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Optimal System of Subalgebras and Invariant Solutions for the Black-Scholes Equation

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Abstract

The main purpose of this thesis is to use modern goal-oriented adaptive methods of Lie group analysis to construct the optimal system of Black-Scholes equation. We will show in this thesis how to obtain all invariant solutions by constructing what has now become so popular, optimal system of sub-algebras, the main Lie algebra admitted by the Black-Scholes equation. First, we obtain the commutator table of already calculated symmetries of the Black-Scholes equation. We then followed with the calculations of transformation of the generators with the Lie algebra L_6 which provides one-parameter group of linear transformations for the operators. Here we make use of the method of Lie equations to solve the partial differential equations. Next, we consider the construction of optimal systems of the Black-Scholes equation where the method requires a simplification of a vector to a general form to each of the transformations of the generators.

Further, we construct the invariant solutions for each of the optimal system. This study is motivated by the analysis of Lie groups which is being taken to another level by ALGA here in Blekinge Institute Technology, Sweden. We give a practical and in-depth steps and explanation of how to construct the commutator table, the calculation of the transformation of the generators and the construction of the optimal system as well as their invariant solutions.

Keywords:
Black-Scholes Equation, commutators, commutator table, Lie equations, invariant solution, optimal system, generators, Airy equation, structure constant,

We dedicate this humble effort to our
beloved parents
who made it possible for us
to reach this far

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Contents

1 Preamble

We live in a world of interrelated changing entities. The position of the earth changes with time, the velocity of a falling body changes with distance, the arc of circle changes with the size of the radius. On the language of mathematics, changing entities are called variables and the rate of change of one variable with respect to another is a derivative, equations which express a relationship among these variables and their derivatives are called differential equations. Differential equations in a proper sense have appeared in mathematics in 1680s in the work of the creators of the differential and integral calculus Newton and G. W. Leibnitz. The term differential equation was mentioned by Leibnitz for the first time in his letter to Newton (1676) and then used in his publication there after. It is as a result of this differential equation principles that the Black-Scholes Equation was born.

In today's world, finance has become one of the fastest developing areas in the modern banking and cooperate world. In view of this, together with modern sophisticated financial products, provides a rapidly growing impetus for new mathematical models and modern mathematical methods. The celebrated Black-Scholes Equation (developed by Fischer Black and Myron Scholes in 1973) is used in almost all financial markets for the determination of prices of options. Again modern option pricing techniques are often considered among the most mathematically complex of all applied areas of finance. Financial analysts have reached the point where they are able to calculate, with alarming accuracy, the value of a stock option.

Most of the models and techniques employed by today's analysts are rooted in a model developed by Fischer Black and Myron Scholes. Fischer Black and Myron Scholes first made mentioned of the Black-Scholes formula in their 1973 paper, know as "The Pricing of Options and Corporate Liabilities." The foundation for these two men research relied on work developed by scholars such as Jack L. Treynor, Paul Samuelson, A. James Boness, Sheen T. Kassouf, and Edward O. Thorp. The fundamental insight of Black-Scholes is that the option is implicitly priced if the stock is traded. Robert C. Merton was the first to publish a paper expanding the mathematical understanding of the options pricing model and coined the term "Black-Scholes" options pricing model. As a result of this development several models have been made. It is due to this fact that we have also decided to look into the Black-Scholes Equation from the perspective of the great mathematician Sophus Lie. And for us to achieve this we will like to define some concepts and tools that we have extensively use in this work.

1.1 Definitions

Optimal system:

An optimal is defined as the best or of the greatest value, sometimes under certain parameters or restrictions. For example, it is usually up to the human resource department to select the optimal candidate for a job from a group of applicants by deciding which candidate meets the requirement. Hence, the optimal system, in our case, is to find the minimal representatives of each class of similar vectors.

Commutator:

Let $X_\alpha = \xi_\alpha^1(x)\frac{\partial}{\partial x^1} + ... + \xi_\alpha^n(x)\frac{\partial}{\partial x^n}, \alpha = 1, 2, ..., r$ and $X_\beta = \xi_\beta^1(x)\frac{\partial}{\partial x^1} + ... + \xi_\beta^n(x)\frac{\partial}{\partial x^n}, \beta = 1, 2, ..., r$ be any two operators then first order differential operator $[X_\alpha, X_\beta]$ is defined by $[X_\alpha, X_\beta] = \sum_{i=1}^n (X_\alpha(\xi_\beta^i) - X_\beta(\xi_\alpha^i))\frac{\partial}{\partial x^i}$.

One parameter group:

A set G of invertible point transformation in the (x,y) plane $\mathbb{R}^2, \bar{x} = f(x, y, a)$, $\bar{y} = g(x, y, a)$ depending on a parameter a is called a one-parameter continuous group, if G contains the identity transformation (for example $f = x$, $g = y,$ at $a = 0$) as well as the inverse of its elements and their composition.

Lie Algebra of operators:

A Lie algebra is a vector space L of operators

$$X_1 = \xi_1^i(x)\frac{\partial}{\partial x^i}, \qquad X_2 = \xi_2^i(x)\frac{\partial}{\partial x^i}$$

having the property $X_1, X_2 \in L$ and their commutator

$$[X_1, X_2] \equiv X_1X_2 - X_2X_1 = (X_1(\xi_2^i) - X_2(\xi_1^i))\frac{\partial}{\partial x^i} \in L.$$

Dimension of Lie Algebra:

The dimension dim L of the Lie algebra is the dimension of the vector space L. We use the symbol L_r to denote an r-dimensional Lie algebra.

Linear Combination:

Suppose we let $X \in L_r$ be any operator and C^α be any constant then $X = C^\alpha X_\alpha$ is called linear combination.

Basis of the Vector Space:

Let L_r be a finite dimension Lie algebra and suppose $X_\alpha = \xi^i_\alpha(x)\frac{\partial}{\partial x^i}, \alpha = 1, ..., r$ be a basis of the vector space L_r. In particular $[X_\alpha, X_\beta] \in L_r$ hence $[X_\alpha, X_\beta] = C^r_{\alpha\beta} X_r, \alpha, \beta = 1, ..., r$. The constant coefficients $C^r_{\alpha\beta}$ are called the structure constants of the algebra L_r.

Linear Span:

Suppose we have operators $X_1, X_2, ..., X_s$ then their linear span is denoted by $< X_1, ..., X_s >$. For example, Given L_r with basis $X_\alpha = \xi^i_\alpha(x)\frac{\partial}{\partial x^i}, \alpha = 1, 2, ..., r$ denoted by $L_\alpha =< X_1, ..., X_r >$.

Subalgebra:

Suppose that L be a Lie algebra, a subspace $K \subset L$ of a vector space L is called a subalgebra of the Lie algebra L if K is closed under multiplication $[K, K] \subset K$. A Lie algebra $L_r, r = 1$ contains one dimensional subalgebra whilst $L_r, r = 2$ contains a two dimensional subalgebra.

Ideal:

The subalgebra K is called an ideal of L if $[K, L] \subset K$.

Lie Equation:

Let us consider the group transformation $\bar{x}^i = f^i(x, a), i = 1, ..., n$ in n-dimensional space $x = (x^1, ..., x^n)$ with generator $X = \xi^i(x)\frac{\partial}{\partial x^i}$ where $\xi^i(x) = \frac{\partial f^i(x,a)}{\partial a} |_{a=0}$ is defined by integrating the following general ordinary differential equation called Lie equation $\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}), \bar{x}^i |_{a=0}= x^i$.

Symmetry Group:

Mathematical objects such as (functions, differential equations, surfaces, etc) are closely related to the concept of group as well as invariance and symmetry. Let G be set of invertible transformations T, then any given object M on which T acts on does not change. That is $T : M \to M$. Mathematically,

$T : M \to M$ contains the following transformation,
(i). Identity I
(ii). Inverse T^{-1}
(iii). Product $T_1 T_2$
where $T_1, T_2 \in G$ then G is called a group or more precisely a symmetry group of the object M.

Airy Equation:

The linear equation $\frac{d^2y}{dx^2} - xy = 0$ is called the Airy equation.

1.2 Black-Scholes equation

We define the conceptual definition of Black-Scholes equation [1]. Conceptually, we can define the term Black-Scholes Equation in three related fields. These are Black-Scholes model, Black-Scholes partial differential equation and Black-Scholes formula.

(a) The Black-Scholes model is defined as a mathematical model of the market for an equity, in which the equity's price is a stochastic process.

(b) The Black-Scholes partial differential equation (PDE) is an equation which (in the model) must be satisfied by the price of a derivative on the equity.

(c) The Black-Scholes formula on the other hand is the result obtained by solving the Black-Scholes PDE for an European call option.

Option as used in Black-Scholes equation definition is the right, but not the obligation, to buy (for a call option) or sell (for a put option) a specific amount of a given stock, commodity, currency, index, or debt, at a specified price (the strike price) during a specified period of time whilst Equity is defined as ownership interest in a corporation in the form of common stock or preferred stock. It also refers to total assets minus total liabilities, in which case it is also referred to as shareholder's equity or net worth or book value.

The Black-Scholes equation is therefore defined as an equation for option securities prices on the basis of an assumed stochastic process for stock prices.An option on the other hand is a contract which gives the holder of the contract the right to buy or sell a commodity or financial asset for a given price before a specified date. The equations is given in symbolic form as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Or

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0$$

where
S = the price of the stock,
$V(S,t)$ = the price of a derivative as a function of time and stock price,
$C(S,t)$ = the price of a European call and P(S,t) the price of a European put option,
K = the strike of the option,
r = the annualized risk-free interest rate, continuously compounded,
μ = the drift rate of S, annualized,
σ = the volatility of the stock; this is the square root of the quadratic variation of the stock's log price process,
t = a time in years; we generally use now = 0, expiry = T,
R = the accumulated profit or loss following a delta-hedging trading strategy.

1.3 Main goal

The main goal of this thesis is to find an optimal system of invariant solutions to the Black-Scholes equation [1].

The problem of group classification of partial differential equations according to their symmetries was first considered by Sophus Lie (see, e.g. [2]) and further developed by L. V. Ovsyannikov (see, [3], [4]). The general approach to finding invariant solutions of differential equations can be found, for example, in [3], [4], [6], [7], [8].

2 Optimal system of invariant solution

We are to construct the optimal system of one-dimensional subalgebras of L_6 and their corresponding invariant solutions using Black-Scholes equation. The Black-Scholes equation has the form

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0, \tag{1}$$

where A, B, C are constants.

2.1 Symmetries of Black Scholes equation

The equation (1) admits the following six operators or symmetries Gazizov,

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t},\ X_2 = x\frac{\partial}{\partial x},\ X_3 = 2t\frac{\partial}{\partial t} + (\ln x + Pt)x\frac{\partial}{\partial x} + 2Ctu\frac{\partial}{\partial u}, \\ X_4 &= A^2tx\frac{\partial}{\partial x} + (\ln x - Pt)u\frac{\partial}{\partial u},\ X_5 = 2A^2t^2\frac{\partial}{\partial t} + 2A^2tx\ln x\frac{\partial}{\partial x} \\ &\quad + [(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u\frac{\partial}{\partial u},\ X_6 = u\frac{\partial}{\partial u} \end{aligned} \tag{2}$$

where $P = B - \frac{A^2}{2}$.

2.2 Calculation of commutators

We make use of the formula shown below in calculating the commutators

$$\begin{aligned} [X_\alpha, X_\beta] &= X_\alpha X_\beta - X_\beta X_\alpha \\ &= \sum_{i=1}^{n}(X_\alpha(\xi^i_\beta) - X_\beta(\xi^i_\alpha))\frac{\partial}{\partial x^i} \end{aligned} \tag{3}$$

where i = 1,2,3 and α,β = 1,2...,6.
As a demonstration, we will show how to calculate the commutator for the case $[X_1, X_5]$ using equation (3).

Case $[X_1, X_5]$:
Symbolical representation of this case is given as

$$\begin{aligned} [X_1, X_5] &= (X_1(\xi^1_5) - X_5(\xi^1_1))\frac{\partial}{\partial t} + (X_1(\xi^2_5) - X_5(\xi^2_1))\frac{\partial}{\partial x} \\ &\quad + (X_1(\xi^3_5) - X_5(\xi^3_1))\frac{\partial}{\partial u}. \end{aligned} \tag{4}$$

From equation (2) we have,
$\xi_1^1 = 1$, $\xi_1^2 = 0$, $\xi_1^3 = 0$,
$\xi_5^1 = 2A^2t^2$, $\xi_5^2 = 2A^2tx\ln x$ and $\xi_5^3 = [(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u\cdot$
This leads to the following results,

$$\begin{aligned}
[X_1, X_5] &= (X_1(2A^2t^2) - X_5(1))\frac{\partial}{\partial t} \\
&\quad + (X_1(2A^2txlnx) - X_5(0))\frac{\partial}{\partial x} \\
&\quad + (X_1(((\ln x - Pt)^2 + 2A^2Ct^2 - A^2t)u) - X_5(0))\frac{\partial}{\partial u} \\
&= 4A^2t\frac{\partial}{\partial t} + 2A^2x\ln x\frac{\partial}{\partial x} \\
&\quad + (2(\ln x - Pt)(-P) + 4A^2Ct - A^2)u\frac{\partial}{\partial u} \\
&= 2A^2(2t)\frac{\partial}{\partial t} + 2A^2(\ln x + Pt - Pt)x\frac{\partial}{\partial x} + 2A^2(2Ctu)\frac{\partial}{\partial u} \\
&\quad - A^2u\frac{\partial}{\partial u} - 2P(\ln x - Pt)u\frac{\partial}{\partial u} \\
&= 2A^2[2t\frac{\partial}{\partial t} + (\ln x + Pt)x\frac{\partial}{\partial x} + 2Ctu\frac{\partial}{\partial u}] \\
&\quad - A^2u\frac{\partial}{\partial u} + (-2P)[(A^2t)x\frac{\partial}{\partial x} + (\ln x - Pt)u\frac{\partial}{\partial u}].
\end{aligned}$$

From the above equation, we get

$$[X_1, X_5] = 2A^2X_3 - A^2X_6 - 2PX_4\cdot \tag{5}$$

The commutators of the rest of the symmetries (2) of the Black-Scholes equation upon calculation is put in a tabular form as follows

2.3 Commutator table

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	$2X_1 + PX_2 + 2CX_6$	$A^2X_2 - PX_6$	$2A^2X_3 - 2PX_4 - A^2X_6$	0
X_2	0	0	X_2	X_6	$2X_4$	0
X_3	$-2X_1 - PX_2 - 2CX_6$	$-X_2$	0	X_4	$2X_5$	0
X_4	$-A^2X_2+PX_6$	$-X_6$	$-X_4$	0	0	0
X_5	$-2A^2X_3 + 2PX_4 + A^2X_6$	$-2X_4$	$-2X_5$	0	0	0
X_6	0	0	0	0	0	0

(6)

3 Construction of an optimal system of one dimensional subalgebras

The Lie algebra L_6 spanned the given symmetries or operators (2) and it provides a possibility to find invariant solutions of the Black-Scholes equation (1) which is based on any one-dimensional subalgebra of L_6. Again it is clear that there are infinite number of one-dimensional subalgebras of L_6. In view of these we can write an arbitrary operator from L_6 as

$$X = l^1 X_1 + \ldots + l^6 X_6, \tag{7}$$

which depends on the six arbitrary constants $l^1, \ldots, l^6$.
To construct an optimal system of one-dimensional subalgebras of the Lie algebra L_6 we follow the simple method used by Ibragimov [8] and Ovsyannikov [3], (see also [4]). The symmetry group with Lie algebra L_6 when transformed provides a 6-parameter group of linear transformations of the operators

$$l = (l^1, \ldots, l^6), \tag{8}$$

where $l^1, \ldots, l^6$ are taken from the arbitrary operator given by equation (7).

3.1 Construction of linear transformation

To find the linear transformations we will make use of their generators (see [4]) which is given as

$$E_\mu = c^\lambda_{\mu\nu} l^\nu \frac{\partial}{\partial l^\lambda}, \qquad \mu = 1, \ldots, 6, \tag{9}$$

and the structure constants of the Lie algebra L_6 which is given by $c^\lambda_{\mu\nu}$ is defined as a commutator by

$$[X_\mu, X_\nu] = c^\lambda_{\mu\nu} X_\lambda \cdot \tag{10}$$

Case 1: For $\mu = 1$, we have

$$E_1 = c^\lambda_{1\nu} l^\nu \frac{\partial}{\partial l^\lambda}, \qquad [X_1, X_\nu] = c^\lambda_{1\nu} X_\lambda. \tag{11}$$

Setting $\nu = 3$ and $\lambda = 1, 2, 6$, that is row 1, column 3 from equation (6) we have

$$[X_1, X_3] = c^1_{13} X_1 + c^2_{13} X_2 + c^6_{13} X_6. \tag{12}$$

Hence the non-vanishing structure constants ($c^\lambda_{\mu\nu}$) are

$$c^1_{13} = 2, \qquad c^2_{13} = P, \qquad c^6_{13} = 2C.$$

Again setting $\nu = 4$ and $\lambda = 2, 6$, that is row 1, column 4 of equation (6) we have

$$[X_1, X_4] = c^2_{14} X_2 + c^6_{14} X_6. \tag{13}$$

Hence the non-vanishing structure constants are

$$c^2_{14} = A^2, \qquad c^6_{14} = -P.$$

From row 1, column 5 of (6) we set $\nu = 5$ and $\lambda = 3, 4, 6$. This yields

$$[X_1, X_5] = c^3_{15} X_3 + c^4_{15} X_4 + c^6_{15} X_6 \tag{14}$$

Hence the non-vanishing structure constants are

$$c^3_{15} = 2A^2, \qquad c^4_{15} = -2P, \qquad c^6_{15} = -A^2.$$

From equation (11) we have

$$\begin{aligned} E_1 &= c^\lambda_{13} l^3 \frac{\partial}{\partial l^\lambda} + c^\lambda_{14} l^4 \frac{\partial}{\partial l^\lambda} + c^\lambda_{15} l^5 \frac{\partial}{\partial l^\lambda} \\ &= c^1_{13} l^3 \frac{\partial}{\partial l^1} + c^2_{13} l^3 \frac{\partial}{\partial l^2} + c^6_{13} l^3 \frac{\partial}{\partial l^6} + c^2_{14} l^4 \frac{\partial}{\partial l^2} \\ &\quad + c^6_{14} l^4 \frac{\partial}{\partial l^6} + c^3_{15} l^5 \frac{\partial}{\partial l^3} + c^4_{15} l^5 \frac{\partial}{\partial l^4} + c^6_{15} l^5 \frac{\partial}{\partial l^6}. \end{aligned} \tag{15}$$

Setting in the values of the non-vanishing structure constants we have

$$\begin{aligned} E_1 &= 2l^3 \frac{\partial}{\partial l^1} + (Pl^3 + A^2 l^4) \frac{\partial}{\partial l^2} + 2A^2 l^5 \frac{\partial}{\partial l^3} \\ &\quad - 2Pl^5 \frac{\partial}{\partial l^4} + (2Cl^3 - Pl^4 - A^2 l^5) \frac{\partial}{\partial l^6}. \end{aligned} \tag{16}$$

Case 2: For $\mu = 2$, we have

$$E_2 = c^\lambda_{2\nu} l^\nu \frac{\partial}{\partial l^\lambda}, \qquad [X_2, X_\nu] = c^\lambda_{2\nu} X_\lambda. \tag{17}$$

Setting $\nu = 3$ and $\lambda = 2$, that is row 2, column 3 from (6) we have

$$[X_2, X_3] = c^2_{23} X_2, \qquad c^2_{23} = 1. \tag{18}$$

Again setting $\nu = 4$ and $\lambda = 6$, that is row 2, column 4 from (6) we have

$$[X_2, X_4] = c^6_{24} X_6, \qquad c^6_{24} = 1. \tag{19}$$

From row 2, column 5 of (6) we set $\nu = 5$ and $\lambda = 4$. This yields

$$[X_2, X_5] = c^4_{25} X_4, \qquad c^4_{25} = 2. \tag{20}$$

Hence equation (17) yields

$$E_2 = l^3 \frac{\partial}{\partial l^2} + 2l^5 \frac{\partial}{\partial l^4} + l^4 \frac{\partial}{\partial l^6}. \tag{21}$$

Case 3: For $\mu = 3$, we have

$$E_3 = c^\lambda_{3\nu} l^\nu \frac{\partial}{\partial l^\lambda}, \qquad [X_3, X_\nu] = c^\lambda_{3\nu} X_\lambda. \tag{22}$$

Setting $\nu = 1$ and $\lambda = 1, 2, 6$, that is row 3, column 1 from (6) we have

$$[X_3, X_1] = c^1_{31} X_1 + c^2_{31} X_2 + c^6_{31} X_6 \tag{23}$$

Hence the non-vanishing structure constants are

$$c^1_{31} = -2, \qquad c^2_{31} = -P, \qquad c^6_{31} = -2C.$$

Again setting $\nu = 2$ and $\lambda = 2$, that is row 3, column 2 from (6) we have

$$[X_3, X_2] = c^2_{32} X_2, \qquad c^2_{32} = -1. \tag{24}$$

Again from row 3, column 4 of (6) we set $\nu = 4$ and $\lambda = 4$. This yields

$$[X_3, X_4] = c^4_{34} X_4, \qquad c^4_{34} = 1. \tag{25}$$

Again from row 3, column 5 of (6) we set $\nu = 5$ and $\lambda = 5$. This yields

$$[X_3, X_5] = c^5_{35} X_5, \qquad c^5_{35} = 2 \tag{26}$$

Therefore, equation (22) becomes

$$\begin{aligned} E_3 &= -2l^1 \frac{\partial}{\partial l^1} - Pl^1 \frac{\partial}{\partial l^2} - 2Cl^1 \frac{\partial}{\partial l^6} \\ &\quad - l^2 \frac{\partial}{\partial l^2} + l^4 \frac{\partial}{\partial l^4} + 2l^5 \frac{\partial}{\partial l^5} \\ &= -2l^1 \frac{\partial}{\partial l^1} - (Pl^1 + l^2) \frac{\partial}{\partial l^2} + l^4 \frac{\partial}{\partial l^4} \\ &\quad + 2l^5 \frac{\partial}{\partial l^5} - 2Cl^1 \frac{\partial}{\partial l^6}. \end{aligned} \tag{27}$$

