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Optimal Solution of Fuzzy Relation Equations

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Abstract

Fuzzy relation equations are becoming extremely important in order to investigate the optimal solution of the inverse problem even though there is a restrictive condition for the availability of the solution of such inverse problems. We discussed the methods for finding the optimal (maximum and minimum) solution of inverse problem of fuzzy relation equation of the form $R \circ Q = T$ where for both cases R and Q are kept unknown interchangeably using different operators (e.g. alpha, sigma etc.). The aim of this study is to make an in-depth finding of best project among the host of projects, depending upon different factors (e.g. capital cost, risk management etc.) in the field of civil engineering. On the way to accomplish this aim, two linguistic variables are introduced to deal with the uncertainty factor which appears in civil engineering problems. Alpha-composition is used to compute the solution of fuzzy relation equation. Then the evaluation of the projects is orchestrated by defuzzifying the obtained results. The importance of adhering to such synopsis, in the field of civil engineering, is demonstrated by an example.

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Chapter 1

Basic Concepts

1.1 Introduction

Mostly our formal computing and reasoning are precise and crisp in character. Crisp means either a statement is true or false. Classically, set theory based on the membership of elements and the membership elements in a set is characterized in binary terms means an element whether belong to the set or not, and in optimization whether a solution is feasible or not, according to the bivalent condition. Fuzzy set theory [1] gives sequential assessment of elements of the membership of a set [0,1]. The concept of fuzzy set theory was first proposed by Lofti A.Zadeh in 1965 at Berkeley University California. Fuzzy set plays a very deep role in modeling of fuzzy control, medical diagnosis and computational intelligence etc.

1.2 Fuzzy Set

Fuzzy set was first defined by Lofti A.Zadeh in 1965. He defined a fuzzy set as a collection of objects with membership values in the interval [0,1]. These membership values represent the grades of membership with the properties and distinct features of the collection.

Fuzzy sets are mathematically defined as follows:

"A subset A of universe X with the membership function $\mu(x)$ which may take any value in the interval [0,1] is called fuzzy set".

$$A = \int_{x \epsilon X} (\mu_A(x)/x) \tag{1.1}$$

where $\mu(x) : X \to [0, 1]$.

1.2.1 Example of a Fuzzy Set

Consider a finite non fuzzy set X which is the set of ages of different peoples. $X = \{10, 15, 20, 25, 30, 35, 40, 45, 50\}$.

A fuzzy subset A of X is also defined as A = "Young Men".

When we assign the membership values to the elements of the set A then it becomes:
$A = \{0/10, 0.5/15, 1/20, 1/25, 1/30, 1/35, 0.5/40, 0/45, 0/50\}$.

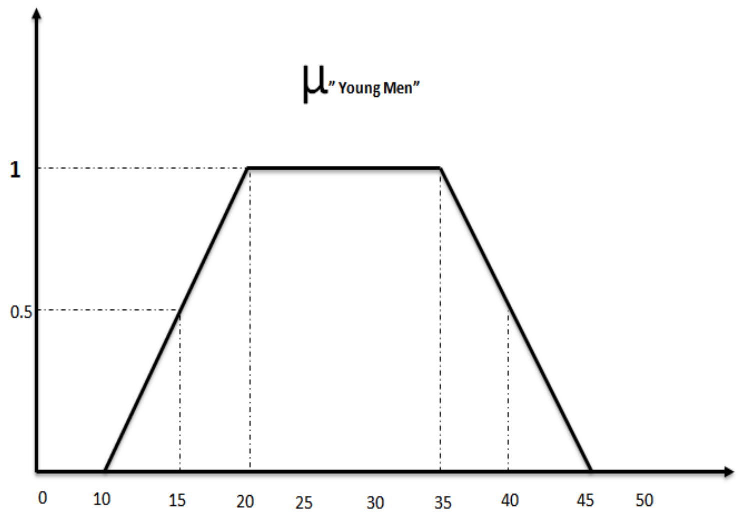

Figure 1.1: Trapezoidal Fuzzy Set "A"

1.3 Fuzzy Relation

Relationship between the objects plays an important role in dynamic system applications. A crisp relation shows the connection of two or more sets. In other sense, fuzzy relation allows "grading" to the connections.

1.3.1 Definition

Let X and Y be two nonempty sets. A fuzzy relation R between X and Y is a fuzzy subset of $X \times Y$ where μ_R: $X \times Y \to [0, 1]$.

If $X = Y$ then R is called a binary fuzzy relation.

A fuzzy relation can be represented in the following manner:

Let $A = a_i$, $i = 1, 2, ..., n$ and

$B = b_j$, $j = 1, 2, ..., m$.
A fuzzy relation $R \subseteq A \times B$ can be represented as

$$R = \{(a_i, b_j), R(a_i, b_j)\}. \tag{1.2}$$

1.3.2 Example of a Fuzzy Relation

Consider A and B are nonempty sets containing the "Heights" of the different people in centimeter (cm).

$A = \{153, 178, 190\}$
$B = \{150, 172, 181\}$

A fuzzy relation $A \times B$ is represented in the following form:

a_i R b_j = "a_i is considerably taller than b_j"

where R is a fuzzy relation.

$$\mu_{\mathbf{Height}}(\mathbf{a_i}, \mathbf{b_j}) = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} 0.1 & 0 & 0 \\ 0.7 & 0.15 & 0 \\ 1 & 0.45 & 0.2 \end{bmatrix} \end{array} \quad \textit{where } i, j = 1, 2, 3.$$

Graphically

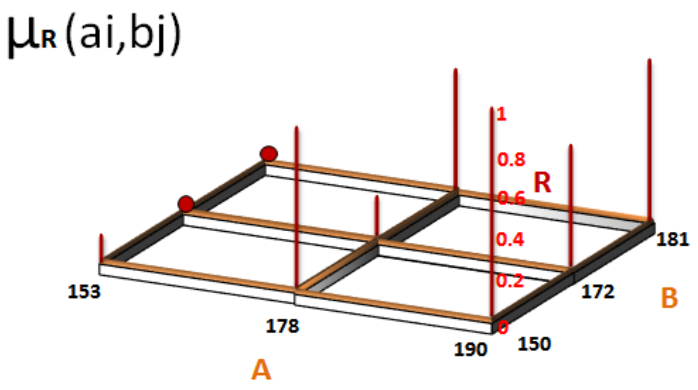

Figure 1.2: 3 D Plot Of Fuzzy Relation "R"

1.3.3 Fuzzy Inverse Relation R^{-1}

A fuzzy relation $R^{-1} \subseteq Y \times X$ is called the inverse of the fuzzy relation $R \subseteq X \times Y$ which is defined as

$$R^{-1}(y,x) = R(x,y) \qquad \forall\ x\ \in\ X\ and\ \forall\ y \in Y \tag{1.3}$$

for all pairs $(y,x)\ \epsilon\ Y \times X$ and

$$\mu_{R^{-1}}(y,x) = \mu_R(x,y) \tag{1.4}$$

where $\mu_{R^{-1}}$ is the membership function of R^{-1}.
R^{-1} is also defined as $R^{-1} = R^t$ and $(R^{-1})^{-1} = R$.

1.3.4 Complement of a Fuzzy Relation $\neg R$

A complement of a fuzzy relation $R \subseteq X \times Y$ is denoted as $\neg R$ and its membership function is expressed as given below:

$$\mu_{\neg R}(x,y) = 1 - \mu_R(x,y) \qquad \forall\ x\ \in\ X\ and\ \forall\ y \in Y \tag{1.5}$$

1.4 Fuzzy Relation Composition Rules

There are special operations for fuzzy relations that are not defined on fuzzy sets. Combination of these operations is known as composition.

1.4.1 Max-Min Composition

Consider two fuzzy relations R_1 and R_2
$R_1(x,y) \subseteq X \times Y$ and
$R_2(y,z) \subseteq Y \times Z$
The max-min composition of R_1 *and* R_2 is given as follows:

$$R_1 \circ R_2 = \{(x,z), max_{y\epsilon Y}(min(\mu_{R_1}(x,y), \mu_{R_2}(y,z)))\} \tag{1.6}$$

or

$$\mu_{R_1 \circ R_2}(x,z) = \bigvee_{y \epsilon Y} \{\mu_{R_1}(x,y) \bigwedge \mu_{R_2}(y,z)\} \tag{1.7}$$

where x ϵ X, y ϵ Y and z ϵ Z.
μ_{R_1} and μ_{R_2} are the membership function of fuzzy relations on fuzzy sets.
"$\bigvee$" denotes the " max" function and " $\bigwedge$" denotes the " min" function.

1.4.2 Example of Max-Min Composition

Consider $X = \{x_1, x_2, x_3\}, \quad Y = \{y_1, y_2, y_3\}$ *and* $Z = \{z_1, z_2\}$.

Let $R_1 \subseteq X \times Y$ and $R_2 \subseteq Y \times Z$ be two fuzzy relations respectively

$$\mathbf{R_1} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.3 & 0.5 & 0 \\ 0.2 & 0.6 & 0.7 \\ 0.4 & 1 & 0.9 \end{bmatrix} \end{array}$$

and

$$\mathbf{R_2} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 \\ 1 & 0.8 \\ 0.5 & 0 \end{bmatrix} \end{array}$$

At first we compute $R_1 \circ R_2$ by using max-min composition.

$$\begin{aligned}
\mu_{R_1 \circ R_2}(x_1, z_1) &= max\{min(0.3, 0.6), min(0.5, 1), min(0, 0.5)\} = 0.5 \\
\mu_{R_1 \circ R_2}(x_1, z_2) &= max\{min(0.3, 0.4), min(0.5, 0.8), min(0, 0)\} = 0.5 \\
\mu_{R_1 \circ R_2}(x_2, z_1) &= max\{min(0.2, 0.6), min(0.6, 1), min(0.7, 0.5)\} = 0.6 \\
\mu_{R_1 \circ R_2}(x_2, z_2) &= max\{min(0.2, 0.4), min(0.6, 0.8), min(0.7, 0.0)\} = 0.6 \\
\mu_{R_1 \circ R_2}(x_3, z_1) &= max\{min(0.4, 0.6), min(1, 1), min(0.9, 0.5)\} = 1 \\
\mu_{R_1 \circ R_2}(x_3, z_2) &= max\{min(0.4, 0.4), min(1, 0.8), min(0.9, 0.0)\} = 0.8
\end{aligned}$$

By using the max-min composition we have the following result.

$$\mathbf{S} = \mathbf{R_1} \circ \mathbf{R_2} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 0.6 & 0.6 \\ 1 & 0.8 \end{bmatrix} \end{array}$$

1.4.3 Max-Product Composition

The max-product composition of two fuzzy relations R_1 and R_2 is defined as follows:

$$R_1 \circ R_2 = \{(x, z), max_{y \epsilon Y}\{\mu_{R_1}(x, y) \bullet \mu_{R_2}(y, z))\} \tag{1.8}$$

$$\mu_{R_1 \circ R_2}(x, z) = \bigvee_{y \in Y} \{\mu_{R_1}(x, y) \bullet \mu_{R_2}(y, z)\} \tag{1.9}$$

1.4.4 Example of Max-Product Composition

Consider $X = \{x_1, x_2, x_3\}, \quad Y = \{y_1, y_2, y_3\}$ *and* $Z = \{z_1, z_2\}$.

