Statistical Modelling and the Fokker-Planck Equation

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ABSTRACT

A **stochastic process** or sometimes called random **process** is the counterpart to a deterministic process in theory. A stochastic process is a random field, whose domain is a region of space, in other words, a random function whose arguments are drawn from a range of continuously changing values. In this case, Instead of dealing only with one possible 'reality' of how the process might evolve under time (as is the case, for example, for solutions of an ordinary differential equation), in a stochastic or random process there is some indeterminacy in its future evolution described by probability distributions. This means that even if the initial condition (or starting point) is known, there are many possibilities the process might go to, but some paths are more probable and others less. However, in discrete time, a stochastic process amounts to a sequence of random variables known as a time series.

Over the past decades, the problems of synergetic are concerned with the study of macroscopic quantitative changes of systems belonging to various disciplines such as natural science, physical science and electrical engineering. When such transition from one state to another take place, fluctuations i.e. (random process) may play an important role. Fluctuations in its sense are very common in a large number of fields and nearly every system is subjected to complicated external or internal influences that are often termed noise or fluctuations.

Fokker-Planck equation has turned out to provide a powerful tool with which the effects of fluctuation or noise close to transition points can be adequately be treated. For this reason, in this thesis work analytical and numerical methods of solving Fokker-Planck equation, its derivation and some of its applications will be carefully treated. Emphasis will be on both for one variable and N- dimensional cases.

Contents

Chapter 1

1. Introduction

Fokker-Planck equation was first used by Andriaan Daniel Fokker (1914) and Max Karl Planck (1917) to describe the Brownian motion of particles. To be acquainted with this equation, we shall later discuss briefly the Brownian motion of particle in its simplest form. Fokker-Planck equations are important stochastic PDE for describing a large number of physical processes. They describe how the probability distribution say **P(x; t)** describing the state of system evolves through time and a set of variables in phase space, **x . A Fokker-Planck equation has the form**

$$\partial/\partial t\, P(x;t) = -\nabla x A(x) P(x;t) + \frac{1}{2}\nabla x \nabla x B(x) P(x;t) \qquad (1.1)$$

However, there are countless ways of deriving Fokker-Planck equations for various processes. One common approach is to derive a FPE corresponding to a given Langevin equation by averaging over the noise (assumed to be white and Gaussian distributed) and constructing a Taylor expansion in powers of the noise [1]. Another approach is to expand a master equation in Kramers-Moyal expansion and truncate it to two terms [2]. One may also directly use the Markovian nature of a process to first derive a Chapman-Kolmogorov equation and then perform a Taylor's expansion about small differences [3].

1.1 Background

The origin of the name ***Fokker-Planck Equation*** is from the work of the famous Physicists **Fokker** and Planck where the former investigated Brownian motion in a radiation field and the latter attempted to build a complete theory of fluctuations based on it. The equation is also known as Kolmogorov forward equation because of Kolmogorov work in developing its rigorous basis [6]. FPE describes the time [1]

[1] Statistical modelling and Fokker-Planck Equation

evolution of the probability density function of position and velocity of a particle, and can be generalised to other observables as well. The equation was first used as a statistical description of Brownian motion of a particle in a fluid.

1.2 Overview of Brownian motion

It was observed that when small pollen grains were suspended in liquid or gas, the grains were found to be in a very animated and irregular state of motion. This was first investigated by Robert Brown in 1827. This observed phenomenon took the name ***Brownian motion*.** The motion is illustrated in Fig. 1.2.

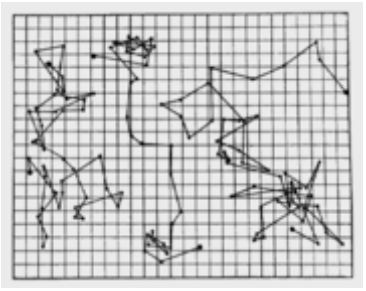

Fig 1.2. Motion of a point undergoing Brownian motion

The mathematical model of Brownian motion has several real-world applications. An often quoted example is the ***stock market*** fluctuations. Another notable example is in the evolution of physical characteristic in the fossil record. The actual explanation of Brownian motion came in 1905 when *Einstein* brought out two major points to the solution of Brownian motion.

- The motion is caused by the exceedingly frequent impacts on the pollen grain of the continuously moving molecules of the liquid in which it is suspended.
- The motion of the molecules is so complicated that its effects on the pollen grain can only be described probabilistically in terms of exceedingly frequent statistically independent impacts.

We should note here that each individual particle executes a motion which is independent of the motions of all other particles. We should also consider that the movements of one and the same particle in different time interval are independent process, as long as time intervals are not chosen too small.[2]

[2] Statistical modelling and Fokker-Planck Equation

Here we introduce a time interval τ and let the total number of particles be $\boldsymbol{n}$ and the in x coordinates the particle will increase by an amount Δ. In this case there is a frequency law for Δ; the number of $\boldsymbol{dn}$ of particles experience a shift which is between Δ and $\Delta+\mathrm{d}\Delta$ can be expressed as

$$\boldsymbol{dn} = \boldsymbol{n}\phi(\Delta)\boldsymbol{d}\Delta \tag{1.2.}$$

where

$$\int_{-\infty}^{\infty}\phi(\Delta)d\Delta = 1 \tag{1.3}$$

We should note that ϕ satisfies the condition that

$$\phi(\Delta) = \phi(-\Delta). \tag{1.4}$$

To investigate the diffusion coefficient which depends on ϕ , Let $\boldsymbol{v} = \boldsymbol{f(x,\ t)}$ be the number of particles per unit volume**,** $\mathbf{t}+\tau$ be the time difference , then the number of particles found at this interval between two planes perpendicular to $\boldsymbol{x}$ axis and passing through point $\boldsymbol{x}$ and $\boldsymbol{x+dx}$**.** We obtain

$$f(x,\ t+\tau)dx = dx\int_{-\infty}^{\infty} f(x+\Delta,t)\phi(\Delta)d\Delta \tag{1.5}$$

