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Fuzzy Relations

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Abstract

One of the most fundamental notions in pure and applied sciences is the concept of a relation. Science has been described as the discovery of relations between objects, states and events. Fuzzy relations generalize the concept of relations in the same manner as fuzzy sets generalize the fundamental idea of sets. This work presents an overview of comparison between classical and fuzzy relations. Some important compositions of fuzzy relations have been described and using these compositions a model for predicting score in cricket is developed. Finally it deals with the restoration and the identification of the causes (diagnosis) through the observed effects (symptoms) on the basis of fuzzy relations.

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1.1 Introduction

In 1965, L. A. Zadeh introduced the concept of fuzzy set theory. Fuzzy set theory is an extension of classical set theory. A logic that is not very precise is called a fuzzy logic. The imprecise way of looking at things and manipulating them is much more powerful than precise way of looking at them and then manipulating them. Fuzzy logic is one of the tools for making computer system capable of solving problems involving imprecision. Fuzzy logic is an attempt to capture imprecision by generalizing the concept of set to fuzzy set.

In every day content most of the problems involve imprecise concept. To handle the imprecise concept, the conventional method of set theory and numbers are insufficient and need to be extended to some other concepts. Fuzzy concept is one of the concepts for this purpose.

A relation is a mathematical description of a situation where certain elements of sets are related to one another in some way. Fuzzy relations are significant concepts in fuzzy theory and have been widely used in many fields such as fuzzy clustering, fuzzy control and uncertainty reasoning. They also play an important role in fuzzy diagnosis and fuzzy modeling. When fuzzy relations are used in practice, how to estimate and compare them is a significant problem. Uncertainty measurements of fuzzy relations have been done by some researchers. Similarity measurement of uncertainty was introduced by Yager who also discussed its application.

1.2 Crisp Relation

To describe the fuzzy relation, first we describe relation by an example of daily life using discrete fuzzy sets. Relationship is described between the colours of a fruit X and the grade of maturity Y. Crisp set X with three linguistic terms is given as

$$X = \{\text{green, yellow, red}\}$$

Similarly the grade of maturity for the other set Y will be

$$Y = \{\text{verdant, half-mature, mature}\}$$

Crisp formulation of a relation $X \rightarrow Y$ between two crisp sets is presented in tabular form

	Verdant	Half-mature	Mature
Green	1	0	0
Yellow	0	1	0
Red	0	0	1

In the above table "0" and "1" describe the grade of membership to this relation. This relation is a new kind of crisp set that is built from the two crisp base set X and Y. This new set is now called R and can be expressed by the rules

1. If the colour of the fruit is green then the fruit is verdant.
2. If the colour of the fruit is yellow then the fruit is half-mature.
3. If the colour of the fruit is red then the fruit is mature.

This crisp relation shows the existence or absence of connection, relations or interconnection between two sets. Now we show the membership grades represented in the fuzzy relation.

	Verdant	Half-mature	Mature
Green	1	0.6	0
Yellow	0.4	1	0.3
Red	0	0.5	1

The table above represents the fuzzy relation.

Crisp relation is defined on the Cartesian product of two universal sets determined as

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

The crisp relation R is defined by its membership function

$$\mu_R(x, y) = \begin{cases} 1, & (x, y) \in R \\ 0, & (x, y) \notin R \end{cases}$$

Here "1" implies complete truth degree for the pair to be in relation and "0" implies no relation.

When the sets are finite the relation is represented by a matrix R called a relation matrix.

1.2.1 Example

Let $X = \{1, 4, 5\}$ and $Y = \{3, 6, 7\}$

Classical matrix for the crisp relation when $R = x < y$ is

$$R = \begin{array}{c} \\ 1 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{matrix} 3 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

1.2.2 Example

Let $A = \{2, 4, 6, 8\}$ and $B = \{2, 4, 6, 8\}$

Classical matrix for the crisp relation $R = x = y$

$$R = \begin{array}{c} \\ 2 \\ 4 \\ 6 \\ 8 \end{array} \begin{array}{c} \begin{matrix} 2 & 4 & 6 & 8 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

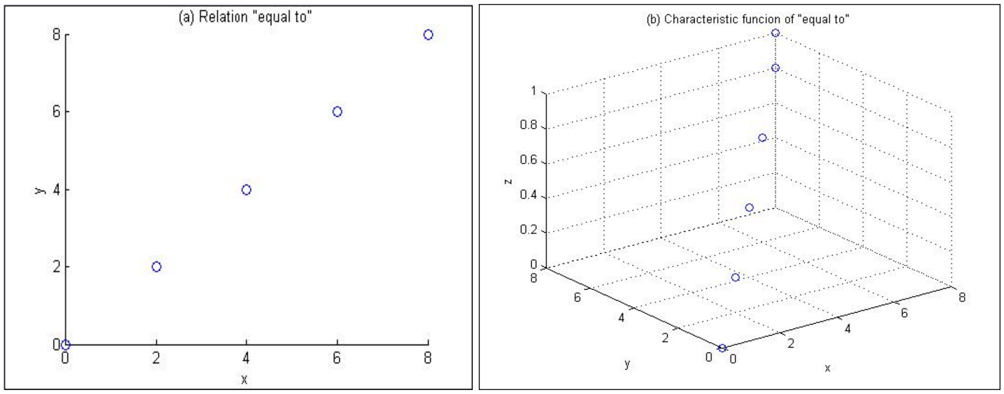

Figure 1.1 Relation “equal to” and its characteristic function

1.3 Fuzzy relation

Let X, $Y \subseteq R$ be universal sets then;

$$R = \left\{ \left((x, y), \mu_R(x, y) \right) \mid (x, y) \in X \times Y \right\}$$

is called a fuzzy relation in $X \times Y \subseteq \mathrm{R}$

or X and Y are two universal sets, the fuzzy relation $R(x, y)$ is given as

$$R(x, y) = \left\{ \frac{\mu_R(x, y)}{(x, y)} \mid (x, y) \in X \times Y \right\}$$

