Application of Ibragimov's method and Noether's theorem for constructing conservation laws of the linear elasticity model

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Thesis for the degree Master of science (two years)
in Mathematical Modeling and Simulation
30 credit points (30 ECTS credits)

June 2014

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Dedicated to my lovely wife who patiently supported me during this work.

Abstract

In this thesis, conservation laws of the Lamé equation describing linear elasticity have been constructed. This has been done by the implementation of two theorems for constructing conservation laws: the new theorem suggested by N H.Ibragimov, and the classical theorem presented by E.Noether.
As a result one can see that the Ibragimov method provides more conservation laws than the Noether theorem, which may suggest the better efficiency of the Ibragimov method.

Keywords: Conservation law; Noether's theorem; Ibragimov's method; Nonlinear self-adjointness; Formal Lagrangian; Linear elasticity; Lamé equation.

Acknowledgment

I would like to express my sincere gratitude to Dr. Raisa Khamitova for her valuable and constructive suggestions during the planning and development of this research work. Her willingness to give her time so generously has been very much appreciated.
My special thanks go to Prof. Nail H. Ibragimov for his courses in the field of PDE and Lie group analysis, which made me familiar with differential system analysis, and his valuable comments during this research work.
Advice given by Dr. Niklas Lavasson and Dr. Andrew Moss has been a great help in development of my insight in research methodology and I am grateful for that.
I thank Dr. Mattias Eriksson for his valuable comments.
Finally, I would like to express my deepest gratitude to my wife and my parents for their support and encouragement throughout my study.

Karlskrona, May 2014

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Contents

1 Introduction

Conservation laws are very important tools for investigating a physical system. They state that a particular measurable property of an isolate physical system does not change as the system evolves (see for example [11]). There are several ideas for constructing conservation laws of a system. One is the direct method used by Laplace for the first time in 1798 [7].
Let us consider a system of $\overline{m}$ differential equations

$$F_{\bar{\alpha}}(x, u, u_{(1)}, \ldots, u_{(s)}) = 0, \quad \bar{\alpha} = 1, \ldots, \overline{m}, \tag{1.1}$$

with n independent variables $x^1, \ldots, x^n$ and m dependent variables $u^1, \ldots, u^m$, and the first, second, $\ldots$, s-th derivatives of u with respect to x denoted as $u_{(1)} = \{u_i^\alpha\}, \ldots, u_{(s)} = \{u^\alpha_{i_1 \ldots i_s}\}$ respectively, $\alpha = 1, 2, \ldots, m$ and other indices change from 1 to n.

A conservation law for Eqs. (1.1) is defined [4] as

$$\Big[D_i(C^i)\Big]_{(1.1)} = 0. \tag{1.2}$$

Here D_i is the total differentiation with respect to x^i:

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots . \tag{1.3}$$

The subscript $|_{(1.1)}$ means the left-hand side of (1.2) is restricted on the solutions of Eqs.(1.1). Here and further, the usual convention of summation in repeated indices is used.

The n-dimensional vector

$$C = (C^1, \ldots, C^n) \tag{1.4}$$

satisfying the equation (1.2) is called a *conserved vector* for the system (1.1). If its components $C^i = C^i(x, u, u_{(1)}, \ldots)$ are functions of x, u and derivatives $u_{(1)}, \ldots,$ of a finite order, conservation vector (1.4) is called a *local conserved vector* and, accordingly, the system (1.1) has a local conservation law.

Vector (1.4) is called a *trivial* conserved vector if

$$D_i(C^i) \equiv 0, \tag{1.5}$$

or components C^i vanish on the solutions of the system (1.1). Two conservation laws that only differ by a trivial conservation law are considered as equivalent [4, 7].

The direct method of constructing conservation laws means that the definition of a conservation law is used.

The works of Jacobi and Klein motivated Emmy Noether to investigate the connection between conservation laws and symmetries of differential equations obtained from variational principle. In 1915 Noether found such a relationship and the theorem was published in 1918 [9]. However, Noether's theorem can be applied only for differential equations with a Lagrangian, so called Euler-Lagrange equations (see Section 2). As an illustration, Noether's theorem is not applicable to evolution equations and to differential equations of an odd order [3]. Furthermore, symmetries of Euler-Lagrange equations should satisfy an additional condition [3, 9].

In order to overcome this problem, in 2007 N.H.Ibragimov presented a remedy based on the concept of a *formal Lagrangian* and proved the new conservation theorem. He introduced the definition of *nonlinear self-adjointness* [3, 4, 5] of differential equations and gave the formula for constructing local conservation laws. Briefly speaking, the Ibragimov theorem state that for any system of differential equations which are nonlinearly self-adjoint, it is possible to use the formal Lagrangian for construction of local conservation laws of a system corresponding to the system's symmetries.

In this work the application of both theorems (see also, for example, [6]) to a system which has the Lagrangian is investigated and the efficiency of the Ibragimov theorem is demonstrated. In order to execute that, the linear elasticity theorem (which is an important and useful theory in physics and mechanics), has been considered and its mathematical representation, the Lamé equation, has been chosen.

1.1 Linear elasticity and the Lamé equation

According to the definition, elasticity is the tendency of a material to temporary deform under an external force and stress and return to the normal form when the load and stress are removed. Elasticity is a useful theory with a wide range of applications, e.g. in design and analysis of the structure of beams, plates and shells, sandwich composites and etc. Besides, it is the basis of a large proportion of the fractional mechanics. The most well-known mathematical representation of the linear elasticity is the equilibrium equations for a homogeneous, isotropic and linearly elastic medium in absence of the body force which is called *Navier's equation* or, alternatively, *elastostatic equation* [2, 10]:

$$\mu\Delta\mathbf{u} + (\mu+\lambda)\operatorname{grad}\operatorname{div}\mathbf{u} = 0,$$

where μ and λ are Lamé moduli, subjected to the restrictions $\mu > 0$, $2\mu + \lambda > 0$ and $\mathbf{u} = (u^1, u^2, u^3)$ and x, y, z are spacial variables.

For the first time in 1962, Günter [1] and Knowles & Sternbery [8] performed a limited investigation of the Noether theorem applications in elasticity [10]. In 1984 Olver [10] presented a first complete systematic implementation of Noether's theorem in elasticity. Olver used elastostatic Navier's equation and by considering tensor calculations, classified conservation laws of the system (one can find the complete method and calculations in Olver's triple published papers [10]).

In this work, I use the Lamé equation which is the non-elastostatic (depends on time t) version of the Navier equation, i.e. $\dfrac{\partial^2 u}{\partial t^2} \neq 0$. Hence

$$\mu\Delta\mathbf{u} + (\mu+\lambda)\operatorname{grad}\operatorname{div}\mathbf{u} = \mathbf{u}_{tt}.$$

Considering $\beta = \dfrac{\mu}{\mu+\lambda}$, the Lamé equation has the following form [2]:

$$\mathbf{u}_{tt} = \operatorname{grad}\operatorname{div}\mathbf{u} + \beta\Delta\mathbf{u}, \tag{1.1.1}$$

where $\mathbf{u} = (u^1, u^2, u^3)$, t is time, and x, y, z are spacial variables.
The equation (1.1.1) has the following symmetries [2]:

$$
\begin{aligned}
X_0 =& \ \frac{\partial}{\partial t}\,, \quad X_1 = \frac{\partial}{\partial x}\,, \quad X_2 = \frac{\partial}{\partial y}\,, \quad X_3 = \frac{\partial}{\partial z}\,,\\
X_4 =& \ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} + u^2\frac{\partial}{\partial u^3} - u^3\frac{\partial}{\partial u^2}\,,\\
X_5 =& \ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + u^3\frac{\partial}{\partial u^1} - u^1\frac{\partial}{\partial u^3}\,,\\
X_6 =& \ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + u^1\frac{\partial}{\partial u^2} - u^2\frac{\partial}{\partial u^1}\,,\\
X_7 =& \ t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\,,\\
X_8 =& \ u^1\frac{\partial}{\partial u^1} + u^2\frac{\partial}{\partial u^2} + u^3\frac{\partial}{\partial u^3}\,,\\
X_\infty =& \ w^1\frac{\partial}{\partial u^1} + w^2\frac{\partial}{\partial u^2} + w^3\frac{\partial}{\partial u^3}\,,
\end{aligned} \tag{1.1.2}
$$

where (w^1, w^2, w^3) is any arbitrary solution of the Lamé equation.
Additionally, the equation (1.1.1) has the Lagrangian

$$
\begin{aligned}
\mathcal{L} = \beta\Big(&(u^1_y + u^2_x)^2 + (u^1_z + u^3_x)^2 + (u^2_z + u^3_y)^2 + 2(u^1_x)^2 + 2(u^2_y)^2 + 2(u^3_z)^2\Big)\\
&+ (1-\beta)(u^1_x + u^2_y + u^3_z)^2 - (u^1_t)^2 - (u^2_t)^2 - (u^3_t)^2\,,
\end{aligned} \tag{1.1.3}
$$

similar to the Lagrangian on the p.342 [2].

Due to the chronological reason, the application of Noether's theorem is presented first and the Ibragimov method and its implementation come after that.

2 Calculation of the conservation laws of the Lamé equation using Noether's theorem

2.1 Introduction

Consider the system (1.1) and suppose the Lagrangian, $\mathcal{L}$, of the system exists. Consider also that the system admits the infinitesimal generator

$$
X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha}\,.
$$

According to the Noether theorem, symmetries which may give a nontrivial conservation law must satisfy the following conditions:

$$
X(\mathcal{L}) + \mathcal{L}\,D_i(\xi^i) = 0 \ \text{ or, alternatively,} \quad X(\mathcal{L}) + \mathcal{L}\,D_i(\xi^i) = D_i(B^i). \tag{2.1.1}
$$

Hence, the condition (2.1.1) should be verified for each symmetry of the system (1.1) first. As it mentioned in Section 1, a corresponding conserved vector for the system (1.1) can be written as

$$
C = (C^1, C^2, ..., C^n). \tag{2.1.2}
$$

Its components C_i have the form

$$C^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) - \cdots \right]$$
$$+ D_j \Big(W^\alpha \Big) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \cdots \right] + D_j D_k \Big(W^\alpha \Big) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \cdots \right], \qquad (2.1.3)$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$, $\alpha = 1, 2, ..., m$, and $i, j, k = 1, \cdots, n$.

If we extend the Lamé equation (1.1.1), we will have a system of differential equations consisting of three equations:

$$\begin{aligned} F_1 &\equiv \beta(u_{xx}^1 + u_{yy}^1 + u_{zz}^1) + u_{xx}^1 + u_{xy}^2 + u_{xz}^3 - u_{tt}^1 = 0, \\ F_2 &\equiv \beta(u_{xx}^2 + u_{yy}^2 + u_{zz}^2) + u_{xy}^1 + u_{yy}^2 + u_{yz}^3 - u_{tt}^2 = 0, \\ F_3 &\equiv \beta(u_{xx}^3 + u_{yy}^3 + u_{zz}^3) + u_{xz}^1 + u_{yz}^2 + u_{zz}^3 - u_{tt}^3 = 0. \end{aligned} \qquad (2.1.4)$$

Here the variables t, x, y, z are *independent* and u^1, u^2, u^3 are *dependent* variables.
A conservation law for the system has the form

$$\Big[D_i(C^i) \Big]_{F_\alpha = 0} = 0 \quad \text{where} \quad \alpha = 1, 2, 3 \, ,$$

$i = 0, 1, 2, 3;$ $x^0 = t,$ $x^1 = x,$ $x^2 = y,$ $x^3 = z$. Therefore we have

$$\Big[D_t(C^0) + D_x(C^1) + D_y(C^2) + D_z(C^3) \Big]_{F_\alpha = 0} = 0 \quad \text{where} \quad \alpha = 1, 2, 3 \, . \qquad (2.1.5)$$

Since the Lagrangian (1.1.3) of the system (2.1.4) includes only first derivatives, the formula (2.1.3) will be simplified to the following form:

$$C^i = \xi^i \mathcal{L} + W^\alpha \frac{\partial \mathcal{L}}{\partial u_i^\alpha} \, , \quad i = 0, 1, 2, 3 \, . \qquad (2.1.6)$$

Hence (2.1.5) can be used for checking the validity of calculations.

Remark. At the end of this thesis, formulas for components of conserved vectors obtained by Noether's theorem are compared with the components obtained by Ibragimov's method. It will be shown that the latter formulas are simpler. Therefore, I do not simplify formulas for components of conserved vectors in Sections 2.2-2.8 and Section 2.11.

2.2 Case $X = \dfrac{\partial}{\partial t}$

In this case $\xi^0 = 1,$ $\xi^1 = \xi^2 = \xi^3 = 0,$ $\eta^\alpha = 0$ and

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha = -u_t^\alpha, \quad \alpha = 1, 2, 3 \, . \qquad (2.2.1)$$

Invoking (2.1.6) we have

$$C_0^0 = \mathcal{L} - u_t^\alpha \frac{\partial \mathcal{L}}{\partial u_t^\alpha} \, ,$$

which gives

$$
\begin{aligned}
C_0^0 =&\beta\Big((u_y^1+u_x^2)^2+(u_z^1+u_x^3)^2+(u_z^2+u_y^3)^2+2(u_x^1)^2+2(u_y^2)^2+2(u_z^3)^2\Big)\\
&+(1-\beta)(u_x^1+u_y^2+u_z^3)^2+(u_t^1)^2+(u_t^2)^2+(u_t^3)^2. \qquad (2.2.2)
\end{aligned}
$$

By applying the formula (2.1.6) for other components we can easily find $C_0^1,\ C_0^2$, and C_0^3:

$$
C_0^1=-u_t^1\Big(4\beta u_x^1+2(1-\beta)(u_x^1+u_y^2+u_z^3)\Big)-u_t^2\Big(2\beta(u_x^2+u_y^1)\Big)-u_t^3\Big(2\beta(u_z^1+u_x^3)\Big), \quad (2.2.3)
$$

$$
C_0^2=-u_t^1\Big(2\beta(u_x^2+u_y^1)\Big)-u_t^2\Big(4\beta u_y^2+2(1-\beta)(u_x^1+u_y^2+u_z^3)\Big)-u_t^3\Big(2\beta(u_z^2+u_y^3)\Big), \quad (2.2.4)
$$

$$
C_0^3=-u_t^1\Big(2\beta(u_z^1+u_x^3)\Big)-u_t^2\Big(2\beta(u_z^2+u_y^3)\Big)-u_t^3\Big(4\beta u_z^3+2(1-\beta)(u_x^1+u_y^2+u_z^3)\Big). \quad (2.2.5)
$$

Now we need to verify if the derived results satisfy the condition (2.1.5). After application of the total differentiations and doing the simplifications, we can easily show that

$$
\begin{aligned}
&\Big[D_t(C^0)+D_x(C^1)+D_y(C^2)+D_z(C^3)\Big]_{F_\alpha=\,0}\\
=&\Big[-2u_t^1F_1-2u_t^2F_2-2u_t^3F_3\Big]_{F_\alpha=\,0}=0, \quad \alpha=1,2,3\,. \qquad (2.2.6)
\end{aligned}
$$

Proposition: The operator X_0 admitted by the system (2.1.4) provides the conserved vector $(C_0^0,C_0^1,C_0^2,C_0^3)$ with the components (2.2.2), (2.2.3), (2.2.4) and (2.2.5).

2.3 Case $X_1=\dfrac{\partial}{\partial x}$

Here $\xi^1=1,\ \xi^0=\xi^2=\xi^3=0,\ \eta^\alpha=0$ and

$$
W^\alpha=\eta^\alpha-\xi^i u_j^\alpha=-u_x^\alpha\,, \quad \alpha=1,2,3\,. \qquad (2.3.1)
$$

Invoking (2.1.6) we obtain:

$$
C_1^0=2u_t^1u_x^1+2u_t^2u_x^2+2u_t^3u_x^3. \qquad (2.3.2)
$$

Additionally, we have

$$
C_1^1=\mathcal{L}-u_x^1\Big(4\beta u_x^1+2(1-\beta)(u_x^1+u_y^2+u_z^3)\Big)-u_x^2\Big(2\beta(u_x^2+u_y^1)\Big)-u_x^3\Big(2\beta(u_z^1+u_x^3)\Big).
$$

Hence using the expression for Lagrangian we obtain:

$$
\begin{aligned}
C_1^1=&\sum_{\alpha=1}^{3}\Big(\beta\Big((u_y^\alpha)^2+(u_z^\alpha)^2-(u_x^\alpha)^2\Big)-(u_t^\alpha)^2\Big)+2\beta(u_z^2u_y^3-u_y^2u_z^3)+2u_y^2u_z^3\\
&-(u_x^1)^2+(u_y^2)^2+(u_z^3)^2. \qquad (2.3.3)
\end{aligned}
$$

Other components are:

$$C_1^2 = -\, u_x^1\Big(2\beta(u_x^2 + u_y^1)\Big) - u_x^2\Big(4\beta u_y^2 + 2(1-\beta)(u_x^1 + u_y^2 + u_z^3)\Big) - u_x^3\Big(2\beta(u_z^2 + u_y^3)\Big),$$

$$C_1^3 = -\, u_x^1\Big(2\beta(u_z^1 + u_x^3)\Big) - u_x^2\Big(2\beta(u_z^2 + u_y^3)\Big) - u_x^3\Big(4\beta u_z^3 + 2(1-\beta)(u_x^1 + u_y^2 + u_z^3)\Big). \quad (2.3.4)$$

Checking (2.1.5) we obtain:

$$\Big[D_t(C^0) + D_x(C^1) + D_y(C^2) + D_z(C^3)\Big]_{F_\alpha = 0}$$
$$= \Big[-2u_x^1 F_1 - 2u_x^2 F_2 - 2u_x^3 F_3\Big]_{F_\alpha = 0} = 0, \quad \alpha = 1,2,3. \quad (2.3.5)$$

Proposition: The operator X_1 admitted by the system (2.1.4) provides the conserved vector $(C_1^0, C_1^1, C_1^2, C_1^3)$ with the components (2.3.2) and (2.3.4).

2.4 Case $X_2 = \dfrac{\partial}{\partial y}$

In this case $\xi^2 = 1,\ \xi^0 = \xi^1 = \xi^3 = 0,\ \eta^\alpha = 0$ and

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha = -u_y^\alpha, \quad \alpha = 1,2,3. \quad (2.4.1)$$

Invoking (2.1.6) we have:

$$C_2^0 = 2u_t^1 u_y^1 + 2u_t^2 u_y^2 + 2u_t^3 u_y^3,$$

$$C_2^1 = -\, u_y^1\Big(4\beta u_x^1 + 2(1-\beta)(u_x^1 + u_y^2 + u_z^3)\Big) - u_y^2\Big(2\beta(u_x^2 + u_y^1)\Big) - u_y^3\Big(2\beta(u_z^1 + u_x^3)\Big),$$

$$C_2^2 = \sum_{\alpha=1}^{3}\Big(\beta\Big((u_x^\alpha)^2 + (u_z^\alpha)^2 - (u_y^\alpha)^2\Big) - (u_t^\alpha)^2\Big) + 2\beta(u_z^1 u_x^3 - u_x^1 u_z^3) + 2u_x^1 u_z^3$$
$$+ (u_x^1)^2 + (u_z^3)^2 - (u_y^2)^2,$$

$$C_2^3 = -\, u_y^1\Big(2\beta(u_z^1 + u_x^3)\Big) - u_y^2\Big(2\beta(u_z^2 + u_y^3)\Big) - u_y^3\Big(4\beta u_z^3 + 2(1-\beta)(u_x^1 + u_y^2 + u_z^3)\Big). \quad (2.4.2)$$

Now we need to verify if the derived results satisfy the condition (2.1.5). After application of the total differentiations and doing the simplifications, we can easily show that

$$\Big[D_t(C^0) + D_x(C^1) + D_y(C^2) + D_z(C^3)\Big]_{F_\alpha = 0}$$
$$= \Big[-2u_y^1 F_1 - 2u_y^2 F_2 - 2u_y^3 F_3\Big]_{F_\alpha = 0} = 0, \quad \alpha = 1,2,3\,.$$

Proposition: The operator X_2 admitted by the system (2.1.4) provides the conserved vector $(C_2^0, C_2^1, C_2^2, C_2^3)$ with the components (2.4.2).

2.5 Case $X_3 = \dfrac{\partial}{\partial z}$

We have $\xi^3 = 1,\ \xi^1 = \xi^2 = \xi^0 = 0,\ \eta^\alpha = 0$ and

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha = -u_z^\alpha, \quad \alpha = 1,2,3. \tag{2.5.1}$$

Invoking (2.1.6) we obtain

$$\begin{aligned}
C_3^0 &= 2u_t^1 u_z^1 + 2u_t^2 u_z^2 + 2u_t^3 u_z^3,\\
C_3^1 &= -\, u_z^1\Big(4\beta u_x^1 + 2(1-\beta)(u_x^1 + u_y^2 + u_z^3)\Big) - u_z^2\Big(2\beta(u_x^2 + u_y^1)\Big) - u_z^3\Big(2\beta(u_z^1 + u_x^3)\Big),\\
C_3^2 &= -\, u_z^1\Big(2\beta(u_x^2 + u_y^1)\Big) - u_z^2\Big(4\beta u_y^2 + 2(1-\beta)(u_x^1 + u_y^2 + u_z^3)\Big) - u_z^3\Big(2\beta(u_z^2 + u_y^3)\Big),\\
C_3^3 &= \sum_{\alpha=1}^{3}\left(\beta\Big((u_x^\alpha)^2 + (u_y^\alpha)^2 - (u_z^\alpha)^2\Big) - (u_t^\alpha)^2\right) + 2\beta(u_y^1 u_x^2 - u_x^1 u_y^2) + 2u_x^1 u_y^2\\
&\quad + (u_x^1)^2 + (u_y^2)^2 - (u_z^3)^2.
\end{aligned} \tag{2.5.2}$$

Accordingly, the equation (2.1.5) has the form:

$$\begin{aligned}
&\Big[D_t(C^0) + D_x(C^1) + D_y(C^2) + D_z(C^3)\Big]_{F_\alpha = 0}\\
=&\Big[-2u_z^1 F_1 - 2u_z^2 F_2 - 2u_z^3 F_3\Big]_{F_\alpha = 0} = 0, \quad \alpha = 1,2,3.
\end{aligned} \tag{2.5.3}$$

Proposition: The operator X_3 admitted by the system (2.1.4) provides the conserved vector $(C_3^0, C_3^1, C_3^2, C_3^3)$ with the components (2.5.2).