Case 4: For $\mu = 4$, we have

$$E_4 = c^{\lambda}_{4\nu} l^{\nu} \frac{\partial}{\partial l^{\lambda}}, \qquad [X_4, X_{\nu}] = c^{\lambda}_{4\nu} X_{\lambda}\cdot \tag{28}$$

Setting $\nu = 1$ and $\lambda = 2, 6$, that is row 4, column 1 from (6) we have

$$[X_4, X_1] = c^2_{41} X_2 + c^6_{41} X_6, \qquad c^2_{41} = -A^2, \qquad c^6_{41} = P. \tag{29}$$

Again setting $\nu = 2$ and $\lambda = 6$ that is row 4, column 2 from (6) we have

$$[X_4, X_2] = c^6_{42} X_6, \qquad c^6_{42} = -1. \tag{30}$$

Again from row 4, column 3 of (6) we set $\nu = 3$ and $\lambda = 4$. This yields

$$[X_4, X_3] = c^4_{43} X_4, \qquad c^4_{43} = -1 \tag{31}$$

Hence, equation (28) becomes

$$\begin{aligned} E_4 &= -A^2 l^1 \frac{\partial}{\partial l^2} + P l^1 \frac{\partial}{\partial l^6} - l^2 \frac{\partial}{\partial l^6} - l^3 \frac{\partial}{\partial l^4} \\ &= -A^2 l^1 \frac{\partial}{\partial l^2} - l^3 \frac{\partial}{\partial l^4} + (P l^1 - l^2) \frac{\partial}{\partial l^6}\cdot \end{aligned} \tag{32}$$

Case 5: For $\mu = 5$, we have

$$E_5 = c^{\lambda}_{5\nu} l^{\nu} \frac{\partial}{\partial l^{\lambda}}, \qquad [X_5, X_{\nu}] = c^{\lambda}_{5\nu} X_{\lambda}\cdot \tag{33}$$

Again setting $\nu = 1$ and $\lambda = 3, 4, 6$, that is row 5, column 1 from (6) we have

$$[X_5, X_1] = c^3_{51} X_3 + c^4_{51} X_4 + c^6_{51} X_6 \tag{34}$$

Hence the non-vanishing structure constants are

$$c^3_{51} = -2A^2, \qquad c^4_{51} = 2P, \qquad c^6_{51} = A^2.$$

Again setting $\nu = 2$ and $\lambda = 4$ that is row 5, column 2 from (6) we have

$$[X_5, X_2] = c^4_{52} X_4, \qquad c^4_{52} = -2. \tag{35}$$

Again from row 5, column 3 of (6) we set $\nu = 3$ and $\lambda = 5$. This yields

$$[X_5, X_3] = c^5_{53} X_5 \qquad c^4_{43} = -2. \tag{36}$$

Therefore from equation (33) we have

$$\begin{aligned} E_5 &= -2A^2l^1\frac{\partial}{\partial l^3} + 2Pl^1\frac{\partial}{\partial l^4} + A^2l^1\frac{\partial}{\partial l^6} - 2l^2\frac{\partial}{\partial l^4} - 2l^3\frac{\partial}{\partial l^5} \\ &= -2A^2l^1\frac{\partial}{\partial l^3} + 2(Pl^1 - l^2)\frac{\partial}{\partial l^4} - 2l^3\frac{\partial}{\partial l^5} + A^2l^1\frac{\partial}{\partial l^6}. \end{aligned} \tag{37}$$

In summary, from equation (16), (21), (27), (32) and (37) we have the following linear transformations

$$\begin{aligned} E_1 &= 2l^3\frac{\partial}{\partial l^1} + (Pl^3 + A^2l^4)\frac{\partial}{\partial l^2} + 2A^2l^5\frac{\partial}{\partial l^3} \\ &\quad - 2Pl^5\frac{\partial}{\partial l^4} + (2Cl^3 - Pl^4 - A^2l^5)\frac{\partial}{\partial l^6} \\ E_2 &= l^3\frac{\partial}{\partial l^2} + 2l^5\frac{\partial}{\partial l^4} + l^4\frac{\partial}{\partial l^6} \\ E_3 &= -2l^1\frac{\partial}{\partial l^1} - (Pl^1 + l^2)\frac{\partial}{\partial l^2} + l^4\frac{\partial}{\partial l^4} + 2l^5\frac{\partial}{\partial l^5} - 2Cl^1\frac{\partial}{\partial l^6} \\ E_4 &= -A^2l^1\frac{\partial}{\partial l^2} - l^3\frac{\partial}{\partial l^4} + (Pl^1 - l^2)\frac{\partial}{\partial l^6} \\ E_5 &= -2A^2l^1\frac{\partial}{\partial l^3} + 2(Pl^1 - l^2)\frac{\partial}{\partial l^4} - 2l^3\frac{\partial}{\partial l^5} + A^2l^1\frac{\partial}{\partial l^6}. \end{aligned} \tag{38}$$

3.2 Construction of Lie equations

We now find the transformations provided by the generators (38). For the generator E_1, the Lie equations with the parameter a_1 are written

$$\begin{aligned} &\frac{d\tilde{l}^1}{da_1} = 2\tilde{l}^3, \qquad \frac{d\tilde{l}^2}{da_1} = (P\tilde{l}^3 + A^2\tilde{l}^4), \qquad \frac{d\tilde{l}^3}{da_1} = 2A^2\tilde{l}^5, \\ &\frac{d\tilde{l}^4}{da_1} = -2P\tilde{l}^5, \quad \frac{d\tilde{l}^5}{da_1} = 0, \quad \frac{d\tilde{l}^6}{da_1} = (2C\tilde{l}^3 - P\tilde{l}^4 - A^2\tilde{l}^5). \end{aligned} \tag{39}$$

We integrate all the six equations of (39) and use the initial condition $\tilde{l}\mid_{a_1=0}= l$. We then proceed as follows:

$$\tfrac{d\tilde{l}^5}{da_1} = 0, \qquad d\tilde{l}^5 = 0, \qquad \tilde{l}^5 = l^5$$

Again $d\tilde{l}^3 = 2A^2\tilde{l}^5 da_1$. Setting the value of $\tilde{l}^5 = l^5$ and integrating it we get $\tilde{l}^3 = 2A^2l^5a_1 + C_1$. Using the initial condition, we have $l^3 = C_1$. Hence

$$\tilde{l}^3 = 2A^2l^5a_1 + l^3. \tag{40}$$

Furthermore, we have $d\tilde{l}^4 = -2P\tilde{l}^5 da_1$ and setting in the equation $\tilde{l}^5 = l^5$ and integrating it we obtain $\tilde{l}^4 = -2Pl^5 a_1 + C_2$. Using the initial condition, we have $l^4 = C_2$. Hence

$$\tilde{l}^4 = -2Pl^5 a_1 + l^4. \tag{41}$$

Furthermore, from equation (39) we have $d\tilde{l}^1 = 2\tilde{l}^3 da_1$. Putting the value for $\tilde{l}^3$ and integrating it yields, $\tilde{l}^1 = 2A^2 l^5 a_1^2 + 2a_1 l^3 + C_3$. From the initial condition once again we have $C_3 = l^1$. Hence

$$\tilde{l}^1 = 2A^2 l^5 a_1^2 + 2a_1 l^3 + l^1. \tag{42}$$

Again we have $d\tilde{l}^2 = (P\tilde{l}^3 + A^2\tilde{l}^4)da_1$ and substituting in the values of $\tilde{l}^3$ and $\tilde{l}^4$ we get $d\tilde{l}^2 = Pl^3 da_1 + A^2 l^4 da_1$. Integrating this equation and applying the initial condition we get$\tilde{l}^2 = Pl^3 a_1 + A^2 l^4 a_1 + C_4$ and $C_4 = l^2$. Hence

$$\tilde{l}^2 = (Pl^3 + A^2 l^4)a_1 + l^2. \tag{43}$$

Finally from (39) we have $d\tilde{l}^6 = (2C\tilde{l}^3 - P\tilde{l}^4 - A^2\tilde{l}^5)da_1$ setting in $\tilde{l}^3$, $\tilde{l}^4$ and $\tilde{l}^5$ we obtain $d\tilde{l}^6 = (4CA^2 l^5 a_1)da_1 + 2Cl^3 + 2P^2 l^5 a_1 - Pl^4 - A^2 l^5)da_1$. Integrating and applying the initial condition, we get $\tilde{l}^6 = 2A^2 C l^5 a_1^2 + 2Cl^3 a_1 + P^2 l^5 a_1^2 - Pl^4 a_1 - A^2 l^5 a_1 + C_5$ where $C_5 = l^6$. Hence

$$\tilde{l}^6 = l^5(2A^2C + P^2)a_1^2 + (2Cl^3 - Pl^4 - A^2 l^5)a_1 + l^6. \tag{44}$$

Therefore for E_1 we have the following transformations

$$\begin{aligned} E_1 : \tilde{l}^1 &= 2A^2 l^5 a_1^2 + 2a_1 l^3 + l^1, \qquad \tilde{l}^2 = (Pl^3 + A^2 l^4)a_1 + l^2, \\ \tilde{l}^3 &= 2A^2 l^5 a_1 + l^3, \qquad \tilde{l}^4 = -2Pl^5 a_1 + l^4, \qquad \tilde{l}^5 = l^5, \\ \tilde{l}^6 &= l^5(2A^2C + P^2)a_1^2 + (2Cl^3 - Pl^4 - A^2 l^5)a_1 + l^6. \end{aligned} \tag{45}$$

Considering the other operators from (38) we obtain the following transformations:

$$\begin{aligned} E_2 : \tilde{l}^1 &= l^1, \qquad \tilde{l}^2 = l^3 a_2 + l^2, \qquad \tilde{l}^3 = l^3, \qquad \tilde{l}^5 = l^5, \\ \tilde{l}^4 &= 2l^5 a_2 + l^4, \qquad \tilde{l}^6 = l^5 a_2^2 + l^4 a_2 + l^6. \end{aligned} \tag{46}$$

$$\begin{aligned} E_3 : \tilde{l}^1 &= l^1 a_3^{-2}, \qquad \tilde{l}^2 = Pl^1 a_3^{-2} + (l^2 - Pl^1)a_3^{-1}, \qquad \tilde{l}^3 = l^3, \\ \tilde{l}^4 &= l^4 a_3, \qquad \tilde{l}^5 = l^5 a_3^2, \qquad \tilde{l}^6 = Cl^1(a_3^{-2} - 1) + l^6. \end{aligned} \tag{47}$$

$$\begin{aligned} E_4 : \tilde{l}^1 &= l^1, \qquad \tilde{l}^2 = -A^2 l^1 a_4 + l^2, \qquad \tilde{l}^3 = l^3, \qquad \tilde{l}^5 = l^5, \\ \tilde{l}^4 &= -l^3 a_4 + l^4, \qquad \tilde{l}^6 = \frac{1}{2}A^2 l^1 a_4^2 + (Pl^1 - l^2)a_4 + l^6. \end{aligned} \tag{48}$$

$$
\begin{aligned}
E_5 : \tilde{l}^1 = l^1, \quad & \tilde{l}^3 = -2A^2 l^1 a_5 + l^3, \quad \tilde{l}^5 = 2A^2 l^1 a_5^2 - 2l^3 a_5 + l^5, \\
\tilde{l}^2 = l^2, \quad & \tilde{l}^4 = 2(Pl^1 - l^2)a_5 + l^4, \quad \tilde{l}^6 = A^2 l^1 a_5 + l^6.
\end{aligned} \tag{49}
$$

$$
E_6 : \tilde{l}^1 = l^1, \quad \tilde{l}^2 = l^2, \quad \tilde{l}^3 = l^3, \quad \tilde{l}^4 = l^4, \quad \tilde{l}^5 = l^6, \quad \tilde{l}^2 = l^2. \tag{50}
$$

It is worthy to note that the transformations (45) - (50) map the vector $X \in L_6$ given by (7) to the vector $\tilde{X} \in L_6$ given by the following formula

$$
\tilde{X} = \tilde{l}^1 X_1 + \ldots + \tilde{l}^6 X_6. \tag{51}
$$

3.3 One functionally invariant

We will prove the transformations (45) - (50) having invariant $J(l^1, \ldots, l^6)$. The reckoning shows that the 6×6 matrix $\|c^{\lambda}_{\mu\nu} l^{\nu}\|$ of the coefficients of the operators (38) has the rank five. It means that the transformations (45) - (50) have precisely one functionally invariant. The integration of the equations

$$
E_{\mu}(J) = 0, \qquad \mu = 1, \ldots, 6,
$$

shows that the invariant is

$$
J = (l^3)^2 - 2A^2 l^1 l^5. \tag{52}
$$

The invariant J (52) simplifies further calculations immediately.Solving the quadratic equation $2A^2 l^1 a_5^2 - 2l^3 a_5 + l^5 = 0$ from E_5 for a_5 we obtain

$$
a_5 = \frac{l^3 \pm \sqrt{J}}{2A^2 l^1}. \tag{53}
$$

3.4 Cases

We will now begin to construct the optimal system.
(**1**). **The case** $l^1 = 0$:
We will bifurcate this case into the following subcases namely, (a) $l^5 \neq 0$, (b) $l^5 = 0$,

(**a**). First we discuss the case when: $l^1 = 0, \quad l^5 \neq 0$,
Considering the vectors (8), we take the form

$$
(0, l^2, l^3, l^4, l^5, l^6), \qquad \text{where} \quad l^5 \neq 0.
$$

Using l^5 we reduce the vector given above. From equation (45) we have $2A^2l^5a_1 + l^3 = 0$ therefore $a_1 = -\frac{l^3}{2A^2l^5}$. Since $l^5 \neq 0$ then $l^3 = 0$.The above vector therefore reduces to the form

$$(0, l^2, 0, l^4, l^5, l^6).$$

We use the transformation (46) $2l^5a_2 + l^4 = 0$ hence $a_2 = \frac{-l^4}{2l^5}$ which yields $l^4 = 0$. This reduce the former vector to

$$(0, l^2, 0, 0, l^5, l^6).$$

(**a**)(i).If $l^2 \neq 0$ then considering (48) we obtain $\frac{1}{2}A^2l^1a_4^2 + (Pl^1 - l^2)a_4 + l^6 = 0$. This yields $a_4 = \frac{l^6}{l^2}$ hence $l^6 = 0$.We can make $l^5 = \pm 1$ using (47). We therefore obtain the following representative for the optimal system.

$$X_2 + X_5 \qquad \text{and} \qquad X_2 - X_5. \tag{54}$$

(**a**)(ii). If we assume that $l^2 = 0$ then our reduced vector becomes

$$(0, 0, 0, 0, l^5, l^6).$$

If we consider $l^6 \neq 0$ and dividing through by l^5 we have the reduced vector

$$(0, 0, 0, 0, 1, k).$$

If $l^6 = 0$, we obtain a new reduced vectors given by

$$(0, 0, 0, 0, l^5, 0).$$

This produces the following representative for the optimal system

$$X_5 + kX_6 \qquad \text{and} \qquad X_5. \tag{55}$$

(**b**). Secondly, considering the case $l^1 = 0$, and $l^5 = 0$, we will have the vector

$$(0, l^2, l^3, l^4, 0, l^6).$$

(**b**)(i). Suppose that $l^3 \neq 0$ then from equation (46) we have $l^3a_2 + l^2 = 0$ hence $a_2 = \frac{-l^2}{l^3}$, which yields $l^2 = 0$. Again from equation (48) we have $l^4 - l^3a_4 = 0$ hence $a_4 = \frac{l^4}{l^3}$ which yields $l^4 = 0$. Furthermore, from (45) we have $a_1 = -\frac{l^6}{2Cl^3}$ which yields $l^6 = 0$. Hence our new vector becomes

$$(0, 0, l^3, 0, 0, 0).$$

From the above vector we obtain another representative for the optimal system as

$$X_3. \tag{56}$$

(**b**)(ii). Now we consider the case when $l^3 = 0$:
If $l^3 = 0$, we consider the vector of the form

$$(0, l^2, 0, l^4, 0, l^6).$$

Base on the above vector, we consider that if $l^4 \neq 0$, then using the transformation (46) we have $l^5 a_2^2 + l^4 a_2 + l^6 = 0$ and since $l^5 = 0$, we obtain $a_2 = -\frac{l^6}{l^4}$. Since $l^4 \neq 0$, it implies that $l^6 = 0$. Again from transformation (45) we have $(Pl^3 + A^2 l^4)a_1 + l^2 = 0$. But $l^3 = 0$, therefore $a_1 = -\frac{l^2}{A^2 l^4}$. Since $l^4 \neq 0$ it means that $l^2 = 0$. Hence we obtain a new reduce vector

$$(0, 0, 0, l^4, 0, 0).$$

The vector above provides the optimal system representative is given by

$$X_4. \tag{57}$$

We will now consider the case when $l^4 = 0$. we have

$$(0, l^2, 0, 0, 0, l^6).$$

(**b**)(ii°).if $l^2 \neq 0$, then from equation (48) we have $\frac{1}{2}A^2 l^1 a_4^2 + (Pl^1 - l^2)a_4 + l^6 = 0$ which yields $a_4 = \frac{l^6}{l^2}$. Since $l^2 \neq 0$, the reckoning shows that $l^6 = 0$.Thus,taking into account the possibility that $l^2 = 0$, hence the new reduced vectors are

$$(0, l^2, 0, 0, 0, 0), \qquad (0, 0, 0, 0, 0, l^6)$$

which yields a representative for the optimal system given by

$$X_2, \qquad X_6. \tag{58}$$

($\mathbf{2}$). **The case** $l^1 \neq 0, J < 0$:
One can clearly see from the condition that $J = (l^3)^2 - 2A^2l^1l^5$ where $(l^3)^2 - 2A^2l^1l^5 < 0$ that $l^5 \neq 0$. We can therefore apply the transformations (49) , (46), (45) and (48) respectively. We have $a_5 = -\frac{l^6}{A^2l^1}$ and since $l^1 \neq 0$ we get $l^6 = 0$. Again, $2l^5a_2 + l^4 = 0$ hence $a_2 = -\frac{l^4}{2l^5}$ and with $l^5 \neq 0$ it implies that $l^4 = 0$. Furthermore, from $2A^2l^5a_1 + l^3 = 0$ we have $a_1 = -\frac{l^3}{2A^2l^5}$ which yields $l^3 = 0$ since $l^5 \neq 0$. Finally, from transformation (48) we obtain $a_4 = \frac{l^2}{A^2l^1}$ and since $l^1 \neq 0$ it shows that $l^2 = 0$. These results $l^2 = l^3 = l^4 = l^6 = 0$ yields the reduced vector

$$(l^1, 0, 0, 0, l^5, 0).$$

The components l^1 and l^5 of the above vector have a common sign since the condition $J < 0$ yields $l^1l^5 > 0$. Therefore using transformation (47) with an appropriate value of the parameter a_3 and invoking that we can multiply the vector l by any constant, we obtain $l^1 = l^5 = 1$ which yields the representative for the optimal system as

$$X_1 + X_5. \tag{59}$$

($\mathbf{3}$). **The case** $l^1 \neq 0, J = 0$:
For this condition,we use equation (53), $a_5 = \frac{l^3}{2A^2l^1}$. If $l^3 \neq 0$ then we can use equation (49) with transformation $a_5 = \frac{l^3}{2A^2l^1}$ and get $\bar{l^3} = 0$. Due to the invariance J we conclude that the equation $J = 0$ yields $(\bar{l^3})^2 - 2A^2\bar{l^1}\bar{l^5} = 0$ since $\bar{l^3} = 0$, it follows that $\bar{l^5} = 0$.Thus we can deal with the vector of the form

$$(l^1, l^2, 0, l^4, 0, l^6). \tag{60}$$

Furthermore , if $l^3 = 0$, we have $J = -2A^2l^1l^5$, and equation $J = 0$ yields $l^5 = 0$, since $l^1 \neq 0$ we get the same vector as (60). Using (48) we have $a_4 = \frac{l^2}{A^2l^1}$ and this yields $l^2 = 0$. Again from (49) we get $2Pl^1a_5 + l^4 = 0$ where $a_5 = -\frac{l^4}{2Pl^1}$ hence $l^4 = 0$. Similarly, from (49) we use $A^2l^1a_5 + l^6 = 0$ which gives $a_5 = -\frac{l^6}{A^2l^1}$ and yields $l^6 = 0$. Taking $l^1 = 1$ the new reduced vector

$$(l^1, 0, 0, 0, 0, 0).$$

Produces representative for the optimal system as

$$X_1. \tag{61}$$

(**4**). **The case** $l^1 \neq 0$**,** $J > 0$:
Since J is an invariant under the transformations (45), (46), (47), (48) and (49) the condition, $J > 0$ yields in the above vector that either $l^3 \neq 0$ or $l^5 \neq 0$.If $l^5 \neq 0$,then l^1 and l^5 should have opposite signs $l^1 l^5 < 0$.
Since l^1 and l^5 having opposite signs for the condition $J > 0$ to hold. Assuming this possibility holds and that $l^1, l^5 \neq 0$, then from (45), (46), (48) and (49). we can deduce that $a_1 = -\frac{l^3}{2A^2 l^5}, a_2 = -\frac{l^4}{2l^5}, a_4 = \frac{l^2}{A^2 l^1}$ and $a_5 = -\frac{l^6}{A^2 l^1}$ respectively. The reckonings show that l^2, l^3, l^4 and l^6 are equal to zero. We therefore have the reduced vector

$$(l^1, 0, 0, 0, l^5, 0)$$

which yields the optimal system representatives

$$X_1 - X_5 \qquad \text{and} \qquad X_5 - X_1. \tag{62}$$

3.5 Optimal system

Finally, the optimal system becomes

$$\begin{array}{ccccc} X_1, & X_2, & X_3, & X_4, & X_5, \\ X_6, & & X_1 + X_5, & & X_1 - X_5, \\ X_2 + X_5, & & X_2 - X_5, & & X_5 + kX_6. \end{array} \tag{63}$$

4 Construction of invariant solutions

For us to construct the optimal system of invariant solutions for the Black-Scholes Equation (1), we have to first of all find the invariant solution for each operator or representative for the optimal system (63). In calculating these invariant solutions for this operators for the optimal system (63) we will consider the positive values of the time t for the sake of simplicity. Readers are to note that the invariance condition for the operator X_6 from the optimal system (63) is $u = u(t)$ and provides a trivial solution that $u =$ constant. We will therefore begin illustrating the construction of invariant solutions with the operator X_1.