Since τ is very negligible we can set

$$f(x,\ t+\tau) = f(x,t) + \tau\ \frac{df}{dt}. \tag{1.6}$$

Therefore,

$$f(x+\Delta,\mathrm{t}) = f(x,t) + \Delta\ \frac{\partial f(x,t)}{\partial x}. + \Delta^2 \,/\, 2!\ \frac{\partial^2 f(x,t)}{\partial x^2} + \ldots\ . \tag{1.7}$$

If we continue in the expansion of the series, it obvious that Δ contribute only small value, hence we obtain

$$f + \frac{\partial f}{\partial \tau}\tau = f\int_{-\infty}^{\infty}\phi(\Delta)\,d\Delta + \frac{\partial f}{\partial x}\int\Delta\phi(\Delta)d\Delta + \frac{\partial^2 f}{\partial x^2}\int_{-\infty}^{\infty}\frac{\Delta^2}{2}\phi(\Delta)d\Delta \tag{1.8}$$

We obtain from equation (1.2.7) by taking into consideration[3]

[3] Statistical modelling and Fokker-Planck Equation

$$\int_{-\infty}^{\infty}\phi(\Delta)d\Delta = 1 \quad \text{and setting } \frac{1}{\tau}\int_{-\infty}^{\infty}\frac{\Delta^2}{2}\phi(\Delta)d\Delta = D, \tag{1.9}$$

And keeping only the first and third terms of the right hand side,

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2} \ \ldots . \tag{1.10}$$

The above equation is known as the diffusion differential equation of diffusion where D is the diffusion coefficient. The equation (1.29) is also a special case of Fokker-Planck equation which describe a very large class of very interesting stochastic processes in which the system has a continuous sample path. In this case, the particle's position, if thought of as obeying a probabilistic law given by solving the diffusion equation (1.2.9) where t is continuous function of time but a random function. This leads us to consider the possibility of describing the dynamics of the system in some direct probabilistic way, so that we would have a random or stochastic differential equation for the path. This idea was initiated by Langevin with the famous equation that bears his name.

Langevin's equation was the first example of the Stochastic differential equation , a differential equation with a random term X and whose solutions is in some sense a random function. Each solutions of Langevin's equation represent a different trajectory and using only properties of X i.e. is fluctuating force, measurable result can be derived.

The general concept of fluctuations describable by such equation has developed very extensively in a very wide range of solutions.

[4] Statistical modelling and Fokker-Planck Equation

Chapter 2

2. Derivation of Fokker-Planck Equation

2.1 Overview

Our derivation of Fokker-Planck starts with an expression of the distribution function known as Kramers-Moyal expression. In this equation, only the Kramers- Moyal coefficients can also be calculated for the non-linear Langevin equations. As a matter of fact, these coefficients vanish for n >= 3 for the Langevin equation with δ correlated Gaussian – distributed Langevin forces, and only the drift and diffusion coefficients enter in the distribution function equation.

Hence, the Kramers-Moyal expression with a infinite number of terms stops after the second term. The equation then is Fokker-Planck equation or the Forward Kolmogorov equation.

Suppose we let $\{X(t): t \geq 0\}$ be a one dimensional stochastic process with $t_1 > t_2 > t_3$. We use $P(X_1, t_1; X_2, t_2)$ to denote the joint probability distribution, i.e., the probability that $X(t_1) = X_1$ and $X(t_2) = X_2$, and $P(X_1; t_1 \mid X_2; t_2)$ to denote the conditional (or transition) probability distribution, i.e., the probability that $X(t_1) = X_1$ given that $X(t_2) = X_2$, defined as $P(X_1, t_1; X_2, t_2) = P(X_1, t_1 \mid X_2, t_2)P(X_2, t_2)$. We will assume $X(t)$ is a Markov process, namely,

$$P(X_1, t_1 \mid X_2, t_2; X_3, t_3) = P(X_1, t_1 \mid X_2, t_2) \quad (2.1)$$

For any continuous state Markov process, the following Chapman Kolmogorov equation is satisfied i.e.

$$P(X_1, t_1 \mid X_3, t_3) = \int P(X_1, t_1 \mid X_2, t_2)\, P(X_2, t_2 \mid X_3, t_3) dX_2 . \quad (2.2)$$

In the following, we will assume that X(t) is time homogeneous:[5]

[5] Statistical modelling and Fokker-Planck Equation

$P(X_1, t_1 + s; X_2, t_2 + s) = P(X_1,t_1,X_2, t_2)$, (2.3)

So that X is invariant with respect to a shift in time. For simplicity of notation, we use $P(X_1, t_1 - t_2 \mid X_2) \equiv P(X_1, t_1 \mid X_2, t_2)$ .The derivation of the Fokker-Planck equation, a partial differential equation for the time evolution of the transition probability density function can then be derived . Let us briefly look at the Kramer- Moyal forward expansion.

2.2 Kramers-Moyal Expansion

These expansions are generally the same as the one used by Einstein to go from Kolmogorov equation to the diffusion equation. The use of this type of approximation, which effectively replaces a process whose sample needs not be continuous with one whose path are continuous.

From the definition of the transition probability, say the probability density $W(x,t+\tau)$ at time $t+\tau$ and the probability density W(x,t) at time t are connected by ($\tau >=0$).

Given,

$$W(x,t+\tau) = \int P(x,t+\tau \mid x',t)W(x',t)dx' \quad (2.4)$$

To derive an expression for the differential $\partial W(x,t) / \partial t$, we must know the transition probability $P(x,t+\tau \mid x',t)$ for small τ .

