G$, and hence $G$ is a $p^{\prime}$-group. By Lemma 4.2(b) we conclude that $p$ divides $|G|$ but this is a contradiction since $G$ is a $p^{\prime}$-group.
(ii) $\Rightarrow$ (i) We consider two cases.

Case 1: $R$ is a division ring. Notice that $R[G]$ is an artinian ring. Indeed, by assumption $G$ is finite and $R$ is artinian. Thus, artinianity of $R[G]$ follows from Theorem 2.2. By Theorem 2.3, $R[G]$ is a principal ideal ring. Hence, by Theorem 1.1, the group ring $R[G]$ is a Köthe ring.
Case 2: $R$ is not a division ring. By assumption, $|G| \cdot 1_{R} \in U(R)$. By Lemma 3.3(a), $R$ is abelian and hence, by Theorem $1.3, R$ is an artinian principal ideal ring. It follows, by Lemma 4.5 , that $G$ is $R$-admissible. Thus, by Theorem 2.5, $R[G]$ is a principal ideal ring and the desired conclusion now follows in the same way as for Case 1.

## 5. Proof of the first main result

In this section, we prove our first main result by combining a couple of lemmas with the results of the preceding sections.

The proof of the following lemma follows by Theorem 1.3, Lemma 3.2(a), and Lemma 3.3.
Lemma 5.1. Let $S=S_{1} \times \cdots \times S_{n}$ be a unital ring. Then $S$ is an abelian Köthe ring if and only if $S_{i}$ is an abelian Köthe ring for every $i \in\{1, \ldots, n\}$.

We recall the following result from [12, Proposition 3].
Lemma 5.2. Let $R$ be a left (or right) artinian ring. Then $R$ is a finite product of local rings if and only if every idempotent of $R$ is central in $R$.

We are now ready to prove our first main result.
Proof of Theorem 1.4. (i) $\Rightarrow$ (ii) By Proposition 3.4, $R$ is a Köthe ring (and thus artinian) and $G$ is finite. The existence of $n$ and local rings $\left(R_{1}, M_{1}\right), \ldots,\left(R_{n}, M_{n}\right)$ such that $R=R_{1} \times \cdots \times R_{n}$ follows from Lemmas 3.3(a) and 5.2. Take an arbitrary $i \in\{1, \ldots, n\}$. Notice that $R_{i}[G]$ is a Köthe ring by Lemma 3.2(a), and abelian by Lemma 3.3(a). By Theorem 4.7, $G$ is $p^{\prime}$-by-cyclic $p$ if $\operatorname{char}\left(R_{i} / M_{i}\right)>0$. If $R_{i}$ is not semiprimitive, then $M_{i}=J\left(R_{i}\right) \neq\{0\}$. In that case, $R_{i}$ is not a division ring and it follows from Theorem 4.7 that $|G| \cdot 1_{R_{i}} \in U\left(R_{i}\right)$.
(ii) $\Rightarrow$ (i) By Lemma 3.2(a) $R_{i}$ is a local Köthe ring for every $i \in\{1, \ldots, n\}$. Take $i \in\{1, \ldots, n\}$. By Lemma 3.3(a), the group ring $R_{i}[G]$ is abelian, and by Theorem 4.7 it is a Köthe ring. Using Lemma 5.1, we conclude that $R[G]=\prod_{i=1}^{n} R_{i}[G]$ is a Köthe ring.

## 6. Group rings over division rings

In this section, we characterize Köthe group rings over division rings, both in characteristic zero (see Theorem 6.1) and in prime characteristic (see Theorem 6.4).

Theorem 6.1. Let $R$ be a division ring with $\operatorname{char}(R)=0$ and let $G$ be a group. The following three assertions are equivalent:
(i) The group ring $R[G]$ is a left Köthe ring;
(ii) The group ring $R[G]$ is a right Köthe ring;
(iii) $G$ is a finite group.

Proof. (i) $\Rightarrow$ (iii) This follows immediately from Proposition 3.4(b).
(iii) $\Rightarrow$ (i) The ring $R$ is a division ring and hence artinian. Using that $G$ is finite, Theorem 2.2 yields that $R[G]$ is artinian. By Theorem 2.3, $R[G]$ is a principal ideal ring. The desired conclusion now follows from Theorem 1.1.
(ii) $\Leftrightarrow$ (iii) The proof is analogous to the proof of $(\mathrm{i}) \Leftrightarrow$ (iii).