2.6 Case $X_4 = y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y} + u^2\dfrac{\partial}{\partial u^3} - u^3\dfrac{\partial}{\partial u^2}$

In this case $\xi^0 = \xi^1 = 0\,,\ \xi^2 = -z\,,\ \xi^3 = y,$ and $\eta^1 = 0\,,\ \eta^2 = -u^3,\ \eta^3 = u^2$. Firstly, we need to verify if the condition (2.1.1) is satisfied. Hence we need to prolong the generator:

$$\tilde{X_4} = X_4 + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha},$$

where

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j). \tag{2.6.1}$$

After applying the prolongation formula (2.6.1) we obtain

$$\begin{aligned}
\tilde{X_4} = {}& y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} + u^2\frac{\partial}{\partial u^3} - u^3\frac{\partial}{\partial u^2} - u_z^1\frac{\partial}{\partial u_y^1} + u_y^1\frac{\partial}{\partial u_z^1} - u_t^3\frac{\partial}{\partial u_t^2} - u_x^3\frac{\partial}{\partial u_x^2} - (u_y^3 + u_z^2)\frac{\partial}{\partial u_y^2}\\
&- (u_z^3 - u_y^2)\frac{\partial}{\partial u_z^2} + u_t^2\frac{\partial}{\partial u_t^3} + u_x^2\frac{\partial}{\partial u_x^3} + (u_y^2 - u_z^3)\frac{\partial}{\partial u_y^3} + (u_z^2 + u_y^3)\frac{\partial}{\partial u_z^3}\,.
\end{aligned} \tag{2.6.2}$$

Using the prolonged operator for verification of the condition (2.1.1) we get

$$\tilde{X}_4(\mathcal{L}) + \mathcal{L}\, D_i(\xi^i) = \tilde{X}_4(\mathcal{L}) = 0.$$

Hence the symmetry X_4 satisfies the condition (2.1.1) and, accordingly, there exists a conservation law corresponding to this symmetry. In this case we have

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha = \eta^\alpha + z u_y^\alpha - y u_z^\alpha, \quad \alpha = 1, 2, 3. \tag{2.6.3}$$

Accordingly, using the formulae (2.1.6) for components of the conserved vector, we obtain

$$C_4^0 = -2(zu_y^1 - yu_z^1)u_t^1 - 2(-u^3 + zu_y^2 - yu_z^2)u_t^2 - 2(u^2 + zu_y^3 - yu_z^3)u_t^3,$$

$$C_4^1 = (zu_y^1 - yu_z^1)\Big(4\beta u_x^1 + 2(1-\beta)(u_x^1 + u_y^2 + u_z^3)\Big) + (-u^3 + zu_y^2 - yu_z^2)\Big(2\beta(u_x^2 + u_y^1)\Big)$$
$$+ (u^2 + zu_y^3 - yu_z^3)\Big(2\beta(u_z^1 + u_x^3)\Big),$$

$$C_4^2 = \sum_{\alpha=1}^{3}\left[\beta\left(z\Big[(u_y^\alpha)^2 - (u_x^\alpha)^2 - (u_z^\alpha)^2\Big] - 2yu_y^\alpha u_z^\alpha\right) + z(u_t^\alpha)^2\right] + z(u_y^2)^2 - z(u_x^1)^2 - z(u_z^3)^2$$
$$+ 2\beta\Big(z(u_x^1u_z^3 - u_z^1u_x^3) - y(u_z^1u_x^2 - u_x^1u_z^2) + u^3(u_x^1 - u_y^2 + u_z^3) + u^2(u_z^2 + u_y^3)\Big)$$
$$- 2zu_x^1u_z^3 - 2(u^3 + yu_z^2)(u_x^1 + u_y^2 + u_z^3),$$

$$C_4^3 = \sum_{\alpha=1}^{3}\left[\beta\left(y\Big[(u_y^\alpha)^2 + (u_x^\alpha)^2 - (u_z^\alpha)^2\Big] + 2zu_y^\alpha u_z^\alpha\right) - y(u_t^\alpha)^2\right] + y(u_x^1)^2 + y(u_y^2)^2 - y(u_z^3)^2$$
$$+ 2\beta\Big(y(u_y^1u_x^2 - u_x^1u_y^2) + z(u_y^1u_x^3 - u_x^1u_y^3) - u^2(u_x^1 + u_y^2 - u_z^3) - u^3(u_z^2 + u_y^3)\Big)$$
$$+ 2yzu_x^1u_y^2 + 2(u^2 + zu_y^3)(u_x^1 + u_y^2 + u_z^3). \tag{2.6.4}$$

Checking the equation (2.1.5) we obtain

$$\Big[D_t(\tilde{C}^0) + D_x(\tilde{C}^1) + D_y(\tilde{C}^2) + D_z(\tilde{C}^3)\Big]_{F_\alpha = 0} = \Big[2W^\alpha F_\alpha\Big]_{F_\alpha = 0} = 0, \quad \alpha = 1, 2, 3. \tag{2.6.5}$$

Proposition: The operator X_4 admitted by the system (2.1.4) provides the conserved vector $(C_4^0, C_4^1, C_4^2, C_4^3)$ with the components (2.6.4).

2.7 Case $X_5 = z\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial z} + u^3\dfrac{\partial}{\partial u^1} - u^1\dfrac{\partial}{\partial u^3}$

Here $\xi^0 = \xi^2 = 0,\ \xi^1 = z,\ \xi^3 = -x,$ and $\eta^2 = 0,\ \eta^1 = u^3,\ \eta^3 = -u^1$. Firstly, we need to verify if the condition (2.1.1) is satisfied. Hence we need to prolong the generator:

$$\tilde{X}_5 = X_5 + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha},$$

where

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j). \tag{2.7.1}$$

After applying the prolongation formula (2.7.1) we obtain:

$$\tilde{X}_5 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + u^3\frac{\partial}{\partial u^1} - u^1\frac{\partial}{\partial u^3} + u^3_t\frac{\partial}{\partial u^1_t} + u^3_y\frac{\partial}{\partial u^1_y} + u^2_z\frac{\partial}{\partial u^2_x} - u^2_x\frac{\partial}{\partial u^2_z} + (u^3_x + u^1_z)\frac{\partial}{\partial u^1_x}$$
$$+ (u^3_z - u^1_x)\frac{\partial}{\partial u^1_z} - u^1_t\frac{\partial}{\partial u^3_t} - u^1_y\frac{\partial}{\partial u^3_y} - (u^1_x - u^3_z)\frac{\partial}{\partial u^3_x} - (u^1_z + u^3_x)\frac{\partial}{\partial u^3_z}\,. \quad (2.7.2)$$

Using the prolonged operator for verification of the condition (2.1.1) we get:

$$\tilde{X}_5(\mathcal{L}) + \mathcal{L}\, D_i(\xi^i) = \tilde{X}_5(\mathcal{L}) = 0.$$

Hence the symmetry X_5 satisfies the condition (2.1.1) and, accordingly, there exists a conservation law corresponding to this symmetry. In this case we have:

$$W^\alpha = \eta^\alpha - \xi^i u^\alpha_j = \eta^\alpha - zu^\alpha_x + xu^\alpha_z, \quad \alpha = 1, 2, 3. \quad (2.7.3)$$

Using the formula (2.1.6) for components of the conserved vector, we get:

$$C^0_5 = \; -2(u^3 - zu^1_x + xu^1_z)u^1_t - 2(-zu^2_x + xu^2_z)u^2_t - 2(-u^1 - zu^3_x + xu^3_z)u^3_t,$$

$$C^1_5 = \sum_{\alpha=1}^{3}\left[\beta\Big(z\Big[(u^\alpha_z)^2 + (u^\alpha_y)^2 - (u^\alpha_x)^2\Big] + 2xu^\alpha_z u^\alpha_x\Big) - z(u^\alpha_t)^2\right] + z(u^2_y)^2 + z(u^3_z)^2 - z(u^1_x)^2$$
$$+ 2\beta\Big(z(u^2_z u^3_y - u^2_y u^3_z) + x(u^2_z u^1_y - u^2_y u^1_z) - u^3(u^2_y + u^3_z - u^1_x) - u^1(u^3_x + u^1_z)\Big)$$
$$+ 2zxu^2_y u^3_z + 2(u^3 + xu^1_z)(u^2_y + u^3_z + u^1_x),$$

$$C^2_5 = (xu^2_z - zu^2_x)\Big(4\beta u^2_y + 2(1-\beta)(u^2_y + u^3_z + u^1_x)\Big) + (-u^1 + xu^3_z - zu^3_x)\Big(2\beta(u^3_y + u^2_z)\Big)$$
$$+ (u^3 + xu^1_z - zu^1_x)\Big(2\beta(u^2_x + u^1_y)\Big),$$

$$C^3_5 = \sum_{\alpha=1}^{3}\left[\beta\Big(x\Big[(u^\alpha_z)^2 - (u^\alpha_y)^2 - (u^\alpha_x)^2\Big] - 2zu^\alpha_z u^\alpha_x\Big) + x(u^\alpha_t)^2\right] + x(u^3_z)^2 - x(u^2_y)^2 - x(u^1_x)^2$$
$$+ 2\beta\Big(x(u^2_x u^1_x - u^2_x u^1_y) - z(u^2_x u^3_y - u^2_y u^3_x) + u^1(u^2_y - u^3_z + u^1_x) + u^3(u^3_x + u^1_z)\Big)$$
$$- 2xu^2_y u^1_x - 2(u^1 + zu^3_x)(u^2_y + u^3_z + u^1_x). \quad (2.7.4)$$

Checking the equation (2.1.5) we obtain:

$$\Big[D_t(\tilde{C}^0) + D_x(\tilde{C}^1) + D_y(\tilde{C}^2) + D_z(\tilde{C}^3)\Big]_{F_\alpha = 0} = \Big[2W^\alpha F_\alpha\Big]_{F_\alpha = 0} = 0, \quad \alpha = 1, 2, 3. \quad (2.7.5)$$

Proposition: The operator X_5 admitted by the system (2.1.4) provides the conserved vector $(C^0_5, C^1_5, C^2_5, C^3_5)$ with the components (2.7.4).

2.8 Case $X_6 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + u^1\frac{\partial}{\partial u^2} - u^2\frac{\partial}{\partial u^1}$

We have $\xi^0 = \xi^3 = 0$, $\xi^1 = -y$, $\xi^2 = x$, and $\eta^1 = -u^2$, $\eta^2 = u^1$, $\eta^3 = 0$. We check the condition (2.1.1). Using the prolonged generator

$$\tilde{X}_6 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + u^1\frac{\partial}{\partial u^2} - u^2\frac{\partial}{\partial u^1} - u^2_t\frac{\partial}{\partial u^1_t} - u^2_z\frac{\partial}{\partial u^1_z} + u^1_t\frac{\partial}{\partial u^2_t} + u^1_z\frac{\partial}{\partial u^2_z} - (u^2_x + u^1_y)\frac{\partial}{\partial u^1_x}$$
$$- (u^2_y - u^1_x)\frac{\partial}{\partial u^1_y} - u^3_y\frac{\partial}{\partial u^3_x} + u^3_x\frac{\partial}{\partial u^3_y} + (u^1_x - u^2_y)\frac{\partial}{\partial u^2_x} + (u^1_y + u^2_x)\frac{\partial}{\partial u^2_y}, \quad (2.8.1)$$

we obtain for the condition (2.1.1) the following:

$$\tilde{X}_6(\mathcal{L}) + \mathcal{L}\, D_i(\xi^i) = \tilde{X}_6(\mathcal{L}) = 0.$$

Hence the symmetry X_6 satisfies the condition (2.1.1) and, accordingly, there exists a conservation law corresponding to this symmetry. In this case we have

$$W^\alpha = \eta^\alpha - \xi^i u^\alpha_j = \eta^\alpha + y u^\alpha_x - x u^\alpha_y, \quad \alpha = 1, 2, 3. \tag{2.8.2}$$

Using the formula (2.1.6) for components of the conserved vector, we get

$$\begin{aligned}
C^0_6 =& \; -2(-u^2 + y u^1_x - x u^1_y) u^1_t - 2(u^1 + y u^2_x - x u^2_y) u^2_t - 2(y u^3_x - x u^3_y) u^3_t, \\
C^1_6 =& \sum_{\alpha=1}^{3} \left[\beta \Big(y \Big[(u^\alpha_x)^2 - (u^\alpha_z)^2 - (u^\alpha_y)^2 \Big] - 2x u^\alpha_x u^\alpha_y \Big) + y (u^\alpha_t)^2 \right] + y(u^1_x)^2 - y(u^3_z)^2 - y(u^2_y)^2 \\
& + 2\beta \Big(y(u^3_y u^2_y - u^3_y u^2_z) - x(u^3_y u^1_z - u^3_z u^1_y) + u^2(u^3_z - u^1_x + u^2_y) + u^1(u^1_y + u^2_x) \Big) \\
& - 2y u^3_z u^2_y - 2(u^2 + x u^1_y)(u^3_z + u^1_x + u^2_y), \\
C^2_6 =& \sum_{\alpha=1}^{3} \left[\beta \Big(x \Big[(u^\alpha_x)^2 + (u^\alpha_z)^2 - (u^\alpha_y)^2 \Big] + 2y u^\alpha_x u^\alpha_y \Big) - x (u^\alpha_t)^2 \right] + x(u^3_z)^2 + x(u^1_x)^2 - x(u^2_y)^2 \\
& + 2\beta \Big(x(u^3_x u^1_z - u^3_z u^1_x) + y(u^3_x u^2_z - u^3_z u^2_x) - u^1(u^3_z + u^1_x - u^2_y) - u^2(u^1_y + u^2_x) \Big) \\
& + 2xy u^3_z u^1_x + 2(u^1 + y u^2_x)(u^3_z + u^1_x + u^2_y), \\
C^3_6 =& \; (y u^3_x - x u^3_y) \Big(4\beta u^3_z + 2(1-\beta)(u^3_z + u^1_x + u_y) \Big) + (-u^2 + y u^1_x - x u^1_y) \Big(2\beta (u^1_z + u^3_x) \Big) \\
& + (u^1 + y u^2_x - x u^2_y) \Big(2\beta (u^3_y + u^2_z) \Big).
\end{aligned} \tag{2.8.3}$$

Checking the equation (2.1.5) we obtain

$$\Big[D_t(\tilde{C}^0) + D_x(\tilde{C}^1) + D_y(\tilde{C}^2) + D_z(\tilde{C}^3) \Big]_{F_\alpha = \, 0} = \Big[2W^\alpha F_\alpha \Big]_{F_\alpha = \, 0} = 0, \quad \alpha = 1, 2, 3. \tag{2.8.4}$$

Proposition: The operator X_6 admitted by the system (2.1.4) provides the conserved vector $(C^0_6, C^1_6, C^2_6, C^3_6)$ with the components (2.8.3).

2.9 Case $X_7 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$

In this case $\xi^0 = t,\ \xi^1 = x,\ \xi^2 = y,\ \xi^3 = z,$ and $\eta^1 = \eta^2 = \eta^3 = 0$. Firstly, we need to check the condition (2.1.1). Hence we need to prolong the generator

$$\tilde{X}_7 = X_7 + \zeta^\alpha_i \frac{\partial}{\partial u^\alpha_i} \quad \text{where} \;\; \zeta^\alpha_i = D_i(\eta^\alpha) - u^\alpha_j D_i(\xi^j) = -u^\alpha_i. \tag{2.9.1}$$

After applying the prolongation formula (2.9.1) we obtain:

$$\begin{aligned}
\tilde{X}_7 = \; & X_7 - \; u^1_t \frac{\partial}{\partial u^1_t} - u^1_x \frac{\partial}{\partial u^1_x} - u^1_y \frac{\partial}{\partial u^1_y} - u^1_z \frac{\partial}{\partial u^1_z} - \; u^2_t \frac{\partial}{\partial u^2_t} - u^2_x \frac{\partial}{\partial u^2_x} \\
& - \; u^2_y \frac{\partial}{\partial u^2_y} - u^2_z \frac{\partial}{\partial u^2_z} - \; u^3_t \frac{\partial}{\partial u^3_t} - u^3_x \frac{\partial}{\partial u^3_x} - u^3_y \frac{\partial}{\partial u^3_y} - u^3_z \frac{\partial}{\partial u^3_z} \, .
\end{aligned} \tag{2.9.2}$$

Then

$$\tilde{X_7}(\mathcal{L}) + \mathcal{L}\, D_i(\xi^i) = -2\mathcal{L} + 4\mathcal{L} = 2\mathcal{L} \neq D_i(B^i), \tag{2.9.3}$$

thus the condition (2.1.1) is not satisfied. Consequently, there is no conservation law corresponding to this symmetry.

2.10 Case $X_8 = u^1\dfrac{\partial}{\partial u^1} + u^2\dfrac{\partial}{\partial u^2} + u^3\dfrac{\partial}{\partial u^3}$

Here $\xi^0 = \xi^1 = \xi^2 = \xi^3 = 0$, and $\eta^1 = u^1$, $\eta^2 = u^2$, $\eta^3 = u^3$. The prolonged operator has the form

$$\tilde{X_8} = \ X_8 + u_t^\alpha \frac{\partial}{\partial u_t^\alpha} + u_x^\alpha \frac{\partial}{\partial u_x^\alpha} + u_y^\alpha \frac{\partial}{\partial u_y^\alpha} + u_z^\alpha \frac{\partial}{\partial u_z^\alpha}, \tag{2.10.1}$$

and

$$\tilde{X_8}(\mathcal{L}) + \mathcal{L}\, D_i(\xi^i) = \tilde{X_8}(\mathcal{L}) = 2\mathcal{L} \neq D_i(B^i). \tag{2.10.2}$$

Hence, the condition (2.1.1) is not satisfied. Consequently, there is no conservation law corresponding to this symmetry.

2.11 Case $X_\infty = w^1\dfrac{\partial}{\partial u^1} + w^2\dfrac{\partial}{\partial u^2} + w^3\dfrac{\partial}{\partial u^3}$

In this case $\xi^0 = \xi^1 = \xi^2 = \xi^3 = 0$, and $\eta^1 = w^1$, $\eta^2 = w^2$, $\eta^3 = w^3$. Here (w^1, w^2, w^3) is an arbitrary solution of the Lamé equation. Firstly, we need to verify if the condition (2.1.1) is satisfied. Hence we need to prolong the generator:

$$\tilde{X_\infty} = X_\infty + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} \quad \text{where} \quad \zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j)\,. \tag{2.11.1}$$

After applying the prolongation formula (2.11.1) we obtain

$$\tilde{X_\infty} = \ X_\infty + w_t^\alpha \frac{\partial}{\partial u_t^\alpha} + w_x^\alpha \frac{\partial}{\partial u_x^\alpha} + w_y^\alpha \frac{\partial}{\partial u_y^\alpha} + w_z^\alpha \frac{\partial}{\partial u_z^\alpha}\,,\ \alpha = 1, 2, 3. \tag{2.11.2}$$

and

$$\begin{aligned}
\tilde{X_\infty}(\mathcal{L}) + \mathcal{L}\, D_i(\xi^i) =& \tilde{X_\infty}(\mathcal{L}) \\
&= 2\beta\Big((u_y^1 + u_x^2)(w_y^1 + w_x^2) + (u_z^1 + u_x^3)(w_z^1 + w_x^3) \\
&\quad + (u_z^2 + u_y^3)(w_z^2 + w_y^3) + 2(u_x^1)(w_x^1) + 2(u_y^2)(w_y^2) + 2(u_z^3)(w_z^3)\Big) \\
&\quad + 2(1-\beta)(u_x^1 + u_y^2 + u_z^3)(w_x^1 + w_y^2 + w_z^3) - 2(u_t^1)(w_t^1) \\
&\quad - 2(u_t^2)(w_t^2) - 2(u_t^3)(w_t^3) = D_i(B^i) + 2u^\alpha F_\alpha(\mathbf{w}) = D_i(B^i).
\end{aligned} \tag{2.11.3}$$

Here $F_\alpha(\mathbf{w}) = 0$, since $\mathbf{w}$ is a solution of the system. The components B^i have the following form:

$$
\begin{aligned}
B^0 =& -2\sum_{\alpha=1}^{3} u^\alpha w_t^\alpha,\\
B^1 =& 2\beta\Big(u^2(w_y^1 + w_x^2) + u^3(w_z^1 + w_x^3) - u^1(-w_x^1 + w_y^2 + w_z^3)\Big) + 2u^1(w_x^1 + w_y^2 + w_z^3),\\
B^2 =& 2\beta\Big(u^1(w_y^1 + w_x^2) + u^3(w_z^2 + w_y^3) - u^2(w_x^1 - w_y^2 + w_z^3)\Big) + 2u^2(w_x^1 + w_y^2 + w_z^3),\\
B^3 =& 2\beta\Big(u^1(w_z^1 + w_x^3) + u^2(w_z^2 + w_y^3) - u^3(w_x^1 + w_y^2 - w_z^3)\Big) + 2u^3(w_x^1 + w_y^2 + w_z^3). \qquad (2.11.4)
\end{aligned}
$$

Hence, the condition (2.1.1) is satisfied and, accordingly, there exists a conservation law corresponding to this symmetry and the conserved vector has the form:

$$
\bar{C}_\infty = \Big(C^0 - B^0, C^1 - B^1, C^2 - B^2, C^3 - B^3\Big), \qquad (2.11.5)
$$

where C^i are derived from (2.1.6). In this case we have

$$
W^\alpha = w^\alpha, \quad \alpha = 1, 2, 3. \qquad (2.11.6)
$$

Using the formulae (2.1.6) for components of the conserved vector, we get

$$
\begin{aligned}
C_\infty^0 =& -2\sum_{\alpha=1}^{3} w^\alpha u_t^\alpha,\\
C_\infty^1 =& 2\beta\Big(w^2(u_x^2 + u_y^1) + w^3(u_x^3 + u_z^1) - w^1(u_y^2 + u_z^3 - u_x^1)\Big) + 2w^1(u_x^1 + u_y^2 + u_z^3),\\
C_\infty^2 =& 2\beta\Big(w^3(u_y^3 + u_z^2) + w^1(u_y^1 + u_x^2) - w^2(u_z^3 + u_x^1 - u_y^2)\Big) + 2w^2(u_x^1 + u_y^2 + u_z^3),\\
C_\infty^3 =& 2\beta\Big(w^1(u_z^1 + u_x^3) + w^2(u_z^2 + u_y^3) - w^3(u_x^1 + u_y^2 - u_z^3)\Big) + 2w^3(u_x^1 + u_y^2 + u_z^3). \qquad (2.11.7)
\end{aligned}
$$

Accordingly, invoking the equation (2.11.4) and the equation (2.11.7) we obtain the components of the conserved vector from the equation (2.11.5). Omitting the constant coefficient 2 in all the components, we have the following result:

$$
\begin{aligned}
\bar{C}_\infty^0 =& -\sum_{\alpha=1}^{3}\Big(w^\alpha u_t^\alpha - u^\alpha w_t^\alpha\Big),\\
\bar{C}_\infty^1 =& \beta\Big(w^2(u_x^2 + u_y^1) + w^3(u_x^3 + u_z^1) - w^1(u_y^2 + u_z^3 - u_x^1) - u^2(w_y^1 + w_x^2) - u^3(w_z^1 + w_x^3)\\
& + u^1(-w_x^1 + w_y^2 + w_z^3)\Big) + w^1(u_x^1 + u_y^2 + u_z^3) - u^1(w_x^1 + w_y^2 + w_z^3),\\
\bar{C}_\infty^2 =& \beta\Big(w^3(u_y^3 + u_z^2) + w^1(u_y^1 + u_x^2) - w^2(u_z^3 + u_x^1 - u_y^2) - u^1(w_y^1 + w_x^2) - u^3(w_z^2 + w_y^3)\\
& + u^2(w_x^1 - w_y^2 + w_z^3\Big) + w^2(u_x^1 + u_y^2 + u_z^3) - u^2(w_x^1 + w_y^2 + w_z^3),
\end{aligned}
$$

$$\bar{C}^3_\infty = \beta\Big(w^1(u^1_z + u^3_x) + w^2(u^2_z + u^3_y) - w^3(u^1_x + u^2_y - u^3_z) - u^1(w^1_z + w^3_x) - u^2(w^2_z + w^3_y) + u^3(w^1_x + w^2_y - w^3_z)\Big) + w^3(u^1_x + u^2_y + u^3_z) - u^3(w^1_x + w^2_y + w^3_z). \tag{2.11.8}$$

Proposition: The operator X_∞ admitted by the system (2.1.4) provides the conserved vector $(\bar{C}^0_\infty, \bar{C}^1_\infty, \bar{C}^2_\infty, \bar{C}^3_\infty)$ with the components (2.11.8).