The operator

$$X_1 = \frac{\partial}{\partial t}.$$

This operator has the characteristic system

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0}$$

There are two linear equations that can be formed from the above characteristic system. The first equation is

$$\frac{dt}{1} = \frac{dx}{0}$$

Integrating the above equation yields $x = C_1$ where C_1 is constant of integration. Hence, one of the invariants is

$$J_1 = x. \tag{64}$$

Checking of invariant J_1:
Checking for the invariant J_1 we have

$$X_1(J_1) = \frac{\partial J_1}{\partial t} = \frac{\partial x}{\partial t} = 0.$$

This operator satisfies the invariant condition.

Similarly, the integration of the equation

$$\frac{dt}{1} = \frac{du}{0}$$

yields $u = C_2$ where C_2 is the constant of integration. Therefore the second invariant is

$$J_2 = u. \tag{65}$$

Checking of invariant J_2:
Similarly, checking for the invariant J_2 we have

$$X_1(J_2) = \frac{\partial J_2}{\partial t} = \frac{\partial u}{\partial t} = 0$$

which also satisfied the invariant condition. Designating one of the invariants as a function of the other, that is

$$J_2 = \varphi(J_1). \tag{66}$$

we obtain

$$u = \varphi(x). \tag{67}$$

Taking derivative of equation (67) with respect to x and t we obtain

$$u_t = 0, \qquad u_x = \varphi', \qquad u_{xx} = \varphi''$$

Setting the above derivatives into the Black-Scholes equation (1) we obtain

$$\frac{1}{2}A^2x^2\varphi'' + Bx\varphi' - C\varphi = 0.$$

Multiplying both sides by 2 yields

$$A^2x^2\varphi'' + 2Bx\varphi' - 2C\varphi = 0 \tag{68}$$

which is a second order homogeneous equation having variable coefficient and a type of Euler's equation. Using Euler's method we let $x = e^t$ and $t = |lnx|$, hence differentiating φ with respect to x we have

$$\begin{aligned} \frac{d\varphi}{dx} &= \frac{d\varphi}{dt} \cdot \frac{dt}{dx} \\ &= \frac{1}{x}\frac{d\varphi}{dt} \end{aligned}$$

Second derivative of the above result with respect to x we obtain

$$\begin{aligned} \frac{d^2\varphi}{dx^2} &= -\frac{1}{x^2}\frac{d\varphi}{dt} + \frac{1}{x}\frac{d^2\varphi}{dt^2} \cdot \frac{dt}{dx} \\ &= \frac{1}{x^2}\frac{d^2\varphi}{dt^2} - \frac{1}{x^2}\frac{d\varphi}{dt} \\ &= \frac{1}{x^2}\Big(\frac{d^2\varphi}{dt^2} - \frac{d\varphi}{dt}\Big). \end{aligned}$$

Setting the above derivatives in equation (68) we obtain

$$\begin{aligned}(A^2x^2)\frac{1}{x^2}\Big(\frac{d^2\varphi}{dt^2}-\frac{d\varphi}{dt}\Big)+2Bx\Big(\frac{1}{x}\frac{d\varphi}{dt}\Big)-2C\varphi=0\\ A^2\frac{d^2\varphi}{dt^2}+(2B-A^2)\frac{d\varphi}{dt}-2C\varphi=0.\end{aligned}$$

But we know that

$$P=B-\frac{A^2}{2}\qquad\Longrightarrow\qquad 2P=2B-A^2.$$

Setting in the above equation we have

$$A^2\frac{d^2\varphi}{dt^2}+2P\frac{d\varphi}{dt}-2C\varphi=0.$$

The differential equation above reduces to the algebraic equation

$$A^2\lambda^2+2P\lambda-2C=0$$

which is also known as characteristic equation. Solving this characteristic equation yields two distinct real solutions

$$\lambda_1=-\frac{P+\sqrt{P^2+2A^2C}}{A^2},\qquad \lambda_2=-\frac{P-\sqrt{P^2+2A^2C}}{A^2} \tag{69}$$

These yields two linearly independent particular solution solution:

$$\varphi_1(t)=e^{\lambda_1 t},\qquad \varphi_2(t)=e^{\lambda_2 t}$$

which gives a general solution

$$\varphi=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}\qquad\Longrightarrow\qquad \varphi=C_1x^{\lambda_1}+C_2x^{\lambda_2} \tag{70}$$

when we put in $t=|\ln x|$. Setting equation (70) in equation (67) we obtain the solution

$$u=C_1x^{\lambda_1}+C_2x^{\lambda_2} \tag{71}$$

where C_1, C_2 are constants.

The next operator we will like to discuss in constructing the invariant solutions is

$$X_2=x\frac{\partial}{\partial x}.$$

This operator also has the characteristic system

$$\frac{dt}{0} = \frac{dx}{x} = \frac{du}{0}.$$

The first linear equation that can be formed from the characteristic system above is

$$\frac{dt}{0} = \frac{dx}{x}$$

Again integrating the above equation yields $t = C_1$ where C_1 is constant of integration. Therefore we obtain one of the invariants to be

$$J_1 = t. \tag{72}$$

Checking of invariant J_1:
Checking for J_1 we have

$$X_2(J_1) = x\frac{\partial J_1}{\partial x} = x\frac{\partial t}{\partial x} = 0.$$

Hence the invariant condition is satisfied.

Furthermore, the integration of the equation

$$\frac{dx}{x} = \frac{du}{0}$$

yields $u = C_2$ where C_2 is the constant of integration. Hence, the second invariant is

$$J_2 = u. \tag{73}$$

Checking of invariant J_2:
Checking for J_2 we have

$$X_2(J_2) = x\frac{\partial J_2}{\partial x} = \frac{\partial u}{\partial x} = 0$$

which also satisfies the invariant condition. Similarly designating one of the invariants as a function of the other, that is

$$J_2 = \varphi(J_1).$$

we obtain

$$u = \varphi(t). \tag{74}$$

Taking derivative of equation (74) with respect to x and t we obtain

$$u_t = \varphi', \qquad u_x = 0, \qquad u_{xx} = 0$$

Putting the above derivatives into the derivatives of the Black-Scholes Equation (1), we obtain

$$\varphi' - C\varphi = 0 \implies \frac{d\varphi}{dJ_1} = C\varphi \implies ln\varphi = CJ_1 + K \implies \varphi = Ke^{CJ_1}.$$

Hence

$$\varphi = Ke^{Ct}. \tag{75}$$

Setting equation (75) in (74) we obtain

$$u = Ke^{Ct}. \tag{76}$$

Checking:
Checking for the invariant u we differentiate u with respect to x and t. That is

$$u_t = KCe^{Ct}, \qquad u_x = 0, \qquad u_{xx} = 0.$$

Setting derivatives above into the Black-Schole equation we obtain

$$KCe^{Ct} - C(Ke^{Ct}) = 0.$$

Hence it satisfied. Therefore, the solution is

$$u = Ke^{Ct}.$$

The next operator we will like to work on is the

$$X_3 = 2t\frac{\partial}{\partial t} + (lnx + Pt)x\frac{\partial}{\partial x} + 2Ctu\frac{\partial}{\partial u}.$$

We obtain independent invariants as first integrals of the characteristic system

$$\frac{dt}{2t} = \frac{dx}{(\ln x + Pt)x} = \frac{du}{2Ctu}.$$

Writing the first equation as a linear equation we obtain

$$\frac{dt}{2t} = \frac{du}{2Ctu}.$$

This yields

$$Ct = \ln u + \ln C_1$$

Solving with respect to the constant of integration C_1 we obtain a first integral $C_1 = ue^{-Ct}$. Hence, one of the invariants is

$$J_1 = ue^{-Ct}. \tag{77}$$

Checking of invariant J_1:
Checking for the invariant J_1 we have

$$\begin{aligned}(X_3)(J_1) &= 2t\frac{\partial J_1}{\partial t} + (\ln x + Pt)x\frac{\partial J_1}{\partial x} + 2Ctu\frac{\partial J_1}{\partial u}. \\ &= -2tCue^{-Ct} + 2Ctue^{-Ct} \\ &= 0.\end{aligned}$$

This satisfies the invariance condition. Again writing the second equation as a linear equation we obtain

$$\frac{dt}{2t} = \frac{dx}{(\ln x + Pt)x}.$$

We will make the following substitution

$$z = \ln x \quad \Longrightarrow \quad \frac{dz}{dx} = \frac{1}{x} \quad \Longrightarrow \quad dz = \frac{1}{x}dx.$$

Hence we have the equation

$$\frac{dt}{2t} = \frac{dz}{(z + Pt)} \quad \Longrightarrow \quad \frac{(z + Pt)}{2t} = \frac{dz}{dt} \quad \Longrightarrow \quad \frac{dz}{dt} = \frac{z}{2t} + \frac{P}{2}$$

$$z' - z\frac{1}{2t} = \frac{P}{2} \tag{78}$$

The above equation is a non-homogeneous first order equation. Solving the homogeneous part we proceed as follows

$$z' - z\frac{1}{2t} = 0 \quad \Longrightarrow \quad \frac{dz}{dt} = \frac{z}{2t} \quad \Longrightarrow \quad \frac{dz}{z} = \frac{dt}{2t}$$

Integration of the above equation yields

$$lnz = \ln\sqrt{t} + \ln C_2 \qquad \Longrightarrow \qquad z = C_2\sqrt{t}. \tag{79}$$

where C_2 is a function of t. Differentiation of z yields

$$z' = C_2'\sqrt{t} + \frac{1}{2\sqrt{t}}C_2.$$

Setting z' in equation (78) we obtain

$$C_2' = \frac{P}{2\sqrt{t}}.$$

Upon integration we obtain

$$C_2 = P\sqrt{t} + C_3$$

Therefore, setting C_2 in equation (79) we have

$$\begin{aligned} z &= \sqrt{t}\Big(P\sqrt{t} + C_3\Big) \\ &= Pt + C_3\sqrt{t} \end{aligned}$$

Replacing z by $\ln x$ we obtain

$$J_2 = P\sqrt{t} - \frac{\ln x}{\sqrt{t}}. \tag{80}$$

Checking of invariant J_2:
The invariant J_2 is checked as follows

$$\begin{aligned} (X_3)(J_2) &= 2t\frac{\partial J_2}{\partial t} + (\ln x + Pt)x\frac{\partial J_2}{\partial x} + 2Ctu\frac{\partial J_2}{\partial u}. \\ &= 2t\Big(\frac{P}{2\sqrt{t}} + \frac{\ln x}{2t^{3/2}}\Big) \\ &\quad + (\ln x + Pt)x\Big(-\frac{1}{x\sqrt{t}}\Big) \\ &= P\sqrt{t} + \frac{\ln x}{\sqrt{t}} - \frac{\ln x}{\sqrt{t}} - P\sqrt{t} \\ &= 0. \end{aligned}$$

Designating the invariants J_2 as a function of J_1, that is, $J_2 = \varphi(J_1)$ we obtain

$$u = e^{Ct}\varphi\Big(P\sqrt{t} - \frac{\ln x}{\sqrt{t}}\Big). \tag{81}$$

Differentiating u with respect to x and t we obtain

$$
\begin{aligned}
u_t &= Ce^{Ct}\varphi + e^{Ct}\Big(\frac{P}{2\sqrt{t}} + \frac{\ln x}{2t^{3/2}}\Big)\varphi' \\
u_x &= e^{Ct}\Big(-\frac{1}{x\sqrt{t}}\Big)\varphi' \\
u_{xx} &= e^{Ct}\Big(\frac{1}{x^2\sqrt{t}}\varphi' + \frac{1}{x^2 t}\varphi''\Big).
\end{aligned}
$$

Setting the above derivatives in the Black-Scholes Equation (1) we obtain

$$
\begin{aligned}
e^{Ct}\Big[C\varphi + \Big(\frac{P}{2\sqrt{t}} + \frac{\ln x}{2t^{3/2}}\Big)\varphi' + \frac{1}{2}A^2x^2\Big(\frac{1}{x^2\sqrt{t}}\varphi' \\
+\frac{1}{x^2 t}\varphi''\Big) + Bx\Big(-\frac{1}{x\sqrt{t}}\varphi'\Big) - C\varphi\Big] &= 0 \\
\frac{A^2}{2t}\varphi'' + \Big(\frac{P}{2\sqrt{t}} + \frac{\ln x}{2t^{3/2}} + \frac{A^2}{2\sqrt{t}} - \frac{B}{\sqrt{t}}\Big)\varphi' &= 0 \\
\frac{A^2}{2t}\varphi'' + \Big(\frac{P}{2\sqrt{t}} + \frac{\ln x}{2t^{3/2}} - \frac{P}{2\sqrt{t}}\Big)\varphi' &= 0
\end{aligned}
$$

Multiplying $2t$ on both sides we obtain

$$
\begin{aligned}
A^2\varphi'' - \Big(P\sqrt{t} - \frac{\ln x}{\sqrt{t}}\Big)\varphi' &= 0 \\
A^2\varphi'' - J_2\varphi' &= 0.
\end{aligned}
$$

Therefore from the above calculations we obtain

$$
\varphi'' - \frac{J_2}{A^2}\varphi' = 0. \tag{82}
$$

we use the method of change of variables. Therefore we let $\varphi' = \upsilon(J_2)$. Hence equation (82) becomes

$$
\upsilon' - \frac{J_2}{A^2}\upsilon = 0 \implies \upsilon' = \frac{J_2}{A^2}\upsilon \implies \frac{d\upsilon}{\upsilon} = \frac{J_2}{A^2}dJ_2.
$$

Upon integration we obtain

$$
\ln \upsilon = \frac{J_2^2}{2A^2} + \ln K.
$$

where K is the constant of integration. Therefore

$$
\upsilon = K\exp\Big(\frac{1}{2A^2}J_2^2\Big).
$$

Replacing v by φ' we obtain $\varphi' = K\exp(\frac{J_2^2}{2A^2})$ hence we have

$$d\varphi = K\exp\Big(\frac{J_2^2}{2A^2}\Big)dJ_2.$$

Upon integration we obtain

$$\varphi = KA\sqrt{\frac{\pi}{2}}erfi\Big(\frac{J_2}{A\sqrt{2}}\Big) + K_1$$

where $erfi$ is imaginary error function and K_1 is also a constant of integration. Setting value of φ in equation (81) we obtain

$$u = Ke^{Ct}\int\exp\Big(\frac{J_2^2}{2A^2}\Big)dJ_2 + K_1e^{Ct}$$

Or

$$u = AKe^{Ct}\sqrt{\frac{\pi}{2}}erfi\Big(\frac{J_2}{A\sqrt{2}}\Big) + K_1e^{Ct}. \tag{83}$$

Checking:
Checking for the invariant u using equation (83), we proceed as follows

$$\begin{aligned}
u_t &= KCe^{Ct}A\sqrt{\frac{\pi}{2}}erfi\Big(\frac{J_2}{A\sqrt{2}}\Big) + Ke^{Ct}\exp\Big(\frac{J_2^2}{2A^2}\Big)\frac{dJ_2}{dt} + K_1Ce^{Ct}\\
&= KCe^{Ct}A\sqrt{\frac{\pi}{2}}erfi\Big(\frac{J_2}{A\sqrt{2}}\Big) + Ke^{Ct}\exp\Big(\frac{J_2^2}{2A^2}\Big)\\
&= \times\Big(\frac{P}{2\sqrt{t}} + \frac{k}{2t^{2/3}} + \frac{\ln x}{2t^{3/2}}\Big) + K_1e^{Ct}\\
u_x &= Ke^{Ct}\exp\Big(\frac{J_2^2}{2A^2}\Big)\frac{dJ_2}{dx}\\
&= Ke^{Ct}\exp\Big(\frac{J_2^2}{2A^2}\Big)(-\frac{1}{x\sqrt{t}})\\
&= -\frac{K}{x\sqrt{t}}e^{Ct}\exp\Big(\frac{J_2^2}{2A^2}\Big)\\
u_{xx} &= Ke^{Ct}\Big[\frac{1}{x^2\sqrt{t}}\exp\Big(\frac{J_2^2}{2A^2}\Big) - \frac{1}{x\sqrt{t}}\frac{J_2}{A^2}\exp\Big(\frac{J_2^2}{2A^2}\Big)\frac{dJ_2}{dx}\Big]\\
&= Ke^{Ct}\Big[\frac{\exp(\frac{J_2^2}{2A^2})}{x^2\sqrt{t}} - \frac{\exp(\frac{J_2^2}{2A^2})}{x\sqrt{t}A^2}\Big(P\sqrt{t} - \frac{1}{\sqrt{t}}(k+\ln x)\Big)\Big(-\frac{1}{x\sqrt{t}}\Big)\Big]\\
&= Ke^{Ct}\Big[\frac{\exp(\frac{J_2^2}{2A^2})}{x^2\sqrt{t}} - \frac{\exp(\frac{J_2^2}{2A^2})}{x^2tA^2}\Big(P\sqrt{t} - \frac{1}{\sqrt{t}}(k+\ln x)\Big)\Big]
\end{aligned}$$

Setting the above derivatives in equation (1) we obtain

$$\begin{aligned} u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu =& Ke^{Ct}\Big[CA\sqrt{\frac{\pi}{2}}erfi\Big(\frac{J_2}{A^2\sqrt{2}}\Big) + \frac{K_1C}{K} \\ &+ \frac{P\exp\left(\frac{J_2^2}{2A^2}\right)}{2\sqrt{t}} + \frac{k\exp\left(\frac{J_2^2}{2A^2}\right)}{2t^{3/2}} + \frac{\ln x\exp\left(\frac{J_2^2}{2A^2}\right)}{2t^{3/2}} + \frac{A^2\exp\left(\frac{J_2^2}{2A^2}\right)}{2} \\ &+ \frac{P\exp\left(\frac{J_2^2}{2A^2}\right)}{2\sqrt{t}} - \frac{k\exp\left(\frac{J_2^2}{2A^2}\right)}{2t^{3/2}} - \frac{\ln x\exp\left(\frac{J_2^2}{2A^2}\right)}{2t^{3/2}} - \frac{B\exp\left(\frac{J_2^2}{2A^2}\right)}{\sqrt{t}} \\ &- \frac{K_1C}{K} - CA\sqrt{\frac{\pi}{2}}erfi\Big(\frac{J_2}{A^2\sqrt{2}}\Big) \\ &= 0 \end{aligned}$$

which satisfies the invariant condition. Hence the solution is

$$u = Ke^{Ct}\int \exp\left(\frac{J_2^2}{2A^2}\right)dJ_2 + K_1e^{Ct}$$

Or

$$u = AKe^{Ct}\sqrt{\frac{\pi}{2}}erfi\Big(\frac{J_2}{A\sqrt{2}}\Big) + K_1e^{Ct}.$$

Similarly, the operator

$$X_4 = A^2tx\frac{\partial}{\partial x} + (\ln x - Pt)u\frac{\partial}{\partial x}$$

is the next to be discussed in constructing the invariant solutions. The characteristic system of the independent invariants as a first integrals is

$$\frac{dt}{0} = \frac{dx}{A^2tx} = \frac{du}{(\ln x - Pt)u}.$$

Written as a linear equation, the first equation

$$\frac{dt}{0} = \frac{dx}{A^2tx}$$

yields $t = C_1$ where C_1 is constant of integration. Hence, one of the invariants is

$$J_1 = t. \tag{84}$$

Checking of invariant J_1:
Checking this invariant clearly shows that it satisfies the invariant condition.