First of all, we assume that we know all the moments (n>=1)

i.e. $M_n(x',t,\tau) = \left\langle \left[\xi(t+\tau) - \xi(t)\right]^n \right\rangle |_{\xi(t)=x'}$

$$= \int (x-x')^n P(x,t+\tau \mid x',t)dx \quad (2.5)$$

[6]

[6] Statistical modelling and Fokker-Planck Equation

Where $|_{\xi(t)=x'}$ means that at time t the random variable has the sharp value x'. Let us now derive a general expansion of the transition probability as follows:

Let $\Delta = x - x'$, the integrand in (2.4) can be expanded in Taylor series according to

$$\int P(x,t+\tau \mid x',t)W(x',t) = P(x-\Delta+\Delta,t+\tau \mid x-\Delta,t)W(x-\Delta,t)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Delta^n \left(\frac{\partial}{\partial x}\right)^n P(x+\Delta,t+\tau \mid x,t)W(x,t).$$

Now inserting this expression in (2.4) and integrating over Δ we will obtain

$$\sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n \left[M_n(x,t,\tau)/n!\right] W(x,t) \tag{2.6}$$

Note: The negative sign of the differential $\mathrm{d}\Delta = -dx'$ may be absorbed into the integration boundaries. We now assume that the moment M_n can be expanded into Taylor series with respect to $\tau (n \geq 1)$

$$M_n(x,t,\tau)/n! = D^{(n)}(x,t)\tau + O(\tau^2) \tag{2.7}$$

The term with τ^0 must vanish because for τ =0 the transition probability P has the initial value

$$P(x,t \mid x',t) = \partial(x-x'), \tag{2.8}$$

This then leads to vanishing moments (2.5). By taking into account only the linear terms in τ we thus have

$$\frac{\partial W(x,t)}{\partial t} = \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n D^{(n)}(x,t)W(x,t) = L_{KM}W, \tag{2.9}$$

where the differential symbol acts on $D^{(n)}$ and W (x,t). The Kramers- Moyal operator L_{KM} is defined by

$$L_{KM}(x,t) = \sum_{n=1}^{\infty} (-\partial/\partial x)^n D^{(n)}(x,t) \tag{2.10}$$

The equation (2.2.6) is called the Kramers-Moyal expansion.

[7]

[7] Statistical modelling and Fokker-Planck Equation

2.3 Derivations of FPE

Let {X (t): t >=0} be a one dimensional stochastic process with $t_1>t_2>t_3$. We use $P(X_1,t_1;X_2,t_2)$ to denote the joint probability distribution, i.e. the probability that $X(t_1)=X_1$ and $X(t_2)=X_2$ and $P(X_1,t_1 \mid X_2,t_2)$ to denote the conditional probability distribution i.e., the probability that $X(t_1)=X_1$ given that $X(t_2)=X_2$, defined as $P(X_1,t_1;X_2,t_2)=P(X_1,t_1 \mid X_2,t_2)\ P(X_2,t_2)$. Let us assume that X (t) is a Markov process namely,

$$P(X_1,t_1 \mid X_2,t_2;X_3,t_3)=P(X_1,t_1 \mid X_2,t_2) \tag{2.11}$$

For any continuous state Markov process, the following Chapman-Kolmogorov equation is satisfied i.e.

$$P(X_1,t_1 \mid X_3,t_3)=\int P(X_1,t_1 \mid X_2,t_2)P(X_2,t_2 \mid X_{3,}t_3)dX_2 \tag{2.12}$$

Likewise, we will also assume X(t) is time Homogeneous:

$$P(X_1,t_1+s;X_2,t_2+s)=P(X_1,t_1,X_2,t_2) \tag{2.13}$$

So that X is invariant with respect to a shift in time. For simplicity of use $P(X_1,t_1-t_2 \mid X_2)\equiv P(X_{1,}t_1 \mid X_2,t_2).$

Now let us outline the derivation of the Fokker-Planck equation, a partial differential equation for the time evolution of the transition probability density function..

Consider,

$$\int_{-\infty}^{\infty} h(Y)\frac{\partial P(Y,t \mid X)}{\partial t}dY, \tag{2.14}$$

Where *h(Y)* is any smooth function with compact support. Writing

$$\frac{\partial P(Y,t \mid X)}{\partial t}dY=\lim_{\Delta t\to 0}\frac{P(Y,t+\Delta t \mid X)-P(Y,t \mid X)}{\Delta t} \tag{2.15}$$

and interch[8]anging the limit with the integral, it follows that

[8] Mathematical modelling and Fokker-Planck Equation

$$\int_{-\infty}^{\infty} h(Y)\frac{\partial P(Y,t\mid X)}{\partial t}dY = \lim_{\Delta t\to 0}\int h(Y)\Big[\ \frac{P(Y,t+\Delta t\mid X)-P(Y,t\mid X)}{\Delta t}\ \Big]dY. \qquad (2.16)$$

Now applying the Chapman-Kolmogorov identity equation (2.13), the right hand side of equation (2.16) can be written as

$$\lim_{\Delta t\to}\frac{1}{\Delta t}\left[\int_{-\infty}^{\infty}h(Y)\int_{-\infty}^{\infty}P(Y,\Delta t\mid Z)P(Z,t\mid X)dZdY-\int_{-\infty}^{\infty}h(Y)P(Y,t\mid X)dY\right] \qquad (2.17)$$

Interchanging the limits of integration in the first term of equation (2.17), letting $Y\to Z$ in the second term, and using the identity $\int_{-\infty}^{\infty}P(Y,\Delta t\mid Z)dY=1$,we have

$$\lim_{\Delta t\leftarrow 0}\frac{1}{\Delta t}\left[\int_{-\infty}^{\infty}P(Z,t\mid X)\int_{-\infty}^{\infty}P(Y,\Delta t\mid Z)(h(Y)-h(Z))dYdZ\right] \qquad (2.18)$$