3 Investigation for nonlinear self-adjointness of the Lamé equation

Now we want to investigate if the system of differential equations (2.1.4)

$$\begin{aligned} F_1 &\equiv \beta(u^1_{xx} + u^1_{yy} + u^1_{zz}) + u^1_{xx} + u^2_{xy} + u^3_{xz} - u^1_{tt} = 0,\\ F_2 &\equiv \beta(u^2_{xx} + u^2_{yy} + u^2_{zz}) + u^1_{xy} + u^2_{yy} + u^3_{yz} - u^2_{tt} = 0,\\ F_3 &\equiv \beta(u^3_{xx} + u^3_{yy} + u^3_{zz}) + u^1_{xz} + u^2_{yz} + u^3_{zz} - u^3_{tt} = 0, \end{aligned} \tag{3.1}$$

is nonlinearly self-adjoint. We must construct the *formal Lagrangian* for the above system. According to N.Ibragimov [4, 5], it will have the following form:

$$\mathcal{L} = v^1 F_1 + v^2 F_2 + v^3 F_3, \tag{3.2}$$

where v^1, v^2, v^3 are new dependent variables. According to the definition on p.18 [4], the system (3.1) is said to be nonlinearly self-adjoint if the adjoint equations

$$F^*_\alpha(\mathbf{x},\ \mathbf{u},\ \mathbf{v},\ \mathbf{u}_{(1)}, \mathbf{v}_{(1)}, \mathbf{u}_{(2)}, \mathbf{v}_{(2)}) \equiv \frac{\delta\mathcal{L}}{\delta u^\alpha} = 0, \qquad \alpha = 1,2,3, \tag{3.3}$$

are satisfied for all solutions $\mathbf{u}$ of the original system upon a substitution

$$v^\alpha = \varphi^\alpha(\mathbf{x},\ \mathbf{u}), \quad \alpha = 1,2,3, \tag{3.4}$$

such that not all components in

$$\varphi(\mathbf{x},\ \mathbf{u}) = \Big(\varphi^1(\mathbf{x},\ \mathbf{u}), \varphi^2(\mathbf{x},\ \mathbf{u}), \varphi^3(\mathbf{x},\ \mathbf{u})\Big),$$

are equal to zero simultaneously.
Here $\mathbf{x}= (t, x, y, z)$, $\mathbf{u}= (u^1, u^2, u^3)$, $\mathbf{v}= (v^1, v^2, v^3)$; $\mathbf{u}_{(1)}, \mathbf{u}_{(2)}$ and $\mathbf{v}_{(1)}, \mathbf{v}_{(2)}$ are first and second derivatives; and $\frac{\delta}{\delta u^\alpha}$ is the variational derivative:

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - D_i\frac{\partial}{\partial u^\alpha_i} + D_i D_j \frac{\partial}{\partial u^\alpha_{ij}} + \cdots, \quad i,j = 0,1,2,3. \tag{3.5}$$

Hence, for the system (3.1), the adjoint equations has the following form:

$$\begin{aligned} F^*_1 &\equiv \frac{\delta\mathcal{L}}{\delta u^1} = D^2_x\frac{\partial\mathcal{L}}{\partial u^1_{xx}} + D^2_y\frac{\partial\mathcal{L}}{\partial u^1_{yy}} + D^2_z\frac{\partial\mathcal{L}}{\partial u^1_{zz}} + D^2_t\frac{\partial\mathcal{L}}{\partial u^1_{tt}} + D_xD_y\frac{\partial\mathcal{L}}{\partial u^1_{xy}} + D_xD_z\frac{\partial\mathcal{L}}{\partial u^1_{xz}} = 0,\\ F^*_2 &\equiv \frac{\delta\mathcal{L}}{\delta u^2} = D^2_x\frac{\partial\mathcal{L}}{\partial u^2_{xx}} + D^2_y\frac{\partial\mathcal{L}}{\partial u^2_{yy}} + D^2_z\frac{\partial\mathcal{L}}{\partial u^2_{zz}} + D^2_t\frac{\partial\mathcal{L}}{\partial u^2_{tt}} + D_xD_y\frac{\partial\mathcal{L}}{\partial u^2_{xy}} + D_yD_z\frac{\partial\mathcal{L}}{\partial u^2_{yz}} = 0,\\ F^*_3 &\equiv \frac{\delta\mathcal{L}}{\delta u^3} = D^2_x\frac{\partial\mathcal{L}}{\partial u^3_{xx}} + D^2_y\frac{\partial\mathcal{L}}{\partial u^3_{yy}} + D^2_z\frac{\partial\mathcal{L}}{\partial u^3_{zz}} + D^2_t\frac{\partial\mathcal{L}}{\partial u^3_{tt}} + D_xD_z\frac{\partial\mathcal{L}}{\partial u^3_{xz}} + D_yD_z\frac{\partial\mathcal{L}}{\partial u^3_{yz}} = 0. \end{aligned} \tag{3.6}$$

Now we simplify the equations and derive:

$$F_1^* \equiv D_x^2((\beta+1)v^1) + D_y^2(\beta v^1) + D_z^2(\beta v^1) + D_t^2(-v^1) + D_xD_y(v^2) + D_xD_z(v^3) = 0,$$

$$F_2^* \equiv D_x^2(\beta v^2) + D_y^2((\beta+1)v^2) + D_z^2(\beta v^2) + D_t^2(-v^2) + D_xD_y(v^1) + D_yD_z(v^3) = 0,$$

$$F_3^* \equiv D_x^2(\beta v^3) + D_y^2(\beta v^3) + D_z^2((\beta+1)v^3) + D_t^2(-v^3) + D_xD_z(v^1) + D_yD_z(v^2) = 0. \quad (3.7)$$

According to the definition of nonlinear self-adjointness [4], the following equations hold after using the substitution (3.4):

$$F_\alpha^*(\mathbf{x},\ \mathbf{u}, \varphi, \mathbf{u}_{(1)}, \varphi_{(1)}, \mathbf{u}_{(2)}, \varphi_{(2)}) \equiv A_\alpha F_1 + B_\alpha F_2 + C_\alpha F_3, \quad \alpha = 1,2,3. \quad (3.8)$$

where $A_\alpha, B_\alpha, C_\alpha$ are undetermined coefficients and $\varphi_{(1)}, \varphi_{(2)}$ are respectively first and second derivatives of $\varphi(\mathbf{x}, \mathbf{u})$. Hence, we have

$$F_1^* = A_1F_1 + B_1F_2 + C_1F_3\,. \quad (3.9)$$

Using substitution (3.4) the equation $F_1^* = 0$ in (3.7) yields

$$\begin{aligned} F_1^* \equiv \quad & D_x\left[(\beta+1)(\varphi^1_x + u^1_x\varphi^1_{u^1} + u^2_x\varphi^1_{u^2} + u^3_x\varphi^1_{u^3})\right] \\ & + D_y\left[\beta(\varphi^1_y + u^1_y\varphi^1_{u^1} + u^2_y\varphi^1_{u^2} + u^3_y\varphi^1_{u^3})\right] \\ & + D_z\left[\beta(\varphi^1_z + u^1_z\varphi^1_{u^1} + u^2_z\varphi^1_{u^2} + u^3_z\varphi^1_{u^3})\right] \\ & + D_x(\varphi^2_y + u^1_y\varphi^2_{u^1} + u^2_y\varphi^2_{u^2} + u^3_y\varphi^2_{u^3}) \\ & + D_x(\varphi^3_z + u^1_z\varphi^3_{u^1} + u^2_z\varphi^3_{u^2} + u^3_z\varphi^3_{u^3}) \\ & - D_t(\varphi^1_t + u^1_t\varphi^1_{u^1} + u^2_t\varphi^1_{u^2} + u^3_t\varphi^1_{u^3}) = 0. \end{aligned}$$

Application of the operators of total differentiations in the above equation leads to the following:

$$\begin{aligned} F_1^* \equiv \quad & (\beta+1)\left[\varphi^1_{xx} + u^1_x\varphi^1_{xu^1} + u^2_x\varphi^1_{xu^2} + u^3_x\varphi^1_{xu^3}\right. \\ & + u^1_{xx}\varphi^1_{u^1} + u^1_x(\varphi^1_{u^1x} + u^1_x\varphi^1_{u^1u^1} + u^2_x\varphi^1_{u^1u^2} + u^3_x\varphi^1_{u^1u^3}) \\ & + u^2_{xx}\varphi^1_{u^2} + u^2_x(\varphi^1_{u^2x} + u^1_x\varphi^1_{u^2u^1} + u^2_x\varphi^1_{u^2u^2} + u^3_x\varphi^1_{u^2u^3}) \\ & \left. + u^3_{xx}\varphi^1_{u^3} + u^3_x(\varphi^1_{u^3x} + u^1_x\varphi^1_{u^3u^1} + u^2_x\varphi^1_{u^3u^2} + u^3_x\varphi^1_{u^3u^3})\right] \\ & + \beta\left[\varphi^1_{yy} + u^1_y\varphi^1_{yu^1} + u^2_y\varphi^1_{yu^2} + u^3_y\varphi^1_{yu^3}\right. \\ & + u^1_{yy}\varphi^1_{u^1} + u^1_y(\varphi^1_{u^1y} + u^1_y\varphi^1_{u^1u^1} + u^2_y\varphi^1_{u^1u^2} + u^3_y\varphi^1_{u^1u^3}) \\ & + u^2_{yy}\varphi^1_{u^2} + u^2_y(\varphi^1_{u^2y} + u^1_y\varphi^1_{u^2u^1} + u^2_y\varphi^1_{u^2u^2} + u^3_y\varphi^1_{u^2u^3}) \\ & \left. + u^3_{yy}\varphi^1_{u^3} + u^3_y(\varphi^1_{u^3y} + u^1_y\varphi^1_{u^3u^1} + u^2_y\varphi^1_{u^3u^2} + u^3_y\varphi^1_{u^3u^3})\right] \\ & + \beta\left[\varphi^1_{zz} + u^1_z\varphi^1_{zu^1} + u^2_z\varphi^1_{zu^2} + u^3_z\varphi^1_{zu^3}\right. \\ & + u^1_{zz}\varphi^1_{u^1} + u^1_z(\varphi^1_{u^1z} + u^1_z\varphi^1_{u^1u^1} + u^2_z\varphi^1_{u^1u^2} + u^3_z\varphi^1_{u^1u^3}) \\ & + u^2_{zz}\varphi^1_{u^2} + u^2_z(\varphi^1_{u^2z} + u^1_z\varphi^1_{u^2u^1} + u^2_z\varphi^1_{u^2u^2} + u^3_z\varphi^1_{u^2u^3}) \\ & \left. + u^3_{zz}\varphi^1_{u^3} + u^3_z(\varphi^1_{u^3z} + u^1_z\varphi^1_{u^3u^1} + u^2_z\varphi^1_{u^3u^2} + u^3_z\varphi^1_{u^3u^3})\right] \\ & + \varphi^2_{yx} + u^1_x\varphi^2_{yu^1} + u^2_x\varphi^2_{yu^2} + u^3_x\varphi^2_{yu^3} \\ & + u^1_{yx}\varphi^2_{u^1} + u^1_y(\varphi^2_{u^1x} + u^1_x\varphi^2_{u^1u^1} + u^2_x\varphi^2_{u^1u^2} + u^3_x\varphi^2_{u^1u^3}) \\ & + u^2_{yx}\varphi^2_{u^2} + u^2_y(\varphi^2_{u^2x} + u^1_x\varphi^2_{u^2u^1} + u^2_x\varphi^2_{u^2u^2} + u^3_x\varphi^2_{u^2u^3}) \\ & + u^3_{yx}\varphi^2_{u^3} + u^3_y(\varphi^2_{u^3x} + u^1_x\varphi^2_{u^3u^1} + u^2_x\varphi^2_{u^3u^2} + u^3_x\varphi^2_{u^3u^3}) \end{aligned}$$

$$
\begin{aligned}
&+\varphi^3_{zx}+u^1_x\varphi^3_{zu^1}+u^2_x\varphi^3_{zu^2}+u^3_x\varphi^3_{zu^3}\\
&+u^1_{zx}\varphi^3_{u^1}+u^1_z(\varphi^3_{u^1x}+u^1_x\varphi^3_{u^1u^1}+u^2_x\varphi^3_{u^1u^2}+u^3_x\varphi^3_{u^1u^3})\\
&+u^2_{zx}\varphi^3_{u^2}+u^2_z(\varphi^3_{u^2x}+u^1_x\varphi^3_{u^2u^1}+u^2_x\varphi^3_{u^2u^2}+u^3_x\varphi^3_{u^2u^3})\\
&+u^3_{zx}\varphi^3_{u^3}+u^3_z(\varphi^3_{u^3x}+u^1_x\varphi^3_{u^3u^1}+u^2_x\varphi^3_{u^3u^2}+u^3_x\varphi^3_{u^3u^3})\\
&-\big[\varphi^1_{tt}+u^1_t\varphi^1_{tu^1}+u^2_t\varphi^1_{tu^2}+u^3_t\varphi^1_{tu^3}\\
&+u^1_{tt}\varphi^1_{u^1}+u^1_t(\varphi^1_{u^1t}+u^1_t\varphi^1_{u^1u^1}+u^2_t\varphi^1_{u^1u^2}+u^3_t\varphi^1_{u^1u^3})\\
&+\underline{\underline{u^2_{tt}}}\varphi^1_{u^2}+u^2_t(\varphi^1_{u^2t}+u^1_t\varphi^1_{u^2u^1}+u^2_t\varphi^1_{u^2u^2}+u^3_t\varphi^1_{u^2u^3})\\
&+\underline{\underline{u^3_{tt}}}\varphi^1_{u^3}+u^3_t(\varphi^1_{u^3t}+u^1_t\varphi^1_{u^3u^1}+u^2_t\varphi^1_{u^3u^2}+u^3_t\varphi^1_{u^3u^3})\Big]\\
=\;& A_1\left[\beta(u^1_{xx}+u^1_{yy}+u^1_{zz})+u^1_{xx}+u^2_{xy}+u^3_{xz}-u^1_{tt}\right]\\
&+B_1\left[\beta(u^2_{xx}+u^2_{yy}+u^2_{zz})+u^1_{xy}+u^2_{yy}+\underline{u^3_{yz}}-u^2_{tt}\right]\\
&+C_1\left[\beta(u^3_{xx}+u^3_{yy}+u^3_{zz})+u^1_{xz}+\underline{u^2_{yz}}+u^3_{zz}-u^3_{tt}\right].
\end{aligned}
\tag{3.10}
$$

Here, A_1, B_1, C_1 do not depend on second derivatives and it is possible to split the equation. The terms with u^3_{yz} and u^2_{yz} are only on the right-hand side of the above equation therefore the coefficients in front of them must be equal to zero. Moreover, the terms with u^2_{tt} and u^3_{tt} are only on the left-hand side of the equation. Hence

$$
u^3_{yz}: B_1=0, \qquad u^2_{yz}: C_1=0,
$$

$$
u^2_{tt}: \varphi^1_{u^2}=0, \qquad u^3_{tt}: \varphi^1_{u^3}=0.
$$

The last two equations show that φ^1 does not depends on u^2, u^3, i.e.

$$
\varphi^1=\varphi^1(t,x,y,z,u^1).
$$

The equation (3.10) will be simplified as follows:

$$
\begin{aligned}
&(\beta+1)\left[\varphi^1_{xx}+u^1_{xx}\varphi^1_{u^1}+u^1_x(2\varphi^1_{u^1x}+u^1_x\varphi^1_{u^1u^1})\right]\\
&+\beta\left[\varphi^1_{yy}+u^1_{yy}\varphi^1_{u^1}+u^1_y(2\varphi^1_{u^1y}+u^1_y\varphi^1_{u^1u^1})\right]\\
&+\beta\left[\varphi^1_{zz}+u^1_{zz}\varphi^1_{u^1}+u^1_z(2\varphi^1_{u^1z}+u^1_z\varphi^1_{u^1u^1})\right]\\
&+\varphi^2_{yx}+u^1_x\varphi^2_{yu^1}+u^2_x\varphi^2_{yu^2}+u^3_x\varphi^2_{yu^3}\\
&+\underline{u^1_{xy}}\varphi^2_{u^1}+u^1_y(\varphi^2_{u^1x}+u^1_x\varphi^2_{u^1u^1}+u^2_x\varphi^2_{u^1u^2}+u^3_x\varphi^2_{u^1u^3})\\
&+u^2_{yx}\varphi^2_{u^2}+u^2_y(\varphi^2_{u^2x}+u^1_x\varphi^2_{u^2u^1}+u^2_x\varphi^2_{u^2u^2}+u^3_x\varphi^2_{u^2u^3})\\
&+\underline{u^3_{yx}}\varphi^2_{u^3}+u^3_y(\varphi^2_{u^3x}+u^1_x\varphi^2_{u^3u^1}+u^2_x\varphi^2_{u^3u^2}+u^3_x\varphi^2_{u^3u^3})\\
&+\varphi^3_{zx}+u^1_x\varphi^3_{zu^1}+u^2_x\varphi^3_{zu^2}+u^3_x\varphi^3_{zu^3}\\
&+\underline{u^1_{zx}}\varphi^3_{u^1}+u^1_z(\varphi^3_{u^1x}+u^1_x\varphi^3_{u^1u^1}+u^2_x\varphi^3_{u^1u^2}+u^3_x\varphi^3_{u^1u^3})\\
&+\underline{u^2_{zx}}\varphi^3_{u^2}+u^2_z(\varphi^3_{u^2x}+u^1_x\varphi^3_{u^2u^1}+u^2_x\varphi^3_{u^2u^2}+u^3_x\varphi^3_{u^2u^3})\\
&+u^3_{zx}\varphi^3_{u^3}+u^3_z(\varphi^3_{u^3x}+u^1_x\varphi^3_{u^3u^1}+u^2_x\varphi^3_{u^3u^2}+u^3_x\varphi^3_{u^3u^3})\\
&-\left[\varphi^1_{tt}+u^1_t\varphi^1_{tu^1}+\underline{\underline{u^1_{tt}}}\varphi^1_{u^1}+u^1_t(\varphi^1_{u^1t}+u^1_t\varphi^1_{u^1u^1})\right]\\
=\;& A_1\left[\beta(u^1_{xx}+u^1_{yy}+u^1_{zz})+u^1_{xx}+u^2_{xy}+u^3_{xz}-\underline{\underline{u^1_{tt}}}\right].
\end{aligned}
\tag{3.11}
$$

Considering the coefficients of u^1_{tt}, u^1_{xy}, u^1_{xz}, u^2_{zx} and u^3_{yx} we will obtain:

$$
u^1_{tt}: A_1=\varphi^1_{u^1}, \quad u^1_{xy}: \varphi^2_{u^1}=0, \quad u^1_{xz}: \varphi^3_{u^1}=0, \quad u^2_{zx}: \varphi^3_{u^2}=0, \quad u^3_{yx}: \varphi^2_{u^3}=0,
$$

whence

$$\varphi^2 = \varphi^2(t,x,y,z,u^2), \quad \varphi^3 = \varphi^3(t,x,y,z,u^3).$$

Hence, the equation (3.11) will be simplified further:

$$\begin{aligned}
&(\beta+1)\left[\varphi^1_{xx} + \underline{\underline{\underline{u^1_x}}}(2\varphi^1_{u^1x} + u^1_x\varphi^1_{u^1u^1})\right] \\
&+\beta\left[\varphi^1_{yy} + \underline{\underline{\underline{u^1_y}}}(2\varphi^1_{u^1y} + u^1_y\varphi^1_{u^1u^1})\right] \\
&+\beta\left[\varphi^1_{zz} + \underline{\underline{\underline{u^1_z}}}(2\varphi^1_{u^1z} + u^1_z\varphi^1_{u^1u^1})\right] \\
&+\varphi^2_{yx} + u^2_x\varphi^2_{yu^2} + \underline{u^2_{yx}}\varphi^2_{u^2} + u^2_y(\varphi^2_{u^2x} + u^2_x\varphi^2_{u^2u^2}) \\
&+\varphi^3_{zx} + u^3_x\varphi^3_{zu^3} + \underline{\underline{u^3_{zx}}}\varphi^3_{u^3} + u^3_z(\varphi^3_{u^3x} + u^3_x\varphi^3_{u^3u^3}) \\
&-\left[\varphi^1_{tt} + u^1_t\varphi^1_{tu^1} + u^1_{tt}\varphi^1_{u^1} + u^1_t(\varphi^1_{u^1t} + u^1_t\varphi^1_{u^1u^1})\right] \\
= {}&\varphi^1_{u^1}\left[\underline{u^2_{xy}} + \underline{\underline{u^3_{xz}}}\right].
\end{aligned} \tag{3.12}$$

Comparing to the coefficients of u^2_{xy} and u^3_{xz} we get from the equation (3.12) the following results:

$$\left.\begin{aligned} u^2_{xy} &: \varphi^1_{u^1} = \varphi^2_{u^2} \\ u^3_{xz} &: \varphi^1_{u^1} = \varphi^3_{u^3} \end{aligned}\right\} \quad \Longrightarrow \quad \varphi^1_{u^1} = \varphi^2_{u^2} = \varphi^3_{u^3} = a(t,x,y,z), \tag{3.13}$$

where a is an arbitrary function.

On the other hand we have

$$u^1_t : \varphi^1_{tu^1} = 0, \quad u^1_y : \varphi^1_{yu^1} = 0, \quad u^1_z : \varphi^1_{zu^1} = 0,$$

and if we suppose that $\beta + 1 \neq 0$, we also get

$$u^1_x : \varphi^1_{xu^1} = 0.$$

Thus a is a constant. Now from the equation (3.13) it follows:

$$\begin{aligned}
\varphi^1 &= au^1 + \psi^1(t,x,y,z), \\
\varphi^2 &= au^2 + \psi^2(t,x,y,z), \\
\varphi^3 &= au^3 + \psi^3(t,x,y,z).
\end{aligned} \tag{3.14}$$

The remaining terms of the equation (3.12) are

$$\beta(\psi^1_{xx} + \psi^1_{yy} + \psi^1_{zz}) + \psi^1_{xy} + \psi^2_{xy} + \psi^3_{xz} - \psi^1_{tt} = 0. \tag{3.15}$$

Consequently, the corresponding new variables are

$$\begin{aligned}
v^1 &= au^1 + \psi^1(t,x,y,z)\,, \\
v^2 &= au^2 + \psi^2(t,x,y,z)\,, \\
v^3 &= au^3 + \psi^3(t,x,y,z)\,.
\end{aligned} \tag{3.16}$$

We have

$$F^*_2 \equiv D^2_x(\beta v^2) + D^2_y(\beta v^2 + v^2) + D^2_z(\beta v^2) + D^2_t(-v^2) + D_xD_y(v^1) + D_yD_z(v^3) = 0.$$

Hence

$$F_2^* \equiv \beta v_{xx}^2 + (\beta+1)v_{yy}^2 + \beta v_{zz}^2 + v_{xy}^1 + v_{yz}^3 - v_{tt}^2 = 0.$$

Since $v^\alpha = au^\alpha + \psi^\alpha(t,x,y,z)$, we obtain

$$\begin{aligned} F_2^* &\equiv \beta(au_{xx}^2 + \psi_{xx}^2) + (\beta+1)(au_{yy}^2 + \psi_{yy}^2) + \beta(au_{zz}^2 + \psi_{zz}^2) \\ &\quad + \underline{au_{xy}^1} + \psi_{xy}^1 + au_{yz}^3 + \psi_{yz}^1 - au_{tt}^2 - \psi_{tt}^2 \\ &= A_2\left[\beta(u_{xx}^1 + u_{yy}^1 + u_{zz}^1) + u_{xx}^1 + u_{xy}^2 + u_{xz}^3 - \underline{u_{tt}^1}\right] \\ &\quad + B_2\left[\beta(u_{xx}^2 + u_{yy}^2 + u_{zz}^2) + \underline{u_{xy}^1} + u_{yy}^2 + u_{yz}^3 - u_{tt}^2\right] \\ &\quad + C_2\left[\beta(u_{xx}^3 + u_{yy}^3 + u_{zz}^3) + u_{xz}^1 + u_{yz}^2 + u_{zz}^3 - \underline{u_{tt}^3}\right]. \end{aligned} \tag{3.17}$$

From the latter equation it follows:

$$u_{tt}^1 : A_2 = 0\,, \qquad u_{tt}^3 : C_2 = 0\,, \qquad u_{xy}^1 : B_2 = a.$$

Thus, the remaining equation is

$$\beta(\psi_{xx}^2 + \psi_{yy}^2 + \psi_{zz}^2) + \psi_{xy}^1 + \psi_{yy}^2 + \psi_{yz}^3 - \psi_{tt}^2 = 0. \tag{3.18}$$

Applying the same procedure for F_3^* we obtain

$$\beta(\psi_{xx}^3 + \psi_{yy}^3 + \psi_{zz}^3) + \psi_{xz}^1 + \psi_{yz}^2 + \psi_{zz}^3 - \psi_{tt}^3 = 0. \tag{3.19}$$

Hence according to the equations (3.15), (3.18) and (3.19), (ψ^1, ψ^2, ψ^3) is an arbitrary solution for the system (3.1). In this case $\mathbf{v}$ is a linear combination of two solutions of the Lamé equation. Therefore, we can choose substitution $\mathbf{v} = \mathbf{u}$ and use it for constructing conservation laws.

4 Calculation of conservation laws of the Lamé equation using Ibragimov's theorem

4.1 Introduction

Methodology presented by N.H.Ibragimov [3, 4, 5] is as follows. Consider a system of m PDEs:

$$F_{\bar\alpha}(x, u, u_{(1)}, ..., u_{(s)}) = 0, \qquad \bar\alpha = 1, 2, ..., \bar m, \tag{4.1.1}$$

where $x = (x^1, x^2, ..., x^n)$ and $u = (u^1, u^2, ..., u^m)$ and $u_{(i)}$ is the partial derivative of u with respect to ith component of variable x. The adjoint equations are given by

$$F_\alpha^*(x, u, v, u_{(1)}, v_{(1)}, ..., u_{(s)}, v_{(s)}) \equiv \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha} = 0, \qquad \alpha = 1, 2, ..., m.$$

According to N. Ibragimov [3], a formal Lagrangian can be introduced:

$$\mathcal{L} = v^{\bar\alpha} F_{\bar\alpha}, \tag{4.1.2}$$

where $v^{\bar\alpha}$ are new variables. If the system (4.1.1) admits an operator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}\,,$$

and it is nonlinearly self-adjoint, then we can replace $v^{\bar{\alpha}}$ by functions of x, u and use the same formulae (2.1.3) for calculating components of a conserved vector:

$$C^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) - \cdots \right] + D_j\left(W^\alpha\right) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \cdots \right] + D_j D_k\left(W^\alpha\right) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \cdots \right], \tag{4.1.3}$$

where $\quad W^\alpha = \eta^\alpha - \xi^j u_j^\alpha \;,\;\; \alpha = 1, 2, ..., m,$ and $\; i, j, k = 1, ..., n.$

Remark. If

$$C^1 = \tilde{C}^1 + D_2(H^2) + \cdots + D_n(H^n), \tag{4.1.4}$$

we can bring the conserved vector to the following form

$$\tilde{C} = \left(\tilde{C}^1, \tilde{C}^2, \cdots, \tilde{C}^n\right) \tag{4.1.5}$$

where the components are

$$\tilde{C}^1,\; \tilde{C}^2 = C^2 + D_1(H^2), \ldots, \tilde{C}^n = C^n + D_1(H^n). \tag{4.1.6}$$

Then the conservation law (1.2) can be written in the equivalent form as

$$\left[D_i(\tilde{C}^i) \right]_{F_{\bar{\alpha}} = 0} = 0. \tag{4.1.7}$$

Similarly, we can simplify $\tilde{C}^2, \ldots, \tilde{C}^{n-1}$.