Let $R_1 \subseteq X \times Y$ and $R_2 \subseteq Y \times Z$ be the fuzzy relations respectively

$$\mathbf{R_1} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.3 & 0.5 & 0 \\ 0.2 & 0.6 & 0.7 \\ 0.4 & 1 & 0.9 \end{bmatrix} \end{array}$$

and

$$\mathbf{R_2} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 \\ 1 & 0.8 \\ 0.5 & 0 \end{bmatrix} \end{array}$$

Now we use the max-product rule in order to compose the fuzzy relation R_1 and R_2

$\mu_{R_1 \circ R_2}(x_1, z_1) = max\{(0.3 \bullet 0.6), (0.5 \bullet 1), (0.0 \bullet 0.5)\} = 0.5$
$\mu_{R_1 \circ R_2}(x_1, z_2) = max\{(0.3 \bullet 0.4), (0.5 \bullet 0.8), (0.0 \bullet 0.0)\} = 0.4$
$\mu_{R_1 \circ R_2}(x_2, z_1) = max\{(0.2 \bullet 0.6), (0.6 \bullet 1), (0.7 \bullet 0.5)\} = 0.6$
$\mu_{R_1 \circ R_2}(x_2, z_2) = max\{(0.2 \bullet 0.4), (0.6 \bullet 0.8), (0.7 \bullet 0.0)\} = 0.8$
$\mu_{R_1 \circ R_2}(x_3, z_1) = max\{(0.4 \bullet 0.6), (1 \bullet 1), (0.9 \bullet 0.5)\} = 1$
$\mu_{R_1 \circ R_2}(x_3, z_2) = max\{(0.4 \bullet 0.4), (1 \bullet 0.8), (0.9 \bullet 0.0)\} = 0.8$

$$\mathbf{S} = \mathbf{R_1} \circ \mathbf{R_2} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 \\ 0.6 & 0.8 \\ 1 & 0.8 \end{bmatrix} \end{array}$$

Chapter 2

Fuzzy Relation Equations and Solution Operators

2.1 Introduction

Since 1970's fuzzy set theory has played an important role in science and scientific community by producing different models in science and technology and by developing models in such fields as: fuzzy control, fuzzy logic, data mining, image analysis and medicine.

Using fuzzy relation equations, we can solve a lot of optimization problems in different ways. One of the scientists, Elie Sanchez, works on it and defines operators for the solution of fuzzy relation equations in all theoretical and practical aspects.

Fuzzy Relation Equations convert a multi-variable problem in the form of a single input of fuzzy relation equation.

2.2 Fuzzy Relation Equation

Fuzzy relation equations (FREs) have the form

$$T = R \circ Q \tag{2.1}$$

where R,Q and T are the fuzzy relations and "$\circ$" is the max-min composition.

2.2.1 Inverse Fuzzy Relation Equation

Consider a FRE of the form $T = R \circ Q$, the inverse fuzzy relationship is defined as:

$$T^{-1} = (R \circ Q)^{-1} \tag{2.2}$$

As

$$(R \circ Q)^{-1} = Q^{-1} \circ R^{-1} \tag{2.3}$$

then inverse fuzzy relationship is

$$T^{-1} = Q^{-1} \circ R^{-1} \tag{2.4}$$

2.2.2 Example of Fuzzy Relation Equation

Suppose $X = \{x_1, x_2\},\ \ Y = \{y_1, y_2, y_3\}\ and\ Z = \{z_1, z_2\}$.
Consider two fuzzy relations $R \subseteq X \times Y$ and $T \subseteq X \times Z$ with membership degrees $\mu_R(x, y)$and $\mu_T(x, z)$ respectively. We have to compute a fuzzy relation "Q" by using the FRE of the form

$$T = R \circ Q$$

taking R^{-1} on both sides we have

$$R^{-1} \circ T = Q$$

then

$$Q = R^{-1} \circ T$$

where $R^{-1} \subseteq Y \times X$ is the inverse relation of R $\subseteq$ of $X \times Y$ with

$$\mu_{R^{-1}}(y, x) = \mu_R(x, y)$$

The given data is

$$\mathbf{T} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0.2 & 1 \\ 0 & 0.5 \end{bmatrix} \end{array}$$

and

$$\mathbf{R} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0 & 0.9 & 0.1 \\ 0.5 & 1 & 0.3 \end{bmatrix} \end{array}$$

$$\mathbf{R^{-1}} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 0 & 0.5 \\ 0.9 & 1 \\ 0.1 & 0.3 \end{bmatrix} \end{array}$$

where $R^{-1} \in Y \times X$.
Now we compute "Q" by using the operation "$\circ$".
As $\mu_Q(y,z) = \bigvee_{x \in X}\{\mu_{R^{-1}}(y,x) \bigwedge \mu_T(x,z)\}$, so

$$\begin{aligned}
\mu_Q(y_1,z_1) &= \bigvee\{0 \bigwedge 0.2, 0.5 \bigwedge 0\} = 0 \\
\mu_Q(y_1,z_2) &= \bigvee\{0 \bigwedge 1, 0.5 \bigwedge 0.5\} = 0.5 \\
\mu_Q(y_2,z_1) &= \bigvee\{0.9 \bigwedge 0.2, 1 \bigwedge 0\} = 0.2 \\
\mu_Q(y_2,z_2) &= \bigvee\{0.9 \bigwedge 1, 1 \bigwedge 0.5\} = 0.9 \\
\mu_Q(y_3,z_1) &= \bigvee\{0.1 \bigwedge 0.2, 0.3 \bigwedge 0\} = 0.1 \\
\mu_Q(y_3,z_2) &= \bigvee\{0.1 \bigwedge 1, 0.3 \bigwedge 0.5\} = 0.3
\end{aligned}$$

$$\mathbf{Q} = \mathbf{R^{-1}} \circ \mathbf{T} = \begin{matrix} \\ y_1 \\ y_2 \\ y_3 \end{matrix} \begin{matrix} \begin{matrix} z_1 & z_2 \end{matrix} \\ \begin{bmatrix} 0 & 0.5 \\ 0.2 & 0.9 \\ 0.1 & 0.3 \end{bmatrix} \end{matrix}$$

where "$Q \in Y \times Z$".

2.2.3 Definition(Lattice)

A *lattice* is a partially ordered set (poset) L in which any two elements "x" and "y" have a greatest lower bound (inf) denoted by $x \bigwedge y = min(x,y)$ and a least upper bound (sup) denoted by $x \bigvee y = max(x,y)$.

2.2.4 Definition(Brouwerian lattice)

A *brouwerian lattice* is a lattice L [2] in which for any given elements "a" and "b", the set of all $x \in L$ such that $a \bigwedge x \leq b$ contains a greatest element, denoted "$a\ \alpha\ b$", the relative pseudo complement of a in b.

2.3 α - Operator

For any given a and b in lattice L $\in$ [0,1], α - operator is defined as

$$a\ \alpha\ b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases} \tag{2.5}$$

It is also called Sanchez operator.

2.3.1 Example of α - Operator

For different values of a and b , α - operator will respond as:

$$0.8\alpha 0.5 = 0.5, \qquad 0.5\alpha 0.6 = 1$$
$$0.3\alpha 0.3 = 1, \qquad 0.6\alpha 0.3 = 0.3$$
.

2.3.2 Properties of α - Operator

1. If b = 0 then "a α b" will be given as:

$$a\alpha 0 = \begin{cases} 1 & \text{if a = b} \\ 0 & \text{if } a > b \end{cases} \tag{2.6}$$

2. If a = 0 then "a α b" will be given as:

$$0 \ \alpha \ \text{b} = 1$$

3. If b = 1 then "a α b" will be given as:

$$\text{a} \ \alpha \ 1 = 1$$

4. If a = 1 then "a α b" will be given as:

$$1 \ \alpha \ \text{b} = \text{b}$$

5. α - operator is not commutative.

$$\text{a} \ \alpha \ \text{b} \neq \text{b} \ \alpha \ \text{a} \qquad \text{(Not Commutative)}$$

e.g. if a = 0.5 and b = 0.7, then

$$0.5 \ \alpha \ 0.7 \neq 0.7 \ \alpha \ 0.5$$
$$1 \neq 0.5$$

6. α - operator is not associative.

$$\text{a} \ \alpha \ (\text{b} \ \alpha \ \text{c}) \neq (\text{a} \ \alpha \ \text{b}) \ \alpha \ \text{c} \qquad \text{(Not Associative)}$$

e.g. if a = 0.5, b = 0.7 and c = 0.6, then

$$0.5 \ \alpha \ (0.7 \ \alpha \ 0.6) \neq (0.5 \ \alpha \ 0.7) \ \alpha \ 0.6$$
$$0.5 \ \alpha \ 0.6 \neq 1 \ \alpha \ 0.6$$
$$1 \neq 0.5$$

Let us point out some other properties of α - operator as follows:

$$a \wedge (a@b) = a \wedge b \leq b, \tag{2.7}$$

$$a\alpha b \geq b, \tag{2.8}$$

$$a\alpha(b \vee c) \geq a@b, \tag{2.9}$$

$$a@(a \wedge b) \geq b \tag{2.10}$$

$$a\alpha(b \vee c) \geq a@c \tag{2.11}$$

2.4 γ - Operator

For any given a and b in lattice L $\in$ [0,1], γ - operator is defined as

$$a \;\gamma\; b = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if a} \neq \text{b} \end{cases} \tag{2.12}$$

γ - operator is also known as equality operator.

2.4.1 Example of γ - Operator

For different values of a and b, γ - operator will respond as:

$$\begin{array}{ll} 0.8\alpha 0.5 = 0, & 0.2\alpha 0.2 = 1 \\ 0.5\alpha 0.9 = 0, & 0.7\alpha 0.7 = 1 \end{array}$$

.

2.4.2 Properties of γ - Operator

1. γ - operator holds the commutative property.

$$\text{a } \gamma \text{ b} = \text{b } \gamma \text{ a} \qquad \text{(Commutative)}$$

e.g. if a = 0.5 and b = 0.7, then

$$\begin{array}{c} 0.5 \;\gamma\; 0.7 = 0.7 \;\gamma\; 0.5 \\ 0 = 0 \end{array}$$

2. γ - operator does not hold the associative property.

$$\text{a } \gamma \text{ (b } \gamma \text{ c)} \neq \text{(a } \gamma \text{ b) } \gamma \text{ c} \qquad \text{(Not Associative)}$$

e.g. if a = 0.5, b = 0.5 and c = 1, then

$$\begin{array}{c} 0.5 \;\gamma\; (0.5 \;\gamma\; 1) \neq (0.5 \;\gamma\; 0.5) \;\gamma\; 1 \\ 0.5 \;\gamma\; 0 \neq 1 \;\gamma\; 1 \\ 0 \neq 1 \end{array}$$

2.5 σ - Operator

For any given a and b in lattice L $\in$ [0,1], σ - operator is defined as [3]

$$a\sigma b = \begin{cases} 0 & \text{if a < b} \\ b & \text{if a} \geq \text{b} \end{cases} \tag{2.13}$$

2.5.1 Example of σ - Operator

For different values of a and b, σ - operator will respond as:

$$\begin{array}{cc} 0.2\alpha 0.5 = 0, & 0.3\alpha 0.3 = 0.3 \\ 0.7\alpha 1 = 0, & 1\alpha 0.6 = 0.6 \end{array}$$

.

2.5.2 Properties of σ - Operator

1. σ - operator does not hold the commutative property.

   $$\text{a } \sigma \text{ b} \neq \text{b } \sigma \text{ a} \qquad \text{(Not Commutative)}$$

   e.g. if a = 0.2 and b = 0.5 , then

   $$\begin{array}{c} 0.2 \ \sigma \ 0.5 \neq 0.5 \ \sigma \ 0.2 \\ 0 \neq 0.2 \end{array}$$

2. σ - operator does not hold the associative property.

   $$\text{a } \sigma \text{ (b } \sigma \text{ c)} \neq \text{(a } \sigma \text{ b) } \sigma \text{ c} \qquad \text{(Not Associative)}$$

   e.g. if a = 0.7, b = 0.9 and c = 0.1, then

   $$\begin{array}{c} 0.7 \ \sigma \ (0.9 \ \sigma \ 0.1) \neq (0.7 \ \sigma \ 0.9) \ \sigma \ 0.1 \\ 0.7 \ \sigma \ 0.1 \neq 0 \ \sigma \ 0.1 \\ 0.1 \neq 0 \end{array}$$

2.6 ε - Operator

For any given a and b in lattice L $\in$ [0,1] , ε - operator is defined as

$$a\varepsilon b = \begin{cases} b & \text{if a < b} \\ 0 & \text{if a} \geq \text{b} \end{cases} \tag{2.14}$$

Operator ε is dual to Operator σ.