Taylor expanding h(Y) about Z gives

$$\lim_{\Delta t\leftarrow 0}\frac{1}{\Delta t}\left[\int_{-\infty}^{\infty}P(Z,t\mid X)\int_{-\infty}^{\infty}P(Y,\Delta t\mid Z)\sum_{n=1}^{\infty}h^{(n)}(Z)\frac{(Y-Z)^n}{n!}dYdZ\right] \qquad (2.19)$$

Defining the jump moment as

$$D^{(n)}(Z)=\frac{1}{n!}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\infty}^{\infty}(Y-Z)^n P(Y,\Delta t\mid Z)dY \qquad (2.20)$$

It follows that

$$\int_{-\infty}^{\infty}h(Y)\frac{\partial P(Y,t\mid X)}{\partial t}dY=\int_{-\infty}^{\infty}P(Z,t\mid X)\sum_{n=1}^{\infty}D^{(n)}(Z)h^{(n)}(Z)dZ. \qquad (2.21)$$

Integrating each term on the right hand side of the equation (2.21) by parts n times and using the assumption on *h,* after moving to the left hand side, it follows that

$$\int_{-\infty}^{\infty}h(Z)\left(\frac{\partial P(Z,t\mid X)}{\partial t}-\sum_{n=1}^{\infty}(-\frac{\partial}{\partial Z})^n\left[D^{(n)}(Z)P(Z,t\mid X)\right]\right)dZ=0 \qquad (2.22)$$

[9] Statistical modelling and Fokker-Planck Equation

Now because h is an arbitrary function, it is necessary that

$$\frac{\partial P(Z,t\mid X)}{\partial t}=\sum_{n=1}^{\infty}(-\frac{\partial}{\partial Z})^{n}\left[D^{(n)}(Z)P(Z,t\mid X)\right]. \tag{2.23}$$

We define the probability distribution function *P(X,t) of X(t)* as the solution of equation (2.23) with initial condition given by a δ -distribution at $t = 0$. In this case,
P(X,t) $P(X,t)\equiv P(X,t\mid X_{0},0)$ and we can as well write equation (2.23) as

$$\frac{\partial P(X,t)}{\partial t}=\sum_{n=1}^{\infty}\left(-\frac{\partial}{\partial x}\right)^{n}\left[D^{(n)}(X)P(X,t)\right], \tag{2.24}$$

With

$$D^{(n)}(X_{0})=\frac{1}{n!}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\langle\left[X(t+\Delta t)-X(t)\right]^{n}\right\rangle|_{t=0} \tag{2.25}$$

This commonly called the Kramers-Moyal expansion. Now if we assume $D^{(n)}(X)=0$ for n > 2, then we have the Fokker-Planck equation:

$$\frac{\partial P(X,t)}{\partial t}=-\frac{\partial}{\partial X}\left[V(X)P(X,t)\right]+\frac{\partial^{2}}{\partial X^{2}}\left[D(X)P(X,t)\right], \tag{2.26}$$

Where, V(X) $\equiv D^{(1)}(X)$ is the drift coefficient and *D(X)* $\equiv D^{(2)}(X)>0$ *i*s the diffusion coefficient, which can be written as

$$\mathrm{V}\,(X_{0})=\frac{\partial\left\langle X(t;X_{0})\right\rangle}{\partial t}\Big|_{t=0},D(X_{0})=\frac{1}{2}\frac{\partial\sigma^{2}(t;X_{0})}{\partial t}|_{t=0} \tag{2.27}$$

Where angular brackets denote ensemble averaging σ^{2} denotes the variance of X and $X(t;X_{0})$ denotes a realization with X (0) = X_{0} . Any Stochastic process X (t) whose probability distribution function satisfies the Fokker-Planck equation is known mathematically as a diffusion process [7].

[10] Statistical modelling and Fokker-Planck Equation

2.4 Fokker-Planck Equation in One Dimension

Considering an equation of motion for the distribution function W (v, t) for one dimensional Brownian motion with one variable x: the general Fokker-Planck equation has the form

$$\frac{\partial W}{\partial t} = \left[-\frac{\partial}{\partial x} D^{(1)}(x) + \frac{\partial^2}{\partial x^2} D^{(2)}(x) \right] W \qquad (2.28)$$

In the above equation $D^{(2)}(x) > 0$ is referred to as diffusion coefficient and $D^{(1)}(x)$ is the drift coefficient. The drift and diffusion may also depend of time. There are some cases where the drift coefficient is linear and the diffusion coefficient is constant. Mathematically, equation (2.1.1) is a linear second order partial Differential equation of parabolic type and is also called forward Kolmogorov equation in most mathematical literature as early mentioned.

2.5 Fokker-Planck Equation for N variables/ Several Dimensions

The generalization of the equation (2.4) to N variables $x_1, x_2, \ldots x_n$ has the form

$$\frac{\partial W}{\partial t} = \left[-\sum_{i=1}^{N} \frac{\partial}{\partial x} D_i^{(1)}(\{x\}) + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} D_{i,j}^{(2)}(\{x\}) \right] W \qquad (2.29)$$

Note: The drift vector $D_i^{(1)}$ and the diffusion tensor Dij2 generally depend on N variables $x_1, x_2, \ldots, x_n = (\{x\})$ .The above equation (2.4) is an equation for the distribution function W({x},t) of N macroscopic variables {x} which may be variables of different kinds for instance position and velocity.