Let us investigate the system (2.1.4):

$$\begin{aligned} F_1 &\equiv \beta(u^1_{xx} + u^1_{yy} + u^1_{zz}) + u^1_{xx} + u^2_{xy} + u^3_{xz} - u^1_{tt} = 0, \\ F_2 &\equiv \beta(u^2_{xx} + u^2_{yy} + u^2_{zz}) + u^1_{xy} + u^2_{yy} + u^3_{yz} - u^2_{tt} = 0, \\ F_3 &\equiv \beta(u^3_{xx} + u^3_{yy} + u^3_{zz}) + u^1_{xz} + u^2_{yz} + u^3_{zz} - u^3_{tt} = 0. \end{aligned} \tag{4.1.8}$$

As it has been shown in the previous chapter, the system is nonlinearly self-adjoint. We choose $\mathbf{v} = \mathbf{u}$, then the formal Lagrangian $\mathcal{L} = v^1 F_1 + v^2 F_2 + v^3 F_3$ has the following symmetrized form:

$$\begin{aligned} \mathcal{L} = \quad & \beta u^1(u^1_{xx} + u^1_{yy} + u^1_{zz}) + u^1 u^1_{xx} + \frac{1}{2} u^1(u^2_{xy} + u^2_{yx}) + \frac{1}{2} u^1(u^3_{xz} + u^3_{zx}) - u^1 u^1_{tt} \\ & + \beta u^2(u^2_{xx} + u^2_{yy} + u^2_{zz}) + \frac{1}{2} u^2(u^1_{xy} + u^1_{yx}) + u^2 u^2_{yy} + \frac{1}{2} u^2(u^3_{yz} + u^3_{zy}) - u^2 u^2_{tt} \\ & + \beta u^3(u^3_{xx} + u^3_{yy} + u^3_{zz}) + \frac{1}{2} u^3(u^1_{xz} + u^1_{zx}) + \frac{1}{2} u^3(u^2_{yz} + u^2_{zy}) + u^3 u^3_{zz} - u^3 u^3_{tt}. \end{aligned} \tag{4.1.9}$$

On the other hand, invoking (4.1.3) we obtain

$$C^i = \xi^i \mathcal{L} + W^\alpha \left[- D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) \right] + D_j\left(W^\alpha\right) \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right). \tag{4.1.10}$$

Thus we have

$$
\begin{aligned}
C^0 &= \xi^0\mathcal{L} + \sum_{\alpha=1}^{3}\Big(- W^{\alpha}u_t^{\alpha} + D_t\Big(W^{\alpha}\Big)u^{\alpha}\Big),\\
C^1 &= \xi^1\mathcal{L} - \ W^1\left(D_x\Big(\frac{\partial\mathcal{L}}{\partial u^1_{xx}}\Big) + D_y\Big(\frac{\partial\mathcal{L}}{\partial u^1_{xy}}\Big) + D_z\Big(\frac{\partial\mathcal{L}}{\partial u^1_{xz}}\Big)\right)\\
&\quad - \ W^2\left(D_x\Big(\frac{\partial\mathcal{L}}{\partial u^2_{xx}}\Big) + D_y\Big(\frac{\partial\mathcal{L}}{\partial u^2_{xy}}\Big)\right)\\
&\quad - \ W^3\left(D_x\Big(\frac{\partial\mathcal{L}}{\partial u^3_{xx}}\Big) + D_y\Big(\frac{\partial\mathcal{L}}{\partial u^3_{xz}}\Big)\right) + D_x\Big(W^{\alpha}\Big)\left(\frac{\partial\mathcal{L}}{\partial u^{\alpha}_{xx}}\right)\\
&\quad + D_y\Big(W^1\Big)\left(\frac{\partial\mathcal{L}}{\partial u^1_{xy}}\right) + D_y\Big(W^2\Big)\left(\frac{\partial\mathcal{L}}{\partial u^2_{xy}}\right) + D_z\Big(W^1\Big)\left(\frac{\partial\mathcal{L}}{\partial u^1_{xz}}\right) + D_z\Big(W^3\Big)\left(\frac{\partial\mathcal{L}}{\partial u^3_{xz}}\right),\\
C^2 &= \xi^2\mathcal{L} - \ W^1\left(D_x\Big(\frac{\partial\mathcal{L}}{\partial u^1_{yx}}\Big) + D_y\Big(\frac{\partial\mathcal{L}}{\partial u^1_{yy}}\Big)\right)\\
&\quad - \ W^2\left(D_x\Big(\frac{\partial\mathcal{L}}{\partial u^2_{yx}}\Big) + D_y\Big(\frac{\partial\mathcal{L}}{\partial u^2_{yy}}\Big) + D_z\Big(\frac{\partial\mathcal{L}}{\partial u^2_{yz}}\Big)\right)\\
&\quad - \ W^3\left(D_y\Big(\frac{\partial\mathcal{L}}{\partial u^3_{yy}}\Big) + D_z\Big(\frac{\partial\mathcal{L}}{\partial u^3_{yz}}\Big)\right) + D_y\Big(W^{\alpha}\Big)\left(\frac{\partial\mathcal{L}}{\partial u^{\alpha}_{yy}}\right)\\
&\quad + D_x\Big(W^1\Big)\left(\frac{\partial\mathcal{L}}{\partial u^1_{yx}}\right) + D_x\Big(W^2\Big)\left(\frac{\partial\mathcal{L}}{\partial u^2_{yx}}\right) + D_z\Big(W^2\Big)\left(\frac{\partial\mathcal{L}}{\partial u^2_{yz}}\right) + D_z\Big(W^3\Big)\left(\frac{\partial\mathcal{L}}{\partial u^3_{yz}}\right),\\
C^3 &= \xi^3\mathcal{L} - \ W^1\left(D_x\Big(\frac{\partial\mathcal{L}}{\partial u^1_{zx}}\Big) + D_z\Big(\frac{\partial\mathcal{L}}{\partial u^1_{zz}}\Big)\right)\\
&\quad - \ W^2\left(D_y\Big(\frac{\partial\mathcal{L}}{\partial u^2_{zy}}\Big) + D_z\Big(\frac{\partial\mathcal{L}}{\partial u^2_{zz}}\Big)\right)\\
&\quad - \ W^3\left(D_x\Big(\frac{\partial\mathcal{L}}{\partial u^3_{zx}}\Big) + D_y\Big(\frac{\partial\mathcal{L}}{\partial u^3_{zy}}\Big) + D_z\Big(\frac{\partial\mathcal{L}}{\partial u^3_{zz}}\Big)\right) + D_z\Big(W^{\alpha}\Big)\left(\frac{\partial\mathcal{L}}{\partial u^{\alpha}_{zz}}\right)\\
&\quad + D_x\Big(W^1\Big)\left(\frac{\partial\mathcal{L}}{\partial u^1_{zx}}\right) + D_x\Big(W^3\Big)\left(\frac{\partial\mathcal{L}}{\partial u^3_{zx}}\right) + D_y\Big(W^2\Big)\left(\frac{\partial\mathcal{L}}{\partial u^2_{zy}}\right) + D_y\Big(W^3\Big)\left(\frac{\partial\mathcal{L}}{\partial u^3_{zy}}\right).
\end{aligned}
\tag{4.1.11}
$$

Thus, by calculating derivatives and substituting in the above equation, we obtain

$$
C^0 = \xi^0\mathcal{L} + \sum_{\alpha=1}^{3}\Big(- W^{\alpha}u_t^{\alpha} + D_t\Big(W^{\alpha}\Big)u^{\alpha}\Big), \tag{4.1.12}
$$

For the second component we have:

$$
\begin{aligned}
C^1 &= \xi^1\mathcal{L} - W^1\Big((\beta+1)u^1_x + \frac{1}{2}u^2_y + \frac{1}{2}u^3_z\Big) - W^2\Big(\beta u^2_x + \frac{1}{2}u^1_y\Big) - W^3\Big(\beta u^3_x + \frac{1}{2}u^1_z\Big)\\
&\quad + D_x\Big(W^1\Big)\Big((\beta+1)u^1\Big) + D_x\Big(W^2\Big)\Big(\beta u^2\Big) + D_x\Big(W^3\Big)\Big(\beta u^3\Big) + D_y\Big(W^1\Big)\Big(\frac{1}{2}u^2\Big)
\end{aligned}
$$

$$+ D_y\Big(W^2\Big)\Big(\frac{1}{2}u^1\Big) + D_z\Big(W^1\Big)\Big(\frac{1}{2}u^3\Big) + D_z\Big(W^3\Big)\Big(\frac{1}{2}u^1\Big)$$

The four last terms can be rewritten as

$$\begin{aligned}
&D_y\Big(W^1\Big)\Big(\frac{1}{2}u^2\Big) + D_y\Big(W^2\Big)\Big(\frac{1}{2}u^1\Big) + D_z\Big(W^1\Big)\Big(\frac{1}{2}u^3\Big) + D_z\Big(W^3\Big)\Big(\frac{1}{2}u^1\Big)\\
=&D_y\Big(\frac{1}{2}(W^1u^2 + W^2u^1)\Big) + D_z\Big(\frac{1}{2}(W^1u^3 + W^3u^1)\Big)\\
&-\frac{1}{2}(W^1u^2_y + W^2u^1_y) - \frac{1}{2}(W^1u^3_z + W^3u^1_z).
\end{aligned}$$

Hence C^1 can be simplified:

$$\begin{aligned}
C^1 = \xi^1\mathcal{L} &- W^1\Big((\beta+1)u^1_x + u^2_y + u^3_z\Big) - W^2\Big(\beta u^2_x + u^1_y\Big) - W^3\Big(\beta u^3_x + u^1_z\Big)\\
&+ D_x\Big(W^1\Big)\Big((\beta+1)u^1\Big) + D_x\Big(W^2\Big)\Big(\beta u^2\Big) + D_x\Big(W^3\Big)\Big(\beta u^3\Big).
\end{aligned} \tag{4.1.13}$$

Then we obtain new components

$$\begin{aligned}
\tilde{C}^2 &= C^2 + D_x\Big(\frac{1}{2}(W^1u^2 + W^2u^1)\Big),\\
\tilde{C}^3 &= C^3 + D_x\Big(\frac{1}{2}(W^1u^3 + W^3u^1)\Big).
\end{aligned}$$

On the other hand

$$\begin{aligned}
C^2 = \xi^1\mathcal{L} &- w^1\Big(\frac{1}{2}u^2_x + \beta u^1_y\Big) - w^2\Big(\frac{1}{2}u^1_x + (\beta+1)u^2_y + \frac{1}{2}u^3_z\Big) - w^3\Big(\beta u^3_y + \frac{1}{2}u^2_z\Big)\\
&+ D_x\Big(W^1\Big)\Big(\frac{1}{2}u^2\Big) + D_x\Big(W^2\Big)\Big(\frac{1}{2}u^1\Big) + D_y\Big(W^1\Big)\Big(\beta u^1\Big) + D_y\Big(W^2\Big)\Big((\beta+1)u^2\Big)\\
&+ D_y\Big(W^3\Big)\Big(\beta u^3\Big) + D_z\Big(W^2\Big)\Big(\frac{1}{2}u^3\Big) + D_z\Big(W^3\Big)\Big(\frac{1}{2}u^2\Big),
\end{aligned}$$

also has terms which can be transferred to the $\tilde{C}^3$:

$$D_z\Big(W^2\Big)\Big(\frac{1}{2}u^3\Big) + D_z\Big(W^3\Big)\Big(\frac{1}{2}u^2\Big) = D_z\Big(\frac{1}{2}(W^2u^3 + W^3u^2)\Big) - \frac{1}{2}(W^2u^3_z + W^3u^2_z).$$

After simplifications, we obtain the following formula for calculating the third component:

$$\begin{aligned}
C^2 = \xi^1\mathcal{L} &- W^1\Big(\beta u^1_y\Big) - W^2\Big((\beta+1)u^2_y + u^3_z\Big) - W^3\Big(\beta u^3_y + u^2_z\Big)\\
&+ D_x\Big(W^1\Big)u^2 + D_x\Big(W^2\Big)u^1 + D_y\Big(W^1\Big)\Big(\beta u^1\Big) + D_y\Big(W^2\Big)\Big((\beta+1)u^2\Big)\\
&+ D_y\Big(W^3\Big)\Big(\beta u^3\Big).
\end{aligned} \tag{4.1.14}$$

Now the fourth component has the form

$$\tilde{C}^3 + D_y\Big(\frac{1}{2}(W^2u^3 + W^3u^2)\Big) = C^3 + D_x\Big(\frac{1}{2}(W^1u^3 + W^3u^1)\Big) + D_y\Big(\frac{1}{2}(W^2u^3 + W^3u^2)\Big),$$

where

$$C^3 = \xi^3\mathcal{L} - w^1\left(\frac{1}{2}u^3_x + \beta u^1_z\right) - w^2\left(\beta u^2_z + \frac{1}{2}u^3_y\right) - w^3\left(\frac{1}{2}u^1_x + \frac{1}{2}u^2_y + (\beta+1)u^3_z\right)$$
$$+ D_x\left(W^1\right)\left(\frac{1}{2}u^3\right) + D_x\left(W^3\right)\left(\frac{1}{2}u^1\right) + D_y\left(W^2\right)\left(\frac{1}{2}u^3\right) + D_y\left(W^3\right)\left(\frac{1}{2}u^2\right)$$
$$+ D_z\left(W^1\right)\left(\beta u^1\right) + D_z\left(W^2\right)\left(\beta u^2\right) + D_z\left(W^3\right)\left((\beta+1)u^3\right).$$

After simplifications we arrive at the following formula for the fourth component:

$$C^3 = \xi^3\mathcal{L} - W^1\left(\beta u^1_z\right) - W^2\left(\beta u^2_z\right) - W^3\left((\beta+1)u^3_z\right)$$
$$+ D_x\left(W^1\right)u^3 + D_x\left(W^3\right)u^1 + D_y\left(W^2\right)u^3 + D_y\left(W^3\right)u^2$$
$$+ D_z\left(W^1\right)\left(\beta u^1\right) + D_z\left(W^2\right)\left(\beta u^2\right) + D_z\left(W^3\right)\left((\beta+1)u^3\right). \tag{4.1.15}$$

Hence, one can construct conservation laws of the system (4.1.8) corresponding to its symmetries by using the formulas (4.1.12), (4.1.13) (4.1.14) and (4.1.15).

4.2 Translation of time: $X_0 = \dfrac{\partial}{\partial t}$

In this case $\xi^0 = 1,\ \xi^1 = \xi^2 = \xi^3 = 0,\ \eta^\alpha = 0$ and

$$W^\alpha = \eta^\alpha - \xi^i u^\alpha_j = -u^\alpha_t. \tag{4.2.1}$$

Hence, using (4.1.12) we obtain the density of the conserved vector

$$C^0_0 = \mathcal{L} - \left(u^1_t\right)^2 - \left(u^2_t\right)^2 - \left(u^3_t\right)^2 + u^1u^1_{tt} + u^2u^2_{tt} + u^3u^3_{tt}. \tag{4.2.2}$$

However, by substituting the Lagrangian and using the (4.1.6) we can rewrite it as

$$\begin{aligned} C^0_0 = &-\sum_{\alpha=1}^{3}\beta\big((u^\alpha_x)^2 + (u^\alpha_y)^2 + (u^\alpha_z)^2\big) + (u^\alpha_t)^2 \\ &- (u^1_x)^2 - (u^2_y)^2 - (u^3_z)^2 - 2u^1_zu^3_x - 2u^1_yu^2_x - 2u^2_zu^3_y \\ &+ D_x(\beta(u^1u^1_x + u^2u^2_x + u^3u^3_x) + u^1u^1_x + u^2u^1_y + u^3u^1_z) \\ &+ D_y(\beta(u^1u^1_y + u^2u^2_y + u^3u^3_y) + u^1u^2_x + u^2u^2_y + u^3u^2_z) \\ &+ D_z(\beta(u^1u^1_z + u^2u^2_z + u^3u^3_z) + u^1u^3_x + u^2u^3_y + u^3u^3_z). \end{aligned} \tag{4.2.3}$$

Hence, we have

$$C^0_0 = \tilde{C}^0_0 + D_x(H_1) + D_y(H_2) + D_z(H_3), \tag{4.2.4}$$

where

$$\begin{aligned} \tilde{C}^0_0 = &-\sum_{\alpha=1}^{3}\left[\beta\big((u^\alpha_x)^2 + (u^\alpha_y)^2 + (u^\alpha_z)^2\big) + (u^\alpha_t)^2\right] \\ &- (u^1_x)^2 - (u^2_y)^2 - (u^3_z)^2 - 2u^1_zu^3_x - 2u^1_yu^2_x - 2u^2_zu^3_y, \end{aligned} \tag{4.2.5}$$

is chosen as the first component of the conserved vector and

$$
\begin{aligned}
H_1 =& \beta(u^1u^1_x + u^2u^2_x + u^3u^3_x) + u^1u^1_x + u^2u^1_y + u^3u^1_z,\\
H_2 =& \beta(u^1u^1_y + u^2u^2_y + u^3u^3_y) + u^1u^2_x + u^2u^2_y + u^3u^2_z,\\
H_3 =& u^1u^1_z + u^2u^2_z + u^3u^3_z) + u^1u^3_x + u^2u^3_y + u^3u^3_z.
\end{aligned}
$$

The last three terms in (4.2.4) will be transferred to the next components of the conserved vector later. For example, $\tilde{C}^1_0$ takes the form

$$\tilde{C}^1_0 = C^1_0 + D_t(H_1). \tag{4.2.6}$$

From the formula (4.1.13) for the second component of a conserved vector it follows:

$$
\begin{aligned}
C^1_0 = \quad & u^1_t\Big((\beta+1)u^1_x + u^2_y + u^3_z\Big) + u^2_t\Big(\beta u^2_x + u^1_y\Big) + u^3_t\Big(\beta u^3_x + u^1_z\Big)\\
& - u^1_{tx}\Big((\beta+1)u^1\Big) - u^2_{tx}\Big(\beta u^2\Big) - u^3_{tx}\Big(\beta u^3\Big).
\end{aligned} \tag{4.2.7}
$$

Applying the same technique to the second component and recalling (4.2.6) we have

$$\tilde{C}^1_0 = 2\beta\sum_{\alpha=1}^{3} u^\alpha_t u^\alpha_x + 2u^1_tu^1_x + 2u^2_tu^1_y + 2u^3_tu^1_z + D_y(H_4) + D_z(H_5). \tag{4.2.8}$$

Here

$$H_4 = \frac{1}{2}u^2u^1_t - \frac{1}{2}u^1u^2_t, \qquad H_5 = \frac{1}{2}u^3u^1_t - \frac{1}{2}u^1u^3_t.$$

The terms $D_y(H_4) + D_z(H_5)$ will be transferred to the next components. Thus, the second component of the conservation vector is

$$\tilde{C}^1_0 = 2\beta\sum_{\alpha=1}^{3} u^\alpha_t u^\alpha_x + 2u^1_tu^1_x + 2u^2_tu^1_y + 2u^3_tu^1_z\,. \tag{4.2.9}$$

Considering the formula (4.1.14) for the third component we obtain

$$
\begin{aligned}
C^2_0 = \quad & u^1_t\Big(\beta u^1_y\Big) + u^2_t\Big((\beta+1)u^2_y + u^3_z\Big) + u^3_t\Big(\beta u^3_y + u^2_z\Big)\\
& - u^1_{tx}\Big(u^2\Big) - u^1_{ty}\Big(\beta u^1\Big) - u^2_{tx}\Big(u^1\Big) - u^2_{ty}\Big((\beta+1)u^2\Big) - u^3_{ty}\Big(\beta u^3\Big).
\end{aligned} \tag{4.2.10}
$$

Accordingly, we have

$$\tilde{C}^2_0 = C^2_0 + D_x(H_2) + D_y(H_4). \tag{4.2.11}$$

Whence

$$\tilde{C}^2_0 = 2\beta\sum_{\alpha=1}^{3} u^\alpha_t u^\alpha_y + 2u^1_tu^2_x + 2u^2_tu^2_y + 2u^3_tu^2_z + D_z(H_6), \tag{4.2.12}$$

where $H_6 = \frac{1}{2}u^3u^2_t - \frac{1}{2}u^2u^3_tu^3u^2_t$. Thus the third component of the conserved vector is

$$\tilde{C}^2_0 = 2\beta\sum_{\alpha=1}^{3} u^\alpha_t u^\alpha_y + 2u^1_tu^2_x + 2u^2_tu^2_y + 2u^3_tu^2_z\,. \tag{4.2.13}$$

Eventually, using (4.1.15) for the last component of a conserved vector we have

$$
\begin{aligned}
C_0^3 = \quad & u_t^1\Big(\beta u_z^1\Big) + u_t^2\Big(\beta u_z^2\Big) + u_t^3\Big((\beta+1)u_z^3\Big) \\
& - u_{tx}^1\Big(u^3\Big) - u_{tz}^1\Big(\beta u^1\Big) - u_{ty}^2\Big(u^3\Big) - u_{tz}^2\Big(\beta u^2\Big) \\
& - u_{tx}^3\Big(u^1\Big) - u_{ty}^3\Big(u^2\Big) - u_{tz}^3\Big((\beta+1)u^3\Big).
\end{aligned} \tag{4.2.14}
$$

Hence, we have

$$
\tilde{C}_0^3 = C_0^3 + D_t(H_3) + D_x(H_5) + D_y(H_6). \tag{4.2.15}
$$

Whence

$$
\tilde{C}_0^3 = 2\beta \sum_{\alpha=1}^{3} u_t^\alpha u_z^\alpha + 2u_t^1 u_x^3 + 2u_t^2 u_y^3 + 2u_t^3 u_z^3. \tag{4.2.16}
$$

Checking (2.1.5) we obtain:

$$
\Big[D_t(\tilde{C}_0^0) + D_x(\tilde{C}_0^1) + D_y(\tilde{C}_0^2) + D_z(\tilde{C}_0^3)\Big]_{F_\alpha=\,0} = \Big[2u_t^1 F_1 + 2u_t^2 F_2 + 2u_t^3 F_3\Big]_{F_\alpha=\,0} = 0. \tag{4.2.17}
$$

Proposition: The operator X_0 admitted by the system (4.1.8) provides the conserved vector $(\tilde{C}_0^0, \tilde{C}_0^1, \tilde{C}_0^2, \tilde{C}_0^3)$, where components of the conserved vector are

$$
\begin{aligned}
\tilde{C}_0^0 =& - \sum_{\alpha=1}^{3} \Big[\beta\big((u_x^\alpha)^2 + (u_y^\alpha)^2 + (u_z^\alpha)^2\big) + (u_t^\alpha)^2\Big] \\
& - (u_x^1)^2 - (u_y^2)^2 - (u_z^3)^2 - 2u_z^1 u_x^3 - 2u_y^1 u_x^2 - 2u_z^2 u_y^3, \\
\tilde{C}_0^1 =& 2\beta \sum_{\alpha=1}^{3} u_t^\alpha u_x^\alpha + 2u_t^1 u_x^1 + 2u_t^2 u_y^1 + 2u_t^3 u_z^1, \\
\tilde{C}_0^2 =& 2\beta \sum_{\alpha=1}^{3} u_t^\alpha u_y^\alpha + 2u_t^1 u_x^2 + 2u_t^2 u_y^2 + 2u_t^3 u_z^2, \\
\tilde{C}_0^3 =& 2\beta \sum_{\alpha=1}^{3} u_t^\alpha u_z^\alpha + 2u_t^1 u_x^3 + 2u_t^2 u_y^3 + 2u_t^3 u_z^3.
\end{aligned} \tag{4.2.18}
$$

4.3 Translation in x direction: $X_1 = \dfrac{\partial}{\partial x}$

Here $\xi^1 = 1,\ \xi^0 = \xi^2 = \xi^3 = 0,\ \eta^\alpha = 0$ and

$$
W^\alpha = \eta^\alpha - \xi^i u_j^\alpha = -u_x^\alpha. \tag{4.3.1}
$$

Hence, using (4.1.12) we have

$$C_1^0 = -u_x^1 u_t^1 - u_x^2 u_t^2 - u_x^3 u_t^3 + u^1 u_{xt}^1 + u^2 u_{xt}^2 + u^3 u_{xt}^3. \tag{4.3.2}$$

However, using (4.1.6) and doing some simplifications it can be rewritten as

$$\tilde{C}_1^0 = -2 \sum_{\alpha=1}^{3} u_t^\alpha u_x^\alpha, \tag{4.3.3}$$

and the remaining terms in (4.3.2) will be transferred to the next components.