2.6.1 Example of ε - Operator

For different values of a and b, ε - operator will respond as:

$$0.1\alpha 0.9 = 0.9, \qquad 0.7\alpha 0 = 0$$
$$0.2\alpha 0.2 = 0, \qquad 0.2\alpha 0.5 = 0.5$$

.

2.6.2 Properties of ε - Operator

1. ε - operator does not hold the commutative property.

$$\text{a } \varepsilon \text{ b} \neq \text{b } \varepsilon \text{ a (Not Commutative)}$$

e.g. if a = 0.2 and b = 0.5, then

$$0.2\ \varepsilon\ 0.5 \neq 0.5\ \varepsilon\ 0.2$$
$$0.5 \neq 0$$

2. ε - operator does not hold the associative property.

$$\text{a } \varepsilon \text{ (b } \varepsilon \text{ c)} \neq \text{(a } \varepsilon \text{ b) } \varepsilon \text{ c (Not Associative)}$$

e.g. if a = 0.9, b = 0.7 and c = 0.5, then

$$0.9\ \varepsilon\ (0.7\ \varepsilon\ 0.5) \neq (0.9\ \varepsilon\ 0.7)\ \varepsilon\ 0.5$$
$$0.9\ \varepsilon\ 0 \neq 0\ \varepsilon\ 0.5$$
$$0 \neq 0.5$$

2.7 σ - Product of two Fuzzy Sets

Consider two fuzzy sets A $\subseteq X$ *and* $B \subseteq$ Y , then the 'σ' product between these two sets is denoted as " a σ b"
and its membership function is defined as ;

$$\mu_{A\sigma B}(x, y) = \mu_A(x)\ \sigma\mu_B\ (y) \qquad \forall\ x \in X\ and\ y \in Y \tag{2.15}$$

2.7.1 Example of σ - Product

Consider $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and
$A = \{0.2/x_1, 0.5/x_2, 0.9/x_3\}$
$B = \{0.7/y_1, 0.1/y_2, 0.8/y_3\}$.
Then

A σ B $= 0/(x_1, y_1) + 0.1/(x_1, y_2) + 0/(x_1, y_3) + 0/(x_2, y_1) + 0.1/(x_2, y_2) + 0/(x_2, y_3) +$
. $0.7/(x_3, y_1) + 0.1/(x_3, y_2) + 0.8/(x_3, y_3)$

$A\ \sigma\ B = 0.1/(x_1, y_2) + 0.1/(x_2, y_2) + 0.7/(x_3, y_1) + 0.1/(x_3, y_2) + 0.9/(x_3, y_3)$

2.8 Composition of the @ - Operator Type

Consider two fuzzy relations $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$. Relationship between these two fuzzy relations when using @ composition is defined as
$R\ @\ Q \subseteq X \times Z$
with the membership function defined as:

$$\mu_{R@Q}(x, z) = \bigwedge_{y \in Y} \{\mu_R(x, y) @ \mu_Q(y, z)\} \qquad \forall\ x \in X,\ y \in Y and\ z \in Z \quad (2.16)$$

2.8.1 Example of @ Composition

let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $Z = \{z_1, z_2, z_3\}$

Consider two fuzzy relations $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$ which are given below respectively .

We are to compute $T \subseteq X \times Z$ using "@" composition

$$\mathbf{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.1 & 0 & 0.5 \\ 1 & 0.7 & 0.8 \\ 0.4 & 0.3 & 0.1 \end{bmatrix} \end{array}$$

$$\mathbf{Q} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 1 & 0.9 & 0.6 \\ 0.5 & 0 & 0.2 \\ 0 & 0.7 & 0.3 \end{bmatrix} \end{array}$$

$T(x, z) = R(x, y)\ @\ Q(y, z) \qquad \forall\ x \in X\ and\ z \in Z$

$$\mathbf{R@Q} = \begin{bmatrix} 0.1 & 0 & 0.5 \\ 1 & 0.7 & 0.8 \\ 0.4 & 0.3 & 0.1 \end{bmatrix} @ \begin{bmatrix} 1 & 0.9 & 0.6 \\ 0.5 & 0 & 0.2 \\ 0 & 0.7 & 0.3 \end{bmatrix}$$

$$\mu_{R@Q}(x_1, z_1) = \bigwedge_{y\in Y}\{(0.1\alpha 1), (0\alpha 0.5), (0.5\alpha 0)\} = 0.5$$
$$\mu_{R@Q}(x_1, z_2) = \bigwedge_{y\in Y}\{(0.1\alpha 0.9), (0\alpha 0), (0.5\alpha 0.7)\} = 1$$
$$\mu_{R@Q}(x_1, z_3) = \bigwedge_{y\in Y}\{(0.1\alpha 0.6), (0\alpha 0.2), (0.5\alpha 0.3)\} = 0.3$$
$$\mu_{R@Q}(x_2, z_1) = \bigwedge_{y\in Y}\{(1\alpha 1), (0.7\alpha 0.5), (0.8\alpha 0)\} = 0$$
$$\mu_{R@Q}(x_2, z_2) = \bigwedge_{y\in Y}\{(1\alpha 0.9), (0.7\alpha 0), (0.8\alpha 0.7)\} = 0$$
$$\mu_{R@Q}(x_2, z_3) = \bigwedge_{y\in Y}\{(1\alpha 0.6), (0.7\alpha 0.2), (0.8\alpha 0.3)\} = 0.2$$
$$\mu_{R@Q}(x_3, z_1) = \bigwedge_{y\in Y}\{(0.4\alpha 1), (0.3\alpha 0.5), (1\alpha 0)\} = 0$$
$$\mu_{R@Q}(x_3, z_2) = \bigwedge_{y\in Y}\{(0.4\alpha 0.9), (0.3\alpha 0), (1\alpha 0.7)\} = 0$$
$$\mu_{R@Q}(x_3, z_3) = \bigwedge_{y\in Y}\{(0.4\alpha 0.6), (0.3\alpha 0.2), (1\alpha 0.3)\} = 0.2$$

$$\mathbf{T}(\mathbf{x},\mathbf{z}) = \mathbf{R}@\mathbf{Q} = \begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & 0.5 & 1 & 0.3 \\ x_2 & 0 & 0 & 0.2 \\ x_3 & 0 & 0 & 0.2 \end{matrix}$$

2.9 Composition of γ Operator

Consider two fuzzy relations $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$. Relationship between these two fuzzy relations when using "γ" composition is defined as
$R\ (\gamma)\ Q \subseteq X \times Z$
with the membership function is defined as:

$$\mu_{R(\gamma)Q}(x, z) = \bigwedge_{y\in Y}\{\mu_R(x, y)\ \gamma\ \mu_Q(y, z)\} \qquad \forall\ x \in X and\ y \in Y \tag{2.17}$$

2.9.1 Example of γ Composition

let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $Z = \{z_1, z_2\}$.
Consider two fuzzy relations $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$ are given below respectively .

We are to compute $T \subseteq X \times Z$ using "(γ)" composition

$$\mathbf{R} = \begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & 0.5 & 0.6 & 0.9 \\ x_2 & 0.4 & 0.2 & 0.8 \end{matrix}$$

$$\mathbf{Q} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \left[\begin{array}{cc} 0.9 & 0.5 \\ 0.7 & 0.0.6 \\ 0.8 & 0.9 \end{array} \right] \end{array}$$

$$T(x,z) = R(x,y)\ (\gamma)\ Q(y,z) \qquad \forall\ \ x\ \in X and\ z \in Z$$

$$\mathbf{R}(\gamma)\mathbf{Q} = \left[\begin{array}{ccc} 0.5 & 0.6 & 0.9 \\ 0.4 & 0.2 & 0.8 \end{array} \right] (\gamma) \left[\begin{array}{cc} 0.9 & 0.5 \\ 0.7 & 0.6 \\ 0.8 & 0.9 \end{array} \right]$$

$$\mu_{R(\gamma)Q}(x_1, z_1) = \bigwedge_{y \in Y} \{(0.5\gamma 0.9), (0.6\gamma 0.7), (0.9\gamma 0.8)\} = 0$$
$$\mu_{R(\gamma)Q}(x_1, z_2) = \bigwedge_{y \in Y} \{(0.5\gamma 0.5), (0.6\gamma 0.6), (0.9\gamma 0.9)\} = 1$$
$$\mu_{R(\gamma)Q}(x_1, z_3) = \bigwedge_{y \in Y} \{(0.4\gamma 0.9), (0.2\gamma 0.7), (0.8\gamma 0.8)\} = 0$$
$$\mu_{R(\gamma)Q}(x_2, z_1) = \bigwedge_{y \in Y} \{(0.4\gamma 0.5), (0.2\gamma 0.6), (0.8\gamma 0.9)\} = 0$$

$$\mathbf{R}(\gamma)\mathbf{Q} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right] \end{array}$$

Chapter 3

Maximum Solution Of Fuzzy Relation Equations

In this chapter we will discuss the methods of finding the maximal solution of FRE with respect to unknowns R and Q. We will discuss the methods for finding the maximal "Q" and maximal "R" respectively for FRE of the form R ∘ Q = T. Here "∇" denotes the maximal solution.

3.1 Determination of Maximal Q

For determination of maximal Q, first, we will discuss some results.

3.1.1 Lemma

If we have two fuzzy relations $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$ then the following inclusion will hold:

$$Q \subseteq R^{-1} @ (R \circ Q) \tag{3.1}$$

where "∘" denotes the max-min composition and "@" is the composition made by α - operator.

PROOF:

Let $A = R^{-1} @ (R \circ Q) \subseteq Y \times Z.$
Then by using (1.4) and (2.11), we have

$$\begin{aligned} \mu_A(y,z) &= \bigwedge_{x \in X} \{\mu_{R^{-1}}(y,x) \ \alpha \ \mu_{R \circ Q}(x,z)\} \\ &= \bigwedge_{x \in X} \{\mu_R(x,y) \ \alpha \ \mu_{R \circ Q}(x,z)\} \\ &= \bigwedge_{x \in X} (\mu_R(x,y) \ \alpha \ (\bigvee_{t \epsilon Y} \{\mu_R(x,t) \bigwedge \mu_Q(t,z)\})) \end{aligned}$$

. $= \bigwedge_{x\in X}(\mu_R(x,y)\ \alpha\ (\mu_R(x,y) \wedge \mu_Q(y,z) \wedge \bigvee_{t\epsilon Y t\neq y}(\mu_R(x,t) \wedge \mu_Q(t,z)))$
so it becomes

$$\mu_A(y,z) \geq \bigwedge_{x\in X}\{\mu_R(x,y)\ \alpha\ (\mu_R(x,y) \wedge \mu_Q(y,z))\}$$

as we know that

$$a\ \alpha\ (a \wedge b) \geq b \tag{3.2}$$

then by using (3.2), we have

$$\mu_A(y,z) \geq \mu_Q(y,z) \qquad \forall\ \ y \in Y\ \ and\ \ Z \in Z$$

3.1.2 Example of Lemma 3.1.1

Consider $X = \{x_1, x_2, x_3, x_4\}, Y = \{y_1, y_2, y_3\}\ and\ Z = \{z_1, z_2, z_3\}$.

$$\mathbf{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.1 & 0 & 0.9 \\ 0.5 & 0.8 & 1 \\ 0.6 & 0.3 & 0.2 \\ 1 & 0.1 & 0.4 \end{bmatrix} \end{array}$$

$$\mathbf{Q} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.4 & 0.1 & 0.6 \\ 0 & 0.3 & 0.2 \\ 0.6 & 1 & 0.8 \end{bmatrix} \end{array}$$

$$\mathbf{R} \circ \mathbf{Q} = \begin{bmatrix} 0.6 & 0.9 & 0.8 \\ 0.6 & 1 & 0.8 \\ 0.4 & 0.3 & 0.6 \\ 0.4 & 0.4 & 0.6 \end{bmatrix} \quad \textbf{and} \quad \mathbf{R^{-1}} = \begin{bmatrix} 0.1 & 0.5 & 0.6 & 1 \\ 0 & 0.8 & 0.3 & 0.1 \\ 0.9 & 1 & 0.2 & 0.4 \end{bmatrix}$$

$$\mathbf{R^{-1}}@(\mathbf{R} \circ \mathbf{Q}) = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.4 & 0.3 & 0.6 \\ 0.6 & 1 & 1 \\ 0.6 & 1 & 0.8 \end{bmatrix} \end{array}$$

So this example shows that $Q \subset R^{-1}@(R \circ Q)$. Hence it clearly satisfies the lemma 3.1.1.

