Mathematical Analysis of Financial Markets and Price Behaviour

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In the loving memory of my grandmother

Josefina Gutiérrez S. (1925-2006)

To my mother Martha J. Gómez for being the best example of hard work,
dedication and success; for all her love and support over the years,
without her I wouldn't be writing this thesis.
To my father Arturo Gutiérrez, to my sister Virna, and to my family
for their love and for being always there.
To Mari Carmen Albero for her love and constant cheering.

We have not succeeded in answering all our problems,
the answers we have found only serve to raise a whole set
of new questions. In some ways we feel we are as confused
as ever, but we believe we are confused on a higher level
and about more important things.

Posted outside the mathematics reading room,
Tromsø University, Norway

Contents

Chapter One
Introduction

Since ancient times a lot of different cultures had been concern about numbers and how to use them for different purposes, either if it was to establish patterns for agriculture, medicine, physics, chemistry or astrology or solving more complex problems. Mathematical knowledge has been developed trough time; and in our time, we still use this knowledge like the number pi π discovered by the Egyptians.

Mathematics is a discipline encompassing quantity, structure, space and change; it evolved trough the use of abstraction, and logical reason, from counting, calculating, measuring and studying the behavior of shapes and motions.

Together with mathematics, the civilizations have change and developed markets and money. A monetary unit used for the purchase of goods and services has been transformed from seeds or coins made of metal into a more complicated subject that people can find and use on a daily basis.

Goods and services can be exchanged for money. It's normal for every person to try to satisfied different need by using money, but sometimes the income is not enough to cover all expenses.

Some of the people's income can become savings; other part will be spent on goods and services while other part can be also invested in order to have a higher amount of money in the future.

At this point is exactly where mathematics and finance start to have a closer relationship. Finance study the ways in which individuals, business and organizations raise, allocate and use monetary and even not monetary resources over time and the management and control of the assets; based primary in the use and develop of new mathematical models.

Together with mathematical models in finance, financial markets have changed drastically over time. In our days we can still find "old fashioned" street markets where we can buy food, but we can also find new markets to buy or sell financial instruments like stocks, bonds or derivatives. The question is how to understand this markets and how to use different numbers and formulas for our own good.

Over the time, enormous theories of how investments should be treated and how they have to be analyzed have been developed. A new type of mathematics has been created to help people deal with investments and money, called Mathematical Finance. The importance of mathematics in finance is amazingly high and we have to understand where everything comes from.

In this work, different topics regarding mathematical finance will be discussed, starting with time value of money and going deeper to different valuation models for different kind of financial instruments and also, the study of financial markets.

One most important parts of this work is chapter eight, in which a new valuation model is presented. The model is called Stochastic Market Price Estimator Model – SMPE – and was created entirely by the author of this work as a contribution to mathematical finance.

The aim of this work is to explain mathematical finance in the most simple and complete way, helping mathematicians to have a better understanding of finance and financiers to have a deeper knowledge regarding the mathematical thought that goes together with financial issues.

Chapter Two
Financial Markets: an overview

2.1 Market concept

We can define a market as a physical space where buyers and sellers are placed together with the purpose of transferring goods and/or services. A market must not have to have a specific or physic location; as long as buyers and seller can communicate to each others and establish the transaction conditions we can speak about a market. Other important fact of the markets is that they don't need to own the goods and/or services traded. In the specific case of financial markets, most of the financial instruments traded are not owned by the market makers; the participants just provide the physical location and the administrative and electronic tools to help the market works. Is important to mention that any kind of good or service can be sold or purchased in a market; and there are specific markets for every single instrument traded in the case of financial markets.

The participant in the market entered it to sell or buy a good or service at a price justified by the economic law of supply and demand. This price is properly determined having accurate and timely information regarding the amount of good that is being sold or bought at the moment, the historic prices and the characteristics of last deals such as unusual high or low prices or high number of goods traded.

Financial markets are commonly liquid, meaning that any instrument can be bought or sold at a known price quickly, having no great change in prices from one trade to another. Current and potential buyers and sellers are willing to trade with instruments placing better buy or sell offers making the prices change and providing dynamism to the market.