Again, also written as a linear equation the second equation becomes

$$\frac{dx}{A^2tx} = \frac{du}{(\ln x - Pt)u} \quad \Longrightarrow \quad \Big(\frac{\ln x}{A^2tx} - \frac{Pt}{A^2tx}\Big)dx = \frac{du}{u}.$$

Integration of the above equation yields

$$\frac{(\ln x)^2}{2A^2t} - \frac{P\ln x}{A^2} = \ln u + \ln C_2.$$

Solving with respect to the constant of integration we obtain

$$exp\Big(\frac{(\ln x)^2}{2A^2t} - \frac{P\ln x}{A^2}\Big) = uC_2 \quad \Longrightarrow \quad C_2 = u\exp\Big(-\frac{(\ln x)^2}{2A^2t} + \frac{P\ln x}{A^2}\Big)$$

We therefore obtain the second invariant

$$J_2 = u\exp\Big(-\frac{(\ln x)^2}{2A^2t} + \frac{P\ln x}{A^2}\Big). \tag{85}$$

Checking of invariant J_2:
Checking for the invariant J_2 we have

$$\begin{aligned} X_4(J_2) &= A^2tx\frac{\partial J_2}{\partial x} + (\ln x - Pt)u\frac{\partial J_2}{\partial x} \\ &= A^2tx\Big(-\frac{\ln x}{A^2tx} + \frac{P}{A^2x}\Big)u\exp\Big(-\frac{(\ln x)^2}{2A^2t} + \frac{P\ln x}{A^2}\Big) \\ &\quad + (\ln x - Pt)u\exp\Big(-\frac{(\ln x)^2}{2A^2t} + \frac{P\ln x}{A^2}\Big) \\ &= 0. \end{aligned}$$

Taking J_2 as function of J_1, that is $J_2 = \varphi(J_1)$ we obtain the invariant

$$u = \exp\Big(\frac{(\ln x)^2}{2A^2t} - \frac{P\ln x}{A^2}\Big)\varphi(t) \tag{86}$$

As usual we take derivative of u with respect to x and t. This yields

$$
\begin{aligned}
u_t &= -\frac{(\ln x)^2}{2t^2A^2}\exp\Big(\frac{(\ln x)^2}{2A^2t}-\frac{P\ln x}{A^2}\Big)\varphi+\exp\Big(\frac{(\ln x)^2}{2A^2t}-\frac{P\ln x}{A^2}\Big)\varphi' \\
u_x &= \Big(\frac{\ln x}{xtA^2}-\frac{P}{A^2x}\Big)\exp\Big(\frac{(\ln x)^2}{2A^2t}-\frac{P\ln x}{A^2}\Big)\varphi \\
u_{xx} &= \Big[\frac{1}{tA^2}\Big(\frac{x(\frac{1}{x})-\ln x}{x^2}\Big)+\frac{P}{A^2x^2}\Big]\exp\Big(\frac{(\ln x)^2}{2A^2t}-\frac{P\ln x}{A^2}\Big)\varphi \\
&\quad+\Big(\frac{\ln x}{xtA^2}-\frac{P}{A^2x}\Big)^2\exp\Big(\frac{(\ln x)^2}{2A^2t}-\frac{P\ln x}{A^2}\Big)\varphi \\
&= \Big(\frac{1-\ln x}{A^2tx^2}+\frac{P}{A^2x^2}\Big)\exp\Big(\frac{(\ln x)^2}{2A^2t}-\frac{P\ln x}{A^2}\Big)\varphi \\
&\quad+\Big(\frac{(\ln x-Pt)^2}{x^2t^2A^4}\Big)\exp\Big(\frac{(\ln x)^2}{2A^2t}-\frac{P\ln x}{A^2}\Big)\varphi
\end{aligned}
$$

Setting the above derivatives in equation (1) we have

$$
\begin{aligned}
&u_t+\frac{1}{2}A^2x^2u_{xx}+Bxu_x-Cu=-\frac{(\ln x)^2}{2t^2A^2}\varphi+\varphi'+\frac{1}{2}A^2x^2\Big[\Big(\frac{1-\ln x}{A^2tx^2} \\
&\quad+\frac{P}{A^2x^2}\Big)+\frac{(\ln x-Pt)^2}{A^4x^2t^2}\Big]\varphi+Bx\Big(\frac{\ln x}{xtA^2}-\frac{P}{A^2x}\Big)\varphi-C\varphi \\
&\quad=\varphi'+\Big(\frac{1-\ln x}{2t}+\frac{P}{2}+\frac{(\ln x-Pt)^2}{2A^2t^2}-\frac{(\ln x)^2}{2t^2A^2}+\frac{B\ln x}{tA^2}-\frac{BP}{A^2}-C\Big)\varphi \\
&\quad=\varphi'+\Big(\frac{1}{2t}-\frac{\ln x}{2t}+\frac{P}{2}+\frac{(\ln x)^2}{2A^2t^2}-\frac{P\ln x}{A^2t}+\frac{P^2}{2A^2}-\frac{(\ln x)^2}{2A^2t^2} \\
&\quad+\frac{B\ln x}{tA^2}-\frac{BP}{A^2}-C\Big)\varphi \\
&\quad=\varphi'+\Big(\frac{1}{2t}+\frac{P}{2}+\frac{P^2}{2A^2}-\frac{BP}{A^2}-C\Big)\varphi.
\end{aligned}
$$

But we know that $P=B-\frac{A^2}{2}$ and therefore,

$$
\begin{aligned}
\varphi'+\Big(\frac{1}{2t}+\frac{P}{2}+\frac{P^2}{2A^2}-\frac{P^2}{A^2}-\frac{P^2}{2}-C\Big)\varphi=0 \\
\varphi'+\Big(\frac{1}{2t}-\frac{P^2}{2A^2}-C\Big)\varphi=0.
\end{aligned}
$$

The above first order differential gives us

$$
\begin{aligned}
\frac{d\varphi}{dJ_1}&=\Big(-\frac{1}{2J_1}+\frac{P^2}{2A^2}+C\Big)\varphi \\
\frac{d\varphi}{\varphi}&=\Big(-\frac{1}{2J_1}+\frac{P^2}{2A^2}+C\Big)dJ_1.
\end{aligned}
$$

Upon integration we obtain

$$\begin{aligned}\ln\varphi &= -\frac{1}{2}lnJ_1 + \frac{P^2}{2A^2}J_1 + CJ_1 + K\\ \varphi &= \frac{K}{\sqrt{J_1}}\exp\Big(\frac{P^2}{2A^2} + C\Big)J_1\\ &= \frac{K}{\sqrt{t}}\exp\Big((\frac{P^2}{2A^2} + C)t\Big)\end{aligned}$$

since $J_1 = t$. Setting φ in equation (86) we obtain

$$u = \frac{K}{\sqrt{t}}\exp\Big(\frac{(\ln x)^2}{2A^2t} - \frac{P\ln x}{A^2} + t(\frac{P^2}{2A^2} + C)\Big). \tag{87}$$

Checking:
Again we check for the invariant condition for the invariant u

$$\begin{aligned}u_t &= -\frac{K}{2t^{3/2}}\exp\Big(\frac{(\ln x)^2}{2A^2t} - \frac{P\ln x}{A^2} + t(\frac{P^2}{2A^2} + C)\Big)\\ &\quad + \frac{K}{\sqrt{t}}\Big(- \frac{(\ln x)^2}{2A^2t^2} + \frac{P^2}{2A^2} + C\Big)\exp\Big(\frac{(\ln x)^2}{2A^2t} - \frac{P\ln x}{A^2} + t(\frac{P^2}{2A^2} + C)\Big)\\ u_x &= \frac{K}{\sqrt{t}}\Big(\frac{\ln x}{xtA^2} - \frac{P}{A^2x}\Big)\exp\Big(\frac{(\ln x)^2}{2A^2t} - \frac{P\ln x}{A^2} + t(\frac{P^2}{2A^2} + C)\Big)\\ u_{xx} &= \frac{K}{\sqrt{t}}\Big[\Big(\frac{1}{A^2t}(\frac{1-\ln x}{x^2}) + \frac{P}{A^2x^2}\Big) + \Big(\frac{\ln x}{xtA^2} - \frac{P}{A^2x}\Big)^2\\ &\quad \times\exp\Big(\frac{(\ln x)^2}{2A^2t} - \frac{P\ln x}{A^2} + t(\frac{P^2}{2A^2} + C)\Big)\Big].\end{aligned}$$

Setting in equation (1) we obtain

$$\begin{aligned}u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu =& \frac{K}{\sqrt{t}}\Big(- \frac{1}{2t} - \frac{(\ln x)^2}{2A^2t} + \frac{P^2}{2A^2} + C + \frac{1}{2t}\\ & - \frac{\ln x}{2t} + \frac{P}{2} + \frac{(\ln x)^2}{2A^2t^2} - \frac{P\ln x}{A^2t} + \frac{P^2}{2A^2} + \frac{B\ln x}{tA^2} - \frac{BP}{A^2} - C\Big)\\ & \times\exp\Big(\frac{(\ln x)^2}{2A^2t} - \frac{P\ln x}{A^2} + t(\frac{P^2}{2A^2} + C)\Big)\\ & = \Big(\frac{P^2}{A^2} + \frac{P}{2} - \frac{\ln x}{2t} - \frac{B\ln x}{A^2t} + \frac{\ln x}{2t} + \frac{B\ln x}{tA^2} - \frac{P^2}{A^2} - \frac{P}{2}\Big)\\ & \times\exp\Big(\frac{(\ln x)^2}{2A^2t} - \frac{P\ln x}{A^2} + t(\frac{P^2}{2A^2} + C)\Big)\\ & = 0.\end{aligned}$$

This satisfies the invariant condition. Hence the answer is

$$u = \frac{K}{\sqrt{t}} \exp\Big(\frac{(\ln x)^2}{2A^2 t} - \frac{P \ln x}{A^2} + t(\frac{P^2}{2A^2} + C)\Big).$$

Furthermore, the operator

$$X_5 = 2A^2t^2\frac{\partial}{\partial t} + 2A^2 tx \ln x\frac{\partial}{\partial x} + [(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u\frac{\partial}{\partial u}$$

which also yields the characteristic system

$$\frac{dt}{2A^2t^2} = \frac{dx}{2A^2tx\ln x} = \frac{du}{[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u}.$$

is the next we will like to discuss. We again write the first equation as a linear equation

$$\frac{dt}{2A^2t^2} = \frac{dx}{2A^2tx\ln x}.$$

This yields the following results

$$\frac{dt}{t} = \frac{dx(\frac{1}{x})}{\ln x} \quad \Longrightarrow \quad \ln t = \ln(\ln x) lnC_1 \quad \Longrightarrow \quad t = xC_1 \ln x.$$

We can also say that $lnx = C_1 t$. Hence, we have the first invariant as

$$J_1 = \frac{1}{t}\ln x. \tag{88}$$

Checking of invariant J_1:
The invariant J_1 is checked as follows

$$\begin{aligned}(X_5)(J_1) &= 2A^2t^2\frac{\partial J_1}{\partial t} + 2A^2tx\ln x\frac{\partial J_1}{\partial x} + [(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u\frac{\partial J_1}{\partial u} \\ &= 2A^2t^2\Big(-\frac{1}{t^2}\ln x\Big) + 2A^2tx\ln x\Big(\frac{1}{xt}\Big) \\ &= 0\end{aligned}$$

which shows that the invariant condition is satisfied.

Similarly, the second equation written as linear equation

$$\frac{dt}{2A^2t^2} = \frac{du}{[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u}$$

is evaluated as follows

$$
\begin{aligned}
\Big[\frac{(C_1t-Pt)^2}{2A^2t^2}+\frac{2A^2Ct^2}{2A^2t^2}-\frac{A^2t}{2A^2t^2}\Big]dt &= \frac{du}{u}\\
\Big(\frac{(C_1-P)^2}{2A^2}+C-\frac{1}{2t}\Big)dt &= \ln u + lnC_2\\
\exp\Big(\frac{(C_1-P)^2}{2A^2}t+Ct\cdot\exp(\ln t^{-1/2})\Big) &= uC_2\\
t^{-1/2}\exp\Big(\frac{(C_1-P)^2}{2A^2}t+Ct\Big) &= uC_2
\end{aligned}
$$

Therefore, the second invariant is

$$
J_2=\sqrt{t}u\exp\Big(-\frac{t}{2A^2}(\frac{1}{t}\ln x-P)^2-Ct\Big). \tag{89}
$$

Checking of invariant J_2:
Checking for the invariant J_2 we have

$$
\begin{aligned}
X_5(J_2) &= 2A^2t^2\frac{\partial J_2}{\partial t}+2A^2tx\ln x\frac{\partial J_2}{\partial x}+[(\ln x-Pt)^2+2A^2Ct^2-A^2t]u\frac{\partial J_2}{\partial u}\\
&= 2A^2t^2u\Big[\frac{1}{2\sqrt{t}}+\sqrt{t}\Big(-\frac{1}{2A^2t^2}(\ln x-Pt)^2\\
&\quad -\frac{2t}{2A^2t}(\ln x-Pt)(-\frac{1}{t^2}\ln x)-C\Big)\Big]\exp\Big(-\frac{t}{2A^2}(\frac{1}{t}\ln x-P)^2-Ct\Big)\\
&\quad +2A^2tx\ln x\sqrt{t}u\Big(-\frac{2t}{2A^2t}(\ln x-Pt)(\frac{1}{xt})\Big)\exp\Big(-\frac{t}{2A^2}(\frac{1}{t}\ln x-P)^2-Ct\Big)\\
&\quad +\Big((\ln x-Pt)^2+2A^2Ct^2-A^2t\Big)u\sqrt{t}\exp\Big(-\frac{t}{2A^2}(\frac{1}{t}\ln x-P)^2-Ct\Big)\\
&= u\sqrt{t}\exp\Big(-\frac{t}{2A^2}(\frac{1}{t}\ln x-P)^2-Ct\Big)\Big(A^2t-(\ln x-Pt)^2+2\ln x(\ln x-Pt)\\
&\quad -2A^2t^2C-2\ln x(\ln x-Pt)+(\ln x-Pt)^2+2A^2Ct^2-A^2t\Big)\\
&= 0.
\end{aligned}
$$

The condition for invariance is therefore certified.

Once again, expressing the invariant J_2 as a function of J_1, that is $J_2=\varphi(J_1)$ we obtain

$$
u=\frac{1}{\sqrt{t}}\exp\Big(\frac{1}{2A^2t}(\ln x-P)^2+Ct\Big)\varphi(\frac{1}{t}\ln x). \tag{90}
$$

Again, we will have to take derivatives of u with respect to t and x and substitute the results in the Black-Scholes Equation (1). Taking the derivatives we have

$$
\begin{aligned}
u_t &= -\frac{1}{2t^{3/2}} \exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\varphi + \frac{1}{\sqrt{t}}\Big(- \frac{1}{2A^2t^2}(\ln x - Pt)^2 \\
&\quad - \frac{P}{tA^2}(\ln x - Pt) + C\Big) \exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\varphi \\
&\quad - \frac{1}{t^2\sqrt{t}} \ln x\Big[\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\varphi'\Big] \\
&= \frac{1}{\sqrt{t}} \exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\Big[- \frac{1}{2t}\varphi - \frac{1}{2A^2t^2}(\ln x - Pt)^2\varphi \\
&\quad - \frac{P}{tA^2}(\ln x - Pt)\varphi + C\varphi - \frac{1}{t^2}\ln x\varphi'\Big]
\end{aligned}
$$

$$
\begin{aligned}
u_x &= \frac{1}{\sqrt{t}}\Big[\frac{1}{A^2tx}(\ln x - Pt)\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\varphi \\
&\quad + \frac{1}{tx}\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\varphi'\Big] \\
&= \frac{1}{\sqrt{t}}\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\Big(\frac{\ln x - Pt}{A^2tx}\varphi + \frac{1}{tx}\varphi'\Big); \\
u_{xx} &= \frac{1}{\sqrt{t}}\Big[- \frac{1}{A^2tx^2}(\ln x - Pt)\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\varphi \\
&\quad + \frac{1}{A^4t^2x^2}(\ln x - Pt)^2\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\varphi \\
&\quad + \frac{1}{A^2tx^2}\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\varphi + \frac{1}{A^2t^2x^2}(\ln x - Pt) \\
&\quad * \exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\varphi' - \frac{1}{tx^2}\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\varphi' \\
&\quad + \frac{1}{A^2t^2x^2}(\ln x - Pt)\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\varphi' \\
&\quad + \frac{1}{t^2x^2}\exp\Big(\frac{t}{2A^2}(\frac{1}{t}\ln x - Pt)^2 + Ct\Big)\varphi''\Big] \\
&= \frac{1}{\sqrt{t}}\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\Big[- \frac{1}{A^2tx^2}(\ln x - Pt)\varphi \\
&\quad + \frac{1}{A^4t^2x^2}(\ln x - Pt)^2\varphi + \frac{1}{A^2tx^2}\varphi + \frac{2}{A^2t^2x^2}(\ln x - Pt)\varphi' \\
&\quad - \frac{1}{tx^2}\varphi' + \frac{1}{t^2x^2}\varphi''\Big].
\end{aligned}
$$

Upon substitution we obtain

$$
\begin{aligned}
u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu &= \frac{1}{\sqrt{t}}\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\Big[-\frac{1}{2t}\varphi \\
&+ C\varphi - \frac{1}{2A^2t^2}(\ln x - Pt)^2\varphi - \frac{P}{tA^2}(\ln x - Pt)\varphi - \frac{1}{t^2}\ln x\varphi' \\
&+ \frac{1}{2}\Big(-\frac{1}{t}(\ln x - Pt)\varphi + \frac{1}{A^2t^2}(\ln x - Pt)^2\varphi + \frac{1}{t}\varphi + \frac{2}{t^2}(\ln x \\
&- Pt)\varphi' - \frac{A^2}{t}\varphi' + \frac{A^2}{t^2}\varphi''\Big) + \frac{Bx}{x}\Big(\frac{\ln x - Pt}{A^2t}\varphi + \frac{1}{t}\varphi'\Big) - C\varphi\Big] \\
&= \frac{A^2}{2t^2}\varphi'' + \Big(\frac{1}{t^2}(\ln x - Pt) - \frac{1}{t^2}\ln x - \frac{A^2}{2t} + \frac{B}{t}\Big)\varphi' \\
&+ \Big(C - \frac{1}{2t} - \frac{1}{2A^2t^2(\ln x - Pt)^2} - \frac{P}{tA^2}(\ln x - Pt) - \frac{1}{2t}(\ln x \\
&- Pt) + \frac{1}{2A^2t^2}(\ln x - Pt)^2 + \frac{1}{2t} + \frac{B}{A^2t}(\ln x - Pt) - C\Big)\varphi.
\end{aligned}
$$

But since we know that $P = B - \frac{A^2}{2}$ it implies that $-\frac{A^2}{2t} + \frac{B}{t} = \frac{1}{t}P$. Therefore, the Black-Scholes Equation (1) becomes

$$
\begin{aligned}
u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu &= \frac{A^2}{2t^2}\varphi'' + \Big(\frac{1}{t^2}(\ln x - Pt) - \frac{1}{t^2}(\ln x - Pt)\Big)\varphi' \\
&+ \Big(-\frac{P}{tA^2}(\ln x - Pt) - \frac{1}{2t}(\ln x - Pt) + \frac{B}{A^2t}(\ln x - Pt)\Big) \\
= \frac{A^2}{2t^2}\varphi'' + \Big[\Big(\frac{A^2}{2} - B\Big)&\frac{(\ln x - Pt)}{tA^2} - \frac{1}{2t}(\ln x - Pt) + \frac{B}{A^2t}(\ln x - Pt)\Big]\varphi.
\end{aligned}
$$

The above calculation yields $\varphi'' = 0$ and upon integration we obtain

$$\varphi = K_1J_1 + K_2$$

where K_1, K_2 are constants of integration. Replacing J_1 we obtain

$$\varphi = \frac{K_1}{t}\ln x + K_2. \tag{91}$$

Setting equation (91) in (90) we have

$$u = \frac{1}{\sqrt{t}}\Big(\frac{K_1}{t}\ln x + K_2\Big)\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big) \tag{92}$$

Checking:
We will now check whether the invariant u satisfies the invariant condition. We therefore proceed as follows;

$$\begin{aligned}
u_t &= -\frac{1}{2t^{3/2}}\Big(\frac{K_1}{t}\ln x + K_2\Big)\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big) \\
&\quad + \frac{1}{\sqrt{t}}\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\Big(-\frac{K_1\ln x}{t^2}\Big) \\
&\quad + \frac{1}{\sqrt{t}}\Big(\frac{K_1}{t}\ln x + K_2\Big)\Big(-\frac{1}{2A^2t^2}(\ln x - Pt)^2 \\
&\quad - \frac{P}{tA^2}(\ln x - Pt) + C\Big)\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big) \\
&= \frac{1}{\sqrt{t}}\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\Big[-\frac{1}{2t}\Big(\frac{K_1}{t}\ln x + K_2\Big) \\
&\quad - \Big(\frac{K_1}{t}\ln x + K_2\Big)\Big(-\frac{1}{2A^2t^2}(\ln x - Pt)^2 - \frac{P}{A^2t}(\ln x - Pt) + C\Big)\Big];
\end{aligned}$$

$$\begin{aligned}
u_x &= \frac{1}{\sqrt{t}}\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\Big[\frac{\ln x - Pt}{A^2tx}\Big(\frac{K_1}{t}\ln x + K_2\Big) + \frac{K_1}{xt}\Big]; \\
u_{xx} &= \frac{1}{\sqrt{t}}\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\Big[-\frac{1}{A^2tx^2}(\ln x - Pt)\Big(\frac{K_1}{t}\ln x + K_2\Big) \\
&\quad + \frac{1}{A^4t^2x^2}(\ln x - Pt)^2\Big(\frac{K_1}{t}\ln x + K_2\Big) + \frac{1}{A^2tx^2}\Big(\frac{K_1}{t}\ln x + K_2\Big) \\
&\quad + \frac{2K_1}{A^2t^2x^2}(\ln x - Pt) - \frac{K_1}{tx^2}\Big]
\end{aligned}$$

Setting these derivatives in equation (1), we obtain

$$\begin{aligned}
u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu &= \frac{1}{\sqrt{t}}\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\Big[\Big[-\frac{K_1\ln x}{t^2} \\
&\quad - \frac{1}{2t}\Big(\frac{K_1}{t}\ln x + K_2\Big) + \Big(\frac{K_1}{t}\ln x + K_2\Big)\Big(-\frac{1}{2A^2t^2}(\ln x - Pt)^2 \\
&\quad - \frac{P}{A^2t}(\ln x - Pt)\Big) + C\Big(\frac{K_1}{t}\ln x + K_2\Big)\Big] + \frac{1}{2A^2x^2}A^2x^2\Big[-\frac{1}{t}(\ln x - Pt) \\
&\quad \times\Big(\frac{K_1}{t}\ln x + K_2\Big) + \frac{1}{A^2t^2}(\ln x - Pt)^2\Big(\frac{K_1}{t}\ln x + K_2\Big) + \frac{1}{t}\Big(\frac{K_1}{t}\ln x \\
&\quad + K_2\Big) + \frac{2K_1}{t^2}(\ln x - Pt) - \frac{A^2K_1}{t}\Big] + Bx\Big(\frac{1}{x}\Big)\Big[\frac{\ln x - Pt}{A^2t}\Big(\frac{K_1}{t}\ln x \\
&\quad + K_2\Big) + \frac{K_1}{t}\Big] - C\Big(\frac{K_1}{t}\ln x + K_2\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = &\frac{1}{\sqrt{t}}\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\Big[-\frac{A^2K_1}{2t} \\
&+ \Big(\frac{K_1}{t}\ln x + K_2\Big)\Big[-\frac{1}{2A^2t^2}(\ln x - Pt)^2 - \frac{P}{A^2t}(\ln x - Pt) \\
&- \frac{1}{2t}(\ln x - Pt) + \frac{1}{2A^2t^2}(\ln x - Pt)^2 + \frac{B}{A^2t}(\ln x - Pt)\Big] \\
&- \frac{K_1\ln x}{t^2} + \frac{K_1}{t^2}(\ln x - Pt) + \frac{BK_1}{t}\Big] \\
&= \frac{1}{\sqrt{t}}\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\Big[-\frac{A^2K_1}{2t} + \Big(\frac{K_1}{t}\ln x + K_2\Big) \\
&\times\Big[-\frac{B}{A^2t}(\ln x - Pt) + \frac{1}{2t}(\ln x - Pt) - \frac{1}{2t}(\ln x - Pt) + \frac{B}{A^2t}(\ln x \\
&- Pt)\Big] - \frac{K_1\ln x}{t^2} + \frac{2K_1\ln x}{2t^2} - \frac{K_1}{t}\Big(B - \frac{A^2}{2}\Big) + \frac{BK_1}{t}\Big] \\
&= \frac{1}{\sqrt{t}}\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big)\Big[-\frac{A^2K_1}{2t} \\
&- \frac{K_1B}{t} + \frac{K_1A^2}{2t} + \frac{BK_1}{t}\Big] = 0.
\end{aligned}
$$

The condition is satisfied. So equation (92) is the solution.