Lemma

Assume that we have two fuzzy relations $R \subseteq X \times Y$ and $T \subseteq X \times Z$ then the following inclusion holds:

$$R \circ (R^{-1}@T) \subset T \tag{3.3}$$

where "$\circ$" denotes the max-min composition and "@" is the composition of α -operator.
The proof of this lemma is analogous to the proof of lemma 3.1.1

Lemma

Consider we have two fuzzy relations $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$ then the following inclusion holds:

$$R \subseteq (Q@(R \circ Q)^{-1})^{-1} \tag{3.4}$$

Lemma

Consider we have two fuzzy relations $Q \subseteq Y \times Z$ and $T \subseteq X \times Z$ then the following inclusion holds:

$$(Q@T^{-1})^{-1} \circ Q \subset T \tag{3.5}$$

3.1.3 Theorem

Let $R \subseteq X \times Y$ and $T \subseteq X \times Z$ be the two fuzzy relations, S(Q) be the set of fuzzy relations $Q \in Y \times Z$ such that $R \circ Q = T$.
$S(Q) = \{Q \in Y \times Z \mid R \circ Q = T\} \neq \phi$, if and only if
$R^{-1}@T \in S(Q)$ then "$R^{-1}@T$" is the the greatest element in S(Q).

Theorem

Let $R \subseteq X \times Y$ and $T \subseteq X \times Z$ be the two fuzzy relations, the set of fuzzy relations $Q \in Y \times Z$ such that $R \circ Q \subseteq T$ contains a greatest element $R^{-1}@T$.

PROOF:
Let $S(Q)^* = \{FuzzyQ \in (Y \times Z) \mid R \circ Q \subseteq T\}$ and $S(R)^* \neq \phi$.
because of the null relation

$$0(y, z) = 0 \qquad \forall \ (y, z) \in Y \times Z, \in S(Q)^*$$

let $Q \subseteq S(R)^* : R \circ Q = T$
then we have

$$R^{-1}@(R \circ Q) \subseteq R^{-1}@T,$$

but from lemma 3.1.1, we have

$$Q \subset R^{-1}@(R \circ Q)$$

then it shows that

$$Q \subset R^{-1}@T$$

now from Theorem 3.1.6 we have

$$R^{-1}@T \in S(Q).$$

Then it shows that $R^{-1}@T \in S(Q)^*$, then $R^{-1}@T$ will be the greatest element in $S(Q)^*$. Hence $R^{-1}@T$ be the greatest element in $S(Q)^*$.
Then

$$Q^{\nabla} = R^{-1}@T \tag{3.6}$$

which is the maximum relation "Q" satisfying the equation $R \circ Q = T$.

3.1.4 Necessary Condition For Existence Of Q^{∇}

The necessary condition for the existence of Q^{∇} satisfying the FRE (2.1) is

$$\mu_T(x, z) \leq \bigvee_{y \in Y} \mu_R(x, y) \qquad \forall \ x \in X \ \ and \ \ z \in Z \tag{3.7}$$

3.1.5 Example of Determining the Maximal Q

Consider $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3, y_4\}$ and $Z = \{z_1, z_2, z_3\}$.
Suppose $R \subseteq X \times Y$ and $T \subseteq X \times Z$ are two fuzzy relations given below respectively

$$y_1 \quad y_2 \quad y_3 \quad y_4$$

$$\mathbf{R} = \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \begin{bmatrix} 1 & 0.3 & 0.2 & 0.7 \\ 0.1 & 0 & 0.9 & 0.4 \\ 0.5 & 0.6 & 0 & 0.7 \end{bmatrix}$$

and

$$\mathbf{T} = \begin{matrix} \\ x_1 \\ x_2 \\ x_3 \end{matrix} \begin{matrix} z_1 \quad z_2 \quad z_3 \\ \begin{bmatrix} 0.3 & 0.6 & 0.9 \\ 0.7 & 0.4 & 0.4 \\ 0.6 & 0.6 & 0.5 \end{bmatrix} \end{matrix}$$

We are to compute "Q^{∇}" **?**

First we check the necessary condition for the existence of "Q^{∇}" using (3.7).

$$\mu_T(x,z) \leq \bigvee_{y \in Y} \mu_R(x,y).$$

$$\begin{bmatrix} 0.3 & 0.6 & 0.9 \\ 0.7 & 0.4 & 0.4 \\ 0.6 & 0.6 & 0.5 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 0.9 \\ 0.7 \end{bmatrix}$$

Hence the necessary condition (3.7) is satisfied. So

$$\mathbf{R^{-1}} = \begin{matrix} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{matrix} \begin{matrix} x_1 \quad x_2 \quad x_3 \\ \begin{bmatrix} 1 & 0.1 & 0.5 \\ 0.3 & 0 & 0.6 \\ 0.2 & 0.9 & 0 \\ 0.7 & 0.4 & 0.7 \end{bmatrix} \end{matrix}$$

now we compute $R^{-1}@T$

$$\mathbf{R^{-1}@T} = \begin{bmatrix} 1 & 0.1 & 0.5 \\ 0.3 & 0 & 0.6 \\ 0.2 & 0.4 & 0.7 \\ 0.7 & 0.4 & 0.7 \end{bmatrix} @ \begin{bmatrix} 0.3 & 0.6 & 0.9 \\ 0.7 & 0.4 & 0.4 \\ 0.6 & 0.6 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} \bigwedge(0.3,1,1) & \bigwedge(0.6,1,1) & \bigwedge(0.9,1,1) \\ \bigwedge(1,1,1) & \bigwedge(1,1,1) & \bigwedge(1,1,0.5) \\ \bigwedge(1,0.7,1) & \bigwedge(1,0.4,1) & \bigwedge(1,0.4,1) \\ \bigwedge(0.3,1,0.6) & \bigwedge(0.6,1,0.6) & \bigwedge(1,1,0.5) \end{bmatrix}$$

$$\mathbf{Q}^{\nabla} = \mathbf{R}^{-1}@\mathbf{T} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.3 & 0.6 & 0.9 \\ 1 & 1 & 0.5 \\ 0.7 & 0.4 & 0.4 \\ 0.3 & 0.6 & 0.5 \end{bmatrix} \end{array}$$

Check:
Here we check whether "Q^{∇}" satisfies the FRE (2.1) i.e $R \circ Q^{\nabla} = T$ or nor?

$$\mathbf{R} \circ \mathbf{Q}^{\nabla} = \begin{bmatrix} 1 & 0.3 & 0.2 & 0.7 \\ 0.1 & 0 & 0.9 & 0.4 \\ 0.5 & 0.6 & 0 & 0.7 \end{bmatrix} \circ \begin{bmatrix} 0.3 & 0.6 & 0.9 \\ 1 & 1 & 0.5 \\ 0.2 & 0.4 & 0.7 \\ 0.7 & 0.4 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.6 & 0.9 \\ 0.7 & 0.4 & 0.4 \\ 0.6 & 0.6 & 0.5 \end{bmatrix}$$

$= \mathrm{T}$

3.1.6 Example of Determining the Maximal Q

Consider $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3\}$ *and* $Z = \{z_1, z_2\}$.
Suppose $R \subseteq X \times Y$ and $T \subseteq X \times Z$ are two fuzzy relations given below respectively.
We are to compute "Q^{∇}" when

$$\mathbf{R} = \begin{bmatrix} 0.1 & 0.2 & 1 \\ 0 & 0.9 & 0.3 \\ 0.8 & 0.5 & 0.1 \\ 1 & 0.2 & 0 \end{bmatrix} \quad \textbf{and} \quad \mathbf{T} = \begin{bmatrix} 0.6 & 1 \\ 0.7 & 0.1 \\ 0.2 & 0 \end{bmatrix}$$

Since the condition (3.7) is satisfied for the above given R and T then, by using (3.6), our result will be

$$\mathbf{Q}^{\nabla} = \mathbf{R}^{-1}@\mathbf{T} = \begin{bmatrix} 0.6 & 1 \\ 0.7 & 1 \\ 0.2 & 0.1 \end{bmatrix}$$

Q^{∇} also satisfies the FRE i.e $R \circ Q = T$.

3.1.7 Example of Determining the Maximal Q

Consider $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ *and* $Z = \{z_1, z_2, z_3\}$.
Suppose $R \subseteq X \times Y$ and $T \subseteq X \times Z$ are two fuzzy relations given below respectively. We are to compute "Q" for

$$\mathbf{R} = \begin{bmatrix} 0.1 & 1 & 3 \\ 0.2 & 0 & 0.9 \\ 1 & 0.5 & 0.4 \end{bmatrix} \quad \textbf{and} \quad \mathbf{T} = \begin{bmatrix} 0.3 & 0.6 & 0.5 \\ 0.3 & 0.9 & 0.2 \\ 0.3 & 0.5 & 0.5 \end{bmatrix}$$

Since the condition (3.7) is fulfilled for the above given R and T.
So by using (3.6), our result will be

$$\mathbf{Q}^{\nabla} = \mathbf{R^{-1}} @ \mathbf{T} = \begin{bmatrix} 0.3 & 0.5 & 0.5 \\ 0.3 & 0.6 & 0.5 \\ 0.3 & 1 & 0.2 \end{bmatrix}$$

Q^{∇} also satisfies the FRE i.e $R \circ Q = T$.

3.2 Determination of Maximal R

First we will discuss some results in order to determine the maximal R.

3.2.1 Theorem

Let $Q \subseteq Y \times Z$ and $T \subseteq X \times Z$ be the two fuzzy relations , S(R) be the set of fuzzy relations $R \in X \times Y$ such that $R \circ Q = T$.
$S(R) = \{FuzzyR \in X \times Y \mid R \circ Q = T\} \neq \phi$, if and only if
$(Q @ T^{-1})^{-1} \in S(R)$ and it is the greatest element in S(R).

Theorem

Let $Q \subseteq Y \times Z$ and $T \subseteq X \times Z$ be the two fuzzy relations, then the set of fuzzy relations $R \in X \times Y$ such that $R \circ Q \subseteq T$ contains a greatest element $(Q @ T^{-1})^{-1}$.