One characteristic of the financial markets is the transaction cost. Transaction costs are all those money outcomes derived from trading in the financial markets, including the costs of reaching the market. The less they are the more efficient the market is.

The financial markets can be divided in:

- Primary Markets
- Secondary Markets
- Over the Counter Markets or OTC-Markets

2.2 Primary Markets

The primary market is where new stocks, corporate and government bonds are sold. The sellers in these markets are commonly companies, countries, states and cities who have capital requirements and issue new instruments to cover these monetary needs. The bonds offered have different characteristics regarding maturity, interest rate and payment period. When the bonds are issued by governments they are sold in one of three different ways.

- Competitive Bid: The bond is sold to the buyer who submits the lowest interest rate according to the conditions of the issuer. The competitive bid could be compared to an acquisition where the buyer stating the best offer will get the instrument.

- Negotiated Sales: The instrument is issued by a government helped by a buyer. The buyer will set the price and conditions together with the issuer and will have the right to sell the instrument. The buyer is mostly an investment bank.

- Private Placement: The financial instrument will be sold directly to a small group of investor or a single investor.

The primary market is the one where the funds go directly to the issuers of stocks or bonds; is the first step in which the securities enter the market.

2.3 Secondary Markets

Secondary markets are those where stocks and bonds already sold once are traded. The sellers are those persons or companies holding the instruments, not the issuer of the bonds or stocks. The investors who buy an instrument resell it at the secondary market to obtain a positive cash flow, meaning they change a security for liquidity, or in other words, they trade a good for money to invest in different assets.

The secondary markets have a trading regulation and are supervise by a trading commission to avoid unfair trading and malpractices. These markets have a specific trading place and trading hours. The information received by the investors should be the same in the market avoiding the use of privileged information. Some examples of these markets are the New Stork Stock Exchange (NYSE), The London Stock Exchange and the Tokyo Stock Exchange.

The secondary market is the one where all values are resold and the funds go to the investors and not to the issuers. In this market, the holder can sell the security at any time. The market assures liquidity at all the time.

2.4 Over the Counter Markets (OTC-Markets)

The Over the Counter Markets, also known as OTC-Markets are those who are not a formal trading organization like all other exchanges. As long as there are buyers and sellers or registered dealers willing to trade some securities we will find an OTC-Market. Within the OTC-Market we can find different kind of Market-Makers willing to trade and match buy and sell orders. Is very common for investors to deal directly with the dealers in order to make a market and trade a specific security or instrument.

2.5 Market Efficiency

In efficient markets, prices will reflect the available and opportunely received information by the market and the prices will adjust in a short period of time after receiving new information. Investors can analyze new information rapidly having a direct impact on their buy or sell postures adjusting prices.

A market can have allocation efficiency distributing funds to the most promising investments. The efficiency of markets can also be internal, making brokers and dealers compete fairly with low transaction costs and high speed transactions. Markets are affected by externalities like the availability of information. If new information is distributed quickly and widely the prices should adjust rapidly and in an unbiased manner making the market externally efficient.

In an efficient market the present value of the security's future returns is estimated by the investors and the investment value is equal to the market value at all times. The Fama Market Model explains this assumption:

$$E(p_{j,t+1} \mid \Phi_t) = [1 + E(r_{j,t+1} \mid \Phi_t)] p_{j,t} \tag{2.1}$$

meaning that the expected price for any security $E(p)$ at the en of the period $(t+1)$ is based on the security's expected normal rate of return during that period $E(r_{j,t+1)})$ given the information set at time $t(\Phi)$.