The next operator we will look at is the

$$
\begin{aligned}
X_1 + X_5 = &(2A^2t^2 + 1)\frac{\partial}{\partial t} + 2A^2tx\ln x\frac{\partial}{\partial x} \\
&+ \Big[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t\Big]u\frac{\partial}{\partial u}.
\end{aligned}
$$

This operator also has a characteristic equation given by

$$
\frac{dt}{2A^2t^2 + 1} = \frac{dx}{2A^2tx\ln x} = \frac{du}{[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]}.
$$

This characteristic system also leads us to two forms of linear equations. The first equation is given by

$$
\frac{dt}{2A^2t^2 + 1} = \frac{dx}{2A^2tx\ln x}.
$$

The reckoning equation yields

$$
\begin{aligned}
\frac{2A^2t}{2A^2t^2 + 1}dt &= \frac{1}{x\ln x}dx \\
\frac{1}{2}\ln(2A^2t^2 + 1) &= \ln(\ln x) + \ln C_1
\end{aligned}
$$

which gives an invariant

$$J_1 = \Big(2A^2t^2 + 1\Big)^{-\frac{1}{2}}(\ln x) \tag{93}$$

where we can express $\ln x$ as being equal to $J_1\Big(2A^2t^2+1\Big)^{\frac{1}{2}}$.

Checking of invariant J_1:
Checking for the invariant J_1 we have

$$\begin{aligned}
(X_1+X_5)(J_1) &= (2A^2t^2+1)\frac{\partial J_1}{\partial t} + 2A^2tx\ln x\frac{\partial J_1}{\partial x} \\
&\quad + \Big[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t\Big]u\frac{\partial J_1}{\partial u} \\
&= (2A^2t^2+1)\Big[-\frac{A^2t}{2}\big(2A^2t^2+1\big)^{-\frac{3}{2}}\ln x\Big] \\
&\quad + 2A^2tx\ln x\big[\frac{1}{x}\big(2A^2t^2+1\big)^{\frac{3}{2}}\big] \\
&= -2A^2t\big(2A^2t^2+1\big)^{-\frac{1}{2}}\ln x + 2A^2t\ln x\big(2A^2t^2+1\big)^{-\frac{1}{2}} \\
&= 0.
\end{aligned}$$

The second linear equation from this characteristic system is

$$\frac{dt}{2A^2t^2+1} = \frac{du}{[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u}.$$

Setting in the expression for $\ln x$ we obtain

$$\Big(\frac{\big(J_1(2A^2t^2+1)^{\frac{1}{2}} - Pt\big)^2}{2A^2t^2+1} + \frac{2A^2Ct^2}{2A^2t^2+1} - \frac{A^2t}{2A^2t^2+1}\Big)dt = \frac{du}{u}$$

$$\Big({J_1}^2 - \frac{2J_1Pt}{(2A^2t^2+1)^{\frac{1}{2}}} + \frac{P^2t^2}{2A^2t^2+1} + \frac{2A^2Ct^2}{2A^2t^2+1} - \frac{A^2t}{2A^2t^2+1}\Big)dt = \frac{du}{u}$$

Upon integration, we obtain

$$\begin{aligned}
&{J_1}^2t - \frac{J_1P}{A^2}(2A^2t^2+1)^{\frac{1}{2}} + \frac{P^2t}{2A^2} - \frac{P^2}{2^{\frac{3}{2}}A^3}\tan^{-1}\sqrt{2}At + Ct \\
&\quad - \frac{C}{\sqrt{2}A}\tan^{-1}\sqrt{2}At - \frac{1}{4}\ln(2A^2t^2+1) = \ln u + \ln C_2
\end{aligned}$$

where $\ln C_2$ is the constant of integration. This gives us the invariant

$$\begin{aligned}
J_2 &= \frac{J_1P(2A^2t^2+1)^{\frac{1}{2}}}{A^2} - {J_1}^2t - \frac{P^2t}{2A^2} + \frac{P^2}{2^{\frac{3}{2}}A^3}\tan^{-1}\sqrt{2}At \\
&\quad - Ct + \frac{C}{\sqrt{2}A}\tan^{-1}\sqrt{2}A^2t + \frac{1}{4}\ln(2A^2t^2+1) + \ln u.
\end{aligned}$$

Setting in value of J_1 we obtain the invariant

$$J_2 = \frac{P\ln x}{A^2} - \frac{t(\ln x)^2}{2A^2t^2+1} - \frac{P^2t}{2A^2} + \frac{P^2}{2^{\frac{3}{2}}A^3}tan^{-1}\sqrt{2}At - Ct + \frac{C}{\sqrt{2}A}tan^{-1}\sqrt{2}A^2t + \frac{1}{4}\ln(2A^2t^2+1) + \ln u. \tag{94}$$

Hence equation above is our second invariant equation.

Checking of invariant J_2:
The second invariant (J_2) is checked as follows

$$\begin{aligned}(X_1+X_5)(J_2) =&(2A^2t^2+1)\frac{\partial J_2}{\partial t} + 2A^2tx\ln x\frac{\partial J_2}{\partial x} + \Big[(\ln x - Pt)^2 \\ &+ 2A^2Ct^2 - A^2t\Big]u\frac{\partial}{\partial u} \\ &= (2A^2t^2+1)\Big[-(\ln x)^2\Big(\frac{2A^2t^2+1-4A^2t^2}{(2A^2t^2+1)^2}\Big) - \frac{P^2t^2}{2A^2t^2+1} \\ &- \frac{2A^2Ct^2}{2A^2t^2+1} + \frac{A^2t}{2A^2t^2+1}\Big] + 2A^2tx\ln x\Big[\frac{P}{xA^2} \\ &- \frac{2t\ln x}{x(2A^2t^2+1)}\Big] + \Big[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t\Big] \\ &= -(lnx)^2 + \frac{4A^2t^2(\ln x)^2}{2A^2t^2+1} - P^2t^2 - 2A^2Ct^2 + A^2t + 2Pt\ln x \\ &- \frac{4A^2t^2(\ln x)^2}{2A^2t^2+1} + (\ln x)^2 - 2Pt\ln x + P^2t^2 + 2A^2Ct^2 - A^2t \\ &= 0.\end{aligned}$$

Since our invariant solution is expressed as the equation $J_2 = \varphi(J_1)$ we obtain

$$u = \exp\Big(-\frac{P\ln x}{A^2} + \frac{t(\ln x)^2}{2A^2t^2+1} + \frac{P^2t}{2A^2} - \frac{P^2}{2^{\frac{3}{2}}A^3}\tan^{-1}\sqrt{2}At + Ct - \frac{C}{\sqrt{2}A}\tan^{-1}\sqrt{2}A^2t - \frac{1}{4}ln(2A^2t^2+1)\Big)\varphi(2A^2t^2+1)^{-1/2}\ln x. \tag{95}$$

Differentiating u with respect to t and x we obtain

$$\begin{aligned}u_t =& \exp\Big(-\frac{P\ln x}{A^2} + \frac{t(\ln x)^2}{2A^2t^2+1} + \frac{P^2t}{2A^2} - \frac{P^2}{2^{\frac{3}{2}}A^3}tan^{-1}\sqrt{2}At + Ct \\ &- \frac{C}{\sqrt{2}A}tan^{-1}\sqrt{2}A^2t - \frac{1}{4}\ln(2A^2t^2+1)\Big)\Big[\Big((\ln x)^2\frac{(2A^2t^2+1-4A^2t^2)}{(2A^2t^2+1)^2} \\ &+ \frac{P^2t^2}{2A^2t^2+1} + \frac{2A^2Ct^2}{2A^2t^2+1} - \frac{A^2t}{2A^2t^2+1}\Big)\varphi - \frac{2A^2t}{(2A^2t^2+1)^{3/2}}\ln x\varphi'\Big];\end{aligned}$$

$$u_x = \exp\Big(-\frac{P\ln x}{A^2}+\frac{t(\ln x)^2}{2A^2t^2+1}+\frac{P^2t}{2A^2}-\frac{P^2}{2^{\frac{3}{2}}A^3}tan^{-1}\sqrt{2}At+Ct$$
$$-\frac{C}{\sqrt{2}A}tan^{-1}\sqrt{2}A^2t-\frac{1}{4}ln(2A^2t^2+1)\Big)\Big[\Big(-\frac{P}{A^2x}+\frac{2t\ln x}{x(2A^2t^2+1)}\Big)\varphi$$
$$+\frac{1}{x(2A^2t^2+1)^{1/2}}\varphi'\Big];$$

$$u_{xx} = \exp\Big(-\frac{P\ln x}{A^2}+\frac{t(\ln x)^2}{2A^2t^2+1}+\frac{P^2t}{2A^2}-\frac{P^2}{2^{\frac{3}{2}}A^3}\tan^{-1}\sqrt{2}At+Ct$$
$$-\frac{C}{\sqrt{2}A}\tan^{-1}\sqrt{2}A^2t-\frac{1}{4}\ln(2A^2t^2+1)\Big)\Big[\Big(-\frac{P}{A^2x}+\frac{2t\ln x}{x(2A^2t^2+1)}\Big)^2\varphi$$
$$+\Big(\frac{P}{A^2x^2}+\frac{2t}{2A^2t^2+1}\Big(\frac{1-\ln x}{x^2}\Big)\Big)\varphi+\Big(-\frac{P}{A^2x}+\frac{2t\ln x}{x(2A^2t^2+1)}\Big)$$
$$\times\Big(\frac{1}{x(2A^2t^2+1)^{1/2}}\varphi'\Big)+\frac{1}{x(2A^2t^2+1)^{1/2}}\Big(-\frac{P}{A^2x}+\frac{2t\ln x}{x(2A^2t^2+1)}\Big)\varphi'$$
$$-\frac{1}{x^2(2A^2t^2+1)^{1/2}}\varphi'+\frac{1}{x^2(2A^2t^2+1)}\varphi''\Big]$$

Setting the above derivatives in the Black-Scholes (1) Equation we obtain

$$u_t+\frac{1}{2}A^2x^2u_{xx}+Bxu_x-Cu = \exp\Big(-\frac{P\ln x}{A^2}+\frac{t(\ln x)^2}{2A^2t^2+1}+\frac{P^2t}{2A^2}$$
$$-\frac{P^2}{2^{\frac{3}{2}}A^3}\tan^{-1}\sqrt{2}At+Ct-\frac{C}{\sqrt{2}A}tan^{-1}\sqrt{2}A^2t-\frac{1}{4}\ln(2A^2t^2+1)\Big)$$
$$\times\Big[\Big((\ln x)^2\Big(\frac{2A^2t^2+1-4A^2t^2}{(2A^2t^2+1)^2}+\frac{P^2t^2}{2A^2t^2+1}+\frac{2A^2Ct^2}{2A^2t^2+1}-\frac{A^2t}{2A^2t^2+1}\Big)\varphi$$
$$-\frac{2A^2t}{(2A^2t^2+1)^{3/2}}\ln x\varphi'\Big)+\frac{1}{2}A^2x^2\Big[\Big(-\frac{P}{A^2x}+\frac{2t\ln x}{x(2A^2t^2+1)}\Big)^2\varphi$$
$$+\Big(\frac{P}{A^2x^2}+\frac{2t(1-\ln x)}{x^2(2A^2t^2+1)}\Big)\varphi+\Big(-\frac{2P}{A^2x^2(2A^2t^2+1)^{1/2}}$$
$$+\frac{4t\ln x}{x^2(2A^2t^2+1)^{3/2}}\Big)\varphi'-\frac{1}{x^2(2A^2t^2+1)^{1/2}}\varphi'+\frac{1}{x^2(2A^2t^2+1)}\varphi''\Big]$$
$$+Bx\Big[\Big(-\frac{P}{A^2x}+\frac{2t\ln x}{x(2A^2t^2+1)}\Big)\varphi+\frac{1}{x(2A^2t^2+1)^{1/2}}\varphi'\Big]-C\varphi\Big]$$

$$
\begin{aligned}
u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = &= \frac{A^2}{2(2A^2t^2+1)}\varphi'' + \Big(- \frac{2A^2t\ln x}{(2A^2t^2+1)^{3/2}} \\
&- \frac{P}{(2A^2t^2+1)^{1/2}} + \frac{2A^2t\ln x}{(2A^2t^2+1)^{3/2}} - \frac{A^2}{2(2A^2t^2+1)^{1/2}} + \frac{B}{(2A^2t^2+1)^{1/2}}\Big)\varphi' \\
&+ \Big(\frac{(\ln x)^2}{(2A^2t^2+1)^2} - \frac{2A^2t^2(\ln x)^2}{(2A^2t^2+1)^2} + \frac{P^2t^2}{2A^2t^2+1} + \frac{2A^2Ct^2}{2A^2t^2+1} \\
&- \frac{A^2t}{2A^2t^2+1} + \frac{P^2}{2A^2} - \frac{2Pt\ln x}{2A^2t^2+1} + \frac{2A^2t^2(\ln x)^2}{(2A^2t^2+1)^2} \\
&+ \frac{P}{2} + \frac{A^2t}{2A^2t^2+1} - \frac{A^2t\ln x}{2A^2t^2+1} - \frac{BP}{A^2} + \frac{2Bt\ln x}{2A^2t^2+1} - C\Big)\varphi
\end{aligned}
$$

where $P = B - \frac{A^2}{2}$. Hence our Black-Scholes Equation becomes

$$
\begin{aligned}
u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu =& \frac{A^2}{2(2A^2t^2+1)}\varphi'' + \Big(\frac{(\ln x)^2}{(2A^2t^2+1)^2} - \frac{P^2}{2A^2} \\
&+ \frac{P^2t^2}{2A^2t^2+1} + \frac{2A^2Ct^2}{2A^2t^2+1}\Big)
\end{aligned}
$$

Multiplying both sides of the above result by

$$
\frac{2(2A^2t^2+1)}{A^2}
$$

and equating the equation to zero we obtain

$$
\begin{aligned}
\varphi'' + \Big(\tfrac{2}{A^2}J_1^2 - \tfrac{P^2(2A^2t^2+1)}{A^4} + \tfrac{2P^2t^2}{A^2} + \tfrac{2}{A^2}(2A^2Ct^2) - \tfrac{2}{A^2}(2A^2t^2+1)C\Big)\varphi = 0, \\
\varphi'' + \Big(\tfrac{2}{A^2}{J_1}^2 - \tfrac{2P^2A^2t^2}{A^4} - \tfrac{P^2}{A^4} + \tfrac{2P^2t^2}{A^2} + 4Ct^2 - 4Ct^2 - \tfrac{2C}{A^2}\Big)\varphi = 0, \\
\varphi'' + \tfrac{2}{A^2}\Big(J_1 - \tfrac{P^2}{2A^2 - C}\Big)\varphi = 0.
\end{aligned}
$$

Hence, we have,

$$
\varphi'' + \frac{2}{A^2}\Big(J_1 - K\Big)\varphi = 0 \tag{96}
$$

where $K = \frac{P^2}{2A^2} + C$.

The operator

$$
X_1 - X_5 = -(2A^2t^2 - 1)\frac{\partial}{\partial t} - 2A^2xt\ln x\frac{\partial}{\partial x} - [(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u\frac{\partial}{\partial u}
$$

has a characteristic system

$$-\frac{dt}{2A^2t^2-1} = -\frac{dx}{2A^2xt\ln x} = -\frac{du}{[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u}.$$

The first linear equation from the characteristic system

$$-\frac{dt}{2A^2t^2-1} = -\frac{dx}{2A^2xt\ln x}, \qquad \frac{2A^2t}{2A^2t^2-1}dt = \frac{dx}{x\ln x}$$

$$\frac{1}{2}\ln(2A^2t^2-1) = \ln x(\ln x) + \ln C, \qquad \ln x = J_1(2A^2t^2-1)^{1/2}$$

Hence the first invariant is

$$J_1 = (2A^2t^2-1)^{-1/2}\ln x. \tag{97}$$

Checking of invariant J_1:
Checking for the invariant J_1 we have

$$\begin{aligned}(X_1 - X_5)(J_1) &= (2A^2t^2-1)\frac{\partial J_1}{\partial t} + 2A^2tx\ln x\frac{\partial J_1}{\partial x} \\ &\quad + \Big[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t\Big]u\frac{\partial J_1}{\partial u} \\ &= (2A^2t^2-1)\ln x\Big[-\frac{1}{2}\big(2A^2t^2-1\big)^{-3/2}4A^2t\Big] \\ &\quad + 2A^2tx\ln x\big[\frac{1}{x}\big(2A^2t^2-1\big)^{-1/2}\big] \\ &= -2A^2t\ln x\big(2A^2t^2-1\big)^{-1/2} + 2A^2t\ln x\big(2A^2t^2+1\big)^{-1/2} \\ &= 0.\end{aligned}$$

The second linear equation from this characteristic system is

$$\frac{dt}{2A^2t^2-1} = \frac{du}{[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u}.$$

Setting in the expression for $\ln x$ we obtain

$$\Big(\frac{\big(J_1(2A^2t^2-1)^{1/2} - Pt\big)^2}{2A^2t^2-1} - \frac{2A^2Ct^2}{2A^2t^2-1} - \frac{A^2t}{2A^2t^2-1}\Big)dt = \frac{du}{u}$$

$$\Big(J_1{}^2 - \frac{2J_1Pt}{(2A^2t^2-1)^{1/2}} + \frac{P^2t^2}{2A^2t^2-1} + \frac{2A^2Ct^2}{2A^2t^2-1} - \frac{A^2t}{2A^2t^2-1}\Big)dt = \frac{du}{u}$$

Upon integration, we obtain

$$
\begin{aligned}
&{J_1}^2 t - \frac{J_1 P}{A^2}(2A^2t^2-1)^{1/2} + P^2\Big(\frac{2At-\sqrt{2}\tanh^{-1}\sqrt{2}At}{4A^3}\Big) \\
&+2A^2C\Big(\frac{2At-\sqrt{2}\tanh^{-1}\sqrt{2}At}{4A^3}\Big) - \frac{1}{4}\ln(2A^2t^2-1) = \ln u + \ln C_2
\end{aligned}
$$

where $\ln C_2$ is the constant of integration. This yields

$$
\begin{aligned}
J_2 = &\frac{J_1P(2A^2t^2-1)^{1/2}}{A^2} - {J_1}^2 t - \frac{P^2t}{A^2} + \frac{P^2}{2^{3/2}A^3}\tanh^{-1}\sqrt{2}At \\
&- Ct + \frac{C}{A\sqrt{2}}\tanh^{-1}\sqrt{2}At + \frac{1}{4}\ln(2A^2t^2-1) + \ln u.
\end{aligned}
$$

Setting in value of J_1 we obtain the invariant

$$
\begin{aligned}
J_2 = &\frac{P\ln x}{A^2} - \frac{t(\ln x)^2}{2A^2t^2-1} - \frac{P^2t}{A^2} + \frac{P^2}{2^{3/2}A^3}\tanh^{-1}\sqrt{2}At - Ct \\
&+ \frac{C}{A\sqrt{2}}\tanh^{-1}\sqrt{2}At + \frac{1}{4}\ln(2A^2t^2-1) + \ln u
\end{aligned}
\tag{98}
$$

which is our second invariant equation.