We prepare ourselves for the case of prime characteristic by recalling the following result from [21, Theorem(2), p. 138].

Theorem 6.2 (Nicholson). Let $R$ be a unital local ring and let $G$ be a locally finite p-group. If p $\cdot 1_{R} \in J(R)$, then the group ring $R[G]$ is local.

Corollary 6.3. Let $R$ be a division ring with char $(R)=p>0$ and let $H$ be a $p$-group. If $R[H]$ is a left (or right) Köthe ring, then $H$ is finite and $p^{\prime}$-by-cyclic $p$.

Proof. Suppose that $R[H]$ is a left Köthe ring. By Proposition 3.4, $H$ is a finite $p$-group and $R[H]$ is a left artinian ring. Using Theorem 6.2, we get that $R[H]$ is a local ring and hence, by Lemma 5.2, $R[H]$ is an abelian ring. It follows from Theorem 4.7 that $H$ is $p^{\prime}$-by-cyclic $p$. The right-handed case is treated analogously to the left-handed case.

Recall that a finite group $G$ is said to be lagrangian (see e.g. $[14,19]$ ) if for every factor of $|G|, G$ possesses a subgroup of that order. Also recall that a group $G$ is said to be a Dedekind group if every subgroup of $G$ is normal in $G$. Note that every finite Dedekind group is lagrangian.

Theorem 6.4. Let $R$ be a division ring with $\operatorname{char}(R)=p>0$ and let $G$ be a finite Dedekind group. The following three assertions are equivalent:
(i) The group ring $R[G]$ is a left Köthe ring;
(ii) The group ring $R[G]$ is a right Köthe ring;
(iii) $G$ is $p^{\prime}$-by-cyclic $p$, or $R[G]$ is semisimple.

Proof. (i) $\Rightarrow$ (iii) We consider two mutually exclusive cases.
Case 1: $|G| \cdot 1_{R} \in U(R)$. By Maschke's theorem, $R[G]$ is semisimple.
Case 2: $|G| \cdot 1_{R} \notin U(R)$. By Lemma 4.2(b), we get that $p$ divides $|G|$. There is a positive integer $r$ such that either $|G|=p^{r}$ or $|G|=p^{r} m$, where $p$ and $m$ are relatively prime.

If $|G|=p^{r}$, then $G$ is $p^{\prime}$-by-cyclic $p$, by Corollary 6.3. On the other hand, if $|G|=p^{r} m$, then using that $G$ is Dedekind and lagrangian, we may choose a normal subgroup $N$ of $G$ such that $|N|=m$. Note that $N$ is a $p^{\prime}$-group and that $G / N$ is a $p$-group. By Proposition 3.4, we get that $R[G / N]$ is a left Köthe ring, and hence, by Corollary $6.3, G / N$ is $p^{\prime}$-by-cyclic $p$. But the only normal subgroup of $G / N$ which is a $p^{\prime}$-group is the trivial group. Hence, $G / N$ is a cyclic $p$-group. This shows that $G$ is $p^{\prime}$-by-cyclic $p$.
(iii) $\Rightarrow$ (i) This follows by combining Theorems $2.2,2.3$, and 1.1 , or by the fact that every semisimple ring, viewed as a module over itself, is a direct sum of simple, and thus cyclic, submodules.
(ii) $\Leftrightarrow$ (iii) The proof is analogous to the proof of $(\mathrm{i}) \Leftrightarrow$ (iii).

## 7. Pure semisimplicity of group rings

In Section 7.1, we make an observation about pure projectivity of modules over (not necessarily commutative) group rings and prove our second main result, which is a Maschke type result for pure semisimplicity of group rings (see Theorem 1.5). In Section 7.2, we use our previous observation to provide an alternative proof of Theorem 4.7 in the commutative setting. We also rephrase our first main result in the commutative setting (see Corollary 7.5).

### 7.1. Pure projectivity and pure semisimplicity

We want to emphasize that in this subsection we are dealing with general rings that are not necessarily abelian nor commutative.