Fuzzy relations are often presented in the form of two dimensional tables. A $m \times n$ matrix represents a contented way of entering the fuzzy relation R.

$$R = \begin{array}{c} \\ x_1 \\ \vdots \\ x_m \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & \cdots & y_n \end{array} \\ \begin{bmatrix} \mu_R(x_1, y_1) & \cdots & \mu_R(x_1, y_n) \\ \vdots & \ddots & \vdots \\ \mu_R(x_m, y_1) & \cdots & \mu_R(x_m, y_n) \end{bmatrix} \end{array}$$

1.3.1 Example

Let $X = \{1, 2, 3\}$ and $Y = \{1, 2\}$

If the membership function associated with each order pair (x, y) is given by

$$\mu_R(x, y) = e^{-(x-y)^2}$$

then derive fuzzy relation.

Solution

The fuzzy relation can be defined in two ways using the standard nomenclature we have.

$$R = \left\{ \frac{e^{-(1-1)^2}}{(1,1)}, \frac{e^{-(1-2)^2}}{(1,2)}, \frac{e^{-(2-1)^2}}{(2,1)}, \frac{e^{-(2-2)^2}}{(2,2)}, \frac{e^{-(3-1)^2}}{(3,1)}, \frac{e^{-(3-2)^2}}{(3,2)} \right\}$$

$$R = \left\{ \frac{1.0}{(1,1)}, \frac{0.37}{(1,2)}, \frac{0.37}{(2,1)}, \frac{1.0}{(2,2)}, \frac{0.02}{(3,1)}, \frac{0.37}{(3,2)} \right\}$$

In the second method using the relational matrix, we have

$$R = \begin{bmatrix} 1 & 0.37 \\ 0.37 & 1 \\ 0.02 & 0.37 \end{bmatrix}$$

Thus the membership function describes the closeness between set X and Y. From the relational matrix it is obvious that higher values imply stronger relation.

1.4 The maximum-minimum composition of relations

Let X, Y and Z be universal sets and let R be a relation that relates elements from X to Y, i.e.

$$R = \{((x,y), \mu_R(x,y))\} \quad x \in X, y \in Y, R \subset X \times Y$$

and

$$Q = \{((y,z), \mu_Q(y,z))\} \quad y \in Y, z \in Z, Q \subset Y \times Z$$

Then S will be a relation that relates elements in X that R contains to the elements in Z that Q contains, i.e.

$$S = R \circ Q$$

Here "$\circ$" means the composition of membership degrees of R and Q in the max-min sense.

$$S = \{((x,z), \mu_s(x,z))\} \quad x \in X, z \in Z, S \subset X \times Z$$

max-min composition is then defined as

$$\mu_S(x,z) = \max_{y \in Y}\left(min\left(\mu_R(x,y), \mu_Q(y,z)\right)\right)$$

and max product composition is then defined

$$\mu_S(x,z) = \max_{y \in Y}\left(min\left(\mu_R(x,y) . \mu_Q(y,z)\right)\right)$$

1.4.1 Example

Let $X = \{x_1, x_2\}$ *and* $Y = \{y_1, y_2\}$ *and* $Z = \{z_1, z_2\}$

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then find the max-min composition and max product composition

$$S = R \circ Q$$

$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the max-min composition.

and $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

is the max product composition

For crisp relations max-min composition and max product will yield the same result, when X has three elements, Y has four elements and Z has two elements like

$X = \{x_1, x_2, x_3\}$ *and* $Y = \{y_1, y_2, y_{3,} y_4\}$ *and* $Z = \{z_1, z_2\}$ then for relations

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}$$

$$Q = \begin{matrix} & x_1 & x_2 \\ y_1 & 0 & 1 \\ y_2 & 0 & 0 \\ y_3 & 0 & 1 \\ y_4 & 0 & 0 \end{matrix}$$

the max-min composition is

$$S = \begin{matrix} & z_1 & z_2 \\ x_1 & 0 & 1 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{matrix}$$

In this example max-min composition and max product have the same result.

1.5 Fuzzy max-min composition operation

Let us consider two fuzzy relations R_1 and R_2 defined on a Cartesian space $X \times Y$ and $Y \times Z$ respectively. The max-min composition of R_1 and R_2 is a fuzzy set defined on a cartesian spaces $X \times Z$ as

$$R_1 \circ R_2 = \left[(x,z), \max\left\{ \min\left\{ \mu_{R_1}(x,y), \mu_{R_2}(y,z) \right\} \right\} \right] | x \in X, y \in Y, z \in Z$$

where $R_1 \circ R_2$ is the max-min composition of fuzzy relations R_1 and R_2 and max product composition is defined as

$$\mu_{R_1 \circ R_2} = \max\left[\mu_{R_1}(x,y) . \mu_{R_2}(y,z) \right] | x \in X, y \in Y, z \in Z$$

1.5.1 Example

Let $R_1(x,y)$ and $R_2(x,y)$ be defined as the following relational matrices

$$R_1 = \begin{bmatrix} 0.6 & 0.5 \\ 1 & 0.1 \\ 0 & 0.7 \end{bmatrix} \quad \text{and } R_2 = \begin{bmatrix} 0.7 & 0.3 & 0.4 \\ 0.9 & 0.1 & 0.6 \end{bmatrix}$$

We shall first calculate the max-min composition $R_1 \circ R_2$

$$R_1 \circ R_2 = \begin{bmatrix} 0.6 & 0.5 \\ 1 & 0.1 \\ 0 & 0.7 \end{bmatrix} \circ \begin{bmatrix} 0.7 & 0.3 & 0.4 \\ 0.9 & 0.1 & 0.6 \end{bmatrix}$$