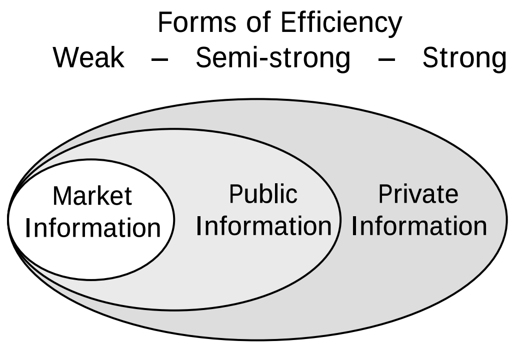
The implication of this model is that if markets are perfectly efficient, investors cannot earn abnormal returns based on the information set because

$$x_{j,t+1} = p_{j,t+1} - E(p_{j,t+1} \mid \Phi_t) \tag{2.2}$$

where $x_{j,t+1}$ is the difference in price at $t+1$ between what the price is and what investors expect.

If new information is received by the market changing Φ_t this information should be incorporated to prices immediately and would have a direct impact in the expected price $E(p_{j,t+1})$, this under the assumption we find ourselves in an efficient market.

Market efficiency has three different forms, the weak form in which current prices reflect all market information; the semi-strong efficiency means that prices adjust rapidly regarding all public information released and the strong efficiency form make the prices content all public and private information available in the market.



Forms of Market Efficiency Exhibit 2.1

Chapter Three
Elementary Mathematical Finance

3.1 Interpretation of Interest Rates

Every person has to face decisions that involve saving money for a future use or borrowing money for current spending. In those cases people need to determine the amount they have to invest or the cost of borrowing. While taking credits for daily consumption or a loan to buy a property or a car, people deal with the concept of money in different periods of time.

Talking about investments, much of the work of analysts involves evaluating transactions with present and future cash flows. Therefore, is important to understand the mathematics of time value of money problems. Money has a time value; one Euro or other currency today will not have the same value in two years and had not the same value one year ago. Time value of money deals with equivalence relationships between cash flows in different periods of time.

The idea of equivalence relationships is relatively simple. If a person pay $50,000 today and in return receive $49,000 today he or she would not accept the arrangement because the two amounts are not equivalent; but if this person receive the $49,000 and pay $50,000 after a year would be fair because $50,000 a year from now would probably worth less than the same amount today. To cut its value based on how much time passes before the money is paid will be know as discount the $50,000. An interest rate r will denote the relationship between cash flows in different dates. If $49,000 today and $50,000 in one year are equivalent, then $50,000 - $49,000 = $1,000 is necessary to compensate for receiving $50,000 in one year and not now. The interest rate will be

$$\frac{1,000}{49,000} = 0.020408 = 2.0408\%$$

Interest rates can be viewed in three different ways:

1. Required rates of return, meaning that an investor must receive a minimum percentage back, additional to the initial money invested to accept the investment.

2. Discount rates, as in the example above an interest rate in necessary to discount future values to find its value today, or in the contrary, as simple interest rates to find an equivalent value in the future from a present value.

3. Opportunity Costs. The opportunity cost is the value of the best forgone alternative. Returning to the example above, if the investor decides to spend the $49,000 today, he forgoes the opportunity to earn 2.0408 percent as the opportunity cost of current consumption.

Interest rates can be determined under a theory of expectations, where investors assume that the long run interest rates today reflex the short run interest rates in the future; but we can also assume that the interest rates will be set according to market segmentation, based on the supply and demand of funds. We can now view an interest rate r as being composed of a real risk-free interest plus a set of four premiums that are required returns or compensation for bearing different types of risk.

r = real risk-free interest rate + inflation premium + default risk premium + liquidity premium + maturity premium

3.1.1 Real Risk-Free Interest Rate

The real risk-free interest rate is the single-period interest rate for a completely risk-free security if no inflation were expected. In an economic perspective, the real risk-free rate reflects the time preferences of individuals for consumption in different periods of time. They could save today to consume tomorrow or borrow to consume today paying tomorrow.

3.1.2 Inflation Premium

The inflation premium compensates investors for expected inflation and reflects the average inflation rate expected over time. Inflation is the loss of purchasing power of a unit of currency; the amount of good and services that could be purchased with one unit of currency will be less when inflation in presented. The opposite case will be called deflation, an increase in the purchasing power. The sum of real risk-free interest rate and the inflation premium is called nominal risk-free interest rate.