Lemma 7.1. Let $S$ be a ring and $T \subseteq S$ a subring. If

$$
\begin{equation*}
0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{4}
\end{equation*}
$$

is a pure exact sequence of left (resp. right) S-modules, then by restriction of scalars

$$
\begin{equation*}
0 \longrightarrow A \xrightarrow{f^{\prime}} B \xrightarrow{g^{\prime}} C \longrightarrow 0 \tag{5}
\end{equation*}
$$

is a pure exact sequence of left (resp. right) $T$-modules.
Proof. Suppose that (4) is pure exact. It is clear that (5) is a short exact sequence of left (resp. right) $T$ modules. By assumption, $f(A)$ is a pure submodule of $B$. Using this and the fact that $T$ is a subring of $S$, it is readily verified that $f^{\prime}(A)$ is a pure submodule of $B$. This shows that (5) is a pure exact sequence of left (resp. right) $T$-modules.

We are now ready to prove our second main result.
Proof of Theorem 1.5. The "if" statement follows immediately from Proposition 3.4(a). Now we show the "only if" statement. Suppose that $R$ is a left pure semisimple ring. Let $M$ be an arbitrary left $R[G]$ module and let

$$
\begin{equation*}
0 \longrightarrow K \xrightarrow{\varphi} L \xrightarrow{\psi} M \longrightarrow 0 \tag{6}
\end{equation*}
$$

be an arbitrary pure exact sequence of left $R[G]$-modules. We are going to show that (6) splits, and thereby that $M$ is pure projective. The desired conclusion will then follow from Proposition 2.7. By restriction of scalars, $K, L$, and $M$ may be viewed as left $R$-modules and, by Lemma 7.1, we obtain a corresponding pure exact sequence

$$
\begin{equation*}
0 \longrightarrow K \xrightarrow{\tilde{\varphi}} L \xrightarrow{\tilde{\psi}} M \longrightarrow 0 \tag{7}
\end{equation*}
$$

of left $R$-modules. Using Proposition 2.7, we conclude that (7) splits. That is, there exists a left $R$-module homomorphism $\widetilde{\phi}: M \rightarrow L$ such that $\widetilde{\psi} \circ \widetilde{\phi}=\operatorname{id}_{M}$.

Now we define the map $\phi: M \rightarrow L, m \mapsto|G|^{-1} \sum_{g \in G} g^{-1} \widetilde{\phi}(g m)$. It is clear that $\phi$ is well-defined and additive. Take $h \in G$ and $m \in M$. We get that

$$
\phi(h m)=|G|^{-1} \sum_{g \in G} g^{-1} \widetilde{\phi}(g h m)=|G|^{-1} \sum_{\lambda \in G} h \lambda^{-1} \widetilde{\phi}(\lambda m)=h\left(|G|^{-1} \sum_{\lambda \in G} \lambda^{-1} \widetilde{\phi}(\lambda m)\right)=h \cdot \phi(m)
$$

and hence $\phi$ is a left $R[G]$-module homomorphism. Moreover, for any $m \in M$ we get that

$$
\begin{aligned}
\psi \circ \phi(m) & =\psi\left(|G|^{-1} \sum_{g \in G} g^{-1} \widetilde{\phi}(g m)\right)=|G|^{-1} \sum_{g \in G} g^{-1}(\psi \circ \widetilde{\phi}(g m)) \\
& =|G|^{-1} \sum_{g \in G} g^{-1}(g m)=\left(|G|^{-1} \sum_{g \in G} g^{-1} g\right) m=1 \cdot m=m .
\end{aligned}
$$

This shows that (6) splits. The right-handed case is treated similarly.
Remark 7.2. If a group ring $R[G]$ is left (or right) pure semisimple, then $G$ is necessarily finite (see Proposition 3.4). However, $|G| \cdot 1_{R} \in U(R)$ is not a necessary condition for pure semisimplicity of $R[G]$. To see this, one can look at e.g. Example 8.2.

### 7.2. Commutative group rings

By combining Theorem 1.5 with Theorem 2.8 we obtain the following result.
Corollary 7.3. Let $R$ be a unital commutative ring and let $G$ be a finite abelian group. Suppose that $|G| \cdot 1_{R} \in$ $U(R)$. Then $R$ is a Köthe ring if and only if $R[G]$ is a Köthe ring.