Now we calculate

$$\mu_{R_1 \circ R_2}(x_1, z_1) = \max(\min(0.6, 0.7), \min(0.5, 0.9)) = \max(0.6, 0.5) = 0.6$$

Similarly we can calculate the other entries. The relational matrix for max-min composition in fuzzy relation is thus

$$R_1 o R_2 = \begin{bmatrix} 0.6 & 0.3 & 0.5 \\ 0.7 & 0.3 & 0.4 \\ 0.7 & 0.1 & 0.6 \end{bmatrix}$$

1.5.2 Example

Let $R_1(x, y)$ *and* $R_2(x, y)$ be defined by the following relational matrix

$$R_1 = \begin{matrix} & y_1 & y_2 & y_3 & y_4 & y_5 \\ x_1 & 0.1 & 0.2 & 0 & 1 & 0.7 \\ x_2 & 0.3 & 0.5 & 0 & 0.2 & 1 \\ x_3 & 0.8 & 0 & 1 & 0.4 & 0.3 \end{matrix}$$

$$R_2 = \begin{matrix} & z_1 & z_2 & z_3 & z_4 \\ y_1 & 0.9 & 0 & 0.3 & 0.4 \\ y_2 & 0.2 & 1 & 0.8 & 0 \\ y_3 & 0.8 & 0 & 0.7 & 1 \\ y_4 & 0.4 & 0.2 & 0.3 & 0 \\ y_5 & 0 & 1 & 0 & 0.8 \end{matrix}$$

we shall first compute the max-min composition $R_1 \circ R_2(x, z)$

$$\mu_{R_1 \circ R_2}(x_1, z_1) = \max(\min(0.1, 0.9), \min(0.2, 0.2), \min(0, 0.8), \min(1, 0.4), \min(0.7, 0))$$

$$= \max(0.1, 0.2, 0, 0.4, 0) = 0.4$$

Similarly we can determine the grades of membership for all pairs

$$(x_i, z_i), i = 1, 2, 3, \; j = 1, \ldots\ldots 4$$

$$R_1 o R_2 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{bmatrix} 0.4 & 0.7 & 0.3 & 0.7 \\ 0.3 & 1 & 0.5 & 0.8 \\ 0.8 & 0.3 & 0.7 & 1 \end{bmatrix} \end{array}$$

for the max product composition, we calculate

$$\mu_{R_1}(x_1, y_1).\mu_{R_2}(y_1, z_1) = 0.1.0.9 = 0.09$$

$$\mu_{R_1}(x_1, y_2).\mu_{R_2}(y_2, z_1) = 0.2.0.2 = 0.04$$

$$\mu_{R_1}(x_1, y_3).\mu_{R_2}(y_3, z_1) = 0.0.8 = 0$$

$$\mu_{R_1}(x_1, y_4).\mu_{R_2}(y_4, z_1) = 1.0.4 = 0.4$$

$$\mu_{R_1}(x_1, y_5).\mu_{R_2}(y_5, z_1) = 0.7.0 = 0$$

hence

$$\mu_{R_1 o R_2}(x_1, z_1) = \max\{0.09, 0.04, 0, 0.4, 0\} = 0.4$$

In the similar way after performing the remaining computation, we obtain

$$R_1 o R_2 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \end{array} \\ \begin{bmatrix} 0.4 & 0.7 & 0.3 & 0.56 \\ 0.27 & 1 & 0.4 & 0.8 \\ 0.8 & 0.3 & 0.7 & 1 \end{bmatrix} \end{array}$$

1.6 Conclusion

It is clear from the example that max-min composition and max product composition of crisp relations will yield the same result, but in fuzzy max-min composition and max product composition have different result.

2.1 Projection of Fuzzy Relation

Let $R = \{((x, y), \mu_R(x, y)) \mid (x, y) \in X \times Y\}$ be a fuzzy relation. The projection of $R(x, y)$ on X denoted by R_1 is given by

$$R_1 = \left\{\left(x, \max_y \mu_R(x, y)\right) \mid (x, y) \in X \times Y\right\}$$

and the projection of $R(x, y)$ on Y denoted by R_2 is given by

$$R_2 = \left\{\left(y, \max_x \mu_R(x, y)\right) \mid (x, y) \in X \times Y\right\}$$

2.1.1 Example

Let R be a fuzzy relation defined by the following relational matrix

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.4 & 0.8 & 1 & 0.8 \\ 0.2 & 0.4 & 0.8 & 1 & 0.8 & 0.6 \\ 0.4 & 0.8 & 1 & 0.8 & 0.4 & 0.2 \end{bmatrix} \end{array}$$

The projection of $R(x, y)$ on X is calculated as, e.g.

$$\mu_{R_1}(x_1) = \max\{0.1, 0.2, 0.4, 0.8, 1, 0.8\} = 1$$

In the similar way can calculate the grades of membership for all pairs, so the X projection is

$$R_1 = \{(x_1, 1), (x_2, 1), (x_3, 1)\}$$

The projection of $R(x, y)$ on Y is calculated as, e.g.

$$\mu_{R_1}(y_1) = \max\{0.1, 0.2, 0.4\} = 0.4$$

In the similar way we can determine the membership grade for all other pairs, so the Y projection

$$R_2 = \{(y_1, 0.4), (y_2, 0.8), (y_3, 1), (y_4, 1), (y_5, 1), (y_6, 0.8)\}$$

2.2 Cylindrical extension of fuzzy relation

The cylindrical extension on $X \times Y$ of a fuzzy set A of X is a fuzzy relation cylA whose membership function is equal to

$$cylA(x, y) = A(x), \qquad \forall\, x \in X, \qquad \forall\, y \in Y$$

Cylindrical extension from *X*-projection means filling all the columns of the related matrix by the X -projection. Similarly cylindrical extension from Y projection means filling all the rows of the relational matrix by the Y -projection.