PROOF:

Let $S(R)^* = \{FuzzyR \in (X \times Y) \mid R \circ Q \subseteq T\}$ and $S(R)^* \neq \phi$
because of the null relation

$$0(x, y) = 0 \qquad \forall \ (x, y) \in Y \times Z \in S(R)^*.$$

Let $R \subseteq S(R)^* : R \circ Q = T$, then we have

$$(Q@(R \circ Q)^{-1})^{-1} \subseteq (Q@T^{-1})^{-1}$$

but from lemma 3.4, we have

$$R \subset (Q@(R \circ Q)^{-1})^{-1}$$

then it shows that

$$R \subset (Q@T^{-1})^{-1}$$

now from theorem 3.2.1, we have

$$(Q@T^{-1})^{-1} \in S(R).$$

Then it shows that $(Q@T^{-1})^{-1} \in S(R)^*$, then $(Q@T^{-1})^{-1}$ will be the greatest element in $S(R)^*$. Hence $(Q@T^{-1})^{-1}$ be the greatest element in $S(R)^*$.
So

$$R^{\nabla} = (Q@T^{-1})^{-1} \tag{3.8}$$

which is the maximum relation for "R" satisfying the equation $R \circ Q = T$.

3.2.2 Necessary Condition for Existence of $\mathbf{R}^{\nabla}$

The necessary condition for the existence of R^{∇} satisfying the FRE (2.1) is

$$\mu_T(x, z) \leq \bigvee_{y \in Y} \mu_Q(y, z) \qquad \forall \ x \in X \ \ and \ \ z \in Z \tag{3.9}$$

3.2.3 Example of Determining the Maximal R

Consider $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3, y_4\}$ *and* $Z = \{z_1, z_2\}$.
Suppose $Q \subseteq Y \times Z$ and $T \subseteq X \times Z$ are two fuzzy relations given below respectively

$$\mathbf{Q} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0.6 & 0 \\ 0.3 & 0.5 \\ 0.7 & 0.1 \\ 0.5 & 1 \end{bmatrix} \end{array}$$

and

$$\mathbf{T} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0.7 & 0.4 \\ 0.5 & 0.7 \\ 0.6 & 0.5 \end{bmatrix} \end{array}$$

For them compute "R^{∇}".
First we check the necessary condition for the existence of "R^{∇}" using (3.9)
Since it is clear from the above given Q and T that $\mu_T(x,z) \leq \bigvee_{y \in Y} \mu_Q(y,z)$.

So now we compute $(Q@T^{-1})^{-1}$.

$$\mathbf{T^{-1}} = \begin{array}{c} \\ z_1 \\ z_2 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0.7 & 0.5 & 0.6 \\ 0.4 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

So

$$\mathbf{Q@T^{-1}} = \begin{bmatrix} 0.6 & 0 \\ 0.3 & 0.5 \\ 0.7 & 0.1 \\ 0.5 & 1 \end{bmatrix} @ \begin{bmatrix} 0.7 & 0.5 & 0.6 \\ 0.4 & 0.7 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} \bigwedge(1,1) & \bigwedge(0.5,1) & \bigwedge(1,1) \\ \bigwedge(1,0.4) & \bigwedge(1,1) & \bigwedge(1,1) \\ \bigwedge(1,1) & \bigwedge(0.5,1) & \bigwedge(0.6,1) \\ \bigwedge(1,0.4) & \bigwedge(1,0.7) & \bigwedge(1,0.5) \end{bmatrix}$$

$$\mathbf{Q@T^{-1}} = \begin{bmatrix} 1 & 0.5 & 1 \\ 0.4 & 1 & 1 \\ 1 & 0.5 & 0.6 \\ 0.4 & 0.7 & 0.5 \end{bmatrix}$$

$$\mathbf{R}^{\nabla} = (\mathbf{Q@T^{-1}})^{-1} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1 & 0.4 & 1 & 0.4 \\ 0.5 & 1 & 0.5 & 0.7 \\ 1 & 1 & 0.6 & 0.5 \end{bmatrix} \end{array}$$

Check:
Here we check whether "R^{∇}" satisfies the FRE (2.1) i.e $R \circ Q = T$ or nor?

$$\mathbf{R}^{\nabla} \circ \mathbf{Q} = \begin{bmatrix} 1 & 0.4 & 1 & 0.4 \\ 0.5 & 1 & 0.5 & 0.7 \\ 1 & 1 & 0.6 & 0.5 \end{bmatrix} \circ \begin{bmatrix} 0.6 & 0 \\ 0.3 & 0.5 \\ 0.7 & 0.1 \\ 0.5 & 1 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.4 \\ 0.5 & 0.7 \\ 0.6 & 0.5 \end{bmatrix} = T$$

Hence, Q^{∇} satisfies the FRE i.e $R \circ Q = T$.

3.2.4 Example of Determining the Maximal R

Consider $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$ *and* $Z = \{z_1, z_2, z_3, z_4, \}$.
Suppose $Q \subseteq Y \times Z$ and $T \subseteq X \times Z$ are two fuzzy relations given below respectively

$$\mathbf{Q} = \begin{bmatrix} 0.2 & 0 & 0.9 & 1 \\ 1 & 0.5 & 0.3 & 0.6 \end{bmatrix} \quad \textbf{and} \quad \mathbf{T} = \begin{bmatrix} 0.3 & 0.3 & 0.9 & 1 \\ 0.7 & 0.5 & 0.3 & 0.6 \\ 0.2 & 0.2 & 0.9 & 0.9 \end{bmatrix}$$

We compute "R^{∇}":
Since the necessary condition (3.9) is satisfied for the above given Q and T.
Then by using (3.8), the result will be

$$\mathbf{R}^{\nabla} = (\mathbf{Q}@\mathbf{T}^{-1})^{-1} = \begin{bmatrix} 1 & 0.3 & 0.9 \\ 0.3 & 0.7 & 0.2 \end{bmatrix}$$

R^{∇} satisfies the FRE i.e $R \circ Q = T$.

Chapter 4

Minimal Solution Of Fuzzy Relation Equations

In this chapter we will discuss the methods to find the minimal solution of FRE with fuzzy membership matrix. We will discuss the methods for finding the minimal "Q" and minmimal"R" respectively for FRE of the form R $\circ$ Q = T. Here "$\triangle$" denotes the minimal solution.

4.1 Determination of Minimal Q

For determination of minimal Q, first, we will discuss some results.

4.1.1 Functional Relations

Consider a relation $R \subseteq X \times Y$ which is said to be functional [4] if and only if ,$\forall\ x \epsilon X$,
there exist $v(u) \in V such\ that$

$$\begin{cases} \mu Q(u, v(u)) & = 1 \\ \mu Q(u, v) & = 0 \end{cases} \tag{4.1}$$

In finite case, functional relation is represented by a matrix such that for every row, there is one and only one element having membership degree **1**, the other elements being equal to 0.

Consider a mapping f from U to V as a functional relation, i.e. $\mu_f(u, v) = 1$ if f(u) = v and $\mu_f(u, v) = 0$ other wise.

Theorem

If $\mathbf{Q} \neq \phi$, then $(\mathbf{Q}, \subseteq)$ is a lattice with a greatest element $R^{-1}@T$ and the smallest element

$$Q^{\triangle} = \neg(R^{-1}@\neg T) \tag{4.2}$$

Proof:

If Q and S are elements of $\mathbf{Q}$, then

$$R \circ (Q \cup S) = (R \circ Q) \cup (R \circ S) \text{ (which always hold)}$$

hence

$$R \circ (Q \cup S) = T \cup T = T) \;\; and \;\; Q \cup S \in \mathbf{Q}$$

now

$$R \circ (Q \cup S) = (Q \cup R) \cap (S \circ R)$$

This holds only because R is functional relation here, hence $Q \cap S \in \mathbf{Q}$. So, $(\mathbf{Q} \subseteq)$ is a lattice. Since $R^{-1}@T$ is the greatest element from (3.6) then let us show that $\neg(R^{-1}@\neg T)$ is the smallest element. $Q \neq \phi$, then $\forall\, q \in Q,\ R \circ Q = T$; R being functional, and we also know that $\neg R \circ Q = \neg T$ and from lemma 3.1.1

$\neg Q \subseteq R^{-1}@(\neg R \circ Q)$
i.e. $\neg Q \subseteq (R^{-1}@\neg T)$ is equivalent to $\neg(R^{-1}@\neg T) \subseteq Q$,

which complete the proof.

4.1.2 Example of Determining the Minimal Q

Consider $X = \{x_1, x_2, x_3, x_4\}, \;\; Y = \{y_1, y_2, y_3, y_4\}$ *and* $Z = \{z_1, z_2\}$.

Suppose $R \subseteq X \times Y$ and $T \subseteq X \times Z$ are two fuzzy relations given below respectively

$$\mathbf{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

and

$$z_1 \quad z_2$$

$$\mathbf{T} = \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} \begin{bmatrix} 0.5 & 0.8 \\ 0.6 & 0.4 \\ 0.7 & 0.3 \\ 0.5 & 0.8 \end{bmatrix}$$

We compute "$Q^{\triangle}$".

$$\mathbf{R^{-1}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \textbf{and} \quad \neg\mathbf{T} = \begin{bmatrix} 0.5 & 0.2 \\ 0.4 & 0.6 \\ 0.3 & 0.7 \\ 0.5 & 0.2 \end{bmatrix}$$

Now we compute $R^{-1}@\neg T$

$$\mathbf{R^{-1}}@\neg\mathbf{T} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} @ \begin{bmatrix} 0.5 & 0.2 \\ 0.4 & 0.6 \\ 0.3 & 0.7 \\ 0.5 & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} \bigwedge(1,1,0.3,1) & \bigwedge(1,1,0.7,1) \\ \bigwedge(1,1,1,1) & \bigwedge(1,1,1,1) \\ \bigwedge(0.5,0,0,0.5) & \bigwedge(0.2,0,0,0.2) \\ \bigwedge(1,0.4,1,1) & \bigwedge(1,0.6,1,1) \end{bmatrix} = \begin{bmatrix} 0.3 & 0.7 \\ 1 & 1 \\ 0.5 & 0.2 \\ 0.4 & 0.62 \end{bmatrix}$$

$$\mathbf{Q}^{\triangle} = \neg(\mathbf{R^{-1}}@\neg\mathbf{T}) = \begin{matrix} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{matrix} \begin{matrix} z_1 \quad z_2 \\ \begin{bmatrix} 0.7 & 0.3 \\ 0 & 0 \\ 0.5 & 0.8 \\ 0.6 & 0.4 \end{bmatrix} \end{matrix}$$

Check:
Here we check whether "$Q^{\triangle}$" satisfies the FRE (2.1) i.e $R \circ Q = T$ or not.

$$\mathbf{R} \circ \mathbf{Q}^{\triangle} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 0.7 & 0.3 \\ 0 & 0 \\ 0.5 & 0.8 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.8 \\ 0.6 & 0.4 \\ 0.7 & 0.3 \\ 0.5 & 0.8 \end{bmatrix} = T$$

4.1.3 Example of Determining the Minimal Q

Consider $X = \{x_1, x_2, x_3, x_4\}, \quad Y = \{y_1, y_2, y_3\}$ *and* $Z = \{z_1, z_2, z_3\}$.