With the use of the above corollary we can provide an alternative proof of Theorem 4.7 for commutative group rings. The alternative proof is identical to the proof of Theorem 4.7, except for $($ ii $) \Rightarrow(i)$, Case 2 , where we can make use of Corollary 7.3 instead of invoking Theorem 2.5.
Remark 7.4. Recall that if $R$ is a commutative artinian ring, then it has only finitely many maximal ideals $M_{1}, \ldots, M_{n}$. Moreover, there is a positive integer $k$ such that $\Pi_{i=1}^{n} M_{i}^{k}=0$. The ideals $M_{1}^{k}, \ldots, M_{n}^{k}$ are pairwise coprime. Consequently, one may show that $R \cong R / M_{1}^{k} \times \cdots \times R / M_{n}^{k}$ (see e.g. [2, Theorem 8.7]). For any $i \in\{1, \ldots, n\}$ we make the following observations:

The ring $R / M_{i}^{k}$ is a local artinian ring. By the third isomorphism theorem, we get that $R / I \cong$ $\left(R / I^{k}\right) /\left(I / I^{k}\right)$ for any ideal $I$ of $R$. In particular, $M_{i} / M_{i}^{k}$ is a maximal ideal of $R / M_{i}^{k}$ if and only if $M_{i}$ is a maximal ideal of $R$.

Using the above observations, Theorem 1.4 takes a slightly more elegant form in the commutative setting.

Corollary 7.5. Let $R$ be a unital commutative ring and let $G$ be an abelian group. Denote by $\operatorname{Max}(R)$ the set of maximal ideals of $R$. The following two assertions are equivalent:
(i) The group ring $R[G]$ is a Köthe ring;
(ii) $R$ is a Köthe ring and $G$ is a finite group which is $p^{\prime}$-by-cyclic $p$ for every $p \in\{\operatorname{char}(R / M) \mid M \in$ $\operatorname{Max}(R)\}$. Moreover, $|G| \cdot 1_{R / M_{i}^{k}} \in U\left(R / M_{i}^{k}\right)$ whenever $R / M_{i}^{k}$, appearing in the decomposition of $R$ (see Remark 7.4), is not semiprimitive.

## 8. Examples

In this section, we present some examples of group rings which are Köthe rings and group rings which are not.

Example 8.1. Let $G:=Q_{8}$ be the quaternion group of order 8 .
(a) Let $R:=\mathbb{Z}$ be the ring of integers. By combining [25, Proposition 3.4] with [22, Lemma 6.4], we immediately conclude that $\mathbb{Z}[G]$ is an abelian ring. However, the integral group ring $\mathbb{Z}[G]$ is not a Köthe ring for at least two reasons; $\mathbb{Z}$ is not artinian, and $G$ is not $p^{\prime}$-by-cyclic $p$ for any prime number $p$.
(b) Let $R:=\mathbb{H}$ be the ring of real quaternions. By Theorem 6.1, we conclude that $\mathbb{H}[G]$ is a Köthe ring. Notice that $R$ is non-commutative and that $G$ is non-abelian.

Example 8.2. Let $K$ be a field of characteristic 2 and let $G:=S_{3}$ be the symmetric group on 3 letters. The alternating group $A_{3}$ is a normal subgroup of $S_{3}$ with $\left|A_{3}\right|=3$, and $S_{3} / A_{3}$ is cyclic of order 2. Thus, $S_{3}$ is $p^{\prime}$-by-cyclic $p$ with $p=2$. However, by [4, p. 69], $K[G]$ is not abelian. Nevertheless, by Theorems 2.2 and 2.3, $K[G]$ is an artinian principal ideal ring and hence, by Theorem 1.1, $K[G]$ is in fact a Köthe ring.

Example 8.3. Let $C_{3}$ be a cyclic group of order 3 , let $D_{8}$ be the dihedral group of order 8 , and consider the finite group $G:=C_{3} \times D_{8}$. Notice that $N:=\left\{e_{C_{3}}\right\} \times D_{8}$ is a normal subgroup of $G$ with $|N|=2^{3}$, and that $G / N \cong C_{3}$ is of order 3. Thus, $G$ is $p^{\prime}$-by-cyclic $p$ with $p=3$.
(a) Let $K$ be a field of characteristic 3 . We notice that the subring $K\left[C_{3}\right]$ contains nonzero nilpotent elements which are central in $K[G]$. Hence, $K[G]$ is not reduced nor semisimple. Nevertheless, by [4, p. 69], $K[G]$ is in fact abelian, and, by Theorem 1.4, $K[G]$ is a Köthe ring.
(b) Let $K$ be a field of characteristic 2 . By [4, p. 69], the group ring $K\left[D_{8}\right]$ is abelian. However, $D_{8}$ is not $p^{\prime}$-by-cyclic $p$ with $p=2$. Thus, by Corollary $6.3, K\left[D_{8}\right]$ is not a left (nor right) Köthe ring.

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