2.2.1 Example

The cylindrical extension of R_2 form the previous example is

$$R_2 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{array} \\ \begin{bmatrix} 0.4 & 0.8 & 1 & 1 & 1 & 0.8 \\ 0.4 & 0.8 & 1 & 1 & 1 & 0.8 \\ 0.4 & 0.8 & 1 & 1 & 1 & 0.8 \end{bmatrix} \end{array}$$

2.3 Reflexive Relation

Let R be a fuzzy relation in $X \times X$ then R is called reflexive, if

$$\mu_R(x, x) = 1 \qquad \forall x \in X$$

2.3.1 Example

Let $X = \{1, 2, 3, 4\}$

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0.9 & 0.6 & 0.2 \\ 0.9 & 1 & 0.7 & 0.3 \\ 0.6 & 0.7 & 1 & 0.9 \\ 0.2 & 0.3 & 0.9 & 1 \end{bmatrix} \end{array}$$

is reflexive relation

2.4 Antireflexive relations

Fuzzy relation $R \subset X \times X$ is antireflexive if

$$\mu_R(x,x) = 0, x \in X$$

2.4.1 Example

$$R_1 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0.6 \\ 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \end{array}$$ is antireflexive relation

2.5 Symmetric Relation

A fuzzy relation R is called symmetric if,

$$\mu_R(x,y) = \mu_R(y,x) \quad \forall x, y \in X$$

2.5.1 Example

Let $X = \{x_1, x_2, x_3\}$

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.7 \\ 0.1 & 1 & 0.6 \\ 0.7 & 0.6 & 0.5 \end{bmatrix} \end{array}$$ is a symmetric relation.

2.6 Antisymmetric Relation

Fuzzy relation $R \subset X \times X$ is antisymmetric iff

$$\text{if } \mu_R(x,y) > 0 \text{ then } \mu_R(y,x) = 0 x, y \in X, x \neq y$$

2.6.1 Example

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0.7 \\ 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix} \end{array} \text{ is antisymmetric relation.}$$

2.7 Transitive Relation

Fuzzy relation $R \subset X \times X$ is transitive in the sense of max-min iff

$$\mu_R(x,z) \geq \max_{y \in X}\left(\min\left(\mu_R(x,y), \mu_R(y,z)\right)\right) x, z \in X$$

since $R^2 = R \circ R$ if

$$\mu_{R^2}(x,z) = \max_{y \in X}\left(\mu_R(x,y), \mu_R(y,z)\right)$$

then R is transitive if $R \circ R = R\left(R \circ R \subseteq R\right)$

and $R^2 \subset R$ means that $\mu_{R^2}(x,y) \leq \mu_R(y,x)$

2.7.1 Example

Let $X = \{x_{1,} x_{2,}, x_3\}$

is $R = \begin{bmatrix} 0.7 & 0.9 & 0.4 \\ 0.1 & 0.3 & 0.5 \\ 0.2 & 0.1 & 0 \end{bmatrix}$ a transitive relation?

Solution

$$R \circ R = \begin{bmatrix} 0.7 & 0.9 & 0.4 \\ 0.1 & 0.3 & 0.5 \\ 0.2 & 0.1 & 0 \end{bmatrix} \circ \begin{bmatrix} 0.7 & 0.9 & 0.4 \\ 0.1 & 0.3 & 0.5 \\ 0.2 & 0.1 & 0 \end{bmatrix}$$

$$R^2 = \begin{bmatrix} 0.7 & 0.7 & 0.5 \\ 0.2 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.2 \end{bmatrix}$$

Since $\mu_{R^2}(x_i, x_j)$ is not always less than or equal to $\mu_R(x_i, x_j)$, hence R is not transitive.

2.7.2 Example

Let $X = \{x_{1,} x_{2,}\}$

$$\text{is} \quad R = \begin{bmatrix} 0.4 & 0.2 \\ 0.7 & 0.3 \end{bmatrix} \text{a transitive relation?}$$

Solution

$$R \circ R = \begin{bmatrix} 0.4 & 0.2 \\ 0.7 & 0.3 \end{bmatrix} \circ \begin{bmatrix} 0.4 & 0.2 \\ 0.7 & 0.3 \end{bmatrix}$$

using max-min composition

$$R^2 = \begin{bmatrix} \max(\min(0.4, 0.4), \min(0.2, 0.7)) & \max(\min(0.4, 0.2), \min(0.2, 0.3)) \\ \max(\min(0.7, 0.4), \min(0.3, 0.7)) & \max(\min(0.7, 0.2), \min(0.3, 0.3)) \end{bmatrix}$$

$$R^2 = \begin{bmatrix} \max(0.4, 0.2) & \max(0.2, 0.2) \\ \max(0.4, 0.3) & \max(0.2, 0.3) \end{bmatrix}$$

$$R^2 = \begin{bmatrix} 0.4 & 0.2 \\ 0.4 & 0.3 \end{bmatrix}$$

$\mu_{R^2}(x_i, x_j)$ is less than or equal to $\mu_R(x_i, x_j)$, so R is transitive.

2.8 Similarity Relations

$R \subset X \times X$ which is reflexive, symmetric and transitive is called the similarity relation.

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{bmatrix} 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ 0.2 & 1 & 0.2 & 0.2 & 0.8 & 0.2 \\ 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ 0.6 & 0.2 & 0.6 & 1 & 0.2 & 0.8 \\ 0.2 & 0.8 & 0.2 & 0.2 & 1 & 0.2 \\ 0.6 & 0.2 & 0.6 & 0.8 & 0.2 & 1 \end{bmatrix} \end{array}$$

is a similarity relation.