Checking of invariant J_2:
The second invariant (J_2) is checked as follows

$$
\begin{aligned}
(X_1 - X_5)(J_2) =&(2A^2t^2-1)\frac{\partial J_2}{\partial t} + 2A^2tx\ln x\frac{\partial J_2}{\partial x} + \Big[(\ln x - Pt)^2 \\
&+ 2A^2Ct^2 - A^2t\Big]u\frac{\partial}{\partial u} \\
&= (2A^2t^2-1)\Big[-(\ln x)^2\Big(\frac{2A^2t^2-1-4A^2t^2}{(2A^2t^2-1)^2}\Big) - \frac{P^2t^2}{2A^2t^2-1} \\
&- \frac{2A^2Ct^2}{2A^2t^2-1} + \frac{A^2t}{2A^2t^2-1}\Big] + 2A^2tx\ln x\Big[\frac{P}{xA^2} \\
&- \frac{2t\ln x}{x(2A^2t^2-1)}\Big] + \Big[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t\Big] \\
&= -(lnx)^2 + \frac{4A^2t^2(\ln x)^2}{2A^2t^2-1} - P^2t^2 - 2A^2Ct^2 + A^2t + 2Pt\ln x \\
&- \frac{4A^2t^2(\ln x)^2}{2A^2t^2-1} + (\ln x)^2 - 2Pt\ln x + P^2t^2 + 2A^2Ct^2 - A^2t \\
&= 0.
\end{aligned}
$$

Expressing the invariant solution as $J_2 = \varphi(J_1)$ we obtain

$$
\begin{aligned}
u = \exp\Big(& -\frac{P\ln x}{A^2} + \frac{t(\ln x)^2}{2A^2t^2-1} + \frac{P^2 t}{A^2} - \frac{P^2}{2^{3/2}A^3}\tanh^{-1}\sqrt{2}At + Ct \\
& - \frac{C}{A\sqrt{2}}\tanh^{-1}\sqrt{2}At - \frac{1}{4}\ln(2A^2t^2-1)\Big)\varphi(2A^2t^2-1)^{-1/2}\ln x.
\end{aligned} \tag{99}
$$

Differentiating u with respect to t and x we obtain

$$
\begin{aligned}
u_t = & \exp\Big(-\frac{P\ln x}{A^2} + \frac{t(\ln x)^2}{2A^2t^2-1} + \frac{P^2 t}{2A^2} - \frac{P^2}{2^{3/2}A^3}\tanh^{-1}\sqrt{2}At + Ct \\
& - \frac{C}{A\sqrt{2}}\tanh^{-1}\sqrt{2}At - \frac{1}{4}\ln(2A^2t^2-1)\Big)\Big[\Big((\ln x)^2\frac{(2A^2t^2-1-4A^2t^2)}{(2A^2t^2-1)^2} \\
& + \frac{P^2t^2}{2A^2t^2-1} + \frac{2A^2Ct^2}{2A^2t^2-1} - \frac{A^2t}{2A^2t^2-1}\Big)\varphi - \frac{2A^2t}{(2A^2t^2-1)^{3/2}}\ln x\varphi'\Big]
\end{aligned}
$$

$$
\begin{aligned}
u_x = & \exp\Big(-\frac{P\ln x}{A^2} + \frac{t(\ln x)^2}{2A^2t^2-1} + \frac{P^2 t}{2A^2} - \frac{P^2}{2^{3/2}A^3}\tanh^{-1}\sqrt{2}At + Ct \\
& - \frac{C}{A\sqrt{2}}\tanh^{-1}\sqrt{2}At - \frac{1}{4}\ln(2A^2t^2-1)\Big)\Big[\Big(-\frac{P}{A^2x} + \frac{2t\ln x}{x(2A^2t^2-1)}\Big)\varphi \\
& + \frac{1}{x(2A^2t^2-1)^{1/2}}\varphi'\Big]
\end{aligned}
$$

$$
\begin{aligned}
u_{xx} = & \exp\Big(-\frac{P\ln x}{A^2} + \frac{t(\ln x)^2}{2A^2t^2-1} + \frac{P^2 t}{2A^2} - \frac{P^2}{2^{3/2}A^3}\tanh^{-1}\sqrt{2}At + Ct \\
& - \frac{C}{A\sqrt{2}}\tanh^{-1}\sqrt{2}At - \frac{1}{4}\ln(2A^2t^2-1)\Big)\Big[\Big(-\frac{P}{A^2x} + \frac{2t\ln x}{x(2A^2t^2-1)}\Big)^2\varphi \\
& + \Big(\frac{P}{A^2x^2} + \frac{2t}{2A^2t^2-1}\Big(\frac{1-\ln x}{x^2}\Big)\Big)\varphi + \Big(-\frac{P}{A^2x} + \frac{2t\ln x}{x(2A^2t^2-1)}\Big) \\
& \times\Big(\frac{1}{x(2A^2t^2-1)^{1/2}}\varphi'\Big) + \frac{1}{x(2A^2t^2-1)^{1/2}}\Big(-\frac{P}{A^2x} + \frac{2t\ln x}{x(2A^2t^2-1)}\Big)\varphi' \\
& - \frac{1}{x^2(2A^2t^2-1)^{1/2}}\varphi' + \frac{1}{x^2(2A^2t^2-1)}\varphi''\Big]
\end{aligned}
$$

Setting the above derivatives in the Black-Scholes (1) Equation we obtain

$$
\begin{aligned}
u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = {} & \exp\Big(- \frac{P\ln x}{A^2} + \frac{t(\ln x)^2}{2A^2t^2-1} + \frac{P^2t}{2A^2} \\
& - \frac{P^2}{2^{3/2}A^3}\tanh^{-1}\sqrt{2}At + Ct - \frac{C}{A\sqrt{2}}\tanh^{-1}\sqrt{2}At - \frac{1}{4}\ln(2A^2t^2-1)\Big) \\
& \times \Big[\Big((\ln x)^2\Big(\frac{2A^2t^2-1-4A^2t^2}{(2A^2t^2-1)^2} + \frac{P^2t^2}{2A^2t^2-1} + \frac{2A^2Ct^2}{2A^2t^2-1} - \frac{A^2t}{2A^2t^2-1}\Big)\varphi \\
& - \frac{2A^2t}{(2A^2t^2-1)^{3/2}}\ln x\varphi'\Big) + \frac{1}{2}A^2x^2\Big[\Big(- \frac{P}{A^2x} + \frac{2t\ln x}{x(2A^2t^2-1)}\Big)^2\varphi \\
& + \Big(\frac{P}{A^2x^2} + \frac{2t(1-\ln x)}{x^2(2A^2t^2-1)}\Big)\varphi + \Big(- \frac{2P}{A^2x^2(2A^2t^2-1)^{1/2}} \\
& + \frac{4t\ln x}{x^2(2A^2t^2-1)^{3/2}}\Big)\varphi' - \frac{1}{x^2(2A^2t^2-1)^{1/2}}\varphi' + \frac{1}{x^2(2A^2t^2-1)}\varphi''\Big] \\
& + Bx\Big[\Big(- \frac{P}{A^2x} + \frac{2t\ln x}{x(2A^2t^2-1)}\Big)\varphi + \frac{1}{x(2A^2t^2-1)^{1/2}}\varphi'\Big] - C\varphi\Big] \\
= {} & \frac{A^2}{2(2A^2t^2-1)}\varphi'' + \Big(- \frac{2A^2t\ln x}{(2A^2t^2-1)^{3/2}} - \frac{P}{(2A^2t^2-1)^{1/2}} \\
& + \frac{2A^2t\ln x}{(2A^2t^2-1)^{3/2}} - \frac{A^2}{2(2A^2t^2-1)^{1/2}} + \frac{B}{(2A^2t^2-1)^{1/2}}\Big)\varphi' \\
& + \Big(- \frac{(\ln x)^2}{(2A^2t^2-1)^2} - \frac{2A^2t^2(\ln x)^2}{(2A^2t^2-1)^2} + \frac{P^2t^2}{2A^2t^2-1} + \frac{2A^2Ct^2}{2A^2t^2-1} \\
& - \frac{A^2t}{2A^2t^2-1} + \frac{P^2}{2A^2} - \frac{2tP\ln x}{2A^2t^2-1} + \frac{2A^2t^2(\ln x)^2}{2A^2t^2-1}^2 + \frac{P}{2} \\
& + \frac{A^2t}{2A^2t^2-1} - \frac{A^2t\ln x}{2A^2t^2-1} - \frac{BP}{A^2} + \frac{2Bt\ln x}{2A^2t^2-1} - C\Big)\varphi
\end{aligned}
$$

where $P = B - \frac{A^2}{2}$. Hence our Black-Scholes Equation becomes (1)

$$
\begin{aligned}
u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = {} & \frac{A^2}{2(2A^2t^2-1)}\varphi'' + \Big(- \frac{(\ln x)^2}{2A^2t^2-1} \\
& + \frac{P^2t^2}{2A^2t^2-1} + \frac{2A^2Ct^2}{2A^2t^2-1} - \frac{P^2}{2A^2} - C\Big)\varphi
\end{aligned}
$$

Multiplying both sides of the above result by

$$\frac{2(2A^2t^2-1)}{A^2}$$

and equating the equation to 0 we obtain

$$\varphi'' - \frac{2}{A^2}(J_1^2 - \frac{P^2}{2A^2} - C)\varphi = 0. \tag{100}$$

We furthermore, consider the operator

$$X_2 + X_5 = 2A^2t^2\frac{\partial}{\partial t} + (2A^2t\ln x + 1)x\frac{\partial}{\partial x}$$
$$+ [(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u\frac{\partial}{\partial u}.$$

This operator has the characteristic system

$$\frac{dt}{2A^2t^2} = \frac{dx}{(2A^2t\ln x + 1)x} = \frac{du}{[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u}$$

and the first equation which is written as a linear equation is

$$\frac{dt}{2A^2t^2} = \frac{dx}{(2A^2t\ln x + 1)x}.$$

To compute this equation we will proceed as follows

$$(2A^2tx\ln x + x)dt - 2A^2t^2dx = 0.$$
$$\frac{dx}{dt} = \frac{x\ln x}{t} + \frac{x}{2A^2t^2}$$
$$t\frac{dx}{dt} = x(\ln x + \frac{1}{2A^2t}).$$

This last equation is of the form $xy' = y[\ln y + F(x)]$. Therefore the above equation admits the symmetry

$$X = xt\frac{\partial}{\partial x}.$$

Now from the equation

$$(2A^2tx\ln x + x)dt - 2A^2t^2dx = 0 \tag{101}$$

let us say that

$$M = 2A^2tx\ln x + x, \qquad N = -2A^2t^2$$

then we have

$$\frac{\partial M}{\partial x} = 2A^2t[1 + \ln x] + 1 \qquad \text{and} \qquad \frac{\partial N}{\partial t} = -4A^2t.$$

This clearly shows that the equation is not integrable since

$$\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}.$$

Now that we have established that the reckoning equation is not exact we will have have to convert it to exact equation. We will do that by multiplying through by an appropriate function. We know that the $Integrating factor$ (Lie's Integrating Factor) is given as

$$\mu = \frac{1}{\xi M + \eta N}.$$

Setting in the expressions of M and N we obtain

$$\mu = \frac{1}{xt(-2A^2t^2)} = -\frac{1}{2A^2t^3x}.$$

where $\xi = 0$ and $\eta = xt$ form the symmetry stated above. Multiplying μ to M and N we will obtain

$$\mu M = -\Big(\frac{2A^2tx\ln x}{2A^2t^3x} + \frac{x}{2A^2t^3x}\Big) = -\frac{\ln x}{t^2} - \frac{1}{2A^2t^3},$$
$$\mu N = -\frac{1}{2A^2t^3x}(-2A^2t^2) = \frac{1}{tx}.$$

Taking derivatives of μM and μN with respect to x and t respectively, we obtain

$$\frac{\partial \mu M}{\partial x} = -\frac{1}{xt^2}, \qquad \frac{\partial \mu N}{\partial t} = -\frac{1}{xt^2}.$$

This clearly shows that

$$\frac{\partial \mu M}{\partial x} = \frac{\partial \mu N}{\partial t}.$$

Equation (101) hence becomes

$$-\Big(\frac{\ln x}{t^2} - \frac{1}{2A^2t^3}\Big)dt + \frac{1}{tx}dx = 0.$$

Integrating the above corresponding exact equation yields

$$\Phi(x,t) = \int \frac{1}{tx}dx + g(t) = \frac{1}{t}\ln x + g(t) \tag{102}$$

Differentiating this results with respect to t and equating it to μ we obtain,

$$-\frac{1}{t^2}\ln x + g'(t) = -\Big(\frac{\ln x}{t^2} + \frac{1}{2A^2t^3}\Big) \Longrightarrow g'(t) = -\frac{1}{2A^2t^3} \Longrightarrow g(t) = \frac{1}{4A^2t^2}.$$

Setting this result in equation (102) we obtain

$$\Phi(x,t) = \frac{1}{t}\ln x + \frac{1}{4A^2t^2}.$$

The above equation yields the first invariant

$$J_1 = \frac{1}{t}\Big(\ln x + \frac{1}{4A^2t}\Big) \tag{103}$$

where we can express lnx and C_1 in terms of each other as

$$\ln x = -\frac{1}{4A^2t} + C_1t, \qquad C_1 = \frac{1}{t}\ln x + \frac{1}{4A^2t^2}. \tag{104}$$

Checking of invariant J_1:
Checking for the invariant J_1 we have

$$\begin{aligned}(X_2+X_5)(J_1) &= 2A^2t^2\frac{\partial J_1}{\partial t} + (2A^2t\ln x + 1)x\frac{\partial J_1}{\partial x}\\ &\quad + [(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u\frac{\partial J_1}{\partial u}\\ &= 2A^2t^2\Big(-\frac{\ln x}{t^2} - \frac{1}{2A^2t^3}\Big) + (1+2A^2t\ln x)x\Big(\frac{1}{xt}\Big)\\ &= -2A^2\ln x - \frac{1}{t} + \frac{1}{t} + 2A^2\ln x\\ &= 0.\end{aligned}$$

Furthermore, the second equation also written as a linear is

$$\frac{dt}{2A^2t^2} = \frac{du}{[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u}.$$

Setting in equation the expression for $\ln x$ we have

$$\frac{dt}{2A^2t^2} = \frac{du}{\Big[\Big(-\frac{1}{4A^2t} + C_1t - Pt\Big)^2 + 2A^2Ct^2 - A^2t\Big]u}$$

$$\Big(\frac{1}{2A^2}\Big[-\frac{1}{4A^2t^2} + (C_1 - P)\Big]^2 + C - \frac{1}{2t}\Big)dt = \frac{du}{u}$$

$$\Big[\frac{1}{2A^2}\Big(\frac{1}{16A^4t^2} - \frac{1}{2A^2t^2}(C_1-P) + (C_1-P)^2\Big) + C - \frac{1}{2t}\Big]dt = \frac{du}{u}.$$

Upon integration we have

$$\Big[\frac{1}{2A^2}\Big(-\frac{1}{48A^4t^3}+\frac{1}{2A^2t}(C_1-P)+t(C_1-P)^2\Big)+Ct-\frac{1}{2}\ln t\Big]dt=\ln u+\ln C_2.$$

This yields the second invariant

$$J_2=\Big[\frac{1}{2A^2}\Big(\frac{1}{48A^4t^3}-\frac{1}{2A^2t}(C_1-P)-t(C_1-P)^2\Big)-Ct+\frac{1}{2}lnt\Big]+\ln u.$$

Setting in the expression for C_1 we have

$$\begin{aligned}J_2=&\Big[\frac{1}{2A^2}\Big(\frac{1}{48A^4t^3}-\frac{1}{2A^2t}\Big(\frac{\ln x}{t}+\frac{1}{4A^2t^2}-P\Big)\\&-t\Big(\frac{\ln x}{t}+\frac{1}{4A^2t^2}-P\Big)^2\Big)-Ct+\frac{1}{2}\ln t\Big]+\ln u\\=&\frac{1}{2A^2}\Big(\frac{1}{48A^4t^3}-\frac{\ln x}{2A^2t}-\frac{1}{8A^4t^3}+\frac{P}{2A^2t}-\frac{(\ln x)^2}{t}-P^2t+2P\ln x\Big)\\&-Ct+\frac{1}{2}\ln t+\ln u.\end{aligned}$$

The second invariant finally becomes

$$\begin{aligned}J_2=&\Big[\frac{1}{2A^2}\Big(-\frac{1}{6A^4t^3}-\frac{\ln x}{A^2t^2}+\frac{P}{A^2t}-\frac{(\ln x)^2}{t}-P^2t+2P\ln x\Big)\\&-Ct+\frac{1}{2}\ln t\Big]+\ln u.\end{aligned}\tag{105}$$

Checking of invariant J_2:

As usual we will check for the invariant condition

$$\begin{aligned}(X_2+X_5)(J_2)=&2A^2t^2\frac{\partial J_2}{\partial t}+(2A^2t\ln x+1)x\frac{\partial J_2}{\partial x}+[(\ln x-Pt)^2\\&+2A^2Ct^2-A^2t]u\frac{\partial J_2}{\partial u}\\=&2A^2t^2\Big[\frac{1}{2A^2}\Big(\frac{1}{2A^4t^4}+\frac{2\ln x}{A^2t^3}-\frac{P}{A^2t^2}+\frac{(\ln x)^2}{t^2}-P^2\Big)-C+\frac{1}{2t}\Big]\\&+(2A^2tx\ln x+x)\Big[\frac{1}{2A^2}\Big(-\frac{1}{xA^2t^2}-\frac{2\ln x}{xt}+\frac{2P}{x}\Big)\Big]\\&+\Big((\ln x)^2-2Pt\ln x+P^2t^2+2A^2Ct^2-A^2t\Big)u\frac{1}{u}\\=&\frac{1}{2A^47^2}+\frac{2\ln x}{A^2t}-\frac{P}{A^2}+(\ln x)^2-P^2t^2-2CA^2t^2+A^2t-\frac{\ln x}{A^2t}\\&-2(\ln x)^2+2Pt\ln x-\frac{1}{2A^4t^2}-\frac{\ln x}{A^2t}+\frac{P}{A^2}\\&+(\ln x)^2-2Pt\ln x+P^2t^2+2A^2Ct^2-A^2t\\=&0.\end{aligned}$$

Taking exponential on both sides of equation (221) we obtain

$$J_2 u = \exp\Big[\frac{1}{2A^2}\Big(-\frac{1}{6A^4t^3}-\frac{\ln x}{A^2t^2}+\frac{P}{A^2t} -\frac{(\ln x)^2}{t}-P^2t+2P\ln x\Big)-Ct+\frac{1}{2}\ln t\Big].$$

Expressing J_2 as function of J_1, that is $J_2=\varphi(J_1)$, we obtain the invariant

$$u=\frac{1}{\sqrt{t}}\exp\Big[\frac{1}{2A^2}\Big(\frac{1}{6A^4t^3}+\frac{\ln x}{A^2t^2}-\frac{P}{A^2t} +\frac{(\ln x)^2}{t}+P^2t-2P\ln x\Big)+Ct\Big]\varphi\Big(\frac{1}{t}(\ln x+\frac{1}{4A^2t})\Big). \tag{106}$$

Letting

$$\mathbf{e}=\exp\Big[\frac{1}{2A^2}\Big(-\frac{1}{6A^4t^3}-\frac{\ln x}{A^2t^2}+\frac{P}{A^2t} -\frac{(\ln x)^2}{t}-P^2t+2P\ln x\Big)-Ct\Big]$$

let us differentiate equation (106) with respect to t and x, we have

$$\begin{aligned}
u_t &= -\frac{1}{2t^{3/2}}\mathbf{e}\varphi+\sqrt{t}\Big[\frac{1}{2A^2}\Big(-\frac{1}{2A^4t^4}-\frac{2\ln x}{A^2t^3}+\frac{P}{A^2t^2}-\frac{(\ln x)^2}{t^2}+P^2\Big) \\
&\quad +C\Big]\mathbf{e}\varphi+\sqrt{t}\Big[-\frac{1}{t^2}\Big(\ln x+\frac{1}{4A^2t}\Big)+\frac{1}{t}\Big(-\frac{1}{4A^2t^2}\Big)\Big]\mathbf{e}\varphi' \\
&= \frac{\mathbf{e}}{2\sqrt{t}A^2}\Big[\Big(-\frac{A^2}{t}-\frac{1}{2A^4t^4}-\frac{2\ln x}{A^2t^3}+\frac{P}{A^2t^2}-\frac{(\ln x)^2}{t^2} \\
&\quad +P^2+2A^2C\Big)\varphi+\Big(-\frac{2A^2\ln x}{t^2}-\frac{1}{t^3}\Big)\varphi'\Big] \\
u_x &= \frac{1}{\sqrt{t}}\Big[\frac{1}{2A^2}\Big(\frac{1}{A^2t^2x}+\frac{2\ln x}{xt}-\frac{2P}{x}\Big)\mathbf{e}\varphi+\Big(\frac{1}{xt}\Big)\mathbf{e}\varphi'\Big] \\
&= \frac{\mathbf{e}}{2A^2\sqrt{t}}\Big[\Big(\frac{1}{A^2t^2x}+\frac{2\ln x}{xt}-\frac{2P}{x}\Big)\varphi+\frac{2A^2}{xt}\varphi'\Big] \\
u_{xx} &= \frac{\mathbf{e}}{2A^2\sqrt{t}}\Big[\frac{1}{A^2x^2}\Big(\frac{1}{2}\Big)\Big(\frac{1}{A^2t^2}+\frac{2\ln x}{t}-2P\Big)^2\varphi \\
&\quad +\Big(-\frac{1}{t^2}+\frac{2A^2}{t}-\frac{2A^2\ln x}{t}+2A^2P\Big)\varphi+\Big(\frac{1}{t^3}+\frac{2A^2\ln x}{t^2} \\
&\quad +\frac{2A^2P}{t}\Big)\varphi'+\Big(\frac{1}{t^3}+\frac{2A^2\ln x}{t^2}+\frac{2A^2P}{t}\Big)\varphi'-\frac{2A^4}{t}\varphi'+\frac{2A^4}{t^2}\varphi''\Big]
\end{aligned}$$