The formula (4.1.13) gives

$$\begin{aligned} C_1^1 = \mathcal{L} + \; & u_x^1 \Big((\beta+1) u_x^1 + u_y^2 + u_z^3 \Big) + u_x^2 \Big(\beta u_x^2 + u_y^1 \Big) + u_x^3 \Big(\beta u_x^3 + u_z^1 \Big) \\ & - u_{xx}^1 \Big((\beta+1) u^1 \Big) - u_{xx}^2 \Big(\beta u^2 \Big) - u_{xx}^3 \Big(\beta u^3 \Big). \end{aligned} \tag{4.3.4}$$

Invoking the same technique for the second component we obtain:

$$\tilde{C}_1^1 = \sum_{\alpha=1}^{3} \Big(\beta \big((u_x^\alpha)^2 - (u_y^\alpha)^2 - (u_z^\alpha)^2 \big) + (u_t^\alpha)^2 \Big) + (u_x^1)^2 - (u_y^2)^2 - (u_z^3)^2 - u_y^2 u_z^3 - u_y^3 u_z^2, \tag{4.3.5}$$

and the rest of the remaining terms will be transferred to the next components.

From the formula (4.1.14) it follows

$$\begin{aligned} C_1^2 = \; & u_x^1 \Big(\beta u_y^1 \Big) + u_x^2 \Big((\beta+1) u_y^2 + u_z^3 \Big) + u_x^3 \Big(\beta u_y^3 + u_z^2 \Big) \\ & - u_{xx}^1 \Big(u^2 \Big) - u_{xy}^1 \Big(\beta u^1 \Big) - u_{xx}^2 \Big(u^1 \Big) - u_{xy}^2 \Big((\beta+1) u^2 \Big) - u_{xy}^3 \Big(\beta u^3 \Big). \end{aligned} \tag{4.3.6}$$

Hence, if we add the transferred terms from previous components and do simplification we obtain:

$$\tilde{C}_1^2 = 2\beta \sum_{\alpha=1}^{3} \left(u_x^\alpha u_y^\alpha \right) + 2 u_x^2 u_y^2 + 2 u_x^1 u_x^2 + u_x^2 u_z^3 + u_x^3 u_z^2, \tag{4.3.7}$$

and the remaining terms will be transferred to the last component.

And finally, the formula (4.1.15) for the last component gives:

$$\begin{aligned} C_1^3 = \; & u_x^1 \Big(\beta u_z^1 \Big) + u_x^2 \Big(\beta u_z^2 \Big) + u_x^3 \Big((\beta+1) u_z^3 \Big) \\ & - u_{xx}^1 \Big(u^3 \Big) - u_{xz}^1 \Big(\beta u^1 \Big) - u_{xy}^2 \Big(u^3 \Big) - u_{xz}^2 \Big(\beta u^2 \Big) \\ & - u_{xx}^3 \Big(u^1 \Big) - u_{xy}^3 \Big(u^2 \Big) - u_{xz}^3 \Big((\beta+1) u^3 \Big). \end{aligned} \tag{4.3.8}$$

Inserting the transferred terms from the previous components to the C_1^3 and doing simplification steps we get:

$$\tilde{C}_1^3 = 2\beta \sum_{\alpha=1}^{3} \left(u_x^\alpha u_z^\alpha \right) + 2 u_x^3 u_z^3 + 2 u_x^1 u_x^3 + u_y^2 u_x^3 + u_y^3 u_x^2. \tag{4.3.9}$$

Checking (2.1.5) we get:

$$\left[D_t(\tilde{C}^0) + D_x(\tilde{C}^1) + D_y(\tilde{C}^2) + D_z(\tilde{C}^3)\right]_{F_\alpha=\,0} = \left[2u_x^\alpha F_\alpha\right]_{F_\alpha=\,0} = 0, \quad \alpha = 1, 2, 3. \tag{4.3.10}$$

Proposition: The operator X_1 admitted by the system (4.1.8) provides the conserved vector $(\tilde{C}_1^0, \tilde{C}_1^1, \tilde{C}_1^2, \tilde{C}_1^3)$, where components of the conserved vector are

$$\tilde{C}_1^0 = -\,2\sum_{\alpha=1}^{3} u_t^\alpha u_x^\alpha,$$

$$\tilde{C}_1^1 = \sum_{\alpha=1}^{3}\left(\beta\big((u_x^\alpha)^2 - (u_y^\alpha)^2 - (u_z^\alpha)^2\big) + (u_t^\alpha)^2\right) + (u_x^1)^2 - (u_y^2)^2 - (u_z^3)^2 - u_y^2 u_z^3 - u_y^3 u_z^2,$$

$$\tilde{C}_1^2 = 2\beta\sum_{\alpha=1}^{3}\left(u_x^\alpha u_y^\alpha\right) + 2u_x^2 u_y^2 + 2u_x^1 u_x^2 + u_x^2 u_z^3 + u_x^3 u_z^2,$$

$$\tilde{C}_1^3 = 2\beta\sum_{\alpha=1}^{3}\left(u_x^\alpha u_z^\alpha\right) + 2u_x^3 u_z^3 + 2u_x^1 u_x^3 + u_y^2 u_x^3 + u_y^3 u_x^2. \tag{4.3.11}$$

4.4 Translation in y direction: $X_2 = \dfrac{\partial}{\partial y}$

We have $\xi^2 = 1,\ \xi^0 = \xi^1 = \xi^3 = 0,\ \eta^\alpha = 0$ and

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha = -u_y^\alpha. \tag{4.4.1}$$

Hence, using the equation (4.1.12) we have

$$C_2^0 = -u_y^1 u_y^1 - u_y^2 u_y^2 - u_y^3 u_y^3 + u^1 u_{yt}^1 + u^2 u_{yt}^2 + u^3 u_{yt}^3. \tag{4.4.2}$$

However, using the (4.1.6) and do some simplifications we can rewrite it as

$$\tilde{C}_2^0 = -2\sum_{\alpha=1}^{3} u_t^\alpha u_y^\alpha, \tag{4.4.3}$$

and the remaining terms will be transferred to the next components.

The formula (4.1.13) gives

$$C_2^1 = u_y^1\Big((\beta+1)u_x^1 u_y^2 + u_z^3\Big) + u_y^2\Big(\beta u_x^2 + u_y^1\Big) + u_y^3\Big(\beta u_x^3 + u_z^1\Big) - u_{yx}^1\Big((\beta+1)u^1\Big) - u_{yx}^2\Big(\beta u^2\Big) - u_{yx}^3\Big(\beta u^3\Big). \tag{4.4.4}$$

Invoking the same technique for the second component we obtain:

$$\tilde{C}_2^1 = 2\beta\sum_{\alpha=1}^{3}\left(u_x^\alpha u_y^\alpha\right) + 2u_x^1 u_y^1 + 2u_y^1 u_y^2 + u_y^1 u_z^3 + u_y^3 u_z^1, \tag{4.4.5}$$

and the rest of the remaining terms will be transferred to the next components.

From the formula (4.1.14) it follows

$$\begin{aligned} C_2^2 = \quad & \mathcal{L} \ + \ u_y^1\Big(\beta u_y^1\Big) + u_y^2\Big((\beta+1)u_y^2 + u_z^3\Big) + u_y^3\Big(\beta u_y^3 + u_z^2\Big) \\ & - u_{yx}^1\Big(u^2\Big) - u_{yy}^1\Big(\beta u^1\Big) - u_{yx}^2\Big(u^1\Big) \\ & - u_{yy}^2\Big((\beta+1)u^2\Big) - u_{yy}^3\Big(\beta u^3\Big). \end{aligned} \tag{4.4.6}$$

Hence, if we add the transferred terms from previous components and do simplification we obtain:

$$\tilde{C}_2^2 = \sum_{\alpha=1}^{3}\Big(\beta\big((u_y^\alpha)^2 - (u_x^\alpha)^2 - (u_z^\alpha)^2\big) + (u_t^\alpha)^2\Big) + (u_y^2)^2 - (u_x^1)^2 - (u_z^3)^2 - u_z^1u_x^3 - u_z^3u_x^1, \tag{4.4.7}$$

and the remaining terms will be transferred to the last component.

And finally, the formula (4.1.15) for the last component gives:

$$\begin{aligned} C_2^3 = \quad & u_y^1\Big(\beta u_z^1\Big) + u_y^2\Big(\beta u_z^2\Big) + u_y^3\Big((\beta+1)u_z^3\Big) \\ & - u_{yx}^1\Big(u^3\Big) - u_{yz}^1\Big(\beta u^1\Big) - u_{yy}^2\Big(u^3\Big) - u_{yz}^2\Big(\beta u^2\Big) \\ & - u_{yx}^3\Big(u^1\Big) - u_{yy}^3\Big(\frac{1}{2}u^2\Big) - u_{yz}^3\Big((\beta+1)u^3\Big), \end{aligned} \tag{4.4.8}$$

Inserting the transferred terms from the previous components to the C_2^3 and doing simplification steps we get:

$$\tilde{C}_2^3 = \ 2\beta\sum_{\alpha=1}^{3}\big(u_y^\alpha u_z^\alpha\big) + 2u_y^3u_z^3 + 2u_y^2u_y^3 + u_x^1u_y^3 + u_y^1u_x^3. \tag{4.4.9}$$

Checking (2.1.5) gives:

$$\Big[D_t(\tilde{C}^0) + D_x(\tilde{C}^1) + D_y(\tilde{C}^2) + D_z(\tilde{C}^3)\Big]_{F_\alpha=\,0} = \Big[2u_y^\alpha F_\alpha\Big]_{F_\alpha=\,0} = 0, \quad \alpha = 1, 2, 3. \tag{4.4.10}$$

Proposition: The operator X_2 admitted by the system (4.1.8) provides the conserved vector $(\tilde{C}_2^0, \tilde{C}_2^1, \tilde{C}_2^2, \tilde{C}_2^3)$, where components of the conserved vector are

$$\begin{aligned} \tilde{C}_2^0 &= -\,2\sum_{\alpha=1}^{3}u_t^\alpha u_y^\alpha, \\ \tilde{C}_2^1 &= 2\beta\sum_{\alpha=1}^{3}\big(u_x^\alpha u_y^\alpha\big) + 2u_x^1u_y^1 + 2u_y^1u_y^2 + u_y^1u_z^3 + u_y^3u_z^1, \\ \tilde{C}_2^2 &= \sum_{\alpha=1}^{3}\Big(\beta\big((u_y^\alpha)^2 - (u_x^\alpha)^2 - (u_z^\alpha)^2\big) + (u_t^\alpha)^2\Big) \\ &\quad + (u_y^2)^2 - (u_x^1)^2 - (u_z^3)^2 - u_z^1u_x^3 - u_z^3u_x^1, \\ \tilde{C}_2^3 &= 2\beta\sum_{\alpha=1}^{3}\big(u_y^\alpha u_z^\alpha\big) + 2u_y^3u_z^3 + 2u_y^2u_y^3 + u_x^1u_y^3 + u_y^1u_x^3. \end{aligned} \tag{4.4.11}$$

4.5 Translation in z direction: $X_3 = \dfrac{\partial}{\partial z}$

In this case $\xi^3 = 1,\ \xi^0 = \xi^1 = \xi^2 = 0,\ \eta^\alpha = 0$ and

$$W^\alpha = \eta^\alpha - \xi^i u^\alpha_j = -u^\alpha_z, \tag{4.5.1}$$

Hence, considering (4.1.12) we have

$$C^0_3 = -u^1_z u^1_t - u^2_z u^2_t - u^3_z u^3_t + u^1 u^1_{zt} + u^2 u^2_{zt} + u^3 u^3_{zt}. \tag{4.5.2}$$

However, using the (4.1.6) and applying simplifications we can rewrite it as

$$\tilde{C}^0_3 = -2\sum_{\alpha=1}^{3} u^\alpha_t u^\alpha_z, \tag{4.5.3}$$

and the remaining terms will be transferred to the next components.

The formula (4.1.13) gives

$$\begin{aligned} C^1_3 = \quad & u^1_z\Big((\beta+1)u^1_x + u^2_y + u^3_z\Big) + u^2_z\Big(\beta u^2_x + u^1_y\Big) + u^3_z\Big(\beta u^3_x + u^1_z\Big) \\ & - u^1_{zx}\Big((\beta+1)u^1\Big) - u^2_{zx}\Big(\beta u^2\Big) - u^3_{zx}\Big(\beta u^3\Big). \end{aligned} \tag{4.5.4}$$

Invoking the same technique for the second component we obtain:

$$\tilde{C}^1_3 = 2\beta\sum_{\alpha=1}^{3}\left(u^\alpha_x u^\alpha_z\right) + 2u^1_x u^1_z + 2u^1_z u^3_z + u^1_z u^2_y + u^1_y u^2_z, \tag{4.5.5}$$

and the rest of the remaining terms will be transferred to the next components.

From the formula (4.1.14) it follows

$$\begin{aligned} C^2_3 = \quad & u^1_z\Big(\beta u^1_y\Big) + u^2_z\Big((\beta+1)u^2_y + u^3_z\Big) + u^3_z\Big(\beta u^3_y + u^2_z\Big) \\ & - u^1_{zx}\Big(u^2\Big) - u^1_{zy}\Big(\beta u^1\Big) - u^2_{zx}\Big(u^1\Big) - u^2_{zy}\Big((\beta+1)u^2\Big) \\ & - u^3_{zy}\Big(\beta u^3\Big). \end{aligned} \tag{4.5.6}$$

Hence, if we add the transferred terms from previous components and do simplification we obtain:

$$\tilde{C}^2_3 = 2\beta\sum_{\alpha=1}^{3}\left(u^\alpha_y u^\alpha_z\right) + 2u^2_y u^2_z + 2u^2_z u^3_z + u^1_x u^2_z + u^1_z u^2_x. \tag{4.5.7}$$

and the remaining terms will be transferred to the last component.

And finally, the formula (4.1.15) for the last component gives:

$$\begin{aligned} C^3_3 = \quad & \mathcal{L} + u^1_z\Big(\beta u^1_z\Big) + u^2_z\Big(\beta u^2_z\Big) + u^3_z\Big((\beta+1)u^3_z\Big) \\ & - u^1_{zx}\Big(u^3\Big) - u^1_{zz}\Big(\beta u^1\Big) - u^2_{zy}\Big(u^3\Big) - u^2_{zz}\Big(\beta u^2\Big) \\ & - u^3_{zx}\Big(u^1\Big) - u^3_{zy}\Big(u^2\Big) - u^3_{zz}\Big((\beta+1)u^3\Big), \end{aligned} \tag{4.5.8}$$

Inserting the transferred terms from the previous components to the C_3^3 and doing simplification steps we get:

$$\tilde{C}_3^3 = \sum_{\alpha=1}^{3}\Big(\beta\big((u_z^\alpha)^2 - (u_x^\alpha)^2 - (u_y^\alpha)^2\big) + (u_t^\alpha)^2\Big) + (u_z^3)^2 - (u_x^1)^2 - (u_y^2)^2 - u_y^1 u_x^2 - u_y^2 u_x^1. \tag{4.5.9}$$

Checking (2.1.5) gives:

$$\Big[D_t(\tilde{C}^0) + D_x(\tilde{C}^1) + D_y(\tilde{C}^2) + D_z(\tilde{C}^3)\Big]_{F_\alpha=\,0} = \Big[2u_z^\alpha F_\alpha\Big]_{F_\alpha=\,0} = 0, \quad \alpha = 1,2,3. \tag{4.5.10}$$

Proposition: The operator X_3 admitted by the system (4.1.8) provides the conserved vector $(\tilde{C}_3^0, \tilde{C}_3^1, \tilde{C}_3^2, \tilde{C}_3^3)$, where components of the conserved vector are

$$\begin{aligned}
\tilde{C}_3^0 &= -\,2\sum_{\alpha=1}^{3} u_t^\alpha u_z^\alpha,\\
\tilde{C}_3^1 &= 2\beta\sum_{\alpha=1}^{3}\big(u_x^\alpha u_z^\alpha\big) + 2u_x^1 u_z^1 + 2u_z^1 u_z^3 + u_z^1 u_y^2 + u_y^1 u_z^2,\\
\tilde{C}_3^2 &= 2\beta\sum_{\alpha=1}^{3}\big(u_y^\alpha u_z^\alpha\big) + 2u_y^2 u_z^2 + 2u_z^2 u_z^3 + u_x^1 u_z^2 + u_z^1 u_x^2,\\
\tilde{C}_3^3 &= \sum_{\alpha=1}^{3}\Big(\beta\big((u_z^\alpha)^2 - (u_x^\alpha)^2 - (u_y^\alpha)^2\big) + (u_t^\alpha)^2\Big)\\
&\quad + (u_z^3)^2 - (u_x^1)^2 - (u_y^2)^2 - u_y^1 u_x^2 - u_y^2 u_x^1.
\end{aligned} \tag{4.5.11}$$

4.6 Rotation: $X_4 = y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y} + u^2\dfrac{\partial}{\partial u^3} - u^3\dfrac{\partial}{\partial u^2}$

Here $\xi^0 = \xi^1 = 0,\ \xi^2 = -z,\ \xi^3 = y,\ \eta^1 = 0,\ \eta^2 = -u^3,\ \eta^3 = u^2$ and

$$\begin{aligned}
W^1 &= zu_y^1 - yu_z^1,\\
W^2 &= zu_y^2 - u^3 - yu_z^2,\\
W^3 &= u^2 + zu_y^3 - yu_z^3.
\end{aligned} \tag{4.6.1}$$

Hence, considering (4.1.12) we have:

$$\begin{aligned}
C_4^0 = \quad & (zu_y^1 - yu_z^1)u_t^1 + (zu_y^2 - u^3 - yu_z^2)u_t^2 + (u^2 + zu_y^3 - yu_z^3)u_t^3 - (zu_{yt}^1 - yu_{zt}^1)u^1\\
& - (zu_{yt}^2 - u_t^3 - yu_{zt}^2)u^2 - (u_t^2 + zu_{yt}^3 - yu_{zt}^3)u^3.
\end{aligned} \tag{4.6.2}$$

However, using the (4.1.6) and applying some simplifications we can rewrite it as

$$\tilde{C}_4^0 = 2\sum_{\alpha=1}^{3}\Big(zu_t^\alpha u_y^\alpha - yu_t^\alpha u_z^\alpha\Big) + 2u^2 u_t^3 - 2u^3 u_t^2, \tag{4.6.3}$$

and the remaining terms will be transferred to the next components.

The formula (4.1.13) gives

$$\begin{aligned} C_4^1 = \ & -(zu_y^1 - yu_z^1)\Big((\beta+1)u_x^1 + u_y^2 + u_z^3\Big) - (zu_y^2 - u^3 - yu_z^2)\Big(\beta u_x^2 + u_y^1\Big) \\ & - (u^2 + zu_y^3 - yu_z^3)\Big(\beta u_x^3 + u_z^1\Big) + (zu_{yx}^1 - yu_{zx}^1)\Big((\beta+1)u^1\Big) \\ & + (zu_{yx}^3 + u_x^2 - yu_{zx}^3)\Big(\beta u^3\Big) + (zu_{yx}^2 - u_x^3 - yu_{zx}^2)\Big(\beta u^2\Big). \end{aligned} \tag{4.6.4}$$

Invoking the same technique for the second component we obtain:

$$\begin{aligned} \tilde{C}_4^1 = 2\beta\Big(\sum_{\alpha=1}^{3}\big(yu_x^\alpha u_z^\alpha - zu_x^\alpha u_y^\alpha\big) - u^2u_x^3 + u^3u_x^2\Big) + 2yu_x^1u_z^1 - 2zu_x^1u_y^1 + 2yu_z^1u_z^3 \\ - 2zu_y^1u_y^2 - zu_y^1u_z^3 - zu_y^3u_z^1 + yu_y^1u_z^2 - u^2u_z^1 + u^3u_y^1 + yu_y^2u_z^1, \end{aligned} \tag{4.6.5}$$

and the rest of the remaining terms will be transferred to the next components.

From the formula (4.1.14) it follows

$$\begin{aligned} C_4^2 = \ & -z\mathcal{L} - (zu_y^1 - yu_z^1)\Big(\beta u_y^1\Big) - (zu_y^2 - u^3 - yu_z^2)\Big((\beta+1)u_y^2 + u_z^3\Big) \\ & - (u^2 + zu_y^3 - yu_z^3)\Big(\beta u_y^3 + u_z^2\Big) + (zu_{yx}^1 - yu_{zx}^1)\Big(u^2\Big) + (zu_{yy}^1 - u_z^1 - yu_{zy}^1)\Big(\beta u^1\Big) \\ & + (zu_{yx}^2 - u_x^3 - yu_{zx}^2)\Big(u^1\Big) + (zu_{yy}^2 - u_y^3 - u_z^2 - yu_{zy}^2)\Big((\beta+1)u^2\Big) \\ & + (zu_{yy}^3 + u_y^2 - u_z^3 - yu_{zy}^3)\Big(\beta u^3\Big). \end{aligned} \tag{4.6.6}$$

Hence, if we add the transferred terms from previous components and do simplification we obtain:

$$\begin{aligned} \tilde{C}_4^2 = \sum_{\alpha=1}^{3}\Big(\beta\big(2yu_y^\alpha u_z^\alpha - z(u_y^\alpha)^2 + z(u_x^\alpha)^2 + z(u_z^\alpha)^2\big) - z(u_t^\alpha)^2\Big) + \beta\big(2u^3u_y^2 - 2u^2u_y^3\big) \\ + z(u_x^1)^2 + z(u_z^3)^2 - z(u_y^2)^2 + u^3u_x^1 + 2u^3u_y^2 + u^3u_z^3 - u^2u_z^2 + yu_x^1u_z^2 + 2yu_y^2u_z^2 \\ + 2yu_z^2u_z^3 + zu_x^1u_z^3 + zu_z^1u_x^3 + yu_x^2u_z^1, \end{aligned} \tag{4.6.7}$$

and the remaining terms will be transferred to the last component.

And finally, the formula (4.1.15) for the last component gives:

$$\begin{aligned} C_4^3 = y\mathcal{L} \ & - (zu_y^1 - yu_z^1)\Big(\beta u_z^1\Big) - (zu_y^2 - u^3 - yu_z^2)\Big(\beta u_z^2\Big) \\ & - (u^2 + zu_y^3 - yu_z^3)\Big((\beta+1)u_z^3\Big) + (zu_{yx}^1 - yu_{zx}^1)\Big(u^3\Big) \\ & + (zu_{yy}^2 - u_y^3 - u_z^2 - yu_{zy}^2)\Big(u^3\Big) + (zu_{yz}^2 + u_y^2 - u_z^3 - yu_{zz}^2)\Big(\beta u^2\Big) \\ & + (zu_{yy}^3 + u_y^2 - u_z^3 - yu_{zy}^3)\Big(u^2\Big) + +(zu_{yx}^3u_x^2 - yu_{zx}^3)\Big(u^1\Big) \\ & + (u_y^1 + zu_{yz}^1 - yu_{zz}^1)\Big(\beta u^1\Big) + (zu_{yz}^3 + u_z^2 + u_y^3 - yu_{zz}^3)\Big((\beta+1)u^3\Big), \end{aligned} \tag{4.6.8}$$

Inserting the transferred terms from the previous components to the C_4^3 and doing simplification steps we get:

$$\begin{aligned}\tilde{C}_4^3 = &\sum_{\alpha=1}^{3}\Big(\beta(y(u_z^\alpha)^2 - y(u_x^\alpha)^2 - y(u_y^\alpha)^2 - 2zu_y^\alpha u_z^\alpha) + y(u_t^\alpha)^2\Big) - \beta\big(2u^3u_x^2 - 2u^2u_x^3\big)\\ &- zu_x^3u_y^1 - zu_x^1u_y^3 - yu_x^2u_y^1 - yu_x^1u_y^2 - 2zu_y^3u_z^3 - 2zu_y^2u_y^3 + y(u_z^3)^2 - y(u_x^1)^2\\ &- y(u_y^2)^2 - 2u^2u_z^3 - u^2u_x^1 + u^3u_y^3 - u^2u_y^2.\end{aligned} \tag{4.6.9}$$

Checking (2.1.5) yields:

$$\Big[D_t(\tilde{C}^0) + D_x(\tilde{C}^1) + D_y(\tilde{C}^2) + D_z(\tilde{C}^3)\Big]_{F_\alpha=\,0} = \Big[-2W^\alpha F_\alpha\Big]_{F_\alpha=\,0} = 0, \quad \alpha = 1,2,3, \tag{4.6.10}$$

where W^α's are presented in (4.6.1).