Suppose $R \subseteq X \times Y$ and $T \subseteq X \times Z$ are two fuzzy relations given below respectively

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \textbf{and} \quad \mathbf{T} = \begin{bmatrix} 0.5 & 0.4 & 0.7 \\ 0.6 & 0.5 & 0.6 \\ 0.5 & 0.7 & 0.9 \\ 0.6 & 0.5 & 0.6 \end{bmatrix}$$

Here we compute "$Q^{\triangle}$".

$$\mathbf{R^{-1}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{and} \quad \neg\mathbf{T} = \begin{bmatrix} 0.5 & 0.6 & 0.3 \\ 0.4 & 0.5 & 0.4 \\ 0.5 & 0.3 & 0.1 \\ 0.4 & 0.5 & 0.4 \end{bmatrix}$$

Now we compute $R^{-1}@\neg T$

$$\mathbf{R^{-1}}@\neg\mathbf{T} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} @ \begin{bmatrix} 0.7 & 0.6 & 0.3 \\ 0.4 & 0.5 & 0.4 \\ 0.5 & 0.3 & 1 \\ 0.4 & 0.5 & 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.3 & 0.1 \\ 0.7 & 0.6 & 0.3 \\ 0.4 & 0.5 & 0.4 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{Q}^{\triangle} = \neg(\mathbf{R^{-1}}@\neg\mathbf{T}) = \begin{bmatrix} 0.5 & 0.7 & 0.9 \\ 0.3 & 0.4 & 0.7 \\ 0.6 & 0.5 & 0.6 \\ 0 & 0 & 0 \end{bmatrix}$$

Check:
Here we check whether "$Q^{\triangle}$" satisfies the FRE (2.1) i.e $R \circ Q = T$ or not.

$$\mathbf{R} \circ \mathbf{Q}^{\triangle} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 0.5 & 0.7 & 0.9 \\ 0.3 & 0.4 & 0.7 \\ 0.6 & 0.5 & 0.6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.4 & 0.7 \\ 0.6 & 0.5 & 0.6 \\ 0.5 & 0.7 & 0.9 \\ 0.6 & 0.5 & 0.6 \end{bmatrix} = T$$

4.2 Determination of Minimal R

First we will discuss some results in order to determine the minimal R.

Let $A \subseteq X$ and $B \subseteq Y$ be two fuzzy sets and $R \subseteq X \times Y$, then the following relational equation is defined as

$$A \circ R = B \tag{4.3}$$

where A and B are the fuzzy sets and R is the fuzzy relation.
We denote a collection of solutions of the equation by:

$$S(R) = \{R \subseteq X \times Y \mid A \circ R = B\}. \tag{4.4}$$

We introduce a mapping $\Gamma : Y \to M(X)$
where M(X) consists of all the subsets of X. We define $\Gamma(y)$ by

$$\Gamma\{y\} = \{x \in X : \mu_A(x) \geq \mu_B(y)\} \tag{4.5}$$

and complement of "Gamma" is defined as

$$\neg\Gamma\{y\} = \{x \in X : \mu_A(x) < \mu_B(y)\}. \tag{4.6}$$

Union of (4.3) and (4.4) is equal to X and intersection of (4.3) and (4.4) is equal to ϕ.

Theorem

Consider $S(R) = \{R \subseteq X \times Y : A \circ R = B\}$
If $S(R) \neq \phi$, then

$$A(\sigma)B \in S(R) \tag{4.7}$$

where (σ) is the σ-product between two fuzzy sets.

Theorem

Necessary and sufficient condition for the existence of minimal solution $R^{\triangle}$ belonging to S(R) is

$$| (\Gamma)\{y\} |= 1 \quad or \quad \mu_B(y) = 0 \qquad \forall\, y \in Y \tag{4.8}$$

Then the minimal $R^{\triangle}$ for the equation $A \circ R = B$ is

$$R^{\triangle} = A(\sigma)B \tag{4.9}$$

where A and B are fuzzy sets and
| y | denotes cardinality, or number of elements of the set Y in finite case.

4.2.1 Example of Determining the Minimal R

Consider $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3, y_4\}$, $A = \{0.4/x_1, 0.3/x_2, 0.9/x_3\}$ and $B = \{0.7/y_1, 0.5/y_3, 0.9/y_4\}$

by using (4.5), we have
$\Gamma\{y_1\} = \{x_3\}$ $\Gamma\{y_2\} = \{x_1, x_2, x_3\}$
$\Gamma\{y_3\} = \{x_3\}$ $\Gamma\{y_4\} = \{x_3\}$
so
$|\Gamma\{y_1\}| = 1$ $|\Gamma\{y_2\}| = 3$
$|\Gamma\{y_3\}| = 1$ $|\Gamma\{y_4\}| = 1$

by using the theorem 4.2.2, we have
$\Gamma\{y_2\} \neq 1 \quad and \quad \mu_B(y_2) = 0$

Since condition (4.8) is satisfied, so we will use the (4.9) to compute the minimal solution $R^{\triangle}$

$$\mu_{A\sigma B}(x, y) = \mu_A(x)\sigma\mu_B(y) \qquad \forall \;\; x \in X and \quad y \in Y$$

we have

$$\mathbf{A}(\sigma)\mathbf{B} = \begin{bmatrix} 0.4\sigma0.7 & 0.4\sigma0 & 0.4\sigma0.5 & 0.4\sigma0.9) \\ 0.3\sigma0.7 & 0.3\sigma0 & 0.3\sigma0.5 & 0.3\sigma0.9 \\ 0.9\sigma0.7 & 0.9\sigma0 & 0.9\sigma0.5 & 0.9\sigma0.9 \end{bmatrix}$$

$$\mathbf{R}^{\triangle} = \mathbf{A}(\sigma)\mathbf{B} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0.5 & 0.9 \end{bmatrix} \end{array}$$

Hence it is the required minimal solution for the relation "R" by using sigma operator.
Check:
We can check here that $A \circ R^{\triangle} = B$?
$A \circ R^{\triangle} = \{0.7/y_1, 0.5/y_3, 0.9/y_4\} = B$

4.2.2 Example of Determining the Minimal R

Consider $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3, y_4\}$ $A = \{0.2/x_1, 0.8/x_2, 0.6/x_3\}$ and
$B = \{0.7/y_1, 0.8/y_3, 0.7/y_4\}$

by using (4.5), we have
$\Gamma\{y_1\} = \{x_2\}$ $\Gamma\{y_2\} = \{x_1, x_2, x_3\}$
$\Gamma\{y_3\} = \{x_2\}$ $\Gamma\{y_4\} = \{x_2\}$
so
$|\Gamma\{y_1\}| = 1$ $|\Gamma\{y_2\}| = 3$
$|\Gamma\{y_3\}| = 1$ $|\Gamma\{y_4\}| = 1$

Since condition (4.8) is satisfied, so we will use the (4.9) to compute the minimal solution $R^{\triangle}$

$$\mathbf{R}^{\triangle} = \mathbf{A}(\sigma)\mathbf{B} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0.8 & 0.7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Hence it is the required minimal solution for the relation "R" by using sigma

operator
Check:
We can check here that $A \circ R^{\triangle} = B$ since
$A \circ R^{\triangle} = \{0.7/y_1, 0.1/y_2, 0.8/y_3, 0.4/y_4\} = B$.

Theorem

If S(R) $\neq \phi$ then S(R) has minimal R_i components each of which is a defined by function

$$\mu_{R_i}(x,y) = \begin{cases} c(c \neq 0) & \text{for } \forall \text{ y} \in \text{Y and for one and only one x} \in \text{X such that x} = x_i | x_i \in \Gamma \ \{\text{y} \\ 0 & \text{Otherwise} \end{cases}$$

and the number of minimal elements is equal

$$N_{min} = \prod_{\mu_B(y) \neq 0 y \in Y} \mid \Gamma\{y\} \mid \tag{4.10}$$

Theorem

If S(R) $\neq \phi$ then the union of all minimal elements in S(R):

$$R^* = \cup R_i \tag{4.11}$$

$where \quad i = 1, 2, ..., N_{min}$.
This means that in the case of $|X||Y| < \infty$, then we can find all the solutions for R of equation $R \circ A = B$.

4.2.3 Example of Determining R^*

Consider $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3, y_4\}$ and $A = \{0.2/x_1, 0.8/x_2, 0.6/x_3\}$, $B = \{0.7/y_1, 0.1/y_2, 0.8/y_3, 0.4/y_4\}$.
Here we determine all the elements of the minimum solution. According to theorem 4.2.5 we have the following pairs (x,y), for whom $\mu_R(x,y) \neq 0$
$\Gamma\{y_1\} = \{x_2\}$ $\qquad$ $\Gamma\{y_2\} = \{x_1, x_2, x_3\}$
$\Gamma\{y_3\} = \{x_2\}$ $\qquad$ $\Gamma\{y_4\} = \{x_2, x_3\}$
so
$|\Gamma\{y_1\}| = 1$ $\qquad$ $|\Gamma\{y_2\}| = 3$

$|\Gamma\{y_3\}| = 1$ $\qquad$ $|\Gamma\{y_4\}| = 2$

$$N_{min} = (|\ \Gamma\{y_1\}\ |)(|\ \Gamma\{y_2\}\ |)(|\ \Gamma\{y_3\}\ |)(|\ \Gamma\{y_4\}\ |) = (1)(3)(1)(2) = 6.$$

So there will be 6 minimal solutions for this problem which are given below:

$$\mathbf{R_1^\triangle} = \begin{bmatrix} 0 & 0.1 & 0 & 0 \\ 0.7 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{R_2^\triangle} = \begin{bmatrix} 0 & 0.1 & 0 & 0 \\ 0.7 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0.4 \end{bmatrix} \mathbf{R_3^\triangle} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.7 & 0.1 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{R_4^\triangle} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.7 & 0.1 & 0.8 & 0 \\ 0 & 0 & 0 & 0.4 \end{bmatrix} \mathbf{R_5^\triangle} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0.8 & 0.4 \\ 0 & 0.8 & 0 & 0 \end{bmatrix} \mathbf{R_6^\triangle} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0.8 & 0 \\ 0 & 0.8 & 0 & 0.4 \end{bmatrix}$$

now by using the (4.12) the union of all the six minimal solutions is as follows:

$$\mathbf{R_1^\triangle} \cup \mathbf{R_2^\triangle} \cup \mathbf{R_3^\triangle} \cup \mathbf{R_4^\triangle} \cup \mathbf{R_5^\triangle} \cup \mathbf{R_6^\triangle} = \begin{matrix} \\ x_1 \\ x_2 \\ x_3 \end{matrix} \begin{matrix} y_1 \quad y_2 \quad y_3 \quad y_4 \\ \begin{bmatrix} 0 & 0.1 & 0 & 0 \\ 0.7 & 0.1 & 0.8 & 0.4 \\ 0 & 0. & 0 & 0.4 \end{bmatrix} \end{matrix} = \mathbf{R}^*.$$

Chapter 5

NEWS

Many researchers have made research for finding the best solution of fuzzy relational equations. The researcher have been trying to explore the problem and to develop the solution procedures. The notion of fuzzy relational equations based upon the max-min composition was first investigated by Sanchez. He studied conditions and theoretical methods to resolve fuzzy relations on fuzzy sets defined as mappings from sets to [0,1]. The solution he obtained by him is only the greatest element derived from the max-min composition of fuzzy relations.
The max-min composition is commonly used when a system requires consetvative solutions in the sense that the goodness of one value cannot compensate the badness of another value.

The researches found that the in some of the cases the minimum solution of FRE didn't exist, was not unique or there was no solution. By observing the nature of the such solution set, researchers yield some methods somehow analytical and numerical to find the optimal solution.

Many researchers have worked on developing the analytical and numerical methods for solution of FREs e.g. Di Nola and Sessa in citeTJ:98:ITB 1983 and Di Nola et al. in 1989 made a contribution on it. Pedrycz 1991, Valente de Oliveira 1993, Blanco 1994, Hirota and Pedrycz 1995, Salehfar 2000, Barajas and Reyes 2005, Ciaramella et al. 2006 have made great contributions to model the FREs numerically with a neural network and then adjust the problem according to the working algorithm.

Many algorithms are developed by using FREs in order to solve the optimization problems. Among them, Branch Point (BP) algorithm [5], Banach bound algorithm and genetic algorithms are frequently used techniques to investigate the solution of optimization problems with the help of FREs. Barajas and Reyes 2005, Pedrycz 1994 also work on genetic algorithms and developed numerical methods to solve FREs which play an important role in finance.
In 1979 E Sanchez provided a method for the solution of fuzzy relational equa-

tions FRE and propose an algorithm in his paper-on Resolution of Composite Fuzzy Relation [2] Equation in which how to calculate the supremum $\wedge$ and also the interval of solutions it provides exactly all the widest solution set which meet our needs . And this algorithm gives an easy way and an easy method to understand and to calculate which meet our expectations

In 2000 Bourke and Grant Fisher discussed four optimizing algorithms for the authenticity of the relation matrix and summarized them. They focused on neuro-based approach to find optimal solution of FREs. Wang 1993 proposed a special kind of neural network in which a changeable membership function is considered for the input and its parameters are selected on the basis BP optimization algorithm.