2.8.1 Theorem

Each equivalence class $R[X]$ is given as

$$R[X] = \bigcup_{\alpha} \alpha \big/ R_{\alpha}[X], \alpha \in [0,1]$$

where $R_{\alpha}[X]$ is the α-cut of $R[\alpha]$.

2.8.2 Definition

$A \subset X$, A is a fuzzy set the α -cut of A is a non fuzzy set denoted by A_{α} and defined by

$$A_{\alpha} = \{x : \mu_A(x) \geq \alpha\}, \quad \alpha \in [0,1]$$

2.8.3 Example

For $R[x_1]$ we have

$$R_{0.2}[x_1] = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$

$$0.2 \big/ R_{0.2}[x_1] = 0.2/x_1 + 0.2/x_2 + 0.2/x_3 + 0.2/x_4 + 0.2/x_5 + 0.2/x_6$$

$$R_{0.6}[x_1] = \{x_1, x_3, x_4, x_6\}$$

$$0.6 \big/ R_{0.6}[x_1] = 0.6/x_1 + 0.6/x_3 + 0.6/x_4 + 0.6/x_6$$

$$R_1[x_1] = \{x_1, x_3\}$$

$$1 \big/ R_1[x_1] = 1/x_1 + 1/x_3$$

Equivalence class for $R[x_1]$

$$R[x_1] = 0.2/x_1 + 0.2/x_2 + 0.2/x_3 + 0.2/x_4 + 0.2/x_5 + 0.2/x_6 + 0.6/x_1 + 0.6/x_3 + 0.6/x_4 + 0.6/x_6 + 1/x_1 + 1/x_3$$

$$R[x_1] = \max(0.2, 0.6, 1)\big/x_1 + 0.2/x_2 + \max(0.2, 0.6, 1)\big/x_3 + \max(0.2, 0.6)\big/x_4 + 0.2/x_5 + \max(0.2, 0.6)\big/x_6$$

$$R[x_1] = 1/x_1 + 0.2/x_2 + 1/x_3 + 0.6/x_4 + 0.2/x_5 + 0.6/x_6$$

2.8.4 Example

Equivalence class for the similarity relation R is

$$R[x_1] = \frac{1}{x_1} + \frac{0.2}{x_2} + \frac{1}{x_3} + \frac{0.6}{x_4} + \frac{0.2}{x_5} + \frac{0.6}{x_6}$$

$$R[x_2] = \frac{0.2}{x_1} + \frac{1}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4} + \frac{0.8}{x_5} + \frac{0.2}{x_6}$$

$$R[x_3] = \frac{1}{x_1} + \frac{0.2}{x_2} + \frac{1}{x_3} + \frac{0.6}{x_4} + \frac{0.2}{x_5} + \frac{0.6}{x_6}$$

$$R[x_4] = \frac{0.6}{x_1} + \frac{0.2}{x_2} + \frac{0.6}{x_3} + \frac{1}{x_4} + \frac{0.2}{x_5} + \frac{0.8}{x_6}$$

$$R[x_5] = \frac{0.2}{x_1} + \frac{0.8}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4} + \frac{1}{x_5} + \frac{0.2}{x_6}$$

$$R[x_6] = \frac{0.6}{x_1} + \frac{0.2}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{0.2}{x_5} + \frac{1}{x_6}$$

2.9 Antisimilarity Relation

If R is a similarity relation then the complement of R is antisimilarity relation.

$R \subset X \times X$ is a antisimilarity relation if

$$\mu_{R'}(x, y) = 1 - \mu_R(x, y)$$

The antisimilarity relation is antireflexive, symmetric and transitive in the sense of max-min, i.e.

$$\mu_{R'}(x, z) \geq \min_{y \in X}\left(\max\left(\mu_{R'}(x, y), \mu_{R'}(y, z)\right)\right) x, z \in R$$

2.9.1 Example

Prove that $R = \begin{bmatrix} 1 & 0.1 & 0.7 \\ 0.1 & 1 & 0.7 \\ 0.7 & 0.7 & 1 \end{bmatrix}$ is antisimilarity relation?

Solution

According to definition of antisimilarity relation

$$\mu_{R'}(x,y) = 1 - \mu_R(x,y)$$

$$\mu_{R'}(x,y) = 1 - \begin{bmatrix} 1 & 0.1 & 0.7 \\ 0.1 & 1 & 0.7 \\ 0.7 & 0.7 & 1 \end{bmatrix}$$

$$\mu_{R'}(x,y) = \begin{bmatrix} 0 & 0.9 & 0.3 \\ 0.9 & 0 & 0.3 \\ 0.3 & 0.3 & 0 \end{bmatrix}$$

This is anti-reflexive, symmetric and transitive, so R is antisimilarity relation.

2.10 Weak Similarity

$R \subset X \times X$ which is reflexive and symmetric is called the relation of weak similarity (not transitive).

$$R = \begin{bmatrix} 1 & 0.1 & 0.8 & 0.2 & 0.3 \\ 0.1 & 1 & 0 & 0.3 & 1 \\ 0.8 & 0 & 1 & 0.7 & 0 \\ 0.2 & 0.3 & 0.7 & 1 & 0.6 \\ 0.3 & 1 & 0 & 0.6 & 1 \end{bmatrix} \quad \text{is weak similarity relation}$$

2.11 Order Relation

An order relation $R \subset X \times X$ is transitive relation in the sense of max-min; i.e

$$\mu_R(x,z) \geq \max_{y \in X}\left(\min\left(\mu_R(x,y), \mu_R(y,z)\right)\right), x, z \in X$$

2.12 Pre Order Relations

A pre order relation $R \subset X \times X$ is reflexive and transitive in the max-min sense e.g.