Inflation affects not only the purchasing power but also the real interest rate. If a person invests expecting to receive the 10% back he will indeed receive this amount but will not have the same value if the inflation rate in the same period is 2%. Therefore is important to calculate the real interest rate using the equation

$$r_{real} = \left(\frac{1+r}{1+\pi}\right) - 1 \tag{3.1}$$

Being r the interest rate and π the expected inflation rate. Using the data from the example above the real interest rate can be calculated

$$r_{real} = \left(\frac{1+0.1}{1+0.02}\right) - 1 = 7.84\%$$

The investor will have a real return of 7.84% instead of 10% as expected.

3.1.3 Default Risk Premium

The default risk premium compensates investors for the possibility that the borrower will fail to make the promised payment at the contracted date and for the contracted amount.

3.1.4 Liquidity Premium

If an investor has a different investment possibility or a sudden need for money he would try to convert the actual investment into cash. This operation is not always possible and the investor could loose the opportunity to invest somewhere else. There is a risk of loss value associated with liquidity, reason why the investor has to be compensated. In other words, the investor will receive an extra premium by forgiving his cash.

3.1.5 Maturity Premium

As the maturity, or ending period for an investment, grows, there is a change in market interest rates having as a consequence an increase in sensitivity to the market value of debt. The investor has to be compensated for this with a premium called maturity premium.

3.1 Time Value of a Single Cash Flow

As it has already been reviewed, an interest rate is compounded by different rates, risk and premiums; it has also been explained money has a different value in time.

If we speak about time value of a single cash flow, we will be talking about an initial investment, or present value, which earns a rate of return (the interest rate per period), denoted as r and its future value, which will be received n number of periods from today. The future value represents the new value of the initial investment which usually is higher than the present value unless we face negative interest rates.

The mathematical expression for the future value is denoted by:

$$FV = PV(1+r) \qquad (3.2)$$

Where

FV = Future Value

PV = Present Value

r = interest rate

Equation 3.2 represent the future value when we face a simple interest rate, this is the rate times the principal. We define principal as the amount of funds originally invested.

On a daily basis, investors or borrowers face not only a simple interest rate but an interest earned on interest. Suppose you have a savings account which will pay you 10% every year and you invest $100 initially.

The future value will be $110 by the end of the first year. If you withdraw the earnings ($10) you would start year two with $100 again and by the end of year two you would receive $10 too. You would be facing the concept of simple interest.

Now, suppose the initial investment or principal is $100 again but this time you are investing in a retirement plan and you are only “keeping” your money at the bank. Let’s suppose the interest rate remains 10%. By the end of the first year your capital will be $110, as in the simple interest rate case.

This time you will not withdraw the return; this time you will not spend any money from this account. You will then start year two with $110 as a principal. This time the profit during year two will be $11. You will then start year three having $121 as a principal, and so on during twenty years. By the end of the twentieth year you will have earned $672.75.

Using simple interest rates, the calculations from the example above will look like:

$$FV_1 = \$100(1+0.1) = \$110$$

$$FV_2 = \$110(1+0.1) = \$121$$

$$FV_3 = \$121(1+0.1) = \$133.10$$

$$FV_4 = \$133.1(1+0.1) = \$146.41$$

....

$$FV_{20} = \$611.59(1+0.1) = \$672.75$$

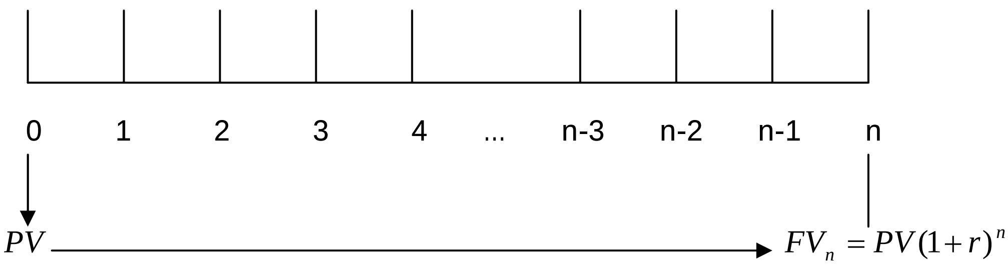
The above calculations can be simplified as following obtaining the new equation for compounding interest

$$FV_{20} = \$100*(1+0.10)*(1+0.10)*(1+0.10)*(1+0.10).....$$

$$FV_{20} = \$100(1+0.10)^{20} = \$672.75$$

$$FV_n = PV(1+r)^n \qquad (3.3)$$

Graphically, the relationship between an initial investment (PV) and its future value (FV) is described in Exhibit 3.1.