In 2005 A.V. Murkowski, worked on FREs with max-product composition and max-min composition covering problem. During his work he has examines examines the characteristics of a solvable equation and attributes of minimal solutions, then reduces the equation to an irreducible form, and then changes the problem into a covering problem.

5.1 Latest news

Fuzzy models for the representation of complex systems have an importance in many areas in approximation, because on a compact domain they approximate the arbitrary accuracy of any continuous mapping. In fuzzy models the main problem is in exponential growth in the possible fuzzy rules in the input domain. Hierarchical fuzzy relational models are the compositions of a series of [6] sub models: is a very efficient way to solve this problem. The main drawback of the fuzzy hierarchical structure of the overall model is the loss of interpretability.

Tatiana Kiseliova [7] in 2000, gives a theoretical comparison of disco and cadiag-II like systems for medical diagnoses using fuzzy approach and this system is characterized in the fuzzy relation based scenario and compared with the cadiag-II-like systems based on fuzzy technologies.
In 2001 Leh Luoh, Wen-June Wang and Yi-Ke Liaw [8] proposed a new algorithm to solve FREs with max-product and max-min composition. The algorithm works systematically on a matrix pattern (array) to get the desired result.

Martina Stepnicka, Bernard De Baets, Lenka Noskov [9] proposed an arithmetic fuzzy model in 2009 in which they use some other fuzzy relations very closely to the Takagi -Sugeno models, under some linear requirements these fuzzy relations change out to be the same system of FRE. The impact of these fuzzy relations is both practical and theoretical according to the FRE system simple solution exist, other than the superior solutions it is easy to apply.

FREs are also used in preventing neuropathy diabetic [10], in which fuzzification of neural networks is used which extends the area of finding the task and applicabilities. First experts dedtection is only based on patients articulate that is compared by medical knowledge, that may lead to various midification and due to patients rejections of certain symptoms may be inappropriate. The proposed detection system uses one committee of Multilayer Perception Neural Networks (MLP) for each one of the entity. Using back propagation algorithm the multilayer percprtron works again and again to remove errors in the network.

S.Jain and K. Lachhwani [11] proposed a methodology for the solution of multi objective programming problem in FRE by introducing constraints, first he found the feasible solution set of FRE and on the basis of this he proposed the algorithm of finding the optimal solution of optimal function by using the computer program that gives an easy way to compute the solution of multi objective programming problem of FRE.

Chapter 6

Evaluation Of Civil Engineering Project Using Fuzzy Relational Equations

The comparative evaluation of civil engineering projects can include a large number of factors but the inclusion of any *soft* factors can weaken the effectiveness of our desirable model. So, in order to design such a model, *soft* factors are usually excluded and *hard* factors are worthwhile. The methodology used in this problem, to evaluate the civil engineering projects, was described by P.N Smith [12] which will help us to recognize a most desirable host of projects suitable in the situation of sudden estimated scheduling depending upon finite non quantitative data. This kind of initial screenings for any projects can also be succeeded by more narrow evaluation of the best subset.
Bilal [13] discusses the different kinds of uncertainty, differentiating among ambiguity and vagueness proposing that civil engineering projects that are usually constructed in a system framework which includes both kinds of uncertainty. Vagueness is defined by fuzzy set theory while ambiguity is defined by probability. Wilhelm and Parsaei [14] suggested a way in order to use the linguistic variables. His method included two linguistic variables which were 'capability' and 'importance'.

In the context of civil engineering project evaluation, every project is described as a number of *factors*. Let us assume two linguistic variables - 'performance' and 'significance'. In this problem, we will denote 'factor' to 'X', 'performance' to 'Y' and 'significance' 'Z'.
Consider that, in a civil engineering project, the 'performance of factors' and 'significance of factors' are known and we are interested to find the 'significance of performances'. 'Significance of performances' usually has a strong effect on a project e.g. if a factor has very high performance but the significance of this factor is nil then one can solve this problem by finding the significance of this

performance using FRE.

'Performance' is described by primary membership values belonging to a base set $Y = [0, 1]$.

Table 6.1

	y_1 0.0	y_2 0.1	y_3 0.2	y_4 0.3	y_5 0.4	y_6 0.5	y_7 0.6	y_8 0.7	y_9 0.8	y_{10} 0.9	y_{11} 1.0
superior	0	0	0	0	0	0	0	0.1	0.25	0.9	1.0
average	0	0.05	0.15	0.4	0.8	1.0	0.8	0.4	0.15	0.05	0.0
poor	1.0	0.8	0.35	0.20	0.05	0	0	0	0	0	0

'Significance' is described by primary membership values belonging to a base set $Z = [0, 1]$.

Table 6.2

	z_1 0.0	z_2 0.1	z_3 0.2	z_4 0.3	z_5 0.4	z_6 0.5	z_7 0.6	z_8 0.7	z_9 0.8	z_{10} 0.9	z_{11} 1.0
critical	0	0	0	0	0	0	0	0.05	0.20	0.85	1.0
important	0	0.10	0.30	0.70	0.90	1.0	0.90	0.70	0.30	0.10	0.0

For the given primary linguistic values, we will define here the secondary linguistic values. For 'performance', secondary linguistic values may be defined as **indeed-superior**, **rather-superior**, **above-average**, **below-average**, **very-poor** and for 'significance' secondary linguistic values may be defined as **indeed-critical**, **rather-critical**, **very-important**, **rather-important**, **not-important**.

In order to define these secondary linguistic values, we denote the primary linguistic values as B(x). So the secondary linguistic values are defined by using the "Baldwin Approach" as showed in Figure 6.1 .

indeed-B(x) = int (B(x))
rather-B(x) = $\sqrt{B(x)}$
very-B(x) = $(B(x))^2$

$$above - B(x) = \begin{cases} \neg B(x) & \text{if } y \geq 0.5 \\ 0 & \text{if } y < 0.5 \end{cases}$$

$$below - B(x) = \begin{cases} \neg B(x) & \text{if } y \leq 0.5 \\ 0 & \text{if } y > 0.5 \end{cases}$$

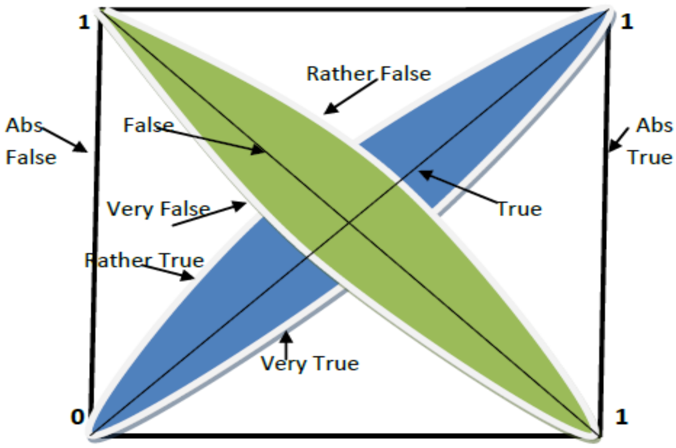

Figure 6.1: Baldwin Approach

The intensification function 'int' is described as

$$int(B(x)) = \begin{cases} 2(B(x))^2 & \text{if } B(x) < 0.5 \\ 1 - 2(1 - B(x))^2 & \text{if } B(x) \geq 0.5 \end{cases}$$

The intensification function increases the high membership values and decreases the low membership values. So the the secondary values for 'performance' and 'significance' are given below in Table 6.3 and 6.4 respectively.

Table 6.3

	y_1 0.0	y_2 0.1	y_3 0.2	y_4 0.3	y_5 0.4	y_6 0.5	y_7 0.6	y_8 0.7	y_9 0.8	y_{10} 0.9	y_{11} 1.0
indeed-superior	0	0	0	0	0	0	0	0.02	0.13	0.98	1.0
rather-superior	0	0	0	0	0	0	0	0.32	0.50	0.95	1.0
above-average	0	0	0	0	0	0	0.20	0.60	0.85	0.95	1.0
below-average	1.0	0.95	0.85	0.60	0.20	0	0	0	0	0	0
very-poor	1.0	0.64	0.12	0.04	0	0	0	0	0	0	0

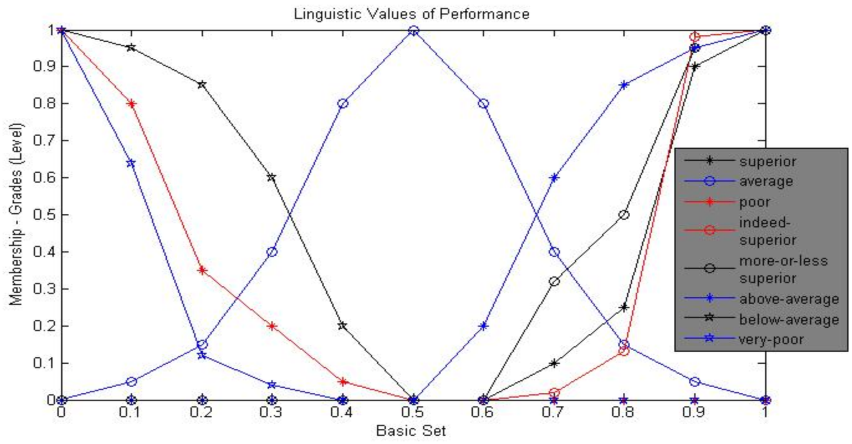

Figure 6.2: Primary and Secondary Linguistic Expressions of "Performance"

Table 6.4

	z_1 0.0	z_2 0.1	z_3 0.2	z_4 0.3	z_5 0.4	z_6 0.5	z_7 0.6	z_8 0.7	z_9 0.8	z_{10} 0.9	z_{11} 1.0
indeed-critical	0	0	0	0	0	0	0	0	0.08	0.96	1.0
rather-critical	0	0	0	0	0	0	0	0.22	0.45	0.92	1.0
very-important	0	0.01	0.09	0.49	0.81	1.0	0.81	0.49	0.09	0.011	0
rather-important	0	0.32	0.55	0.84	0.95	1	0.95	0.84	0.55	0.32	0
not-important	1.0	0.9	0.7	1.3	0.1	0	0.1	0.3	0.7	0.9	1.0

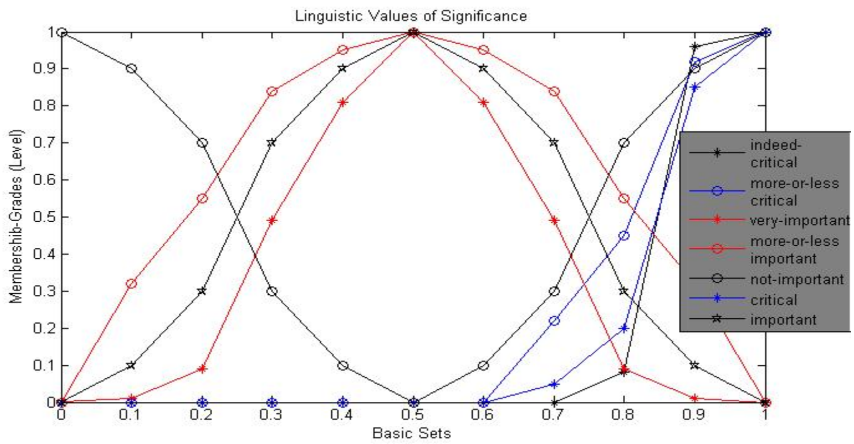

Figure 6.3: Primary and Secondary Linguistic Expressions of "Significance"

Consider the evaluation of best project among 3 civil engineering projects on the basis of 8 factors - capital cost, time management, design and structure, environmental impact, quality control, risk management, human factor, strategic management as follows:

Table 6.5

Factor	Performance of Project 1	Performance of Project 2	Performance of Project 3	Significance
Capital Cost	average	indeed superior	poor	very important
Time Management	indeed superior	below average	superior	critical
Design and Structure	very poor	indeed superior	average	indeed critical
Environmental Impact	below average	poor	superior	rather important
Quality Control	superior	average	poor	important
Risk Management	poor	below average	indeed superior	indeed critical
Human Factor	very poor	above average	superior	important
Strategic Management	above average	poor	average	important

Now we calculate maximum fuzzy relation (Q_{ij}^{∇}) among performance (R_{ij}) of projects i=1,2,3 relative to factors j=1,2,...,8 and the significance (T_j) of factor j, satisfying the FRE $T_j = R_{ij} \circ Q_{ij}$, where "$R_{ij}^{-1}$ @ T_j" is the maximum fuzzy relation.