Setting the above derivatives in equation (1), we obtain

$$
\begin{aligned}
u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu =& \frac{\mathbf{e}}{2A^2\sqrt{t}}\Big[\Big(-\frac{A^2}{t} - \frac{1}{2A^4t^4} - \frac{2\ln x}{A^2t^3} + \frac{P}{A^2t^2} \\
& - \frac{(\ln x)^2}{t^2} + P^2 + 2A^2C\Big)\varphi + \Big(-\frac{2A^2\ln x}{t^2} - \frac{1}{t^3}\Big)\varphi' + \frac{1}{2}\Big[\Big(\frac{1}{2}\Big)\Big(\frac{1}{A^2t^2} \\
& + \frac{2\ln x}{t} - 2P\Big)^2\varphi + \Big(-\frac{1}{t^2} + \frac{2A^2}{t} - \frac{2A^2\ln x}{t} + 2A^2P\Big)\varphi + 2\Big(\frac{1}{t^3} \\
& + \frac{2A^2\ln x}{t^2} + \frac{2A^2P}{t}\Big)\varphi' - \frac{2A^4}{t}\varphi' + \frac{2A^4}{t^2}\varphi''\Big] + B\Big[\Big(\frac{1}{A^2t^2} + \frac{2\ln x}{t} - 2P\Big)\varphi \\
& + \frac{2A^2}{t}\varphi'\Big] - 2CA^2\varphi\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A^4}{t^2}\varphi'' + \Big(-\frac{2A^2\ln x}{t^2} - \frac{1}{t^3} + \frac{1}{t^3} + \frac{2A^2\ln x}{t^2} - \frac{2A^2P}{t} - \frac{A^4}{t} + \frac{2BA^2}{t}\Big)\varphi' \\
&+ \Big(-\frac{A^2}{t} - \frac{1}{2A^4t^4} - \frac{2\ln x}{A^2t^3} + \frac{P}{A^2t^2} - \frac{(\ln x)^2}{t^2} + P^2 + 2A^2C + \frac{1}{4A^4t^4} \\
&+ \frac{(\ln x)^2}{t^2} + P^2 + \frac{\ln x}{A^2t^3} - \frac{2P\ln x}{t} - \frac{P}{A^2t^2} - \frac{1}{2t^2} + \frac{A^2}{t} - \frac{A^2\ln x}{t} \\
&+ A^2P + \frac{B}{A^2t^2} + \frac{2B\ln x}{t} - 2PB - 2A^2C\Big)\varphi \\
&= \frac{A^4}{t^2}\varphi'' + \Big(\frac{1}{4A^4t^4} - \frac{\ln x}{A^2t^3} + 2P^2 - \frac{1}{2t^2} + A^2P + \frac{B}{A^2t^2} - 2PB\Big)\varphi \\
&= \frac{A^4}{t^2}\varphi'' + \Big(\frac{1}{4A^4t^4} - \frac{\ln x}{A^2t^3}\Big) + \Big(2P^2 - \frac{1}{2t^2} + A^2P + \frac{B}{A^2t^2} - 2P(P + \frac{A^2}{2})\Big)\varphi \\
&= \frac{A^4}{t^2}\varphi'' + \Big(\frac{1}{4A^4t^4} - \frac{\ln x}{A^2t^3} + 2P^2 - \frac{1}{2t^2} + A^2P + \frac{B}{A^2} - 2P^2 - A^2P\Big)\varphi \\
&= \frac{1}{A^2t^2}\Big(A^6\varphi'' - \frac{1}{t}\Big(\ln x - \frac{1}{4A^2t^2}\Big) - \frac{A^2}{2} + B\Big) \\
&= A^6\varphi'' - \Big(J_1 - B - \frac{A^2}{2}\Big)
\end{aligned}
$$

Therefore, we have

$$\varphi'' - \frac{1}{A^6}(J_1 - P)\varphi = 0. \tag{107}$$

Letting $z = k(J_1 - P)$ then $\frac{dz}{dJ_1} = k$. Therefore, from

$$\frac{d\varphi}{dJ_1} = \frac{d\varphi}{dz}\frac{dz}{dJ_1}$$

we obtain

$$\frac{d\varphi}{dJ_1} = k\frac{d\varphi}{dz} \qquad \text{or} \qquad \varphi' = k\dot{\varphi}.$$

The second derivative of the above result yields

$$\frac{d^2\varphi}{dJ_1^2} = k\Big[\ddot{\varphi}\frac{dz}{dJ_1}\Big] \qquad \text{or} \qquad \varphi'' = k^2\ddot{\varphi}$$

Setting these derivatives in equation (107) we have

$$k^2\ddot{\varphi} - \frac{1}{A^6}\frac{z}{k}\varphi = 0$$

which gives us

$$\ddot{\varphi} - \frac{z}{A^6k^3}\varphi = 0. \tag{108}$$

But we know from Airy equation that

$$y'' - xy = 0. \tag{109}$$

Hence, comparing the above two equations we obtain $\frac{1}{A^6k^3} = 1$ which yields $k = \frac{1}{A^2}$. Therefore $z = \frac{1}{A^2}(J_1 - P)$. Equation (223) hence becomes

$$\ddot{\varphi} - z\varphi = 0$$

which is also an Airy equation. The general solution for this Airy equation is

$$\varphi = C_1A_i(z) + C_2B_i(z) \tag{110}$$

where C_1, C_2 are constants of the Airy equation and A_i and B_i are defined as follows

$$A_i(z) = \frac{1}{\pi}\int_0^\infty \cos(z\tau + \frac{1}{3}\tau^3)d\tau$$
$$B_i(z) = \frac{1}{\pi}\int_0^\infty \Big[exp(z\tau - \frac{1}{3}\tau^3) + \sin(z\tau + \frac{1}{3}\tau^3)\Big]d\tau.$$

Setting equation (110) in (107) we obtain the solution

$$u = \frac{1}{\sqrt{t}}\Big(C_1A_i(z) + C_2B_i(z)\Big)\exp\Big[\frac{1}{2A^2}\Big(\frac{1}{6A^4t^3} + \frac{\ln x}{A^2t^2} - \frac{P}{A^2t} + \frac{(\ln x)^2}{t} + P^2t - 2P\ln x\Big) + Ct\Big]. \tag{111}$$

Furthermore, we consider the operator

$$X_2 - X_5 = -2A^2t^2\frac{\partial}{\partial t} - (2A^2t\ln x - 1)x\frac{\partial}{\partial x}$$
$$- [(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u\frac{\partial}{\partial u}.$$

This operator also has the characteristic system

$$-\frac{dt}{2A^2t^2} = -\frac{dx}{(2A^2t\ln x - 1)x} = -\frac{du}{[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u}$$

and again the first equation which is written as a linear equation is

$$\frac{dt}{2A^2t^2} = \frac{dx}{(2A^2t\ln x - 1)x}.$$

To compute this equation we will proceed as follows

$$(2A^2tx\ln x - x)dt - 2A^2t^2dx = 0$$
$$t\frac{dx}{dt} = x(\ln x - \frac{1}{2A^2t}).$$

The last equation is of the form $xy' = y[\ln y + F(x)]$. Therefore, the above equation also admits the symmetry

$$X = xt\frac{\partial}{\partial x}. \tag{112}$$

Furthermore, from the equation

$$(2A^2tx\ln x + x)dt - 2A^2t^2dx = 0 \tag{113}$$

we once again let

$$M = 2A^2tx\ln x - x, \qquad N = -2A^2t^2$$

then we have

$$\frac{\partial M}{\partial x} = 2A^2t[1 + \ln x] - 1 \qquad \text{and} \qquad \frac{\partial N}{\partial t} = -4A^2t.$$

We have clearly shown that the equation is not integrable since

$$\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}.$$

Since the reckoning equation is not exact we will have to convert it to exact equation by multiplying through an appropriate function as we did earlier. We also know that the $Integrating factor$ (Lie's Integrating Factor) as we stated earlier is given as

$$\mu = \frac{1}{\xi M + \eta N}.$$

Setting in the expressions of M and N and taking $\xi = 0$ and $\eta = xt$, we obtain

$$\mu = \frac{1}{xt(-2A^2t^2)} = -\frac{1}{2A^2t^3x}.$$

Multiplying μ to M and N, we respectively obtain

$$\mu M = -\Big(\frac{2A^2tx\ln x}{2A^2t^3x} - \frac{x}{2A^2t^3x}\Big) = -\frac{\ln x}{t^2} + \frac{1}{2A^2t^3},$$
$$\mu N = -\frac{1}{2A^2t^3x}(-2A^2t^2) = \frac{1}{tx}.$$

Upon taking derivatives of μM and μN with respect to x and t respectively, we obtain

$$\frac{\partial \mu M}{\partial x} = -\frac{1}{xt^2}, \qquad \frac{\partial \mu N}{\partial t} = -\frac{1}{xt^2}.$$

This clearly shows that the equation is now exact since

$$\frac{\partial \mu M}{\partial x} = \frac{\partial \mu N}{\partial t}.$$

Equation (113) hence becomes

$$-\Big(\frac{\ln x}{t^2} - \frac{1}{2A^2t^3}\Big)dt + \frac{1}{tx}dx = 0.$$

Upon integration of the above exact equation we obtain

$$\Phi(x,t) = \int \frac{1}{tx}dx + g(t) = \frac{1}{t}lnx + g(t) \tag{114}$$

Differentiating equation (114) with respect to t and equating it to μ we obtain,

$$-\frac{1}{t^2}\ln x + g'(t) = -\Big(\frac{\ln x}{t^2} - \frac{1}{2A^2t^3}\Big) \Longrightarrow g'(t) = \frac{1}{2A^2t^3} \Longrightarrow g(t) = -\frac{1}{4A^2t^2}.$$

Setting the above result in equation (114) we obtain

$$\Phi(x,t) = \frac{1}{t}\ln x - \frac{1}{4A^2t^2}.$$

From this equation yields the first invariant

$$J_1 = \frac{1}{t}\Big(\ln x - \frac{1}{4A^2t}\Big) \tag{115}$$

where we can express lnx and C_1 in terms of each other as

$$\ln x = \frac{1}{4A^2t} + C_1t, \qquad C_1 = \frac{1}{t}\ln x - \frac{1}{4A^2t^2}. \tag{116}$$

Checking of invariant J_1:
For we to be satisfied that this invariant J_1 is correct, as usual, we will have to check it. Therefore we have

$$\begin{aligned}
(X_2 - X_5)(J_1) &= -2A^2t^2\frac{\partial J_1}{\partial t} - (2A^2t\ln x - 1)x\frac{\partial J_1}{\partial x} \\
&\quad - [(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u\frac{\partial J_1}{\partial u} \\
&= -2A^2t^2\Big(-\frac{\ln x}{t^2} + \frac{1}{2A^2t^3}\Big) + (1 - 2A^2t\ln x)x\Big(\frac{1}{xt}\Big) \\
&= 2A^2\ln x - \frac{1}{t} + \frac{1}{t} - 2A^2\ln x \\
&= 0
\end{aligned}$$

which satisfies the invariant condition. Additionally, the second equation also written as a linear is

$$-\frac{dt}{2A^2t^2} = -\frac{du}{[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t]u}.$$

Setting in equation (116) once again for $\ln x$ into the above equation we have

$$\begin{aligned}
\frac{dt}{2A^2t^2} &= \frac{du}{\Big[\Big(\frac{1}{4A^2t} + C_1t - Pt\Big)^2 + 2A^2Ct^2 - A^2t\Big]u} \\
\frac{\Big[\Big(\frac{1}{4A^2}t + C_1t - Pt\Big)^2 + 2A^2Ct - A^2t\Big]}{2A^2t^2}dt &= \frac{du}{u} \\
\Big(\frac{1}{2A^2}\Big[\frac{1}{4A^2t^2} + (C_1 - P)\Big]^2 + C - \frac{1}{2t}\Big)dt &= \frac{du}{u} \\
\Big[\frac{1}{2A^2}\Big(\frac{1}{16A^4t^2} + \frac{1}{2A^2t^2}(C_1 - P) + (C_1 - P)^2\Big) + C - \frac{1}{2t}\Big]dt &= \frac{du}{u}.
\end{aligned}$$

Upon integration we again have

$$\Big[\frac{1}{2A^2}\Big(-\frac{1}{48A^4t^3}-\frac{1}{2A^2t}(C_1-P)+t(C_1-P)^2\Big)+Ct-\frac{1}{2}lnt\Big]dt=\ln u+\ln C_2.$$

The second invariant is therefore

$$J_2=\Big[\frac{1}{2A^2}\Big(\frac{1}{48A^4t^3}+\frac{1}{2A^2t}(C_1-P)-t(C_1-P)^2\Big)-Ct+\frac{1}{2}lnt\Big]+\ln u.$$

Setting in the expression for C_1 we have

$$\begin{aligned}J_2=&\Big[\frac{1}{2A^2}\Big(\frac{1}{48A^4t^3}+\frac{1}{2A^2t}\Big(\frac{\ln x}{t}-\frac{1}{4A^2t^2}-P\Big)\\&-t\Big(\frac{\ln x}{t}-\frac{1}{4A^2t^2}-P\Big)^2\Big)-Ct+\frac{1}{2}\ln t\Big]+\ln u\\=&\frac{1}{2A^2}\Big[\frac{1}{48A^4t^3}+\frac{\ln x}{2A^2t}-\frac{1}{8A^4t^3}-t\Big(\frac{(\ln x)^2}{t^2}+\frac{1}{16A^4t^4}+P^2\\&-\frac{\ln x}{2A^2t^3}+\frac{P}{2A^2t^2}-\frac{2P\ln x}{t}\Big)\Big]-Ct+\frac{1}{2}\ln t+\ln u.\end{aligned}$$

The second invariant finally becomes

$$\begin{aligned}J_2=&\frac{1}{2A^2}\Big(-\frac{1}{6A^4t^3}+\frac{\ln x}{A^2t^2}-\frac{P}{2A^2t}-\frac{(\ln x)^2}{t}-P^2t+2P\ln x\Big)\\&-Ct+\frac{1}{2}\ln t+\ln u.\end{aligned}\tag{117}$$

Checking of invariant J_2:

As usual we will check for the invariant condition

$$\begin{aligned}(X_2-X_5)(J_2)=&-2A^2t^2\frac{\partial J_2}{\partial t}+(1-2A^2t\ln x)x\frac{\partial J_2}{\partial x}\\&+[(\ln x-Pt)^2+2A^2Ct^2-A^2t]u\frac{\partial J_2}{\partial u}\\=&-2A^2t^2\Big[\frac{1}{2A^2}\Big(\frac{1}{2A^4t^4}-\frac{2\ln x}{A^2t^3}+\frac{P}{A^2t^2}+\frac{(\ln x)^2}{t^2}-P^2\Big)\\&-C+\frac{1}{2t}\Big]+(1-2A^2tx\ln x)x\Big[\frac{1}{2A^2}\Big(\frac{1}{xA^2t^2}-\frac{2\ln x}{xt}\\&+\frac{2P}{x}\Big)\Big]-\Big((\ln x)^2-2Pt\ln x+P^2t^2+2A^2Ct^2-A^2t\Big)u\frac{1}{u}\\=&-\frac{1}{2A^4t^2}+\frac{2\ln x}{A^2t}-\frac{P}{2A^2}-(\ln x)^2+P^2t^2+2CA^2t^2\\&-A^2t-\frac{\ln x}{A^2t}+2(\ln x)^2-2Pt\ln x+\frac{1}{2A^4t^2}-\frac{\ln x}{A^2t}\\&+\frac{P}{A^2}-(\ln x)^2+2Pt\ln x-P^2t^2-2A^2Ct^2-A^2t\\=&\,0.\end{aligned}$$

Hence invariance for J_2 is satisfied. Taking exponential on both sides of equation (117) we obtain

$$J_2 = u\sqrt{t}\exp\Big[\frac{1}{2A^2}\Big(-\frac{1}{6A^4t^3}+\frac{\ln x}{A^2t^2}-\frac{P}{2A^2t}$$
$$-\frac{(\ln x)^2}{t}-P^2t+2P\ln x\Big)-Ct\Big].$$

Expressing J_2 as function of J_1, that is $J_2=\varphi(J_1)$, we obtain the solution

$$u=\frac{1}{\sqrt{t}}exp\Big[\frac{1}{2A^2}\Big(\frac{1}{6A^4t^3}-\frac{\ln x}{A^2t^2}+\frac{P}{2A^2t}$$
$$+\frac{(\ln x)^2}{t}+P^2t-2P\ln x\Big)+Ct\Big]\varphi\Big(\frac{1}{t}(\ln x-\frac{1}{4A^2t})\Big). \tag{118}$$

Furthermore, letting

$$\mathbf{e}=\exp\Big[\frac{1}{2A^2}\Big(\frac{1}{6A^4t^3}-\frac{\ln x}{A^2t^2}+\frac{P}{2A^2t}$$
$$+\frac{(\ln x)^2}{t}+P^2t-2P\ln x\Big)+Ct\Big]$$

and differentiating equation (118) with respect to t and x, we have

$$u_t=-\frac{1}{2t^{3/2}}\mathbf{e}\varphi+frac1\sqrt{t}\Big[\frac{1}{2A^2}\Big(-\frac{1}{2A^4t^4}+\frac{2\ln x}{A^2t^3}-\frac{P}{2A^2t^2}-\frac{(\ln x)^2}{t^2}$$
$$+P^2\Big)+C\Big]\mathbf{e}\varphi+\frac{1}{\sqrt{t}}\Big[-\frac{1}{t^2}\Big(\ln x-\frac{1}{2A^2t}\Big)\Big]\mathbf{e}\varphi'$$
$$u_t=\frac{\mathbf{e}}{2\sqrt{t}A^2}\Big[\Big(-\frac{A^2}{t}-\frac{1}{2A^4t^4}+\frac{2\ln x}{A^2t^3}-\frac{P}{A^2t^2}-\frac{(\ln x)^2}{t^2}+P^2+2A^2C\Big)\varphi$$
$$+\Big(-\frac{2A^2\ln x}{t^2}+\frac{1}{t^3}\Big)\varphi'\Big]$$
$$u_x=\frac{1}{\sqrt{t}}\Big[\frac{1}{2A^2}\Big(-\frac{1}{A^2t^2x}+\frac{2\ln x}{xt}-\frac{2P}{x}\Big)\mathbf{e}\varphi+\Big(\frac{1}{xt}\Big)\mathbf{e}\varphi'\Big]$$
$$=\frac{\mathbf{e}}{2A^2\sqrt{t}}\Big[\Big(-\frac{1}{A^2t^2x}+\frac{2\ln x}{xt}-\frac{2P}{x}\Big)\varphi+\frac{2A^2}{xt}\varphi'\Big]$$
$$u_{xx}=\frac{\mathbf{e}}{2A^2\sqrt{t}}\Big[\frac{1}{A^2x^2}\Big(\frac{1}{2}\Big)\Big(\frac{1}{A^2t^2}+\frac{2\ln x}{t}-2P\Big)^2\varphi+\Big(\frac{1}{t^2}+\frac{2A^2}{t}$$
$$-\frac{2A^2\ln x}{t}+2A^2P\Big)\varphi+\Big(-\frac{1}{t^3}+\frac{2A^2\ln x}{t^2}-\frac{2A^2P}{t}\Big)\varphi'$$
$$+\Big(-\frac{1}{t^3}+\frac{2A^2\ln x}{t^2}-\frac{2A^2P}{t}\Big)\varphi'-\frac{2A^4}{t}\varphi'+\frac{2A^4}{t^2}\varphi''\Big]$$

Setting the above derivatives in equation (1), we obtain

$$
\begin{aligned}
u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu =& \frac{\mathbf{e}}{2A^2\sqrt{t}}\Big[\Big(-\frac{A^2}{t} - \frac{1}{2A^4t^4} + \frac{2\ln x}{A^2t^3} - \frac{P}{A^2t^2} \\
&- \frac{(\ln x)^2}{t^2} + P^2 + 2A^2C\Big)\varphi + \Big(-\frac{2A^2\ln x}{t^2} + \frac{1}{t^3}\Big)\varphi' + \frac{1}{2}\Big[\Big(\frac{1}{2}\Big)\Big(-\frac{1}{A^2t^2} \\
&+ \frac{2\ln x}{t} - 2P\Big)^2\varphi + \Big(\frac{1}{t^2} + \frac{2A^2}{t} - \frac{2A^2\ln x}{t} + 2A^2P\Big)\varphi + 2\Big(-\frac{1}{t^3} \\
&+ \frac{2A^2\ln x}{t^2} - \frac{2A^2P}{t}\Big)\varphi' - \frac{2A^4}{t}\varphi' + \frac{2A^4}{t^2}\varphi''\Big] + B\Big[\Big(-\frac{1}{A^2t^2} + \frac{2\ln x}{t} \\
&- 2P\Big)\varphi + \frac{2A^2}{t}\varphi'\Big] - 2CA^2\varphi\Big] \\
&= \frac{A^4}{t^2}\varphi'' + \Big(-\frac{2A^2\ln x}{t^2} + \frac{1}{t^3} - \frac{1}{t^3} + \frac{2A^2\ln x}{t^2} - \frac{2A^2P}{t} - \frac{A^4}{t} + \frac{2BA^2}{t}\Big)\varphi' \\
&+ \Big(-\frac{A^2}{t} - \frac{1}{2A^4t^4} + \frac{2\ln x}{A^2t^3} - \frac{P}{A^2t^2} - \frac{(\ln x)^2}{t^2} + P^2 + 2A^2C + \frac{1}{4A^4t^4} \\
&+ \frac{(\ln x)^2}{t^2} + P^2 - \frac{\ln x}{A^2t^3} - \frac{2P\ln x}{t} + \frac{P}{A^2t^2} + \frac{1}{2t^2} + \frac{A^2}{t} - \frac{A^2\ln x}{t} \\
&+ A^2P - \frac{B}{A^2t^2} + \frac{2B\ln x}{t} - 2PB - 2A^2C\Big)\varphi \\
&= \frac{A^4}{t^2}\varphi'' + \Big(\frac{\ln x}{A^2t^3} - \frac{1}{4A^4t^4} + 2P^2 + \frac{1}{2t^2} + A^2P - \frac{B}{A^2t^2} - 2PB\Big)\varphi \\
&= \frac{A^4}{t^2}\varphi'' + \Big(\frac{\ln x}{A^2t^3} - \frac{1}{4A^4t^4} + 2P^2 + \frac{1}{2t^2} + A^2P - \frac{B}{A^2t^2} - 2P(P + \frac{A^2}{2})\Big)\varphi \\
&= \frac{A^4}{t^2}\varphi'' + \Big(\frac{1}{4A^4t^4} - \frac{\ln x}{A^2t^3} + 2P^2 - \frac{1}{2t^2} + A^2P + \frac{B}{A^2} - 2P^2 - A^2P\Big)\varphi \\
&= \frac{1}{A^2t^2}\Big(A^6\varphi'' + \frac{1}{t}\Big(\ln x - \frac{1}{4A^2t^2}\Big) - \frac{A^2}{2} + B\Big) \\
&= A^6\varphi'' + \Big(J_1 - B - \frac{A^2}{2}\Big).
\end{aligned}
$$