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 0.7 & 0.8 & 0.5 & 0.5 \\ 0 & 1 & 0.3 & 0 & 0.2 \\ 0 & 0.7 & 1 & 0 & 0.2 \\ 0.6 & 1 & 0.9 & 1 & 0.6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

2.13 Half Order Relation

A fuzzy half order is a relation $R \subset X \times X$ which is reflexive

$$\mu_R(x, x) = 1 \qquad \forall x \in X$$

and weakly antisymmetric, i.e.

$$\text{if} \quad \mu_R(x, y) > 0 \text{ and } \mu_R(y, x) > 0 \quad \text{then} \quad x = y$$

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{bmatrix} 1 & 0.8 & 0.2 & 0.6 & 0.6 & 0.4 \\ 0 & 1 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 1 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad \text{is half order relation}$$

3.1 Fuzzy Graph

In 1975, Rosenfeld considered fuzzy relations on fuzzy sets. He developed the theory of fuzzy graphs. Bang and Yeh during the same time introduced various connectedness concepts in fuzzy graph. Inexact information is used in expressing or describing human behaviors and mental process. The information depends upon a person subjectively and it is difficult to process objectively.

Fuzzy information can be analyzed by using a fuzzy graph. Fuzzy graph is an expression of fuzzy relation and thus the fuzzy graph is frequently expressed in fuzzy matrix.

Mathematically a graph is defined as G = $(V,\ E)$ where V denotes the set of vertices and E denotes the set of edges. A graph is called a crisp graph if all the values of arcs are 1 or 0 and a graph is called fuzzy graph if its values is between 0 and 1.Fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma : S \to [0,1]$ where S is the set of vertices and $\mu : S \times S \to [0,1], \forall x, y \in S$.

Fuzzy graph $H = (\tau, v)$ is called a fuzzy subgraph of G if

$$\tau(x) \le \sigma(x), \forall x \in S \quad \text{and} \quad v(x,y) \le \mu(x,y) \forall x, y \in S$$

3.1.1Example

Fuzzy relation is defined by the following fuzzy matrix the corresponding fuzzy graph is shown in the fig (3.1)

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} 0.5 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.5 \\ 1.0 & 1.0 & 0.0 \end{bmatrix} \end{array}$$

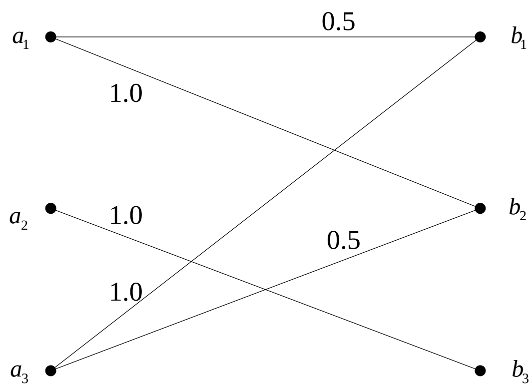

Fig 3.1

Fuzzy graph

3.2 Complement of a Fuzzy Graph

The complement of a fuzzy graph $G:(\sigma,\mu)$ is a fuzzy graph $\overline{G}:(\overline{\sigma},\overline{\mu})$ where $\overline{\sigma}\equiv\sigma$ and

$$\overline{\mu}(u,v)=\sigma(u)\wedge\sigma(v)-\mu(u,v) \quad \forall\ u,v\in V$$

Complement of a fuzzy graph are shown in fig below

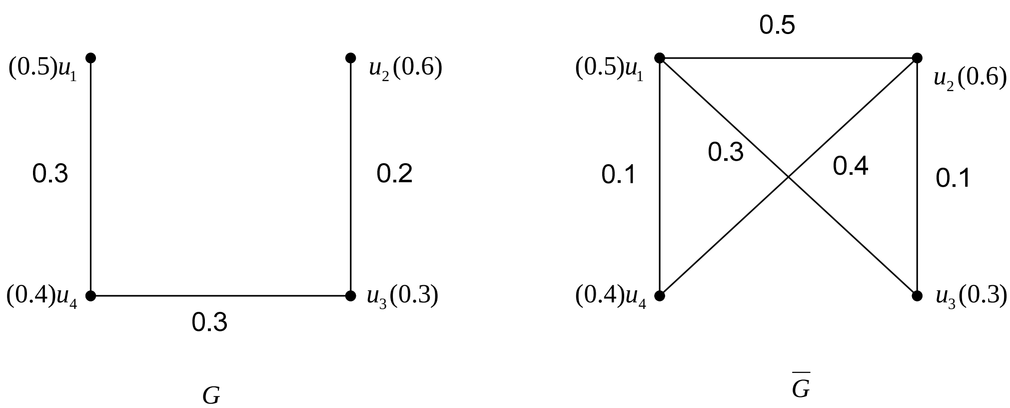

Fig 3.2(a)

Complement of a fuzzy graph

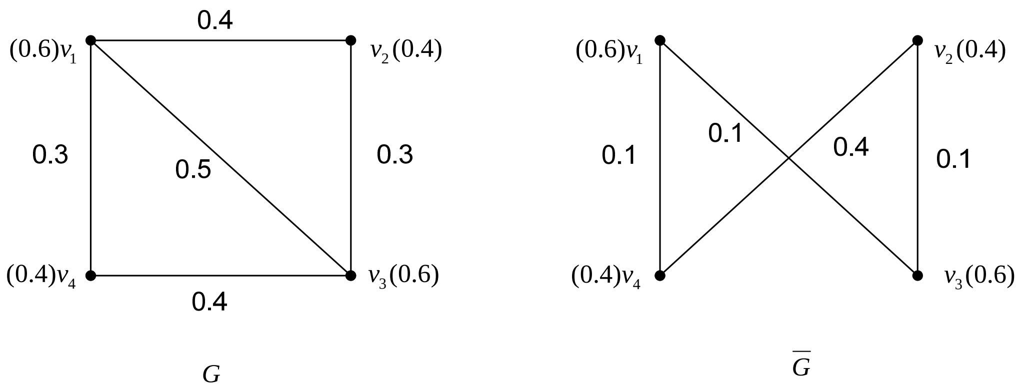

Fig 3.2(b)

Complement of a fuzzy graph

3.3 Model for Predicting Score in Cricket

In this model we can predict score using max-min composition,max product composition and max-av composition.