[11]

As earlier discussed, in the case of Brownian motion, the complete solution of a macroscopic system would consist in solving all the microscopic equations of the

[11] Statistical modelling and Fokker-Planck Equation

system. Since this may be difficult, we use instead a stochastic description i.e. , we describe the system by macroscopic variables which fluctuate in a stochastic way. For a deterministic treatment we neglect the fluctuations of the macroscopic variables. Then equation (2.4) is equivalent to the system of the differential equations *(i* = 1…, N)

$$dx_i/dt = \dot{x}_i = D_i^{(1)}(x_i,...,x_N) = D_i^{(1)}(\{x\}) \qquad (2.30)$$

for the N macrovariables {x}.

The Fokker-Planck Equation is of course not the only equation of motion for the distribution function. There are other equations such as Boltzmann equation and the Master equation. It is only that Fokker-Planck equation is one of the simplest equations for the continuous macroscopic variables. It usually appears for the variables describing macroscopic but small subsystems, like the position and velocity for the Brownian motion of a small particle, a current in an electric circuit, the electrical field in a laser.

However, if the subsystem is larger the fluctuations may be neglected and thus one has a deterministic equation. In these cases where the deterministic equations are not stable, a stochastic description is then necessary for large systems.[8]
By solving the Fokker-Planck equation one obtains distribution functions from which any averages of microscopic variables are obtained by integration. Since application of Fokker-Planck equation is not restricted to systems near thermal equilibrium, we may as well apply it to systems far from thermal equilibrium e.g. the laser. The FPE not only describe stationary properties, but also the dynamics of systems, if the appropriate time- dependent solution is used.

[12] Statistical modelling and Fokker-Planck Equation

Chapter 3

3. Methods of Solutions of the Fokker-Planck Equation

The Fokker-Planck equation, being a partial differential equation can be solved analytically only in special cases. A formal analogy of the Fokker-Planck equation with the Schrödinger equation allows the use of advanced operator techniques known from quantum mechanics for its solution in a number of cases. In many applications, one is only interested in the steady-state probability distribution $f_0(x)$, which can be found from $f_0(x)$ =0. The computation of mean-first-passage times and splitting probabilities can be reduced to the solution of an ordinary differential equation which is intimately related to the Fokker-Planck equation. [9].

For the purpose of this thesis work, we shall be concerned with some methods for solving Fokker-Planck equation and with its application and also look at some few examples. The analytical solutions shall be approached with few applications

However, it is difficult to obtain solutions of Fokker-Planck equation especially if no separation of variables is possible or if the number of variables is large. Other methods such as computer simulations methods; numerical integration methods; analytical solution for certain model potentials for a one variable Fokker-Planck equation; matrix continued fraction solution for a two variable Fokker-Planck equation. All these are various methods and approach in solving the Fokker-Planck Equation and we are going to use one or two of the methods.

3.1 Methods of Solution for one variable

The method for solving the one variable Fokker-Planck equation with time-independent drift and diffusion coefficient.

Suppose we assume $D^{(2)}(X)>0$ i.e. From

[13]

$$\dot{W}(x, t) = L_{FP}W(x,t) \qquad (3.1)$$

[13] Statistical modelling and Fokker-Planck Equation

and

$$L_{FP}(x) = -\frac{\partial}{\partial x} D^{(1)}(x,t) + \frac{\partial^2}{\partial x^2} D^{(2)}(x,t) \qquad (3.2)$$

(3.1) can also be written as

$$\partial W(x,t)/\partial t = L_{FP} W(x,t) = -(\partial/\partial x) S(x,t) \qquad (3.3)$$

Where $S(x,t)$ is the probability current.

Note: Equation (3.1) and (3.2) can also be written as follows

$\frac{\partial W}{\partial t} + \frac{\partial S}{\partial x} = 0$, therefore

$$S(x,t) = \left[D'(x,t) - \frac{\partial}{\partial x} D^{(2)}(x,t) \right] W(x,t) \qquad (3.4)$$

This is a continuous equation for a probability distribution. Hence *S* is termed as a probability current.

3.2 Normalization

By a suitable transformation $x' \equiv y = y(x)$, the x-dependent diffusion coefficient can be transformed by an arbitrary constant D > 0.

For one variable case, using this transformation below

$$D'_{kr} = \frac{\partial x'_k}{\partial x_i} \frac{\partial x'_r}{\partial x_j} D_{ij} \qquad (3.5)$$

i.e.

[14] Statistical modelling and Fokker-Planck Equation

$$D'^{(2)} \equiv D = \left(\frac{dy}{dx}\right)^2 D^{(2)}(x) \tag{3.6}$$

Thus, the transformation is given by

$$y = y(x) = \int_{x_0}^{x} \sqrt{D / D^{(2)}(\xi)}\, d\xi \tag{3.7}$$

Then the transformed drift coefficient takes the form

$$D'_k = \left(\frac{\partial x'_k}{\partial t}\right)_x + \frac{\partial x'_k}{\partial x_i} D_i + \frac{\partial^2 x'_k}{\partial x_i \partial x_j} D_{ij}, \tag{3.8}$$

i.e.

$$\begin{aligned} D'^{(1)}(y) &= \frac{dy}{dx} D^{(1)}(x) + \frac{d^2 y}{dx^2} D^{(2)}(x) \\ &= \sqrt{\frac{D}{D^{(2)}(x)}} \left[D'(x) - \frac{1}{2} \frac{dD^{(2)}(x)}{dx} \right] \end{aligned} \tag{3.9}$$

Hence, the transformed Fokker-Planck equation reads (D = const)

$$\frac{\partial W'(y,t)}{\partial t} = \left[-\frac{\partial}{\partial y} D'^{(1)}(y) + D \frac{\partial^2}{\partial y^2} \right] W'(y,t) \tag{3.10}$$

Where W' is given by

$$W' = J \cdot W = \left(\frac{dy}{dx}\right)^{-1} W = \sqrt{D^{(2)}(x) / DW} \tag{3.11}$$

Therefore, without loss of generality, we may treat the equation with constant diffusion, i.e.

$$\frac{\partial W}{\partial t} = \left[\frac{\partial}{\partial x} f'(x) + D \frac{\partial^2}{\partial x^2} \right] W = -\frac{\partial}{\partial x} S(x,t) \tag{3.12}$$

[15]

Where *S* is the probability current. In this case we have introduced the potential

$$f(x) = -\int^{x} D^{(1)}(x')dx' \tag{3.13}$$

[15] Statistical modelling and Fokker-Planck Equation

Because D is arbitrary, we may use D = 1. This normalization is however not very convenient if the low-noise limit D tends to zero ($D \to 0$) is considered and we therefore retain the constant D.