Proposition: The operator X_4 admitted by the system (4.1.8) provides the conserved vector $(\tilde{C}_4^0, \tilde{C}_4^1, \tilde{C}_4^2, \tilde{C}_4^3)$, where components of the conserved vector are

$$\begin{aligned}\tilde{C}_4^0 =& 2\sum_{\alpha=1}^{3}\Big(zu_t^\alpha u_y^\alpha - yu_t^\alpha u_z^\alpha\Big) + 2u^2u_t^3 - 2u^3u_t^2,\\ \\ \tilde{C}_4^1 =& 2\beta\Big(\sum_{\alpha=1}^{3}\big(yu_x^\alpha u_z^\alpha - zu_x^\alpha u_y^\alpha\big) - u^2u_x^3 + u^3u_x^2\Big) + y\Big(2u_x^1u_z^1 + u_y^1u_z^2 + 2u_z^1u_z^3 + u_y^2u_z^1\Big)\\ &- z\Big(2u_y^1u_y^2 + u_y^1u_z^3 + u_y^3u_z^1 + 2u_x^1u_y^1\Big) - u^2u_z^1 + u^3u_y^1,\\ \\ \tilde{C}_4^2 =& \sum_{\alpha=1}^{3}\Big[\beta\Big(2yu_y^\alpha u_z^\alpha + z\big[(u_x^\alpha)^2 + (u_z^\alpha)^2 - (u_y^\alpha)^2\big]\Big) - z(u_t^\alpha)^2\Big] + \beta\big(2u^3u_y^2 - 2u^2u_y^3\big)\\ &+ z\Big((u_x^1)^2 + (u_z^3)^2 - (u_y^2)^2 + u_x^1u_z^3 + u_z^1u_x^3\Big) + y\Big(u_x^1u_z^2 + 2u_y^2u_z^2 + 2u_z^2u_z^3 + u_x^2u_z^1\Big)\\ &+ u^3u_x^1 + 2u^3u_y^2 + u^3u_z^3 - u^2u_z^2,\\ \\ \tilde{C}_4^3 =& \sum_{\alpha=1}^{3}\Big[\beta\Big(y\big[(u_z^\alpha)^2 - (u_x^\alpha)^2 - (u_y^\alpha)^2\big] - 2zu_y^\alpha u_z^\alpha\Big) + y(u_t^\alpha)^2\Big] - \beta\big(2u^3u_z^2 - 2u^2u_z^3\big)\\ &- z\Big(u_x^3u_y^1 + u_x^1u_y^3 + zu_y^3u_z^3 + 2u_y^2u_y^3\Big) - y\Big(u_x^2u_y^1 + u_x^1u_y^2 - (u_z^3)^2 + (u_y^2)^2 + (u_x^1)^2\Big)\\ &- 2u^2u_z^3 - u^2u_x^1 + u^3u_y^3 - u^2u_y^2.\end{aligned} \tag{4.6.11}$$

4.7 Rotation: $X_5 = z\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial z} + u^3\dfrac{\partial}{\partial u^1} - u^1\dfrac{\partial}{\partial u^3}$

We have $\xi^0 = \xi^2 = 0,\ \xi^1 = z,\ \xi^3 = -x,\ \eta^2 = 0,\ \eta^1 = u^3,\ \eta^3 = -u^1$ and

$$\begin{aligned}W^1 &= u^3 - zu_x^1 + xu_z^1,\\ W^2 &= -zu_x^2 + xu_z^2,\\ W^3 &= -u^1 - zu_x^3 + xu_z^3.\end{aligned} \tag{4.7.1}$$

Hence, considering (4.1.12) we have:

$$\begin{aligned} C_5^0 = \quad & (u^3 - zu_x^1 + xu_z^1)u_t^1 + (-zu_x^2 + xu_z^2)u_t^2 + (-u^1 - zu_x^3 + xu_z^3)u_t^3 \\ & - (u_t^3 - zu_{xt}^1 + xu_{zt}^1)u^1 - (-zu_{xt}^2 + xu_{zt}^2)u^2 - (-u_t^1 - zu_{xt}^3 + xu_{zt}^3)u^3. \end{aligned} \tag{4.7.2}$$

However, using the (4.1.6) and applying some simplifications it can be rewritten as

$$\tilde{C}_5^0 = 2\sum_{\alpha=1}^{3}\Big(xu_t^\alpha u_z^\alpha - zu_t^\alpha u_x^\alpha\Big) + 2u^3u_t^1 - 2u^1u_t^3, \tag{4.7.3}$$

and the remaining terms will be transferred to the next components.

The formula (4.1.13) gives

$$\begin{aligned} C_5^1 = \ z\mathcal{L} \ & - (u^3 - zu_x^1 + xu_z^1)\Big((\beta+1)u_x^1 + u_y^2 + u_z^3\Big) - (-zu_x^2 + xu_z^2)\Big(\beta u_x^2 + u_y^1\Big) \\ & - (-u^1 - zu_x^3 + xu_z^3)\Big(\beta u_x^3 + u_z^1\Big) + (u_x^3 - zu_{xx}^1 + u_z^1 + xu_{zx}^1)\Big((\beta+1)u^1\Big) \\ & + (-zu_{xx}^2 + u_z^2 + xu_{zx}^2)\Big(\beta u^2\Big) + (-u_x^1 - zu_{xx}^3 + u_z^3 + xu_{zx}^3)\Big(\beta u^3\Big). \end{aligned} \tag{4.7.4}$$

Invoking the same technique for the second component we obtain:

$$\begin{aligned} \tilde{C}_5^1 = & \sum_{\alpha=1}^{3}\Big(\beta\big(-2xu_x^\alpha u_z^\alpha - z(u_y^\alpha)^2 + z(u_x^\alpha)^2 - z(u_z^\alpha)^2\big) + z(u_t^\alpha)^2\Big) - \beta\big(2u^3u_x^1 - 2u^1u_x^3\big) \\ & + z(u_x^1)^2 - z(u_z^3)^2 - z(u_y^2)^2 + u^1u_z^1 - 2u^3u_x^1 - u^3u_z^3 - u^3u_y^2 - 2xu_z^1u_z^3 - 2xu_x^1u_z^1 \\ & - xu_z^1u_y^2 - xu_y^1u_z^2 - zu_z^2u_y^3 - zu_y^2u_z^3, \end{aligned} \tag{4.7.5}$$

and the rest of the remaining terms will be transferred to the next components.

From the formula (4.1.14) it follows

$$\begin{aligned} C_5^2 = \ & - (u^3 - zu_x^1 + xu_z^1)\Big(\beta u_y^1\Big) - (-zu_x^2 + xu_z^2)\Big((\beta+1)u_y^2 + u_z^3\Big) \\ & - (-u^1 - zu_x^3 + xu_z^3)\Big(\beta u_y^3 + u_z^2\Big) + (u_x^3 - zu_{xx}^1 + u_z^1 + xu_{zx}^1)\Big(u^2\Big) \\ & + (-zu_{xx}^2 + u_z^2 + xu_{zx}^2)\Big(u^1\Big) + (-u_y^1 - zu_{xy}^3 + xu_{zy}^3)\Big(\beta u^3\Big) \\ & + (u_y^3 - zu_{xy}^1 + xu_{zy}^1)\Big(\beta u^1\Big) + (-zu_{xy}^2 + xu_{zy}^2)\Big((\beta+1)u^2\Big). \end{aligned} \tag{4.7.6}$$

Hence, if we add the transferred terms from previous components and do simplification we obtain:

$$\begin{aligned} \tilde{C}_5^2 = 2\beta\Big(&\sum_{\alpha=1}^{3}\big(zu_x^\alpha u_y^\alpha - xu_y^\alpha u_z^\alpha\big) - u^3u_y^1 + u^1u_y^3\Big) + 2zu_x^2u_y^2 - 2xu_y^2u_z^2 + 2zu_x^2u_x^1 \\ & - 2xu_z^2u_z^3 - xu_z^2u_x^1 - xu_z^1u_x^2 + zu_z^2u_x^3 - u^3u_x^2 + u^1u_z^2 + zu_z^3u_x^2, \end{aligned} \tag{4.7.7}$$

and the remaining terms will be transferred to the last component.

And finally, the formula (4.1.15) for the last component gives:

$$\begin{aligned} C_5^3 = & -x\mathcal{L} \ -(u^3 - zu^1_x + xu^1_z)\Big(\beta u^1_z\Big) - (-zu^2_x + xu^2_z)\Big(\beta u^2_z\Big) \\ & - (-u^1 - zu^3_x + xu^3_z)\Big((\beta+1)u^3_z\Big) + (u^3_x - zu^1_{xx} + u^1_z + xu^1_{zx})\Big(\frac{1}{2}u^3\Big) \\ & + (-zu^2_{xy} + xu^2_{zy})\Big(u^3\Big) + (-u^1_x - zu^3_{xx} + u^3_z - xu^3_{zx})\Big(u^1\Big) \\ & + (-u^1_y - zu^3_{xy} + xu^3_{zy})\Big(u^2\Big) + (-u^2_x - zu^2_{xz} + xu^2_{zz})\Big(\beta u^2\Big) \\ & + (u^3_z - u^1_x - zu^1_{xz} + xu^1_{zz})\Big(\beta u^1\Big) + (-u^1_z - u^3_x - zu^3_{xz} + xu^3_{zz})\Big((\beta+1)u^3\Big), \end{aligned} \tag{4.7.8}$$

Inserting the transferred terms from the previous components to the C_5^3 and doing simplification steps we get:

$$\begin{aligned} \tilde{C}_5^3 = & \sum_{\alpha=1}^{3}\Big(\beta(x(u^\alpha_x)^2 + x(u^\alpha_y)^2 - x(u^\alpha_z)^2 + 2zu^\alpha_x u^\alpha_z) - x(u^\alpha_t)^2\Big) + \beta\big(2u^1u^3_z - 2u^3u^1_z\big) \\ & + zu^2_y u^3_x + xu^2_y u^1_x + xu^2_x u^1_y + zu^3_y u^2_x + 2zu^3_x u^3_z + 2zu^1_x u^3_x + x(u^1_x)^2 + x(u^2_y)^2 \\ & - x(u^3_z)^2 + 2u^1u^3_z + u^1u^1_x + u^1u^2_y - u^3u^3_x. \end{aligned} \tag{4.7.9}$$

Checking (2.1.5) yields:

$$\Big[D_t(\tilde{C}^0) + D_x(\tilde{C}^1) + D_y(\tilde{C}^2) + D_z(\tilde{C}^3)\Big]_{F_\alpha=\,0} = \Big[-2W^\alpha F_\alpha\Big]_{F_\alpha=\,0} = 0, \quad \alpha = 1,2,3, \tag{4.7.10}$$

where W^α's are coefficients presented in (4.7.1).

Proposition: The operator X_5 admitted by the system (4.1.8) provides the conserved vector $(\tilde{C}_5^0, \tilde{C}_5^1, \tilde{C}_5^2, \tilde{C}_5^3)$, where components of the conserved vector are

$$\begin{aligned} \tilde{C}_5^0 = & 2\sum_{\alpha=1}^{3}\Big(xu^\alpha_t u^\alpha_z - zu^\alpha_t u^\alpha_x\Big) + 2u^3u^1_t - 2u^1u^3_t, \\ \tilde{C}_5^1 = & \sum_{\alpha=1}^{3}\Big(\beta\Big[-2xu^\alpha_x u^\alpha_z + z\Big((u^\alpha_x)^2 - (u^\alpha_y)^2 - (u^\alpha_z)^2\Big)\Big] + z(u^\alpha_t)^2\Big) - \beta\big(2u^3u^1_x - 2u^1u^3_x\big) \\ & + z\Big((u^1_x)^2 - (u^3_z)^2 - (u^2_y)^2 - u^2_z u^3_y - u^2_y u^3_z\Big) - x\Big(2u^1_z u^3_z + 2u^1_x u^1_z + u^1_z u^2_y + u^1_y u^2_z\Big) \\ & + u^1u^1_z - 2u^3u^1_x - u^3u^3_z - u^3u^2_y, \\ \tilde{C}_5^2 = & 2\beta\Big(\sum_{\alpha=1}^{3}\big(zu^\alpha_x u^\alpha_y - xu^\alpha_y u^\alpha_z\big) - u^3u^1_y + u^1u^3_y\Big) + z\Big(2u^2_x u^2_y + 2u^2_x u^1_x + u^2_z u^3_x + u^3_z u^2_x\Big) \\ & - x\Big(2u^2_y u^2_z + 2u^2_z u^3_z + u^2_z u^1_x + u^1_z u^2_x\Big) - u^3u^2_x + u^1u^2_z, \\ \tilde{C}_5^3 = & \sum_{\alpha=1}^{3}\Big(\beta\Big[x\Big((u^\alpha_x)^2 + (u^\alpha_y)^2 - (u^\alpha_z)^2\Big) + 2zu^\alpha_x u^\alpha_z\Big] - x(u^\alpha_t)^2\Big) + \beta\big(2u^1u^3_z - 2u^3u^1_z\big) \\ & + z\Big(u^2_y u^3_x + u^3_y u^2_x + 2u^3_x u^3_z + 2u^1_x u^3_x\Big) + x\Big(u^2_y u^1_x + u^2_x u^1_y + (u^1_x)^2 + (u^2_y)^2 - (u^3_z)^2\Big) \\ & + 2u^1u^3_z + u^1u^1_x + u^1u^2_y - u^3u^3_x. \end{aligned} \tag{4.7.11}$$

4.8 Rotation: $X_6 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + u^1\frac{\partial}{\partial u^2} - u^2\frac{\partial}{\partial u^1}$

In the case of this rotation, $\xi^0 = \xi^3 = 0,\ \xi^1 = -y,\ \xi^2 = x,\ \eta^1 = -u^2,\ \eta^2 = u^1,\ \eta^3 = 0$ and

$$\begin{aligned} W^1 &= -u^2 + yu^1_x - xu^1_y, \\ W^2 &= u^1 + yu^2_x - xu^2_y, \\ W^3 &= yu^3_x - xu^3_y. \end{aligned} \tag{4.8.1}$$

Hence, considering (4.1.12) we have

$$\begin{aligned} C^0_6 = \quad & (-u^2 + yu^1_x - xu^1_y)u^1_t + (u^1 + yu^2_x - xu^2_y)u^2_t + (yu^3_x - xu^3_y)u^3_t \\ & - (-u^2_t + yu^1_{xt} - xu^1_{yt})u^1 - (u^1_t + yu^2_{xt} - xu^2_{yt})u^2 - (yu^3_{xt} - xu^3_{yt})u^3. \end{aligned} \tag{4.8.2}$$

However, using the (4.1.6) and applying some simplifications we can rewrite it as

$$\tilde{C}^0_6 = 2\sum_{\alpha=1}^{3}\Big(yu^\alpha_t u^\alpha_x - xu^\alpha_t u^\alpha_y\Big) + 2u^1u^2_t - 2u^2u^1_t, \tag{4.8.3}$$

and the remaining terms will be transferred to the next components.

The formula (4.1.13) gives

$$\begin{aligned} C^1_6 = \ -y\mathcal{L}\ & - (-u^2 + yu^1_x - xu^1_y)\Big((\beta+1)u^1_x + u^2_y + u^3_z\Big) - (u^1 + yu^2_x - xu^2_y)\Big(\beta u^2_x + u^1_y\Big) \\ & - (yu^3_x - xu^3_y)\Big(\beta u^3_x + u^1_z\Big) + (-u^2_x + yu^1_{xx} - u^1_y - xu^1_{yx})\Big((\beta+1)u^1\Big) \\ & + (u^1_x + yu^2_{xx} - u^2_y - xu^2_{yx})\Big(\beta u^2\Big) + (yu^3_{xx} - u^3_y - xu^3_{yx})\Big(\beta u^3\Big). \end{aligned} \tag{4.8.4}$$

Invoking the same technique for the second component we obtain:

$$\begin{aligned} \tilde{C}^1_6 = \ & \sum_{\alpha=1}^{3}\Big(\beta\big(2xu^\alpha_x u^\alpha_y - y(u^\alpha_x)^2 + y(u^\alpha_y)^2 + y(u^\alpha_z)^2\big) - y(u^\alpha_t)^2\Big) + \beta\big(2u^2u^1_x - 2u^1u^2_x\big) \\ & + y(u^2_y)^2 - y(u^1_x)^2 + y(u^3_z)^2 + u^2u^2_y + 2u^2u^1_x + u^2u^3_z - u^1u^1_y + 2xu^1_xu^1_y + 2xu^1_yu^2_y \\ & + xu^1_yu^3_z + yu^3_zu^2_y + yu^2_zu^3_y + xu^1_zu^3_y, \end{aligned} \tag{4.8.5}$$

and the rest of the remaining terms will be transferred to the next components.

From the formula (4.1.14) it follows

$$\begin{aligned} C^2_6 = \ x\mathcal{L}\ & - (-u^2 + yu^1_x - xu^1_y)\Big(\beta u^1_y\Big) - (u^1 + yu^2_x - xu^2_y)\Big((\beta+1)u^2_y + u^3_z\Big) \\ & - (yu^3_x - xu^3_y)\Big(\beta u^3_y + u^2_z\Big) + (-u^2_x + yu^1_{xx} - u^1_y - xu^1_{yx})\Big(u^2\Big) \\ & + (-u^2_y + u^1_x + yu^1_{xy} - xu^1_{yy})\Big(\beta u^1\Big) + (u^1_x + yu^2_{xx} - u^2_y - xu^2_{yx})\Big(u^1\Big) \\ & + (u^1_y + u^2_x + yu^2_{xy} - xu^2_{yy})\Big((\beta+1)u^2\Big) + (u^3_x + yu^3_{xy} - xu^3_{yy})\Big(\beta u^3\Big). \end{aligned} \tag{4.8.6}$$

Hence, if we add the transferred terms from previous components and do simplification we obtain:

$$\begin{aligned}\tilde{C}_6^2 = &\sum_{\alpha=1}^{3}\Big(\beta(x(u_y^\alpha)^2 - x(u_z^\alpha)^2 - x(u_x^\alpha)^2 - 2yu_x^\alpha u_y^\alpha) + x(u_t^\alpha)^2\Big) - \beta\big(2u^2u_y^1 - 2u^1u_y^2\big)\\ &- yu_z^2u_x^3 - yu_z^3u_x^2 - xu_z^1u_x^3 - xu_z^3u_x^1 - 2yu_x^2u_y^2 - 2yu_x^1u_x^2 + x(u_y^2)^2 - x(u_z^3)^2\\ &- x(u_x^1)^2 - 2u^1u_y^2 - u^1u_z^3 + u^2u_x^2 - u^1u_x^1,\end{aligned} \tag{4.8.7}$$

and the remaining terms will be transferred to the last component.

Finally, the formula (4.1.15) for the last component gives:

$$\begin{aligned}C_6^3 = &-(-u^2 + yu_x^1 - xu_y^1)\Big(\beta u_z^1\Big) - (u^1 + yu_x^2 - xu_y^2)\Big(\beta u_z^2\Big)\\ &- (yu_x^3 - xu_y^3)\Big((\beta+1)u_z^3\Big) + (-u_x^2 + yu_{xx}^1 - u_y^1 - xu_{yx}^1)\Big(u^3\Big)\\ &+ (u_y^1 + u_x^2 + yu_{xy}^2 - xu_{yy}^2)\Big(u^3\Big) + (u_z^1 + yu_{xz}^2 - xu_{yz}^2)\Big(\beta u^2\Big)\\ &+ (u_x^3 + yu_{xy}^3 - xu_{yy}^3)\Big(u^2\Big) + (yu_{xx}^3 - u_y^3 - xu_{yx}^3)\Big(u^1\Big)\\ &+ (-u_z^2 + yu_{xz}^1 - xu_{yz}^1)\Big(\beta u^1\Big) + (yu_{xz}^3 - xu_{yz}^3)\Big((\beta+1)u^3\Big),\end{aligned} \tag{4.8.8}$$

Inserting the transferred terms from the previous components to the C_6^3 and doing simplification steps we get:

$$\begin{aligned}\tilde{C}_6^3 = &2\beta\Big(\sum_{\alpha=1}^{3}\big(xu_z^\alpha u_y^\alpha - yu_z^\alpha u_x^\alpha\big) - u^1u_z^2 + u^2u_z^1\Big) + 2xu_z^3u_y^3 - 2yu_z^3u_x^3 + 2xu_y^3u_y^2\\ &- 2yu_x^3u_x^1 - yu_x^3u_y^2 - yu_x^2u_y^3 + xu_x^3u_y^1 - u^1u_y^3 + u^2u_x^3 + xu_x^1u_y^3.\end{aligned} \tag{4.8.9}$$

Checking (2.1.5) gives

$$\Big[D_t(\tilde{C}^0) + D_x(\tilde{C}^1) + D_y(\tilde{C}^2) + D_z(\tilde{C}^3)\Big]_{F_\alpha=\,0} = \Big[-2W^\alpha F_\alpha\Big]_{F_\alpha=\,0} = 0, \quad \alpha = 1,2,3, \tag{4.8.10}$$

where W^α's are coefficients presented in (4.8.1).

Proposition: The operator X_6 admitted by the system (4.1.8) provides the conserved vector $(\tilde{C}_6^0, \tilde{C}_6^1, \tilde{C}_6^2, \tilde{C}_6^3)$, where components of the conserved vector are

$$\tilde{C}_6^0 = 2\sum_{\alpha=1}^{3}\Big(yu_t^\alpha u_x^\alpha - xu_t^\alpha u_y^\alpha\Big) + 2u^1u_t^2 - 2u^2u_t^1,$$

$$\begin{aligned}\tilde{C}_6^1 = &\sum_{\alpha=1}^{3}\Big[\beta\Big(2xu_x^\alpha u_y^\alpha + y\big[(u_y^\alpha)^2 - (u_x^\alpha)^2 + (u_z^\alpha)^2\big]\Big) - y(u_t^\alpha)^2\Big] + \beta\big(2u^2u_x^1 - 2u^1u_x^2\big)\\ &+ y\Big((u_y^2)^2 - (u_x^1)^2 + (u_z^3)^2 + u_z^3u_y^2 + u_z^2u_y^3\Big) + x\Big(2u_x^1u_y^1 + 2u_y^1u_y^2 + u_y^1u_z^3 + u_z^1u_y^3\Big)\\ &+ u^2u_y^2 + 2u^2u_x^1 + u^2u_z^3 - u^1u_y^1,\end{aligned}$$

$$\tilde{C}_6^2 = \sum_{\alpha=1}^{3}\Big[\beta\Big(x\big[(u_y^\alpha)^2 - (u_z^\alpha)^2 - (u_x^\alpha)^2\big] - 2yu_x^\alpha u_y^\alpha\Big) + x(u_t^\alpha)^2\Big] + \beta\big(2u^2u_y^1 - 2u^1u_y^2\big)$$

$$- y\Big(u^2_z u^3_x + u^3_z u^2_x + 2u^2_x u^2_y + 2u^1_x u^2_x\Big) + x\Big((u^2_y)^2 - u^1_z u^3_x - u^3_z u^1_x - (u^3_z)^2 - (u^1_x)^2\Big)$$
$$- 2u^1 u^2_y - u^1 u^3_z + u^2 u^2_x - u^1 u^1_x,$$

$$\tilde{C}^3_6 = 2\beta\Big(\sum_{\alpha=1}^{3}\big(xu^\alpha_z u^\alpha_y - yu^\alpha_z u^\alpha_x\big) - u^1 u^2_z + u^2 u^1_z\Big) + x\Big(2u^3_z u^3_y + 2u^3_y u^2_y + u^3_x u^1_y + u^1_x u^3_y\Big)$$
$$- y\Big(2u^3_z u^3_x + 2u^3_x u^1_x + u^3_x u^2_y + u^2_x u^3_y\Big) - u^1 u^3_y + u^2 u^3_x. \tag{4.8.11}$$

4.9 Scaling: $X_7 = t\dfrac{\partial}{\partial t} + x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + z\dfrac{\partial}{\partial z}$

Here $\xi^0 = t$, $\xi^1 = x,\ \xi^2 = y,\ \xi^3 = z,\ \eta^1 = \eta^2 = \eta^3 = 0$ and

$$\begin{aligned} W^1 &= -(tu^1_t + xu^1_x + yu^1_y + zu^1_z),\\ W^2 &= -(tu^2_t + xu^2_x + yu^2_y + zu^2_z),\\ W^3 &= -(tu^3_t + xu^3_x + yu^3_y + zu^3_z). \end{aligned} \tag{4.9.1}$$

Hence, using (4.1.12) we have:

$$C^0_7 = t\mathcal{L} - (tu^j_t + xu^j_x + yu^j_y + zu^j_z)u^j_t + (u^j_t + tu^j_{tt} + xu^j_{xt} + yu^j_{yt} + zu^j_{zt})u^j, \tag{4.9.2}$$

where $j = 1, 2, 3$.
However, using the (4.1.6) and applying some simplifications we can rewrite it as

$$\tilde{C}^0_7 = \sum_{\alpha=1}^{3}\Big(\beta t\Big((u^\alpha_x)^2 + (u^\alpha_y)^2 + (u^\alpha_z)^2\Big) - (u^\alpha_t)^2 - 2xu^\alpha_t u^\alpha_x - 2yu^\alpha_t u^\alpha_y - 2zu^\alpha_t u^\alpha_z - 2u^\alpha u^\alpha_t\Big)$$
$$- 2tu^1_x u^2_y - 2tu^1_x u^3_z - 2tu^2_y u^3_z - t(u^1_x)^2 - t(u^2_y)^2 - t(u^3_z)^2, \tag{4.9.3}$$

and the remaining terms will be transferred to the next components.