Therefore, we have

$$\varphi'' + \frac{1}{A^6}(J_1 - P)\varphi = 0. \tag{119}$$

Again,letting $z = -k(J_1 - P)$ then $\frac{dz}{dJ_1} = -k$. Therefore, from

$$\frac{d\varphi}{dJ_1} = \frac{d\varphi}{dz}\frac{dz}{dJ_1}$$

we obtain

$$\frac{d\varphi}{dJ_1} = -k\frac{d\varphi}{dz} \qquad \text{or} \qquad \varphi' = -k\dot{\varphi}.$$

The second derivative of the above result yields

$$\frac{d^2\varphi}{dJ_1^2} = k\Big[\ddot{\varphi}\frac{dz}{dJ_1}\Big] \qquad \text{or} \qquad \varphi'' = k^2\ddot{\varphi}$$

Setting these derivatives in equation (119) we have

$$k^2\ddot{\varphi} + \frac{1}{A^6}\Big(\frac{-z}{k}\Big)\varphi = 0$$

which gives us

$$\ddot{\varphi} - \frac{z}{A^6k^3}\varphi = 0. \tag{120}$$

But we know from Airy equation that

$$y'' - xy = 0. \tag{121}$$

Therefore, comparing equations (120) and (121) we obtain $\frac{1}{A^6k^3} = 1$ which yields $k = \frac{1}{A^2}$. Therefore $z = \frac{1}{A^2}(J_1 - P)$. Equation (120) therefore becomes

$$\ddot{\varphi} - z\varphi = 0$$

which is also an Airy equation. The general solution for this Airy equation is again

$$\varphi = C_1A_i(z) + C_2B_i(z) \tag{122}$$

where C_1, C_2 are constants of the Airy equation and A_i and B_i are defined as follows

$$A_i(z) = \frac{1}{\pi}\int_0^\infty \cos(z\tau + \frac{1}{3}\tau^3)d\tau$$
$$B_i(z) = \frac{1}{\pi}\int_0^\infty \Big[\exp(z\tau - \frac{1}{3}\tau^3) + \sin(z\tau + \frac{1}{3}\tau^3)\Big]d\tau$$

Setting equation (122) in (118) we obtain the new solution

$$u = \frac{1}{\sqrt{t}}\Big(C_1A_i(z) + C_2B_i(z)\Big)\exp\Big[\frac{1}{2A^2}\Big(\frac{1}{6A^4t^3} - \frac{\ln x}{A^2t^2} + \frac{P}{2A^2t} + \frac{(\ln x)^2}{t} + P^2t - 2P\ln x\Big) + Ct\Big]. \tag{123}$$

Finally, the operator

$$X_5 + kX_6 = 2A^2t^2\frac{\partial}{\partial t} + 2A^2tx\ln x\frac{\partial}{\partial x} + \Big[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t + k\Big]u\frac{\partial}{\partial u}$$

will be tackled. The characteristic system of this operator is

$$\frac{dt}{2A^2t^2} = \frac{dx}{2A^2tx\ln x} = \frac{du}{\Big[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t + k\Big]u}.$$

Writing the first equation as a linear equation, we obtain

$$\frac{dt}{2A^2t^2} = \frac{dx}{2A^2tx\ln x}$$
$$\frac{dt}{t} = \frac{\frac{1}{x}dx}{\ln x}.$$

This yields $\ln t = \ln(\ln x) + \ln C_1$ which yields the invariant

$$J_1 = \frac{1}{t}\ln x \tag{124}$$

when z is replaced.

Checking of invariant J_1:
The invariant J_1 is checked as follows

$$\begin{aligned}(X_5 + kX_6)(J_1) &= 2A^2t^2\frac{\partial J_1}{\partial t} + 2A^2tx\ln x\frac{\partial J_1}{\partial x} \\ &\quad + \Big[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t + k\Big]u\frac{\partial J_1}{\partial u} \\ &= 2A^2t^2\Big(-\frac{1}{t^2}\ln x\Big) + 2A^2tx\ln x\Big(\frac{1}{tx}\Big) \\ &= 0.\end{aligned}$$

The second equation clearly becomes

$$\frac{dt}{2A^2t^2} = \frac{du}{\Big[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t + k\Big]u}.$$

Setting $\ln x = C_1 t$, we obtain

$$\Big(\frac{(C_1 t - Pt)^2}{2A^2t^2} + \frac{2A^2Ct^2}{2A^2t^2} - \frac{A^2t}{2A^2t^2} + \frac{k}{2A^2t^2}\Big)dt = \frac{du}{u}$$

$$\Big(\frac{t^2(C_1 - P)^2}{2A^2t^2} + C - \frac{1}{2t} + \frac{k}{2A^2t^2}\Big)dt = \frac{du}{u}.$$

Upon integration we obtain

$$\frac{(C_1 - P)^2}{2A^2}t + Ct - \frac{1}{2}\ln t - \frac{k}{2A^2t} = \ln u + \ln C_2,$$

$$\exp\Big(\frac{(C_1 - P)^2 t}{2A^2} + Ct - \frac{k}{2A^2t}\Big) + \exp\Big(\ln(t)^{-1/2}\Big) = uC_2,$$

$$uC_2 = t^{-1/2}\exp\Big(\frac{(C_1 - P)^2 t}{2A^2} + Ct - \frac{k}{2A^2t}\Big).$$

The reckoning shows that

$$\begin{aligned} J_2 &= ut^{1/2}\exp\Big(-\frac{(C_1 - P)^2 t}{2A^2} - Ct + \frac{k}{2A^2t}\Big) \\ &= ut^{1/2}\exp\Big(-\frac{t}{2A^2}(\frac{1}{t}\ln x - P)^2 - Ct + \frac{k}{2A^2t}\Big) \end{aligned} \qquad (125)$$

when C_1 is replaced.

Checking of invariant J_2:
Checking the invariant J_2 we have

$$\begin{aligned} (X_5 + kX_6)(J_2) &= 2A^2t^2\frac{\partial J_2}{\partial t} + 2A^2tx\ln x\frac{\partial J_2}{\partial x} \\ &\quad + \Big[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t + k\Big]u\frac{\partial J_2}{\partial u} \\ &= 2A^2t^2u\Big[\frac{1}{2\sqrt{t}} + \sqrt{t}\Big\{-\frac{1}{2A^2t^2}(\ln x - Pt)^2 \\ &\quad - (\frac{t}{2A^2})\frac{2}{t}(\ln x - Pt)(-\frac{1}{t^2}\ln x) - C - \frac{k}{2A^2t^2}\Big\}\Big]\varrho \\ &\quad + 2A^2tx\ln xu\sqrt{t}\Big[-\frac{2t}{2A^2t}(\ln x - Pt)(\frac{1}{xt})\Big] \\ &\quad + \Big[(\ln x - Pt)^2 + 2A^2Ct^2 - A^2t + k\Big]u\sqrt{t}\varrho \\ &= u\sqrt{t}\varrho\Big[A^2t - (\ln x - Pt)^2 + 2\ln x(\ln x - Pt) \\ &\quad - 2A^2Ct^2 - k - 2\ln x(\ln x - Pt) \\ &\quad + (\ln x - Pt)^2 + 2A^2Ct^2 - A^2t + k\Big] \\ &= 0. \end{aligned}$$

where $\varrho = \exp\Big(-\frac{t}{2A^2}(\frac{1}{t}\ln x - P)^2 - Ct + \frac{k}{2A^2 t}\Big)$. Now expressing J_2 as function of J_1 we obtain the solution

$$u = \frac{1}{\sqrt{t}}\exp\Big(\frac{t}{2A^2}(\frac{1}{t}\ln x - P)^2 + Ct - \frac{k}{2A^2 t}\Big)\varphi\Big(\frac{1}{t}\ln x\Big),$$

or (126)

$$u = \frac{1}{\sqrt{t}}\exp\Big(\frac{1}{2A^2 t}(\ln x - Pt)^2 + Ct - \frac{k}{2A^2 t}\Big)\varphi\Big(\frac{1}{t}\ln x\Big).$$

The reckoning equation, (126), upon differentiation yields

$$\begin{aligned} u_t &= -\frac{1}{2t^{3/2}}\varrho\varphi + \frac{1}{\sqrt{t}}\Big[-\frac{1}{2A^2t^2}(\ln x - Pt)^2 - \frac{2P}{2A^2 t}(\ln x - Pt) \\ &\quad + C + \frac{k}{2A^2t^2}\Big]\varphi\varrho - \frac{1}{t^{5/2}}\ln x\varphi'\varrho \\ &= \frac{1}{\sqrt{t}}\varrho\Big[-\frac{1}{2t}\varphi - \Big(\frac{1}{2A^2t^2}(\ln x - Pt)^2 + \frac{P}{A^2 t}(\ln x - Pt) \\ &\quad - C - \frac{k}{2A^2t^2}\Big)\varphi - \frac{1}{t^2}\ln x\varphi'\Big] \end{aligned}$$

$$\begin{aligned} u_x &= \frac{1}{\sqrt{t}}\Big[\frac{2}{2A^2t}(\ln x - Pt)(\frac{1}{x})\varphi\varrho + \varrho\frac{1}{xt}\varphi'\Big], \\ &= \frac{1}{\sqrt{t}}\Big[\frac{2}{2A^2tx}(\ln x - Pt)\varphi + \frac{1}{xt}\varphi'\Big]\varrho, \\ u_{xx} &= \frac{1}{\sqrt{t}}\Big[-\frac{1}{A^2tx^2}(\ln x - Pt)\varphi + \frac{1}{A^2tx^2}\varphi + \frac{1}{A^2tx^2}\varphi \\ &\quad + \frac{1}{A^2tx}(\ln x - Pt)(\frac{1}{xt})\varphi' + \frac{1}{A^2tx}(\ln x - Pt) \\ &\quad \times\Big(\frac{2}{2A^2t}(\ln x - Pt)(\frac{1}{x})\Big)\varphi - \frac{1}{x^2t}\varphi' \\ &\quad + \frac{1}{x^2t^2}\varphi'' + \frac{2}{2A^2t}(\ln x - Pt)(\frac{1}{x})\varphi'\Big]\varrho. \end{aligned}$$

Setting the above derivatives in (1) we have

$$
\begin{aligned}
u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu =& \frac{\varrho}{\sqrt{t}}\Big[- \Big(\frac{1}{2t} + \frac{1}{2A^2t^2}(\ln x - Pt)^2 + \frac{P}{2A^2t}(\ln x \\
& - Pt) - C - \frac{k}{2A^2t^2}\Big)\varphi - \frac{1}{t^2}\ln x\varphi' + \frac{1}{2}A^2x^2\Big\{ - \frac{1}{A^2tx^2}(\ln x - Pt)\varphi \\
& + \frac{1}{A^2tx^2}\varphi + \frac{2}{A^2x^2t^2}(\ln x - Pt)\varphi' + \frac{1}{A^4t^2x^2}(\ln x - Pt)^2\varphi - \frac{1}{x^2t}\varphi' \\
& + \frac{1}{x^2t^2}\varphi''\Big\} + Bx\Big(\frac{1}{A^2tx}(\ln x - Pt)\varphi + \frac{1}{xt}\varphi'\Big) - C\varphi\Big] \\
\\
=& \frac{A^2}{2t^2}\varphi'' + \Big(\frac{1}{t^2}(\ln x - Pt) - \frac{A^2}{2t} - \frac{1}{t^2}\ln x + \frac{B}{t}\Big)\varphi' \\
& + \Big(C - \frac{1}{2t} - \frac{1}{2A^2t^2}(\ln x - Pt)^2 - \frac{P}{A^2t}(\ln x - Pt) + \frac{k}{2A^2t^2} \\
& - \frac{1}{2t}(\ln x - Pt) + \frac{1}{2t} + \frac{1}{2A^2t^2}(\ln x - Pt)^2 + \frac{B}{A^2t}(\ln x - Pt) - C\Big)\varphi \\
\\
=& \frac{A^2}{2t^2}\varphi'' + \Big(\frac{1}{t^2}(\ln x - Pt) - \frac{A^2}{2t} - \frac{1}{t^2}\ln x + \frac{B}{t}\Big)\varphi' \\
& + \Big(\frac{k}{2A^2t^2} - \frac{P}{A^2t}(\ln x - Pt) - \frac{1}{2t}(\ln x - Pt) + \frac{B}{A^2t}(\ln x - Pt)\Big).
\end{aligned}
$$

Since $P = B - \frac{A^2}{2}$ we can deduce that $\frac{B}{t} - \frac{A^2}{2t} = \frac{1}{t}(B - \frac{A^2}{2} = \frac{P}{t})$ hence we have

$$
\begin{aligned}
\frac{A^2}{2t^2}\varphi'' + \Big(\frac{1}{t^2}(\ln x - Pt) - \frac{1}{t^2}(\ln x - Pt)\Big)\varphi' + \Big[\Big(\frac{k}{2A^2t^2} - \Big(B - \frac{A^2}{2}\Big) \\
\times \Big(\frac{\ln x - Pt}{A^2t}\Big) - \frac{1}{2t}(\ln x - Pt) + \frac{B}{A^2t}(\ln x - Pt)\varphi\Big] = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{A^2}{2t^2}\varphi'' + \Big(\frac{k}{2A^2t^2} - \frac{B}{A^2t}(\ln x - Pt) + \frac{1}{2t}(\ln x - Pt) \\
- \frac{1}{2t}(\ln x - Pt) + \frac{B}{A^2t}(\ln x - Pt)\varphi = 0.
\end{aligned}
$$

The reckoning shows that

$$\frac{A^2}{2t^2}\varphi'' + \frac{k}{2A^2t^2}\varphi = 0$$
$$A^2\varphi'' + \frac{k}{A^2}\varphi = 0$$
$$\varphi'' + \frac{k}{A^4}\varphi = 0.$$

The above equation is a second order homogeneous equation with constant coefficient. The characteristic system for this equation is $\lambda^2 + \alpha\lambda = 0$ which yields $\lambda_1 = 0, \lambda_2 = -\alpha$. Hence the general solution is given by

$$\varphi = k_1 e^{\lambda_1 J_1} + k_2 e^{\lambda_2 J_1},$$
$$\varphi = k_1 + k_2 e^{-\alpha J_1},$$

where $J_1 = \frac{1}{t}\ln x$. Therefore $\varphi = k_1 + k_2 x^{-\alpha/t}$. Setting φ in (126) we obtain

$$u = \frac{1}{\sqrt{t}}(k_1 + k_2 x^{-\alpha/t}) \exp\Big(\frac{t}{2A^2}(\frac{1}{t}\ln x - P)^2 + Ct - \frac{k}{2A^2 t}\Big). \tag{127}$$

4.1 Summary of optimal system of invariant solutions

$$X_1: \quad u = C_1x^{\lambda_1} + C_2x^{\lambda_2};$$
$$\lambda_1 = -\frac{P + \sqrt{P^2 + 2A^2C}}{A^2};$$
$$\lambda_2 = -\frac{P - \sqrt{P^2 + 2A^2C}}{A^2}.$$

$$X_2: \quad u = Ke^{Ct}.$$

$$X_3: \quad u = Ke^{Ct}\int \exp\Big(\frac{J_2^2}{2A^2}\Big)dJ_2 + K_1e^{Ct};$$

Or

$$u = AKe^{Ct}\sqrt{\frac{\pi}{2}}erfi\Big(\frac{J_2}{A\sqrt{2}}\Big) + K_1e^{Ct};$$
$$J_2 = P\sqrt{t} - \frac{\ln x}{\sqrt{t}}.$$

$$X_4: \quad u = \frac{K}{\sqrt{t}}\exp\Big(\frac{(\ln x)^2}{2A^2t} - \frac{P\ln x}{A^2} + t(\frac{P^2}{2A^2} + C)\Big).$$

$$X_5: \quad u = \frac{1}{\sqrt{t}}\Big(\frac{K_1}{t}\ln x + K_2\Big)\exp\Big(\frac{1}{2A^2t}(\ln x - Pt)^2 + Ct\Big).$$

$$X_1 + X_5: \quad u = \exp\Big(- \frac{P\ln x}{A^2} + \frac{t(\ln x)^2}{2A^2t^2 + 1} + \frac{P^2t}{2A^2}$$
$$- \frac{P^2}{2^{\frac{3}{2}}A^3}\tan^{-1}\sqrt{2}At + Ct - \frac{C}{\sqrt{2}A}\tan^{-1}\sqrt{2}A^2t$$
$$- \frac{1}{4}\ln(2A^2t^2 + 1)\Big)\varphi(J_1);$$
$$J_1 = \Big(2A^2t^2 + 1\Big)^{-\frac{1}{2}}(\ln x);$$
$$\varphi'' + \frac{2}{A^2}\Big(J_1 - K\Big)\varphi = 0$$
$$\text{where} K = \frac{P^2}{2A^2} + C.$$

$$X_1 - X_5: \qquad u = \exp\Big(- \frac{P\ln x}{A^2} + \frac{t(\ln x)^2}{2A^2t^2-1} + \frac{P^2 t}{A^2} - \frac{P^2}{2^{3/2}A^3}\tanh^{-1}\sqrt{2}At$$
$$+ Ct - \frac{C}{A\sqrt{2}}\tanh^{-1}\sqrt{2}At - \frac{1}{4}\ln(2A^2t^2-1)\Big)\varphi(J_1);$$
$$J_1 = (2A^2t^2-1)^{-1/2}\ln x;$$
$$\varphi'' - \frac{2}{A^2}(J_1^2 - \frac{P^2}{2A^2} - C)\varphi = 0.$$

$$X_2 + X_5: \qquad u = \frac{1}{\sqrt{t}}\Big(C_1A_i(z) + C_2B_i(z)\Big)\exp\Big[\frac{1}{2A^2}\Big(\frac{1}{6A^4t^3}$$
$$+ \frac{\ln x}{A^2t^2} - \frac{P}{A^2t} + \frac{(\ln x)^2}{t} + P^2t - 2P\ln x\Big) + Ct\Big];$$
$$A_i(z) = \frac{1}{\pi}\int_0^\infty \cos(z\tau + \frac{1}{3}\tau^3)d\tau$$
$$B_i(z) = \frac{1}{\pi}\int_0^\infty \Big[exp(z\tau - \frac{1}{3}\tau^3) + \sin(z\tau + \frac{1}{3}\tau^3)\Big]d\tau.$$

$$X_2 - X_5: \qquad u = \frac{1}{\sqrt{t}}\Big(C_1A_i(z) + C_2B_i(z)\Big)\exp\Big[\frac{1}{2A^2}\Big(\frac{1}{6A^4t^3}$$
$$- \frac{\ln x}{A^2t^2} + \frac{P}{2A^2t} + \frac{(\ln x)^2}{t} + P^2t - 2P\ln x\Big) + Ct\Big];$$
$$A_i(z) = \frac{1}{\pi}\int_0^\infty \cos(z\tau + \frac{1}{3}\tau^3)d\tau$$
$$B_i(z) = \frac{1}{\pi}\int_0^\infty \Big[\exp(z\tau - \frac{1}{3}\tau^3) + \sin(z\tau + \frac{1}{3}\tau^3)\Big]d\tau.$$

$$X_5 + kX_6: \qquad u = \frac{1}{\sqrt{t}}(k_1 + k_2x^{-\alpha/t})\exp\Big(\frac{t}{2A^2}(\frac{1}{t}\ln x - P)^2 + Ct - \frac{k}{2A^2t}\Big).$$

5 Conclusion

By means of Lie group techniques we have studied the construction of optimal system of invariant solutions for the Black-scholes equation. We derived one functionally invariant which aided us in the construction of the optimal system of one-dimensional subalgebra. Again, we have calculated invariant solutions corresponding to the optimal system. Essentially, the optimal system is price process of some interest rate derivatives.

5.1 Future Work

Since MSc thesis is only the beginning of a research life, we will end this thesis with a discussion on open problems that are closely associated to the work presented which will be interested to investigate further. The first among these challenges are the invariant solutions to the following representatives of the optimal system; $X_1 + X_5, X_1 - X_5$,

We will like to urge up and coming researchers to do further work on how to use the optimal system of invariant solution of the Black-Scholes equation. We will also encourage further investigation into both practical and theoretical implications of the optimal system. The researcher should consider using the optimal system of invariant solution to analyze financial data and develop financial algorithms that will lead to a precise output.

References

[1] Black, F. and Scholes, M., *The pricing of options and corporate liabilities*, J. Pol. Econ. **81** 637 - 659, 1973.

[2] Lie, S., *On integration of a class of linear second-order partial differential equations*, Arch. for Math.., Vol. 6(3), 1881, p. 328 - 368. English transl. by N. H. Ibragimov in [5].

[3] Ovsyannikov, L.V., *Group properties of differential equations.* Siberian Branch, USSR Academy of Sciences, Novosibirirsk, 1962. (Russian).

[4] Ovsyannikov, L.V., *Group analysis of differential equations.* Nuaka, Moscow, 1978. English transl., ed W.F. Ames, Academic Press, New York 1982.

[5] Ibragimov, N.H, *CRC Handbook of Lie group analysis of differential equations. Vol. 2: Applications in engineering and physical sciences*, CRC Press Inc., Boca Raton, 1995.

[6] Ibragimov, N.H, *A practical course in differential equations and mathematical modelling*, Third Edition, ALGA Publication, Karlskrona 2006.

[7] Ibragimov, N.H, *Elementary Lie group analysis and ordinary differential equations*, John Wiley and Sons, Chichester, 1999.

[8] Ibragimov, N.H, *Exercises for courses based on Lie group analysis*, ALGA Publication, Karlskrona, 2006.

[9] Gazizov, R. K. and Ibragimov, N. H., *Lie symmetry analysis of differential equations in finance, Nonlinear Dynamics*, Vol. 17(4), 1998, p. 387 - 407.