3.3 Stationary Solution

In the case of Stationary solution the probability current in equation (3.3) must be constant. Therefore, we should note that if the probability current vanishes at some x , the current must be zero for any x. Then for S = 0

$$D^{(!)}(x)W_{st}(x) = \frac{D^{(1)}(x)}{D^{(2)}(x)} D^{(2)}(x)W_{st}(x) = \frac{\partial}{\partial x} D^{(2)}(x)W_{st}(x) \qquad (3.14)$$

Here, we can quickly integrate (3.3.1), yielding

$$W_{st}(x) = \frac{N_0}{D^{(2)}(x)} \exp\left(\int^x \frac{D^{(1)}(x')}{D^{(2)}(x')} dx' \right) = Ne^{-\phi(x)} \ , \qquad (3.15)$$

Where N_0 is the constant of integration which has to be chosen such that W_{st} is normalized?

In (3.15), if we introduced the potential

$$\phi(x) = \ln D^{(2)}(x) - \int^x \frac{D^{(1)}(x')}{D^{(2)}(x')} dx' \qquad (3.16)$$

Note: But since $\phi(x)$ is defined only up to an additive constant $\ln D^{(2)}$ term may be omitted.

[16]

Therefore, the probability current may be written as

$$S(x,t) = -D^{(2)} e^{-\phi(x)} \frac{\partial}{\partial x} \left[e^{\phi(x)} W(x,t) \right] \qquad (3.17)$$

In stationary state, where S is constant, then we have for arbitrary S

[16] Statistical modelling and Fokker-Planck Equation

$$W_{st}(x) = Ne^{-\phi(x)} - Se^{-\phi(x)} \int^{x} \frac{e^{\phi(x')}}{D^{(2)}(x')} dx' \qquad (3.18)$$

Note: One of the integration constant in equation (3.18) is determined by normalization i.e. $\int W_{st}(x)dx = 1$, where others must be determined from the boundary conditions.

3.4 Transformation of Variable

The transformation of variable can be done in such a way that the transformed Fokker-Planck equation can be solved analytically. The transformed coefficients are given by (3.2.1) and (3.2.4). The problem of finding such transformation is, however, as hard as solving the Fokker-Planck equation. This method is usually applied in opposite way. Here one starts with a Fokker-Planck equation whose solution is know e.g., the Fokker-Planck equation for an Ornstein-Uhlenbeck process. In this case if one obtains a nonlinear transformation of variables, a complicated Fokker-Planck equation which of course can then be solved [8].

3.5 Numerical Integration

Numerical integration method is one way of performing a numerical integration on Fokker-Planck equation (3.20) and (3.21) below.

$$\partial W / \partial t = L_{FP} W = -\partial S_i / \partial x_i \qquad (3.20)$$

And

$$L_{FP} = -\frac{\partial}{\partial x} D_i\ (x) + \frac{\partial^2}{\partial x_i \partial x_j} D\ _{ij}(x) \qquad (3.21)$$

[17]

is to use instead of continuous variable {x} the discrete variables {n}, defined by $x_i = \Delta x_i n_i$ (no summation convention), with discrete times $t_m = \Delta tm.$ If the differential are then approximated by differences in a consistent way, solving the Fokker-Planck equation is reduced to iterating a difference equation. The difference equations must

[17] Statistical modelling and Fokker-Planck Equation

be stable in the sense that the probability error does not increase faster than the probability itself, otherwise one does not obtain an approximate solution to the continuous Fokker-Planck equation [8]. It is also very crucial for the stability that the differentials are approximated by appropriate differences [11, 12, and 13].

3.6 WKB Method

WKB theory is a method for approximating the solution of a differential equation whose highest derivative is multiplied by a small parameter ε. In our own case for very small diffusion coefficients one may use a WKB method. This method has been applied to diffusion in one dimensional and and multi-dimensional bistable potentials *by Caroli et al.* [14]. It essentially consists in the following. After indicating the smallest of the diffusion coefficient by a parameter ε >0 we insert in the Fokker-Planck equation as follows:

$$\dot{W} = \left(-\frac{\partial}{\partial x_i} D_i + \varepsilon \frac{\partial^2}{\partial x_i \partial x_j} D_{ij} \right) W \tag{3.22}$$

The ‘ansatz’

$$W = \exp\left(-\frac{1}{\varepsilon} w \right) \tag{3.23}$$

and obtain in lower order

$$\dot{w} = -D_i \frac{\partial w}{\partial x_i} - D_{ij} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} + O(\varepsilon). \tag{3.24}$$

[18]

This first order nonlinear partial differential equation is usually easier to solve than the Fokker-Planck Equation. The method of WKB is well known in quantum mechanics, where it is useful for describing the transition to classical mechanics, and in optics, where it is useful for describing the transition from wave optics to ray optics. We

[18] Statistical modelling and Fokker-Planck Equation

should note that some care has to be taken for the application of WKB method. From the study of quantum mechanics [15] that at a classical turning point the WBK solution is not valid. Near these points the full Schrödinger equation must be used and this solution must then be matched with the WKB solutions. Ofcourse, in certain cases the Fokker-Planck equation can be transformed to Schrödinger equation, the same should also be true for the Fokker-Planck equation. [14].