Relationship between PV and FV Exhibit 3.1

3.3 The Frequency of Compounding

In section 3.2 we assume that the interest paying investment took place just once every year. In this section we will examine different type of interest compounding, which means investments paying interests more than one time per year.

Many banks for example offer a yearly interest rate that compounds 12 times a year, meaning that the interest in pad every month.

Financial institutions quote an annual interest rate instead of the monthly or any other compounding frequency. This annual interest rate is called stated annual interest rate, quoted interest rate or nominal interest rate.

The stated annual interest rate will be denoted by r_s. For example, a bank might state that a particular certificate of deposit pays 5 percent compounded monthly. The stated annual interest rate equals the monthly interest rate multiplied by 12, in this case, the monthly interest rate will be

$$\frac{0.05}{12} = 0.004167 = 0.4167\%$$

If we apply equation 3.3 to this interest rate we will have

$$(1+0.004167)^{12} - 1 = 0.051162 = 5.1162\% \text{ Not } 5\%$$

Analyzing the prior example we can explain the difference between a nominal and an effective interest rate. The nominal interest rate is the one “published” by the banks or bond issues; the effective interest rate is the real return the investor will receive after a specific period of time.

When the compounding period is more than one time per year, the future value equation 3.3 can be expressed as

$$FV_n = PV\left(1+\frac{r_s}{m}\right)^{mn} \qquad (3.4)$$

r_s : Stated annual interest rate

m : Number of compounding periods per year

n : Number of years

$\frac{r_s}{m}$: Stated annual interest rate divided by the number of compounding periods per year which will give the nominal interest rate for the period, if $m = 12$ the interest rate we will have will be the monthly one and so on.

The number of compounding periods, $\frac{r_s}{m}$, and the number of compounding periods, nm, must be compatible.

Example 3.1

For a completely understanding of the concepts reviewed so far in this chapter an example will be made.

Suppose a bank offers you a certificate of deposit (CD) with a four year maturity, a stated fix interest rate of 8 percent compounded quarterly and a feature allowing reinvestment. You decided to invest $10,000.

$$PV = \$10{,}000$$
$$r_s = 8\% = 0.08$$
$$m = 4$$
$$n = 4$$

$$FV_n = PV\left(1+\frac{r_s}{m}\right)^{nm} \qquad \text{(eq. 3.4)}$$

$$FV_4 = \$10{,}000\left(1+\frac{0.08}{4}\right)^{4*4}$$

$$FV_4 = \$10{,}000(1.02)^{16}$$
$$FV_4 = \$10{,}000(1.372786)$$
$$FV_4 = \$13{,}727.86$$

Exhibit 3.1 shows the effect of compounding frequency on future value supposing a stated interest rate of 10 percent and a present value of $1 and an investment period (n) one year

Frequency	$\frac{r_s}{m}$	mn	Future Value of $1
Annual	$\frac{10\%}{1}=10\%$	$1*1=1$	$\$1.00(1.10)=1.10$
Semiannual	$\frac{10\%}{2}=5\%$	$2*1=2$	$\$1.00(1.05)^2=1.1025$
Quarterly	$\frac{10\%}{4}=2.5\%$	$4*1=4$	$\$1.00(1.025)^4=1.103813$
Monthly	$\frac{10\%}{12}=0.8333\%$	$12*1=12$	$\$1.00(1.0083)^{12}=1.104275$
Daily	$\frac{10\%}{365}=0.0274\%$	$365*1=365$	$\$1.00(1.000274)^{365}=1.105167$
Continuous			$\$1.00e^{0.10(1)}=1.105171$

Effect of Compounding Frequency on Future Value Exhibit 3.2

It can be clearly seen that the future value will tend to increase by increasing the compounding frequency.