Speed of bowling = {fast bowling, medium bowling, spin bowling} and
Y= condition on pitches= {good wicket, fair wicket, sporting wicket, green wicket, crumbling wicket, rough wicket}

Let *R* denotes the relationship between speed of bowling and condition on pitch and *Q* denotes the relationship between conditions on pitches and runs on the board.

$$R = \begin{array}{c} \\ fast \\ medium \\ spin \end{array} \begin{array}{c} \begin{array}{cccccc} gd.w & f.w & s.w & gr.w & c.w & r.w \end{array} \\ \begin{bmatrix} 0.6 & 0.5 & 0.4 & 0.1 & 0.9 & 0.5 \\ 0.8 & 0.6 & 0.9 & 0.2 & 0.1 & 0.6 \\ 0.7 & 0.8 & 0.6 & 0.7 & 0.1 & 0.2 \end{bmatrix} \end{array}$$

and

$$Q = \begin{array}{c} \\ gd.w \\ f.w \\ s.w \\ gr.w \\ c.w \\ r.w \end{array} \begin{array}{c} \begin{array}{ccc} low.r & ave.r & hig.r \end{array} \\ \begin{bmatrix} 0.4 & 0.8 & 0.7 \\ 0.3 & 0.8 & 0.8 \\ 0.2 & 0.7 & 0.8 \\ 0.8 & 0.6 & 0.4 \\ 0.7 & 0.5 & 0.4 \\ 0.9 & 0.4 & 0.2 \end{bmatrix} \end{array}$$

$R \circ Q =$ Relationship between speed of the bowling and runs on the board

We calculate $R \circ Q$ by using max-min composition rule

$$\max\{\min(0.6,0.4),\min(0.5,0.3),\min(0.4,0.2),\min(0.1,0.8),\min(0.9,0.7),\min(0.5,0.9)\}$$

$$= \max\{0.4,0.3,0.2,0.1,0.7,0.5\}$$

$$= 0.7$$

Similarly we can calculate the other entries

The relational matrix for max-min composition in fuzzy relational is thus

$$R \circ Q = \begin{array}{c} \\ fast \\ medium \\ spin \end{array} \begin{array}{c} \begin{array}{ccc} low.r & ave.r & hig.r \end{array} \\ \begin{bmatrix} 0.7 & 0.6 & 0.6 \\ 0.6 & 0.8 & 0.8 \\ 0.7 & 0.8 & 0.8 \end{bmatrix} \end{array} \tag{3.1}$$

Max Product composition

Now by using max product composition we find the relationship between speed of the bowling and runs on the board

$R \circ Q =$ Relationship between speed of the bowling and runs on the board

We calculate $R \circ Q$ by using max product composition rule

$$\max(0.6.0.4, 0.5.0.3, 0.4.0.2, 0.1.0.8, 0.9.0.7, 0.5.0.9)$$

$$= \max(0.24, 0.15, 0.08, 0.08, 0.63, 0.45)$$

$$= 0.63$$

Similarly

$$\max(0.48, 0.4, 0.28, 0.06, 0.45, 0.2)$$

$$= 0.48$$

and

$$\max(0.42, 0.4, 0.32, 0.04, 0.36, 0.1)$$

$$= 0.42$$

Similarly we calculate the other entries and the relational matrix for max product composition is

$$R \circ Q = \begin{matrix} & low.r & ave.r & hig.r \\ fast & 0.63 & 0.48 & 0.4 \\ medium & 0.54 & 0.64 & 0.64 \\ spin & 0.56 & 0.64 & 0.64 \end{matrix} \qquad (3.2)$$

Max-av Composition

Now by using max product composition we find the relationship between speed of the bowling and runs on the board

$R \circ Q =$ Relationship between speed of the bowling and runs on the board

We calculate $R \circ Q$ by using max-av composition rule

$$\frac{1}{2}.\max(0.6+0.4, 0.5+0.3, 0.4+0.2, 0.1+0.8, 0.9+0.7, 0.5+0.9)$$

$$= \frac{1}{2}.\max(1, 0.8, 0.6, 0.9, 0.16, 0.14)$$

$$= \frac{1}{2}(0.16)$$

$$= 0.8$$

for the second entry

$$\frac{1}{2}.\max(0.14, 0.13, 0.11, 0.7, 0.14, 0.9)$$

$$= \frac{1}{2}(0.14)$$

$$= 0.7$$

for third entry

$$\frac{1}{2}.\max\left(0.13, 0.13, 0.12, 0.5, 0.13, 0.7\right)$$

$$=\frac{1}{2}\left(0.13\right)$$

$$=0.65$$

Similarly we calculate the other entries and the relational matrix for max-av composition is

$$R \underset{av}{\circ} Q = \begin{array}{r} \\ fast \\ medium \\ spin \end{array} \begin{array}{c} \begin{array}{ccc} low.r & ave.r & hig.r \end{array} \\ \begin{bmatrix} 0.8 & 0.7 & 0.65 \\ 0.85 & 0.8 & 0.85 \\ 0.75 & 0.8 & 0.8 \end{bmatrix} \end{array} \tag{3.3}$$

By analyzing the results of $(3.1),(3.2)$ and (3.3) we conclude that (3.2) is more reliable.