3.7 Some examples and Solutions to Fokker-Planck Equation

Example 1

Let a stochastic process x(t) has an Ito differential equation

$$dx = a(x,t)dt + b(x,t)dW \tag{3.25}$$

Then the Fokker-Planck equation can be derived very easily. To do this derivation we first calculate the differential equation for the mean value of a function *f(x).* The SDE for *f(x)* is

$$df = a(x,t)\frac{df}{dx}dt + \frac{d^2 f}{dx^2}b^2(x,t)dt + \frac{df}{dx}b(x,t)dW \tag{3.26}$$

Taking the averages on both sides gives the differential equation for the mean of *f,* which is

$$d\langle f \rangle = \left\langle a(x,t)\left(\frac{df}{dx}\right)\right\rangle dt + \left\langle \frac{b^2(x,t)}{2}\left(\frac{d^2 f}{dx^2}\right)\right\rangle dt \tag{3.27}$$

Or alternatively,

$$\begin{aligned}\frac{d\langle f \rangle}{dt} &= \left\langle a(x,t)\left(\frac{df}{dx}\right)\right\rangle + \left\langle \frac{b^2(x,t)}{2}\left(\frac{d^2 f}{dx^2}\right)\right\rangle \\ &= \int_{-\infty}^{\infty}\left[a(x,t)\left(\frac{df}{dx}\right) + \frac{b^2(x,t)}{2}\left(\frac{d^2 f}{dx^2}\right)\right]P(x,t)dx \end{aligned} \tag{3.28}$$

Now, Let us integrate by parts. Using the fact that $\lim_{x\to\pm\infty} P(x,t) = 0$, we obtain

$$\frac{d\langle f\rangle}{dt} = \int_{-\infty}^{\infty} f(x)\left[-\frac{\partial}{\partial x}(a(x,t)P(x,t)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(b(x,t)P(x,t))\right]dx \tag{3.29}$$

We know that the mean of *f* is given by,

$$\langle f\rangle = \int_{-\infty}^{\infty} f(x)P(x,t)dx \tag{3.30}$$

So that the derivative of the mean can also be written as

$$\frac{d\langle f\rangle}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} f(x)P(x,t)dx = \int_{-\infty}^{\infty} f(x)\frac{\partial P(x,t)}{\partial t}dx \tag{3.31}$$

Equating Equations (3.29) and (3.31), and realizing that they must be equal for any *f(x),* we have the Fokker-Planck equation for the probability density for *x (t):*

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[a(x,t)P(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[b(x,t)P(x,t)\right] \tag{3.32}$$

From the above Fokker-Planck equation (3.32), we can calculate the stationary solutions for *P(x,t).* If we recall that a stationary solution is one in which *P(x,t)* does not change with time. Often the stationary solution for a given set of boundary conditions is the density that *P(x,t)* will tend to as $t \to \infty$. The equation for the stationary solutions is given by setting $\partial P(x,t)/\partial t$ to zero in the Fokker-Planck equation.

[19]

i.e., $$-\frac{\partial}{\partial x}\left[a(x,t)P(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[b(x,t)P(x,t)\right] = 0 \tag{3.33}$$

Note that the Fokker-Planck equation may be written as

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[-a(x,t)P(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[b(x,t)P(x,t)\right] \equiv \frac{\partial}{\partial x}J(x,t) \tag{3.34}$$

Where J(x, t) is defined as the relation between *P* and *J* given in equation (3.4)

[19] Statistical modelling and Fokker-Planck Equation

It implies that *J(x)* is the *probability current*; *J(x, t)* is the rate at which probability is flowing across the point *x* at time *t.* The equation for the stationary solutions i.e. (3.3) can therefore be written as

$$\frac{\partial J(x,t)}{\partial x} = 0 \qquad (3.35)$$

This means that

$$J(x) = c \qquad (3.36)$$

for some constant c. Note here that the value of c will depend on the boundary conditions. If the particle is confined to a finite interval $x \in [a,b]$ then *J (a, t) = J (b,t) = 0*. Thus *J(x, t)* will be zero everywhere in the interval. In this case the stationary solution is thus given by

$$-a(x)P(x) + \frac{1}{2}\frac{d}{dx}b(x)P(x) = 0 \qquad (3.37)$$

For example if b constant (additive noise), then we have

$$\frac{d}{dx}P(x) = \frac{a(x)}{b}P(x) \qquad (3.38)$$

The solutions to (3.7.14) are trivial by separating the variables and then integrate.
i.e.

$dP(x)/P(x) = \frac{a(x)}{b}dx$, integrating we have

[20]

$\ln P(x) = \frac{1}{b}\int a(x)dx + C$, where C is the constant of integration (3.39)

Then the general solution will be given as

$$P(x) = e^{\frac{1}{b}\int a(x)dx + c} \qquad (3.40)$$

Or

[20] Statistical modelling and Fokker-Planck Equation

$$P(x) = e^{\frac{1}{b}\int a(x)dx} \cdot e^{c}$$

Where $e^{c} \approx N$, therefore we have solution of the form

$$P(x) = Ne^{\frac{1}{b}\int a(x)dx}$$

Example Case Two:

Suppose the Fokker Planck equation (3.7.8) above with the diffusion equation $dx = a(x,t)dt + b(x,t)dW$ has its coefficients *a(x, t) and b (x.t)* to be constant say: *a(x, t) = - 2 and b(x, t) =1.* Then the diffusion equation becomes

dx = -2dt + dW (t) Then the Fokker-Planck equation is

$$\frac{\partial P(x,t)}{\partial t} = 2\frac{\partial}{\partial x}\left[P(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[P(x,t)\right] \qquad (3.41)$$

To get the general solution of equation (3.41) using separation of variables or using the Fourier transforms. If the system is in equilibrium, the derivative with respect to time equals to zero.