Since we have to evaluate 3 projects, so we compute the 'significance of performance' for each project.

For Project i = 1:

$$\mathbf{R}_{11} = \begin{bmatrix} 0 & 0.05 & 0.15 & 0.40 & 0.80 & 1.0 & 0.80 & 0.40 & 0.15 & 0.05 & 0 \end{bmatrix}$$

and

$$\mathbf{T_1} = \begin{bmatrix} 0 & 0.01 & 0.09 & 0.49 & 0.81 & 1.0 & 0.81 & 0.49 & 0.09 & 0.01 & 0 \end{bmatrix}$$

$$\mathbf{R_{11}^{-1}@T_1} = \begin{bmatrix} 0 \\ 0.05 \\ 0.15 \\ 0.40 \\ 0.80 \\ 1.0 \\ 0.80 \\ 0.40 \\ 0.15 \\ 0.05 \\ 0 \end{bmatrix} @ \begin{bmatrix} 0 & 0.01 & 0.09 & 0.49 & 0.81 & 1.0 & 0.81 & 0.49 & 0.09 & 0.01 & 0 \end{bmatrix}$$

$$\mathbf{Q}_{11}^{\nabla} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0.01 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0.01 & 0 \\ 0 & 0.01 & 0.09 & 1 & 1 & 1 & 1 & 1 & 0.09 & 0.01 & 0 \\ 0 & 0.01 & 0.09 & 1 & 1 & 1 & 1 & 1 & 0.09 & 0.01 & 0 \\ 0 & 0.01 & 0.09 & 0.49 & 1 & 1 & 1 & 0.49 & 0.09 & 0.01 & 0 \\ 0 & 0.01 & 0.09 & 0.49 & 0.81 & 1 & 0.81 & 0.49 & 0.09 & 0.01 & 0 \\ 0 & 0.01 & 0.09 & 0.49 & 1 & 1 & 1 & 0.49 & 0.09 & 0.01 & 0 \\ 0 & 0.01 & 0.09 & 1 & 1 & 1 & 1 & 1 & 0.09 & 0.01 & 0 \\ 0 & 0.01 & 0.09 & 1 & 1 & 1 & 1 & 1 & 0.09 & 0.01 & 0 \\ 0 & 0.01 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0.01 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Check:

$$\mathbf{R_{11}} \circ \mathbf{Q_{11}^{\nabla}} = \begin{bmatrix} 0 & 0.01 & 0.09 & 0.49 & 0.81 & 1.0 & 0.81 & 0.49 & 0.09 & 0.01 & 0 \end{bmatrix} = \mathbf{T}_1$$

Similar calculations are done for j=2,3...,8 and the intersection of all these fuzzy relations can be found as $Q_i = \cap_{j=1,2,\dots,8} Q_{ij} = \wedge_{j=1,2,\dots,8} Q_{ij}$, where "$\wedge$" denotes the "min" function.
So $Q_1 = \cap_{j=1,2,\dots,8} Q_{1j}$

$$\mathbf{Q}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.09 & 0.01 & 0 \\ 0 & 0.01 & 0.09 & 0.49 & 0.81 & 1 & 0.81 & 0.49 & 0.09 & 0.01 & 0 \\ 0 & 0.01 & 0.09 & 0.49 & 1 & 1 & 1 & 0.49 & 0.09 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.09 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0.09 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0.20 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0.20 & 0.10 & 0 \end{bmatrix}$$

For Project i = 2:

$$\mathbf{R}_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.02 & 0.13 & 0.98 & 1 \end{bmatrix}$$

and

$$\mathbf{T_1} = \begin{bmatrix} 0 & 0.01 & 0.09 & 0.49 & 0.81 & 1.0 & 0.81 & 0.49 & 0.09 & 0.01 & 0 \end{bmatrix}$$

$$\mathbf{R_{21}^{-1}}@\mathbf{T_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.02 \\ 0.13 \\ 0.98 \\ 1 \end{bmatrix} @ \begin{bmatrix} 0 & 0.01 & 0.09 & 0.49 & 0.81 & 1.0 & 0.81 & 0.49 & 0.09 & 0.01 & 0 \end{bmatrix}$$

$$\mathbf{Q}_{21}^{\nabla} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0.01 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0.01 & 0 \\ 0 & 0.01 & 0.09 & 1 & 1 & 1 & 1 & 1 & 0.09 & 0.01 & 0 \\ 0 & 0.01 & 0.09 & 0.49 & 0.81 & 1 & 0.81 & 0.49 & 0.09 & 0.01 & 0 \\ 0 & 0.01 & 0.09 & 0.49 & 0.81 & 1 & 0.81 & 0.49 & 0.09 & 0.01 & 0 \end{bmatrix}$$

Check:

$$\mathbf{R_{21}} \circ \mathbf{Q_{21}^{\nabla}} = \begin{bmatrix} 0 & 0.01 & 0.09 & 0.49 & 0.81 & 1.0 & 0.81 & 0.49 & 0.09 & 0.01 & 0 \end{bmatrix} = \mathbf{T}_1$$

For i=2, similar calculations are done for j=2,3,...,8 and so the intersection of all these fuzzy relations is $Q_2 = \cap_{j=1,2,...,8} Q_{2j}$.
So

$$\mathbf{Q}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.10 & 0 \\ 0 & 0.10 & 0.30 & 0.70 & 0.90 & 1 & 0.90 & 0.70 & 0.30 & 0.10 & 0 \\ 0 & 0.10 & 0.30 & 0.70 & 1 & 1 & 1 & 0.70 & 0.30 & 0.10 & 0 \\ 0 & 0.01 & 0.30 & 1 & 1 & 1 & 1 & 1 & 0.30 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0.09 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.01 & 0 \end{bmatrix}$$

For Project i = 3:

$$\mathbf{R}_{31} = \begin{bmatrix} 1 & 0.80 & 0.35 & 0.20 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{T_1} = \begin{bmatrix} 0 & 0.01 & 0.09 & 0.49 & 0.81 & 1.0 & 0.81 & 0.49 & 0.09 & 0.01 & 0 \end{bmatrix}$$

$$\mathbf{R_{31}^{-1}} @ \mathbf{T_1} = \begin{bmatrix} 1.0 \\ 0.80 \\ 0.35 \\ 0.20 \\ 0.05 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} @ \begin{bmatrix} 0 & 0.01 & 0.09 & 0.49 & 0.81 & 1.0 & 0.81 & 0.49 & 0.09 & 0.01 & 0 \end{bmatrix}$$

$$\mathbf{Q}_{31}^{\nabla} = \begin{bmatrix} 0 & 0.01 & 0.09 & 0.49 & 0.81 & 1 & 0.81 & 0.49 & 0.09 & 0.01 & 0 \\ 0 & 0.01 & 0.09 & 0.49 & 1 & 1 & 1 & 0.49 & 0.09 & 0.01 & 0 \\ 0 & 0.01 & 0.09 & 1 & 1 & 1 & 1 & 1 & 0.09 & 0.01 & 0 \\ 0 & 0.01 & 0.09 & 1 & 1 & 1 & 1 & 1 & 0.09 & 0.01 & 0 \\ 0 & 0.01 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0.01 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Check:

$$\mathbf{R_{31}} \circ \mathbf{Q_{31}^{\nabla}} = \begin{bmatrix} 0 & 0.01 & 0.09 & 0.49 & 0.81 & 1.0 & 0.81 & 0.49 & 0.09 & 0.01 & 0 \end{bmatrix} = \mathbf{T}_1$$

For i=3, similar calculations are done against j=2,3...,8 and so the intersection of all these fuzzy relations is $Q_3 = \cap_{j=1,2,...,8} Q_{3j}$.
So

$$\mathbf{Q}_3 = \begin{bmatrix} 0 & 0.01 & 0.09 & 0.49 & 0.81 & 1.0 & 0.81 & 0.49 & 0.09 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.09 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.20 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.20 & 0.1 & 0 \end{bmatrix}$$

If the 'significance' of each factor in the project is defined by the administration as "indeed- critical" then we solve the relational equation for i = 1,2,3, where 'i' denotes the number of projects. Now compute the $y^{-1} = Q_i \;@\; (indeed-critical)^{-1}$ for i = 1,2,3. Calculations yield following results:

For Project 1

$y_1^{-1} = Q_1 \;@\; (indeed - critical)^{-1} = [1\ 1\ 1\ 1\ 0.08\ 0\ 0\ 0\ 0\ 0\ 0]^T$
$y_1 = [1\ 1\ 1\ 1\ 0.08\ 0\ 0\ 0\ 0\ 0\ 0]$

For Project 2

$y_2^{-1} = Q_2 \;@\; (indeed - critical)^{-1} = [1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1]^T$
$y_2 = [1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1]$

For Project 3

$y_3^{-1} = Q_3 \;@\; (indeed - critical)^{-1} = [0\ 0.08\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0.08\ 0.08]^T$
$y_3 = [0\ 0.08\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0.08\ 0.08]$

Defuzzification Using Mean Of Maximum Method: Defuzzification's methods are usually used to rank the projects. Mean of maximum method is one of the methods for defuzzifying [15].

$$d_{MM}(F) = \sum_{y_k \epsilon M} (y_k)/|M| \tag{6.1}$$

where $M = \{y_k | F(y_k) = hgt(F)\}$
F(y) is the membership function of a fuzzy set F. "hgt(F)" is the maximum membership value of the fuzzy set F and $|M|$ is known as the cardinality of M.

For project 1, $M = \{0, 0.1, 0.2, 0.3\}$
For project 2, $M = \{0, 0.1, 0.2, 0.3, 0.4, 0.8, 0.9, 1.0\}$
For project 3, $M = \{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$

So mean of maximum method gives us corresponding values

MM1 = 0.15 for project 1
MM2 = 0.46 for project 2
MM3 = 0.50 for project 3

Since MM3 give us the best performance, so this implies that project 3 shows best performance.

Chapter 7

Conclusion

We concludes that we can distinguish an optimal solution of a problem among the bunch of solutions by using FREs. It can be established after performing lot of calculations using FREs, that in our suggested scenarios of civil engineering, the chosen factors (capital cost, human factor, risk management etc.) have shown variable results. These results have led us to pick the best project showing good outcome among the given scenarios and herein the fuzzy relation operations were applied to evaluate the best outcome which is ideal in our case scenario. Each project was evaluated against the aforementioned factors, where the performance and significance of each factor is described in terms of linguistic expressions (defined as a fuzzy set). Then, the results of our problem led us to an optimal project among the chosen projects depending on the quality of outcome.

A possible future research can be led to optimize the factors instead of investigating the best project. In other words, such combination of linguistic variables against the abstracted factors should be focussed, whereon an ideal project can be designed.

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