3.4 The Modus Ponens Law in Medical Diagnosis

The creators of fuzzy set theory, who develop mathematical models applied to different technical domain, have also made representative contributions in medical investigation. To decide an appropriate diagnosis in one patient we introduce three non fuzzy sets

The set of symptoms $S=\{S_1, S_2, \ldots, S_m\}$

The set of diagnosis $D=\{D_1, D_2, \ldots, D_P\}$

The set of patients $P=\{P_1\}$

The symptoms occurring in set S are associated with the diagnosis from set D.The symptoms $S_1, S_2, \ldots, S_n$ that are stated in set S are included in the pairs$(P_1, S_1),(P_1, S_2),\ldots,(P_1, S_n)$ .Fuzzy relation PS as a one row matrix

$$PS = \begin{array}{c} \begin{array}{ccc} S_1 & S_2 \ldots & S_n \end{array} \\ \left[\ \mu_{PS}\left(P_{1,}S_1\right) \quad \mu_{PS}\left(P_{1,}S_2\right)\ldots \quad \mu_{PS}\left(P_{1,}S_n\right)\right] \end{array}$$

where $\mu_{PS}\left(P_1, S_j\right),\ j=1,2,\ldots,n$ is a value of the membership degree providing us with evaluation of the intensity S_j in P_1.

The next relation consists of the pairs$(S_1, D_1),(S_1, D_2),\ldots,(S_n, D_P)$.

$$SD = \begin{matrix} S_1 \\ \vdots \\ S_n \end{matrix} \begin{bmatrix} \mu_{SD}(S_1, D_1) & \cdots & \mu_{SD}(S_1, D_P) \\ \vdots & \ddots & \vdots \\ \mu_{SD}(S_n, D_1) & \cdots & \mu_{SD}(S_n, D_P) \end{bmatrix}$$

The fuzzy relation in which each value of the membership degree tied to the pair

(S_j, D_k), $j = 1, 2, \ldots, n \quad k = 1, 2, \ldots, P$ and patient-diagnosis relation is a matrix

$$\begin{matrix} & D_1 & D_2 & D_P \\ PD = & [\mu_{PD}(P_1, D_1) & \mu_{PD}(P_1, D_2)\ldots & \mu_{PD}(P_1, D_P)] \end{matrix}$$

Finally the relation which allowing us to estimate association between the patient and the considered diagnosi

$$PD = PS \circ SD$$

3.4.1Example

In this example we can find an appropriate diagnosis using relation between patient to symptoms and symptoms to diagnosis. To find this we can choose

The set of patients $P = \{P_1\}$

The set of symptoms $S = \{S_1, S_2, \ldots, S_4\}$

= {smoking, hypertension, increased level of LDL-cholesterol, pain in chest}

The set of diagnosis $D = \{D_1, D_2, D_3\}$

The set of diagnosis D = {high risk of cardiovascular diseases, coronary heart diseases, myocardial infarct}

The purpose is to find one of diagnosis in P_1 who shows the presence of symptoms in a certain degree. PS formed by the questions (asked by a physician or by a questionnaire).

$$PS = [0.913 \quad 0.81 \quad 0.63 \quad 0.2]$$

Now we calculate symptoms-diagnosis relation by introducing linguistic variables

Presence = {never, almost never, very seldom, seldom, rather seldom, moderately, rather often, often, very often, almost always, always}

Numerical description of fuzzy variables in "presence"

Fuzzy variables	x	$\mu_{"common"}(x)$
"never"	7.5	0
"almost never"	15	0.016
"very seldom"	22.5	0.062
"seldom"	30	0.14
"rather seldom"	37.5	0.25
"moderately"	50	0.5
"rather often"	62.5	0.75
"often"	70	0.86
"very often"	77.5	0.938
"almost always"	85	0.984
"always"	92.5	1

So SD can be constructed as

$$SD = \begin{bmatrix} 0.75 & 0.984 & 0.14 \\ 0.86 & 0.5 & 0.938 \\ 0.14 & 0.5 & 0.984 \\ 0.25 & 0.984 & 0.86 \end{bmatrix}$$

The relation PD represents

$$PD = PS \circ SD$$

$$PD = \begin{bmatrix} 0.913 & 0.81 & 0.63 & 0.2 \end{bmatrix} \circ \begin{bmatrix} 0.75 & 0.984 & 0.14 \\ 0.86 & 0.5 & 0.938 \\ 0.14 & 0.5 & 0.984 \\ 0.25 & 0.984 & 0.86 \end{bmatrix}$$

by using max-min composition rule

$$\max\left\{\min(0.913, 0.75), \min(0.81, 0.86), \min(0.63, 0.14), \min(0.2, 0.25)\right\}$$

$$= \max\{0.75, 0.81, 0.14, 0.2\}$$

$$= 0.81$$

Similarly we can calculate the other entries

$$\begin{matrix} & D_1 & D_2 & D_3 \\ PD = & [0.81 & 0.913 & 0.81] \end{matrix}$$

According to the rule, the higher degree the more probable diagnosis, we choose D_2

3.5 Conclusion

Fuzzy relations generalize the concept of fuzzy sets to multidimensional universes and introduce the notion of association degree between the elements of some universe of discourse. Fuzzy relations generalize the concept of relations in the same manner as fuzzy sets generalize the fundamental idea of sets. Operations with fuzzy relations are important to process fuzzy models constructed via fuzzy relations. Relations are associations and remain at the very basis of most methodological approaches of science and engineering. Fuzzy relations are more general constructs than functions; they allow dependencies between several variables to be captured without necessarily committing to any particular directional association of the variables being involved.

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