[21]

Hence the Fokker-Planck equation is given by

$2\frac{\partial}{\partial x}P(x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}P(x) = 0$. This is just an ordinary equation of the first order.

Multiplying through by 2, and let $h(x) = \frac{\partial P(x)}{\partial x}$ we arrive at

$$4h(x) + h'(x) = 0 \qquad (3.42)$$

Then

[21] Statistical modelling and Fokker-Planck Equation

$h'(x) = -\ 4h(x)$, $\frac{dh(x)}{dx} = -4h(x)$ (separation of variables)

Now integrating both sides we have,

$\ln h(x) = -4x + C,$ where C is the constant of integration. Therefore

$$h(x) = e^{-4x+C} \approx Ke^{-4x} \tag{3.43}$$

Hence, $P(x) = \frac{K}{4}e^{-4x} \equiv \tilde{K}e^{-4x}$ (3.44)

Example case 3:

Example of stationary solutions (**Diffusion in a Gravitational Fields**)

Suppose a Brownian particle moving in a constant gravitational field is given by stochastic differential equation:

$dx = -gdt + \sqrt{D}dW(t).$, where the corresponding Fokker-Planck equation is

[22]

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}(gp) + \frac{1}{2}D\frac{\partial^2 p}{\partial x^2} \tag{3.45}$$

on the interval (a, b) with reflecting boundary conditions. The stationary solutions is given by setting J = 0, J (a) = J (b) i.e.

$dJ(x)/dx = 0$ Which has solutions J(x) = 0. Then,

$$\frac{d(gp)}{dx} + \frac{1}{2}D\frac{d^2 p}{dx^2} = 0 \quad , \tag{3.46}$$

Separating of variables and integrating accordingly we have

[22] Statistical modelling and Fokker-Planck Equation

$$g\frac{dp}{dx} = -\frac{1}{2}D\frac{d^2p}{dx^2} \tag{3.47}$$

This is trivial if we use separation of variable and integrate accordingly.

Let

$\frac{dp}{dx} = y \Rightarrow \frac{d^2p}{dx^2} = y'$ i.e.

$gy = -\frac{1}{2}Dy'$ …..using separation of variables we have

$$\frac{2g}{D} = \frac{y'}{y} \tag{3.48}$$

integrating both sides we have

$\ln y = \int -\frac{2g}{D} + C$, where C is the constant of integration.

[23]

Therefore,

$$y = e^{\int -\frac{2g}{D} + C} \tag{3.49}$$

Hence, since $y = \frac{dp}{dx}$

$\Rightarrow \frac{dp}{dx} = e^{\int \frac{-2g}{D} + C}$

$dp = e^{\int \frac{-2g}{D} + C} dx$, Now integrating both sides we have

$$P = \int e^{\int \frac{-2g}{D} + C} dx \tag{3.50}$$

[23] Statistical modelling and Fokker-Planck Equation

$$\Rightarrow P_s(x) = e^{-\frac{2gx}{D}+C}$$

$$P_s(x) = e^{-\frac{2gx}{D}} \cdot e^{C}$$

$$P_s(x) = Ne^{-\frac{2gx}{D}} \qquad (3.51)$$

where *N is the absorbed constant factor.*

Clearly, the solution is normalized on interval (a, b) only if ***a*** is finite, though ***b*** might not be infinite. The results indicate that the particles diffusing in a beaker of fluid will fall down and if the container is infinitely deep, the fluid will never cease falling.

[24] Statistical modelling and Fokker-Planck Equation

Chapter 4

4. Summary and Conclusions

The importance of Fokker-Planck equation in stochastic differential equation; to be precised in the theory of Markov process gives the time evolution of the probability density function for the system. The results of the solutions indicate that there are a large number of exact results for single variable systems, which makes the separate treatment of the systems appropriates.

However, proper construction of appropriate boundary conditions has been seen to be fundamentally important in both single and multivariable process. Results also show that most Fokker-Planck equation of one variable system shows trivial and exact results or solutions while in the case of many variables the results are not explicitly as in one variable system.

We know from various examples that some Stochastic process describe by a conditional probability satisfying the Fokker-Planck equation is equivalent to the Ito Stochastic differential equation (SDE).We have also seen that Fokker-Planck equation is simply a second order parabolic partial differential equation which solutions need an initial boundary conditions.

In general, mathematically stochastic systems can be modelled by stochastic map (in the time discrete case) and by Stochastic differential equation (SDE) in the continuous case. Furthermore, the stochastic equation of motion of a particle in an accelerator are very complicated and cannot be solved analytically, therefore the use of numerical scheme has to be adopted. Investigation reveals that the finite difference method used for the Fokker-Planck equation with two phased space variables with time have been performed in using extensive numerical simulations with finite elements for the Partial differential equation and the Monte Carlo simulations for the stochastic differential equation.[25]

[25] Statistical modelling and Fokker-Planck Equation

The finite difference scheme is simple to implement and very flexible with respect to different boundary conditions and also easy to understand the physical meaning of each term in the scheme, therefore is a good example for solving more complicated and more realistic accelerator models such as higher dimensional problems, non-Gaussian white noise perturbation of Ornstein-Uhlenbeck type and explicitly time dependent coefficients.

[26] Statistical modelling and Fokker-Planck Equation

Notation, Symbols and Definition of some terms

- FPE: Fokker-Planck Equations
- SDE: Stochastic Differential Equation
- W(t): Wiener process
- J(x): Probability current in jump process
- $P_{(x_1,t_1;x_2,t_2;\ldots)}$: Joint probability density
- $\mathcal{N}$: Normalization constant
- WKB: (Wentzel-Kramers-Brillouin) approximation

[27] Statistical modelling and Fokker-Planck Equation

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[28]

[28] Statistical modelling and Fokker-Planck Equation

Appendix

29