The formula (4.1.13) gives

$$\begin{aligned} C^1_7 = {} & x\mathcal{L} + (tu^1_t + xu^1_x + yu^1_y + zu^1_z)\Big((\beta+1)u^1_x + u^2_y + u^3_z\Big)\\ & + (tu^2_t + xu^2_x + yu^2_y + zu^2_z)\Big(\beta u^2_x + u^1_y\Big)\\ & + (tu^3_t + xu^3_x + yu^3_y + zu^3_z)\Big(\beta u^3_x + u^1_z\Big)\\ & - (u^1_x + tu^1_{tx} + xu^1_{xx} + yu^1_{yx} + zu^1_{zx})\Big((\beta+1)u^1\Big)\\ & - (u^2_x + tu^2_{tx} + xu^2_{xx} + yu^2_{yx} + zu^2_{zx})\Big(\beta u^2\Big)\\ & - (u^3_x + tu^3_{tx} + xu^3_{xx} + yu^3_{yx} + zu^3_{zx})\Big(\beta u^3\Big). \end{aligned} \tag{4.9.4}$$

Invoking the same technique for the second component we obtain:

$$\tilde{C}_7^1 = \sum_{\alpha=1}^{3}\left(\beta\Big(2tu_x^\alpha u_t^\alpha + 2yu_x^\alpha u_y^\alpha + 2zu_x^\alpha u_z^\alpha + x(u_x^\alpha)^2 - x(u_z^\alpha)^2 - x(u_y^\alpha)^2 + 2u^\alpha u_x^\alpha\Big) + x(u_t^\alpha)^2\right)$$
$$+ x(u_x^1)^2 - x(u_y^2)^2 - x(u_z^3)^2 + u^1u_z^3 + 2u^1u_x^1 + \frac{3}{2}u^1u_y^2 + \frac{1}{2}u^2u_y^1 - 2xu_y^2u_z^3 + 2tu_t^1u_x^1$$
$$+ 2yu_x^1u_y^1 + 2zu_x^1u_z^1 + 2tu_t^1u_y^2 + 2yu_y^1u_y^2 + 2tu_t^1u_z^3 + 2zu_z^1u_z^3 + zu_y^1u_z^2 + zu_y^2u_z^1 + yu_y^3u_z^1$$
$$+ yu_y^1u_z^3, \tag{4.9.5}$$

and the rest of the remaining terms will be transferred to the next components.

From the formula (4.1.14) it follows

$$C_7^2 = y\mathcal{L} + (tu_t^1 + xu_x^1 + yu_y^1 + zu_z^1)\Big(\beta u_y^1\Big)$$
$$+ (tu_t^2 + xu_x^2 + yu_y^2 + zu_z^2)\Big((\beta+1)u_y^2 + u_z^3\Big)$$
$$+ (tu_t^3 + xu_x^3 + yu_y^3 + zu_z^3)\Big(\beta u_y^3 + u_z^2\Big)$$
$$- (u_x^1 + tu_{tx}^1 + xu_{xx}^1 + yu_{yx}^1 + zu_{zx}^1)\Big(\frac{1}{2}u^2\Big)$$
$$- (u_y^1 + tu_{ty}^1 + xu_{xy}^1 + yu_{yy}^1 + zu_{zy}^1)\Big(\beta u^1\Big)$$
$$- (u_x^2 + tu_{tx}^2 + xu_{xx}^2 + yu_{yx}^2 + zu_{zx}^2)\Big(\frac{1}{2}u^1\Big)$$
$$- (u_y^2 + tu_{ty}^2 + xu_{xy}^2 + yu_{yy}^2 + zu_{zy}^2)\Big((\beta+1)u^2\Big)$$
$$- (u_y^3 + tu_{ty}^3 + xu_{xy}^3 + yu_{yy}^3 + zu_{zy}^3)\Big(\beta u^3\Big). \tag{4.9.6}$$

Hence, if we add the transferred terms from previous components and do simplification we obtain:

$$\tilde{C}_7^2 = \sum_{\alpha=1}^{3}\left(\beta\Big(2tu_y^\alpha u_t^\alpha + 2xu_x^\alpha u_y^\alpha + 2zu_y^\alpha u_z^\alpha + y(u_y^\alpha)^2 - y(u_x^\alpha)^2 - y(u_z^\alpha)^2 + 2u^\alpha u_y^\alpha\Big) + y(u_t^\alpha)^2\right)$$
$$+ y(u_y^1)^2 - y(u_x^2)^2 - y(u_z^3)^2 + 2u^2u_y^2 + 2u^2u_z^3 + \frac{3}{2}u^2u_x^1 + \frac{1}{2}u^1u_x^2 + 2zu_y^2u_z^2 + 2xu_x^2u_z^3$$
$$+ 2zu_z^2u_z^3 + 2xu_x^2u_y^2 + 2xu_x^1u_x^2 + 2tu_t^2u_x^1 + 2tu_t^2u_z^3 + 2tu_t^2u_y^2 + zu_x^1u_z^2 - yu_x^3u_z^1 + zu_x^2u_z^1$$
$$- yu_x^1u_z^3, \tag{4.9.7}$$

and the remaining terms will be transferred to the last component.

Finally, the formula (4.1.15) for the last component gives:

$$C_7^3 = z\mathcal{L} + (tu_t^1 + xu_x^1 + yu_y^1 + zu_z^1)\Big(\beta u_z^1\Big)$$
$$+ (tu_t^2 + xu_x^2 + yu_y^2 + zu_z^2)\Big(\beta u_z^2\Big)$$
$$+ (tu_t^3 + xu_x^3 + yu_y^3 + zu_z^3)\Big((\beta+1)u_z^3\Big)$$
$$- (u_x^1 + tu_{tx}^1 + xu_{xx}^1 + yu_{yx}^1 + zu_{zx}^1)\Big(u^3\Big)$$

$$
\begin{aligned}
&- (u^2_y + tu^2_{ty} + xu^2_{xy} + yu^2_{yy} + zu^2_{zy})\Big(u^3\Big)\\
&- (u^2_z + tu^2_{tz} + xu^2_{xz} + yu^2_{yz} + zu^2_{zz})\Big(\beta u^2\Big)\\
&- (u^3_y + tu^3_{ty} + xu^3_{xy} + yu^3_{yy} + zu^3_{zy})\Big(u^2\Big)\\
&- (u^3_x + tu^3_{tx} + xu^3_{xx} + yu^3_{yx} + zu^3_{zx})\Big(u^1\Big)\\
&- (u^1_z + tu^1_{tz} + xu^1_{xz} + yu^1_{yz} + zu^1_{zz})\Big(\beta u^1\Big)\\
&- (u^3_z + tu^3_{tz} + xu^3_{xz} + yu^3_{yz} + zu^3_{zz})\Big((\beta + 1)u^3\Big), \qquad (4.9.8)
\end{aligned}
$$

Inserting the transferred terms from the previous components to the C^3_7 and doing simplification steps we get:

$$
\begin{aligned}
\tilde{C}^3_7 = &\sum_{\alpha=1}^{3}\bigg(\beta\Big(2tu^\alpha_z u^\alpha_t + 2xu^\alpha_x u^\alpha_z + 2yu^\alpha_y u^\alpha_z + z(u^\alpha_z)^2 - z(u^\alpha_x)^2 - z(u^\alpha_y)^2 + 2u^\alpha u^\alpha_z\Big) + z(u^\alpha_t)^2\bigg)\\
&+ z(u^1_z)^2 - z(u^2_x)^2 - z(u^3_y)^2 + 2u^3u^2_y + 2u^3u^1_x + 2u^3u^3_z + u^1u^3_x + 2xu^1_x u^3_x + 2yu^2_y u^3_y\\
&+ 2tu^3_t u^1_x + 2tu^3_t u^2_y + 2xu^3_x u^2_y + 2tu^3_t u^3_z + 2yu^3_y u^3_z + 2xu^3_x u^3_z - zu^1_x u^2_y - zu^2_x u^1_y\\
&+ yu^1_x u^3_y + yu^1_y u^3_x. \qquad (4.9.9)
\end{aligned}
$$

Checking (2.1.5) yields:

$$
\Big[D_t(\tilde{C}^0) + D_x(\tilde{C}^1) + D_y(\tilde{C}^2) + D_z(\tilde{C}^3)\Big]_{F_\alpha=\,0} = \Big[-2W^\alpha F_\alpha\Big]_{F_\alpha=\,0} = 0, \quad \alpha = 1, 2, 3, \qquad (4.9.10)
$$

where W^α are coefficients presented in (4.9.1).

Proposition: The operator X_7 admitted by the system (4.1.8) provides the conserved vector $(\tilde{C}^0_7, \tilde{C}^1_7, \tilde{C}^2_7, \tilde{C}^3_7)$, where components of the conserved vector are

$$
\begin{aligned}
\tilde{C}^0_7 = &\sum_{\alpha=1}^{3}\bigg[\beta t\Big((u^\alpha_x)^2 + (u^\alpha_y)^2 + (u^\alpha_z)^2\Big) - (u^\alpha_t)^2 - 2u^\alpha_t\Big(xu^\alpha_x + yu^\alpha_y + zu^\alpha_z + 2u^\alpha\Big)\bigg]\\
&- t\Big(2u^1_x u^2_y + 2u^1_x u^3_z + 2u^2_y u^3_z + (u^1_x)^2 + (u^2_y)^2 + (u^3_z)^2\Big),
\end{aligned}
$$

$$
\begin{aligned}
\tilde{C}^1_7 = &\sum_{\alpha=1}^{3}\bigg[\beta\bigg(2u^\alpha_x\Big[tu^\alpha_t + yu^\alpha_y + zu^\alpha_z + u^\alpha\Big] + x\Big[(u^\alpha_x)^2 - (u^\alpha_z)^2 - (u^\alpha_y)^2\Big]\bigg) + x(u^\alpha_t)^2\bigg]\\
&+ x\Big((u^1_x)^2 - (u^2_y)^2 - (u^3_z)^2 - 2u^2_y u^3_z\Big) + y\Big(u^1_y u^3_z + 2u^1_x u^1_y + 2u^1_y u^2_y + u^3_y u^1_z\Big)\\
&+ z\Big(2u^1_x u^1_z + 2u^1_z u^3_z + u^1_y u^2_z + u^2_y u^1_z\Big) + 2t\Big(u^1_t u^1_x + u^1_t u^2_y + u^1_t u^3_z\Big)\\
&+ u^1u^3_z + 2u^1u^1_x + \frac{3}{2}u^1u^2_y + \frac{1}{2}u^2u^1_y,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{C}^2_7 = &\sum_{\alpha=1}^{3}\bigg[\beta\bigg(2u^\alpha_y\Big[tu^\alpha_t + xu^\alpha_x + zu^\alpha_z + 2u^\alpha\Big] + y\Big[(u^\alpha_y)^2 - (u^\alpha_x)^2 - (u^\alpha_z)^2\Big]\bigg) + y(u^\alpha_t)^2\bigg]\\
&+ y\Big((u^1_y)^2 - (u^2_x)^2 - (u^3_z)^2 - u^1_x u^3_z - u^3_x u^1_z\Big) + 2x\Big(u^2_x u^2_y + u^2_x u^3_z + u^1_x u^2_x\Big)
\end{aligned}
$$

$$+ z\Big(2u_z^2u_z^3 + 2u_y^2u_z^2 + u_x^2u_z^1 + u_x^1u_z^2\Big) + 2t\Big(u_t^2u_x^1 + u_t^2u_z^3 + u_t^2u_y^2\Big)$$
$$+ 2u^2u_y^2 + 2u^2u_z^3 + \frac{3}{2}u^2u_x^1 + \frac{1}{2}u^1u_x^2,$$

$$\tilde{C}_7^3 = \sum_{\alpha=1}^{3}\Bigg[\beta\Big(2u_z^\alpha\Big[tu_t^\alpha + 2xu_x^\alpha + 2yu_y^\alpha + 2u^\alpha\Big] + z\Big[(u_z^\alpha)^2 - (u_x^\alpha)^2 - (u_y^\alpha)^2\Big]\Big) + z(u_t^\alpha)^2\Bigg]$$
$$+ z\Big((u_z^1)^2 - (u_x^2)^2 - (u_y^3)^2 - u_x^1u_y^2 - u_x^2u_y^1\Big) + 2x\Big(u_x^1u_x^3 + u_x^3u_y^2 + u_x^3u_z^3\Big)$$
$$+ 2t\Big(u_t^3u_x^1 + u_t^3u_y^2 + u_t^3u_z^3\Big) + y\Big(u_x^1u_y^3 + u_y^1u_x^3 + 2u_y^2u_y^3 + 2u_y^3u_z^3\Big)$$
$$+ 2u^3u_y^2 + 2u^3u_x^1 + 2u^3u_z^3 + u^1u_x^3. \tag{4.9.11}$$

4.10 Scaling: $X_8 = u^1\dfrac{\partial}{\partial u^1} + u^2\dfrac{\partial}{\partial u^2} + u^3\dfrac{\partial}{\partial u^3}$

We have $\xi^0 = \xi^1 = \xi^2 = \xi^3 = 0,$ and $\eta^\alpha = u^\alpha$. Invoking definition of W^α we get

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha = u^\alpha. \tag{4.10.1}$$

Hence, considering the equation (4.1.12) we have

$$C_8^0 = 0. \tag{4.10.2}$$

Using the equation (4.1.13) and invoking the same technique for the second component we obtain:

$$C_8^1 = -\,u^1\Big((\beta+1)u_x^1 + u_y^2 + u_z^3\Big) - u^2\Big(\beta u_x^2 + u_y^1\Big) - u^3\Big(\beta u_x^3 + u_z^1\Big)$$
$$+ u_x^1\Big((\beta+1)u^1\Big) + u_x^2\Big(\beta u^2\Big) + u_x^3\Big(\beta u^3\Big) = 0. \tag{4.10.3}$$

Considering the same method for the third component gives:

$$C_8^2 = -\,u^1\Big(\beta u_y^1\Big) - u^2\Big((\beta+1)u_y^2 + u_z^3\Big) - u^3\Big(\beta u_y^3 + u_z^2\Big)$$
$$+ u_x^1\Big(u^2\Big) + u_y^1\Big(\beta u^1\Big) + u_x^2\Big(u^1\Big) + u_y^2\Big((\beta+1)u^2\Big)$$
$$+ u_y^3\Big(\beta u^3\Big) = 0. \tag{4.10.4}$$

And finally, the formula (4.1.15) for the last component gives:

$$C_8^3 = -\,u^1\Big(\beta u_z^1\Big) - u^2\Big(\beta u_z^2\Big) - u^3\Big((\beta+1)u_z^3\Big)$$
$$+ u_x^1\Big(u^3\Big) + u_z^1\Big(\beta u^1\Big) + u_y^2\Big(u^3\Big) + u_z^2\Big(\beta u^2\Big)$$
$$+ u_x^3\Big(u^1\Big) + u_y^3\Big(u^2\Big) + u_z^3\Big((\beta+1)u^3\Big) = 0. \tag{4.10.5}$$

Hence, all the components of the conserved vector are equal to zero i.e. there is no nontrivial conservation law corresponding to this symmetry.

Proposition: There is no conservation law corresponding to the symmetry X_8 for the system the system (4.1.8).

4.11 Addition of solutions: $X_\infty = w^1\frac{\partial}{\partial u^1} + w^2\frac{\partial}{\partial u^2} + w^3\frac{\partial}{\partial u^3}$

In this case $\xi^0 = \xi^1 = \xi^2 = \xi^3 = 0,\ \eta^\alpha = w^\alpha$ and

$$W^\alpha = \eta^\alpha - \xi^i u^\alpha_j = w^\alpha. \tag{4.11.1}$$

Hence, considering the equation (4.1.12) we have:

$$C^0_\infty = \ w^1 u^1_t + w^2 u^2_t + w^3 u^3_t - w^1_t u^1 - w^2_t u^2 - w^3_t u^3. \tag{4.11.2}$$

Using the equation (4.1.13) and invoking the same technique for the second component we obtain:

$$\begin{aligned} C^1_\infty =& - w^1\Big((\beta+1)u^1_x + u^2_y + u^3_z\Big) - w^2\Big(\beta u^2_x + u^1_y\Big) - w^3\Big(\beta u^3_x + u^1_z\Big) \\ &+ w^1_x\Big((\beta+1)u^1\Big) + w^2_x\Big(\beta u^2\Big) + w^3_x\Big(\beta u^3\Big). \end{aligned} \tag{4.11.3}$$

Considering the same method for the third component and using (4.1.14) gives

$$\begin{aligned} C^2_\infty =& - w^1\Big(\beta u^1_y\Big) - w^2\Big((\beta+1)u^2_y + u^3_z\Big) - w^3\Big(\beta u^3_y + u^2_z\Big) \\ &+ w^1_x\Big(u^2\Big) + w^1_y\Big(\beta u^1\Big) + w^2_x\Big(u^1\Big) + w^2_y\Big((\beta+1)u^2\Big) \\ &+ w^3_y\Big(\beta u^3\Big). \end{aligned} \tag{4.11.4}$$

And finally, the last component is

$$\begin{aligned} C^3_\infty =& - w^1\Big(\beta u^1_z\Big) - w^2\Big(\beta u^2_z\Big) - w^3\Big((\beta+1)u^3_z\Big) \\ &+ w^1_x\Big(u^3\Big) + w^1_z\Big(\beta u^1\Big) + w^2_y\Big(u^3\Big) + w^2_z\Big(\beta u^2\Big) + w^3_x\Big(u^1\Big) \\ &+ w^3_y\Big(u^2\Big) + w^3_z\Big((\beta+1)u^3\Big). \end{aligned} \tag{4.11.5}$$

Checking (2.1.5) yields:

$$\Big[D_t(C^0) + D_x(C^1) + D_y(C^2) + D_z(C^3)\Big]_{F_\alpha = \ 0} = \Big[-2w^\alpha F_\alpha\Big]_{F_\alpha = \ 0} = 0, \quad \alpha = 1,2,3. \tag{4.11.6}$$

Proposition: The operator X_∞ admitted by the system (4.1.8) provides the conserved vector $(\tilde{C}^0_\infty, \tilde{C}^1_\infty, \tilde{C}^2_\infty, \tilde{C}^3_\infty)$, where components of the conserved vector are (4.11.2), (4.11.3), (4.11.4) and (4.11.5).

5 Comparison of results

As it mentioned in Section 1, two conservation laws that only differ by a trivial conservation law are considered as equivalent. Let us compare the conserved vectors obtained by both methods and corresponding to the same symmetry. We denote a conserved vector obtained by Noether's theorem and Ibragimov's method by C and $\tilde{C}$, respectively.

Conservation laws corresponding to the time translation $X = \dfrac{\partial}{\partial t}$

Noether's theorem:

$$
\begin{aligned}
C_0^0 =& -\sum_{\alpha=1}^{3}\Big[\beta\big((u_x^\alpha)^2+(u_y^\alpha)^2+(u_z^\alpha)^2\big)+(u_t^\alpha)^2+(u_\alpha^\alpha)^2\Big]\\
&-2\beta(u_y^1u_x^2-u_x^1u_y^2+u_z^1u_x^3-u_x^1u_z^3+u_z^2u_y^3-u_y^2u_z^3)\\
&-2u_x^1u_y^2-2u_x^1u_z^3-2u_y^2u_z^3,\\
C_0^1 =& -u_t^1\Big(4\beta u_x^1+2(1-\beta)(u_x^1+u_y^2+u_z^3)\Big)-u_t^2\Big(2\beta(u_x^2+u_y^1)\Big)-u_t^3\Big(2\beta(u_z^1+u_x^3)\Big),\\
C_0^2 =& -u_t^1\Big(2\beta(u_x^2+u_y^1)\Big)-u_t^2\Big(4\beta u_y^2+2(1-\beta)(u_x^1+u_y^2+u_z^3)\Big)-u_t^3\Big(2\beta(u_z^2+u_y^3)\Big),\\
C_0^3 =& -u_t^1\Big(2\beta(u_z^1+u_x^3)\Big)-u_t^2\Big(2\beta(u_z^2+u_y^3)\Big)-u_t^3\Big(4\beta u_z^3+2(1-\beta)(u_x^1+u_y^2+u_z^3)\Big).
\end{aligned}
\tag{5.1}
$$

Ibragimov's method:

$$
\begin{aligned}
\tilde{C}_0^0 =& -\sum_{\alpha=1}^{3}\Big[\beta\big((u_x^\alpha)^2+(u_y^\alpha)^2+(u_z^\alpha)^2\big)+(u_t^\alpha)^2\Big]\\
&-(u_x^1)^2-(u_y^2)^2-(u_z^3)^2-2u_z^1u_x^3-2u_y^1u_x^2-2u_z^2u_y^3,\\
\tilde{C}_0^1 =& 2\beta\sum_{\alpha=1}^{3}u_t^\alpha u_x^\alpha+2u_t^1u_x^1+2u_t^2u_y^1+2u_t^3u_z^1,\\
\tilde{C}_0^2 =& 2\beta\sum_{\alpha=1}^{3}u_t^\alpha u_y^\alpha+2u_t^1u_x^2+2u_t^2u_y^2+2u_t^3u_z^2,\\
\tilde{C}_0^3 =& 2\beta\sum_{\alpha=1}^{3}u_t^\alpha u_z^\alpha+2u_t^1u_x^3+2u_t^2u_y^3+2u_t^3u_z^3.
\end{aligned}
\tag{5.2}
$$

Comparing both sets of formulas, one can show that

$$
C = -\tilde{C} + \hat{C}, \tag{5.3}
$$

where the components of the vector $\hat{C}$ are

$$
\begin{aligned}
\hat{C}_0^0 =& 2(\beta-1)\Big(u_y^1u_x^2-u_x^1u_y^2+u_z^1u_x^3-u_x^1u_z^3+u_z^2u_y^3-u_y^2u_z^3\Big),\\
\hat{C}_0^1 =& 2(\beta-1)\Big(u_t^1u_y^2-u_t^2u_y^1+u_t^1u_z^3-u_t^3u_z^1\Big),\\
\hat{C}_0^2 =& 2(\beta-1)\Big(u_t^2u_x^1-u_t^1u_x^2+u_t^2u_z^3-u_t^3u_z^2\Big),\\
\hat{C}_0^3 =& 2(\beta-1)\Big(u_t^3u_x^1-u_t^1u_x^3+u_t^1u_y^2-u_t^2u_y^3\Big).
\end{aligned}
\tag{5.4}
$$

One can easily verify that

$$
D_i(\hat{C}^i) = 0, \tag{5.5}
$$

which means that it is a trivial conservation law. Hence the results of the both methods are equivalent. Since the formulae (5.2) are simpler, we can choose them as the final result. Thus, they defined a conserved vector corresponding to translation of time.

Translation: $X_1 = \dfrac{\partial}{\partial x}$

Noether's theorem:

$$
\begin{aligned}
C_1^0 =& 2\sum_{\alpha=1}^{3} u_t^\alpha u_x^\alpha,\\
C_x^1 =& \sum_{\alpha=1}^{3}\Big(\beta\Big((u_y^\alpha)^2 + (u_z^\alpha)^2 - (u_x^\alpha)^2\Big) - (u_t^\alpha)^2\Big) + 2\beta(u_z^2 u_y^3 - u_y^2 u_z^3)\\
&+ 2u_y^2 u_z^3 - (u_x^1)^2 + (u_y^2)^2 + (u_z^3)^2,\\
C_x^2 =& -2\beta\Big(\sum_{\alpha=1}^{3} u_x^\alpha u_y^\alpha + u_x^3 u_z^2 - u_x^2 u_z^3\Big) - 2u_x^2(u_x^1 + u_y^2 + u_z^3),\\
C_x^3 =& -2\beta\Big(\sum_{\alpha=1}^{3} u_x^\alpha u_3^\alpha + u_x^2 u_y^3 - u_x^3 u_y^2\Big) - 2u_x^3(u_x^1 + u_z^3 + u_y^2).
\end{aligned} \tag{5.6}
$$

Ibragimov's method:

$$
\begin{aligned}
\tilde{C}_1^0 =& -2\sum_{\alpha=1}^{3} u_t^\alpha u_x^\alpha,\\
\tilde{C}_1^1 =& \sum_{\alpha=1}^{3}\Big(\beta\big((u_x^\alpha)^2 - (u_y^\alpha)^2 - (u_z^\alpha)^2\big) + (u_t^\alpha)^2\Big)\\
&+ (u_x^1)^2 - (u_y^2)^2 - (u_z^3)^2 - u_y^2 u_z^3 - u_y^3 u_z^2,\\
\tilde{C}_1^2 =& 2\beta\sum_{\alpha=1}^{3}\big(u_x^\alpha u_y^\alpha\big) + 2u_x^2 u_y^2 + 2u_x^1 u_x^2 + u_x^2 u_z^3 + u_x^3 u_z^2,\\
\tilde{C}_1^3 =& 2\beta\sum_{\alpha=1}^{3}\big(u_x^\alpha u_z^\alpha\big) + 2u_x^3 u_z^3 + 2u_x^1 u_x^3 + u_y^2 u_x^3 + u_y^3 u_x^2.
\end{aligned} \tag{5.7}
$$

In this case the vector $\hat{C}$ has the following components:

$$C_\infty = -\tilde{C}_\infty + \hat{C}_\infty$$

Thus we have

$$
\begin{aligned}
\hat{C}_1^0 =& 0,\\
\hat{C}_1^1 =& (2\beta - 1)(u_z^2 u_y^3 - u_y^2 u_z^3),\\
\hat{C}_1^2 =& (2\beta - 1)(u_x^2 u_z^3 - u_x^3 u_z^2),\\
\hat{C}_1^3 =& (2\beta - 1)(u_x^3 u_y^2 - u_x^2 u_y^3).
\end{aligned} \tag{5.8}
$$

By calculating the divergence of the above one can easily see that

$$D_i(\hat{C}^i) = 0. \tag{5.9}$$

Hence the result of the both methods are equivalent.

Since the results for the translations of y and z are similar to the translation of x, we will accordingly obtain similar results that $C = -\tilde{C} + \hat{C}$.

Conservation laws corresponding to symmetries $X_4 = y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y} + u^2\dfrac{\partial}{\partial u^3} - u^3\dfrac{\partial}{\partial u^2}$

Noether's theorem:

$$\begin{aligned}
C_4^0 =& -2(zu_y^1 - yu_z^1)u_t^1 - 2(-u^3 + zu_y^2 - yu_z^2)u_t^2 - 2(u^2 + zu_y^3 - yu_z^3)u_t^3,\\
C_4^1 =& (zu_y^1 - yu_z^1)\Big(4\beta u_x^1 + 2(1-\beta)(u_x^1 + u_y^2 + u_z^3)\Big) + (-u^3 + zu_y^2 - yu_z^2)\Big(2\beta(u_x^2 + u_y^1)\Big)\\
&+ (u^2 + zu_y^3 - yu_z^3)\Big(2\beta(u_z^1 + u_x^3)\Big),\\
C_4^2 =& \sum_{\alpha=1}^{3}\left[\beta\Big(z\Big[(u_y^\alpha)^2 - (u_x^\alpha)^2 - (u_z^\alpha)^2\Big] - 2yu_y^\alpha u_z^\alpha\Big) + z(u_t^\alpha)^2\right] + z(u_y^2)^2 - z(u_x^1)^2 - z(u_z^3)^2\\
&+ 2\beta\Big(z(u_x^1u_z^3 - u_z^1u_x^3) - y(u_z^1u_x^2 - u_x^1u_z^2) + u^3(u_x^1 - u_y^2 + u_z^3) + u^2(u_z^2 + u_y^3)\Big)\\
&- 2zu_x^1u_z^3 - 2(u^3 + yu_z^2)(u_x^1 + u_y^2 + u_z^3),\\
C_4^3 =& \sum_{\alpha=1}^{3}\left[\beta\Big(y\Big[(u_y^\alpha)^2 + (u_x^\alpha)^2 - (u_z^\alpha)^2\Big] + 2zu_y^\alpha u_z^\alpha\Big) - y(u_t^\alpha)^2\right] + y(u_x^1)^2 + y(u_y^2)^2 - y(u_z^3)^2\\
&+ 2\beta\Big(y(u_y^1u_x^2 - u_x^1u_y^2) + z(u_y^1u_x^3 - u_x^1u_y^3) - u^2(u_x^1 + u_y^2 - u_z^3) - u^3(u_z^2 + u_y^3)\Big)\\
&+ 2yzu_x^1u_y^2 + 2(u^2 + zu_y^3)(u_x^1 + u_y^2 + u_z^3).
\end{aligned} \tag{5.10}$$

Ibragimov's method:

$$\begin{aligned}
\tilde{C}_4^0 =& 2\sum_{\alpha=1}^{3}\Big(zu_t^\alpha u_y^\alpha - yu_t^\alpha u_z^\alpha\Big) + 2u^2u_t^3 - 2u^3u_t^2,\\
\tilde{C}_4^1 =& 2\beta\Big(\sum_{\alpha=1}^{3}\big(yu_x^\alpha u_z^\alpha - zu_x^\alpha u_y^\alpha\big) - u^2u_x^3 + u^3u_x^2\Big) + y\Big(2u_x^1u_z^1 + u_y^1u_z^2 + 2u_z^1u_z^3 + u_y^2u_z^1\Big)\\
&- z\Big(2u_y^1u_y^2 + u_y^1u_z^3 + u_y^3u_z^1 + 2u_x^1u_y^1\Big) - u^2u_z^1 + u^3u_y^1,\\
\tilde{C}_4^2 =& \sum_{\alpha=1}^{3}\left[\beta\Big(2yu_y^\alpha u_z^\alpha + z\Big[(u_x^\alpha)^2 + (u_z^\alpha)^2 - (u_y^\alpha)^2\Big]\Big) - z(u_t^\alpha)^2\right] + \beta\big(2u^3u_y^2 - 2u^2u_y^3\big)\\
&+ z\Big((u_x^1)^2 + (u_z^3)^2 - (u_y^2)^2 + u_x^1u_z^3 + u_z^1u_x^3\Big) + y\Big(u_x^1u_z^2 + 2u_y^2u_z^2 + 2u_z^2u_z^3 + u_x^2u_z^1\Big)\\
&+ u^3u_x^1 + 2u^3u_y^2 + u^3u_z^3 - u^2u_z^2,\\
\tilde{C}_4^3 =& \sum_{\alpha=1}^{3}\left[\beta\Big(y\Big[(u_z^\alpha)^2 - (u_x^\alpha)^2 - (u_y^\alpha)^2\Big] - 2zu_y^\alpha u_z^\alpha\Big) + y(u_t^\alpha)^2\right] - \beta\big(2u^3u_z^2 - 2u^2u_z^3\big)\\
&- z\Big(u_x^3u_y^1 + u_x^1u_y^3 + zu_y^3u_z^3 + 2u_y^2u_y^3\Big) - y\Big(u_x^2u_y^1 + u_x^1u_y^2 - (u_z^3)^2 + (u_y^2)^2 + (u_x^1)^2\Big)
\end{aligned}$$

$$- 2u^2u^3_z - u^2u^1_x + u^3u^3_y - u^2u^2_y. \tag{5.11}$$

Thus we have

$$\begin{aligned}
\hat{C}^0_4 =&0,\\
\hat{C}^1_4 =&(\beta-1)\Big(y(u^1_zu^2_y - u^2_zu^1_y) - z(u^1_yu^3_z - u^3_yu^1_z) - u^3u^1_y + u^2u^1_z\Big),\\
\hat{C}^2_4 =&(\beta-1)\Big(z(u^1_xu^3_z - u^1_zu^3_x) - y(u^1_zu^2_x - u^1_xu^2_z) + u^3u^1_x + u^3u^3_z + u^2u^2_z\Big),\\
\hat{C}^3_4 =&(\beta-1)\Big(y(u^1_xu^2_y - u^1_yu^2_x) - z(u^1_yu^3_x - u^1_xu^3_y) + u^2u^1_x + u^2u^2_y + u^3u^3_y\Big).
\end{aligned} \tag{5.12}$$

Calculating the divergence of the vector $\hat{C}$ with the above components, one can easily see that

$$D_i(\hat{C}^i) = 0. \tag{5.13}$$

The same results can be obtained for X_5 and X_6. Hence the conservation laws derived by the Noether theorem corresponding to the rotation symmetries, are equivalent to the ones obtained by Ibragimov's method.

Conservation laws corresponding to $X_\infty = w^1\frac{\partial}{\partial u^1} + w^2\frac{\partial}{\partial u^2} + w^3\frac{\partial}{\partial u^3}$

Noether's theorem:

$$\begin{aligned}
C^0_\infty =& -\sum_{\alpha=1}^{3}(w^\alpha u^\alpha_t - u^\alpha w^\alpha_t),\\
C^i_\infty =& \sum_{\alpha=1}^{3}\Big(\beta\Big(w^\alpha u^\alpha_i - u^\alpha w^\alpha_i\Big) + w^i u^\alpha_\alpha - u^i w^\alpha_\alpha\Big), \quad i = 1,2,3.
\end{aligned} \tag{5.14}$$

Ibragimov's method:

$$\begin{aligned}
C^0_\infty =& w^1u^1_t + w^2u^2_t + w^3u^3_t - w^1_tu^1 - w^2_tu^2 - w^3_tu^3,\\
C^1_\infty =& -w^1\Big((\beta+1)u^1_x + u^2_y + u^3_z\Big) - w^2\Big(\beta u^2_x + u^1_y\Big) - w^3\Big(\beta u^3_x + u^1_z\Big)\\
&+ w^1_x\Big((\beta+1)u^1\Big) + w^2_x\Big(\beta u^2\Big) + w^3_x\Big(\beta u^3\Big),\\
C^2_\infty =& -w^1\Big(\beta u^1_y\Big) - w^2\Big((\beta+1)u^2_y + u^3_z\Big) - w^3\Big(\beta u^3_y + u^2_z\Big)\\
&+ w^1_x\Big(u^2\Big) + w^1_y\Big(\beta u^1\Big) + w^2_x\Big(u^1\Big) + w^2_y\Big((\beta+1)u^2\Big)\\
&+ w^3_y\Big(\beta u^3\Big),\\
C^3_\infty =& -w^1\Big(\beta u^1_z\Big) - w^2\Big(\beta u^2_z\Big) - w^3\Big((\beta+1)u^3_z\Big)
\end{aligned}$$

$$+ w^1_x\Big(u^3\Big) + w^1_z\Big(\beta u^1\Big) + w^2_y\Big(u^3\Big) + w^2_z\Big(\beta u^2\Big) + w^3_x\Big(u^1\Big)$$
$$+ w^3_y\Big(u^2\Big) + w^3_z\Big((\beta+1)u^3\Big). \tag{5.15}$$

In this case we get $C_\infty = -\tilde{C}_\infty + \hat{C}_\infty$, where

$$\begin{aligned}
\hat{C}^0_\infty &= 0,\\
\hat{C}^1_\infty &= -2(u^1w^2_y + w^2u^1_y + u^1w^3_z + w^3u^1_z),\\
\hat{C}^2_\infty &= 2(u^1w^2_x + w^2u^1_x - u^2w^3_z - w^3u^2_z),\\
\hat{C}^1_\infty &= 2(u^1w^3_x + w^3u^1_x + u^2w^3_y + w^3u^2_y).
\end{aligned} \tag{5.16}$$

Accordingly we show that

$$D_i(\hat{C}^i) = 0. \tag{5.17}$$

Thus, the conservation law derived by the Noether theorem corresponding to X_∞, is equivalent to the conservation law obtained by the Ibragimov method.

6 Summary

In this work the application of Noether's theorem and Ibragimov's theorem for constructing conservation laws for the Lamé equation have been investigated. It has been shown that in all cases when it is possible to construct a conserved vector using Noether's theorem, it is equivalent to the conserved vector obtained by the Ibragimov method. Moreover one can see that the Ibragimov method provides more conservation laws than the Noether theorem, which may suggests the better efficiency of the Ibragimov method.
Let us denote $\mathbf{x} = (x, y, z)$, and $x^1 = x,\ x^2 = y$ and $x^3 = z$. In the formulas below there is no summation in repeated indices. The following conserved vectors have been constructed for the system (1.1.1):

Time translation: $X = \dfrac{\partial}{\partial t}$

$$\begin{aligned}
\tilde{C}^0_0 = &-\sum_{\alpha=1}^{3}\Big[\beta\big((u^\alpha_x)^2 + (u^\alpha_y)^2 + (u^\alpha_z)^2\big) + (u^\alpha_t)^2 + (u^\alpha_\alpha)^2\Big]\\
&- 2u^1_zu^3_x - 2u^1_yu^2_x - 2u^2_zu^3_y\,,
\end{aligned}$$

$$\tilde{C}^j_0 = 2\sum_{\alpha=1}^{3}\Big[\beta u^\alpha_t u^\alpha_j + u^\alpha_t u^j_\alpha\Big]. \tag{6.1}$$

where $i, j, k = 1, 2, 3$ and $i \neq j \neq k$.

Spacial translation symmetries: $X_i = \dfrac{\partial}{\partial x^i}$

$$\tilde{C}_i^0 = -\,2\sum_{\alpha=1}^{k} u_t^\alpha u_i^\alpha\,,$$

$$\begin{aligned}\tilde{C}_i^i = &\sum_{\alpha=1}^{k}\Big(\beta\big((u_i^\alpha)^2-(u_j^\alpha)^2-(u_k^\alpha)^2\big)+(u_t^\alpha)^2\Big)\\ &+(u_i^i)^2-(u_j^j)^2-(u_k^k)^2-u_j^j u_k^k-u_j^k u_k^j\,,\end{aligned}$$

$$\tilde{C}_i^j = 2\beta\sum_{\alpha=1}^{k}\left(u_i^\alpha u_j^\alpha\right)+2u_i^j u_j^j+2u_i^i u_i^j+u_i^j u_k^k+u_i^k u_k^j\,, \tag{6.2}$$

where $i,j,k=1,2,3$ and $i\neq j\neq k$.

Rotations: $X_s = x^j\dfrac{\partial}{\partial x^k}-x^k\dfrac{\partial}{\partial x^j}+u^j\dfrac{\partial}{\partial u^k}-u^k\dfrac{\partial}{\partial u^j}$

$$\tilde{C}_s^0 = r\left[2\sum_{\alpha=1}^{3}\Big(x^k u_t^\alpha u_j^\alpha - x^j u_t^\alpha u_k^\alpha\Big)+2u^j u_t^k-2u^k u_t^j\right],$$

$$\begin{aligned}\tilde{C}_s^i = r\Bigg[&2\beta\Bigg(\sum_{\alpha=1}^{3}\left(x^j u_i^\alpha u_k^\alpha - x^k u_i^\alpha u_j^\alpha\right)-u^j u_i^k+u^k u_i^j\Bigg)+x^j\Big(2u_i^i u_k^i+2u_k^i u_k^k+u_j^i u_k^j+u_j^j u_k^i\Big)\\ &-x^k\Big(2u_i^i u_j^i+2u_j^i u_j^j+u_j^i u_k^k+u_j^k u_k^i\Big)-u^j u_k^i+u^k u_j^i\Bigg],\end{aligned}$$

$$\begin{aligned}\tilde{C}_s^j = r\Bigg[&\sum_{\alpha=1}^{3}\Big(\beta\big(2x^j u_j^\alpha u_k^\alpha - x^k(u_j^\alpha)^2+x^k(u_i^\alpha)^2+x^k(u_k^\alpha)^2\big)-x^k(u_t^\alpha)^2\Big)+\beta\big(2u^k u_j^j-2u^j u_j^k\big)\\ &+x^k\Big((u_i^i)^2+(u_k^k)^2-(u_j^j)^2+u_i^i u_k^k+u_k^i u_i^k\Big)+x^j\Big(u_i^i u_k^j+2u_j^j u_k^j+2u_k^j u_k^k+u_i^j u_k^i\Big)\\ &+u^k u_i^i+2u^k u_j^j+u^k u_k^k-u^j u_k^j\Bigg],\end{aligned} \tag{6.3}$$

where

$$\begin{aligned}&s=i+3, \qquad i,j,k=1,2,3, \qquad i\neq j\neq k,\\ &r=\begin{cases}(-1)^{i+j+1} & \text{if}\quad j>i,\\ (-1)^{i+j} & \text{if}\quad j<i.\end{cases}\end{aligned} \tag{6.4}$$

Scaling: $X_7 = t\dfrac{\partial}{\partial t}+x\dfrac{\partial}{\partial x}+y\dfrac{\partial}{\partial y}+z\dfrac{\partial}{\partial z}$

This conserved vector has been obtained by the Ibragimov method. In this case, Noether's theorem gives no conservation law.

$$\begin{aligned}
\tilde{C}_7^0 =& \sum_{\alpha=1}^{3}\left[\beta t\Big((u_x^\alpha)^2+(u_y^\alpha)^2+(u_z^\alpha)^2\Big)-(u_t^\alpha)^2-2u_t^\alpha\Big(xu_x^\alpha+yu_y^\alpha+zu_z^\alpha+2u^\alpha\Big)\right]\\
&-t\Big(2u_x^1u_y^2+2u_x^1u_z^3+2u_y^2u_z^3+(u_x^1)^2+(u_y^2)^2+(u_z^3)^2\Big),
\end{aligned}$$

$$\begin{aligned}
\tilde{C}_7^1 =& \sum_{\alpha=1}^{3}\left[\beta\Big(2u_x^\alpha\Big[tu_t^\alpha+yu_y^\alpha+zu_z^\alpha+u^\alpha\Big]+x\Big[(u_x^\alpha)^2-(u_z^\alpha)^2-(u_y^\alpha)^2\Big]\Big)+x(u_t^\alpha)^2\right]\\
&+x\Big((u_x^1)^2-(u_y^2)^2-(u_z^3)^2-2u_y^2u_z^3\Big)+y\Big(u_y^1u_z^3+2u_x^1u_y^1+2u_y^1u_y^2+u_y^3u_z^1\Big)\\
&+z\Big(2u_x^1u_z^1+2u_z^1u_z^3+u_y^1u_z^2+u_y^2u_z^1\Big)\ +2t\Big(u_t^1u_x^1+u_t^1u_y^2+u_t^1u_z^3\Big)\\
&+u^1u_z^3+2u^1u_x^1+\frac{3}{2}u^1u_y^2+\frac{1}{2}u^2u_y^1,
\end{aligned}$$

$$\begin{aligned}
\tilde{C}_7^2 =& \sum_{\alpha=1}^{3}\left[\beta\Big(2u_y^\alpha\Big[tu_t^\alpha+xu_x^\alpha+zu_z^\alpha+2u^\alpha\Big]+y\Big[(u_y^\alpha)^2-(u_x^\alpha)^2-(u_z^\alpha)^2\Big]\Big)+y(u_t^\alpha)^2\right]\\
&+y\Big((u_y^1)^2-(u_x^2)^2-(u_z^3)^2-u_x^1u_z^3-u_x^3u_z^1\Big)+2x\Big(u_x^2u_y^2+u_x^2u_z^3+u_x^1u_x^2\Big)\\
&+z\Big(2u_z^2u_z^3+2u_y^2u_z^2+u_x^2u_z^1+u_x^1u_z^2\Big)+2t\Big(u_t^2u_x^1+u_t^2u_z^3+u_t^2u_y^2\Big)\\
&+2u^2u_y^2+2u^2u_z^3+\frac{3}{2}u^2u_x^1+\frac{1}{2}u^1u_x^2,
\end{aligned}$$

$$\begin{aligned}
\tilde{C}_7^3 =& \sum_{\alpha=1}^{3}\left[\beta\Big(2u_z^\alpha\Big[tu_t^\alpha+2xu_x^\alpha+2yu_y^\alpha+2u^\alpha\Big]+z\Big[(u_z^\alpha)^2-(u_x^\alpha)^2-(u_y^\alpha)^2\Big]\Big)+z(u_t^\alpha)^2\right]\\
&+z\Big((u_z^1)^2-(u_x^2)^2-(u_y^3)^2-u_x^1u_y^2-u_x^2u_y^1\Big)+2x\Big(u_x^1u_z^3+u_x^3u_y^2+u_x^3u_z^3\Big)\\
&+2t\Big(u_t^3u_x^1+u_t^3u_y^2+u_t^3u_z^3\Big)+y\Big(u_x^1u_y^3+u_y^1u_x^3+2u_y^2u_y^3+2u_y^3u_z^3\Big)\\
&+2u^3u_y^2+2u^3u_x^1+2u^3u_z^3+u^1u_x^3. \qquad (6.5)
\end{aligned}$$

Addition of solutions: $X_\infty = w^1\dfrac{\partial}{\partial u^1}+w^2\dfrac{\partial}{\partial u^2}+w^3\dfrac{\partial}{\partial u^3}$

$$\begin{aligned}
C_\infty^0 =& -2\sum_{\alpha=1}^{3}(w^\alpha u_t^\alpha-u^\alpha w_t^\alpha),\\
C_\infty^i =& 2\sum_{\alpha=1}^{3}\left(\beta\Big(w^\alpha u_i^\alpha-u^\alpha w_i^\alpha\Big)+w^iu_\alpha^\alpha-u^iw_\alpha^\alpha\right), \qquad (6.6)
\end{aligned}$$

where w is any arbitrary solution of the system (1.1.1) except u and $i = 1, 2, 3$.

Remark. As it mentioned in the section 2.10 and the section 4.10, there is no conservation law corresponding to X_8.

References

[1] Wilhelm Günther. Über einige randintegrale der elastomechanik. *Abh. Brauchschw. Wiss. Ges*, 14:53–72, 1962.

[2] Nail H. Ibragimov. *CRC handbook of Lie group analysis of differential equations. Vol. 2, Applications in engineering and physical sciences.* CRC press, Boca Raton, 1995.

[3] Nail H. Ibragimov. A new conservation theorem. *Journal of Mathematical Analysis and Applications*, 333(1):311–328, 2007.

[4] Nail H. Ibragimov. Nonlinear self-adjointness in constructing conservation laws. *Archives of ALGA*, 7/8:1–86, 2010.

[5] Nail H. Ibragimov. Nonlinear self-adjointness and conservation laws. *Journal of Physics A: Mathematical and Theoretical*, 44(43):432002, 2011.

[6] Nail H. Ibragimov and Raisa Khamitova. Conservation Laws in Thomas's Model of Ion Exchange in a Heterogeneous Solution. *Discontinuity, Nonlinearity and Complexity*, 2(2):147–158, 2013.

[7] Raisa Khamitova. *Symmetries and conservation laws.* PhD thesis, LNU, 2009.

[8] James K. Knowles and Eli Sternberg. On a class of conservation laws in linearized and finite elastostatics. *Archive for Rational Mechanics and Analysis*, 44(3):187–211, 1972.

[9] Emmy Noether. Invariante variationsprobleme. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, mathematisch-physikalische Klasse*, Heft 2:235–257, 1918.

[10] Peter J. Olver. Conservation laws in elasticity. *Archive for rational mechanics and analysis*, 85(2):111–129, 1984.

[11] Victor J. Stenger. *Timeless reality: symmetry, simplicity, and multiple universes.* Prometheus Books, 2000.