From future value equation 3.3 and the future value with different compounding periods 3.4, the present value equation can be derived if we need to calculate the initial required investment to obtain a specific future value. The present value equation for different compounding periods looks like

$$PV=FV_n\left(1+\frac{r_s}{m}\right)^{-mn} \tag{3.5}$$

3.4 The Equivalence of Interest Rates

Some banks, financial institutions or even corporations offer financial instruments with particular characteristics. One of those is the interest rate, which can be with different frequency of compounding. If we want to make an analysis comparing two or more investment options we have to compare the interest rates, but first of all, we have to have this interest rates being equivalent.

Suppose you have two different investment options, a CD that offers you 4.89% compounded monthly or a bond that offers you 5% annually, which interest rate would you take? It is not that easy to say because both interest rates have a different compounding frequency, but we can make interest rates to be equivalent by applying the following equation:

$$E\% = \left[\left(1+\left(\frac{r_m m}{360}\right)\right)^{\left(\frac{m_{eq}}{m}\right)} - 1\right] * \frac{360}{m_{eq}} \tag{3.6}$$

$E\%$: Equivalent interest rate

r_m : Known interest rate with a compounding frequency m

m : Actual time expressed in days

m_{eq} : Equivalent (desired) time expressed in days

We will now apply the example above in equation 3.6; we have an annual interest rate of 5% and want to have the equivalent monthly rate:

$$E\% = \left[\left(1+\left(\frac{0.05*360}{360}\right)\right)^{\left(\frac{30}{360}\right)} - 1\right] * \frac{360}{30} = 4.8889\%$$

We can see with this result that the two interest rates are equivalent, they have a different compounding frequency but at the end the amount of money received (or paid) will be equal.

3.5 Unknown Interest Rates According to Maturity

Banks and financial institutions often need for their analyses interest rates for periods that are not given by the market. Interest rates can be given for periods of thirty, ninety, one hundred eighty or one year, but what happen when the rate needed is for a period of two hundred eight days?

Suppose two interest rates are given by the market, one for a period of 90 days being 5 percent and a second one being 6.8 percent for a period of 180 days; for the calculation an interest rate for one hundred days is needed. We can proceed to calculate the interest rate by interpolation

$$r_{mm} = \left[\left[\left(\frac{1+\left(\frac{r_{lm} * m_l}{360}\right)}{1+\left(\frac{r_{sm} * m_s}{360}\right)}\right)^{\frac{m_m - m_s}{m_l - m_s}} * \left[1+\left(\frac{r_{sm} * m_s}{360}\right)\right]\right] - 1\right] * \frac{360}{m_m} \qquad (3.7)$$

r_{sm} : Interest rate short maturity

r_{mm} : Interest rate middle maturity (seek interest rate)

r_{lm} : Interest rate long maturity

m_s : Short maturity

m_m : Middle maturity (period we are looking for)

m_l : Long maturity

Applying the data from our example

$$r_{mm} = \left[\left[\left(\frac{1+\left(\frac{0.068 * 180}{360}\right)}{1+\left(\frac{0.05 * 90}{360}\right)}\right)^{\frac{100-90}{180-90}} * \left[1+\left(\frac{0.05 * 90}{360}\right)\right]\right] - 1\right] * \frac{360}{100} = 0.05352 = 5.352\%$$

Chapter Four
Basic Statistical and Probability Concepts In Mathematical Finance

4.1 Fundaments of Statistics

The term statistics can be related to two different meanings, one referring to data and the other to method. Statistical methods include descriptive statistics and statistical inference or inferential statistics.

Descriptive statistics is the study of how data can be summarized effectively to describe the important aspects of large data set. Data will be transformed into information by consolidating a large amount of numerical details.

Statistical inference involves making forecast, estimations or judgments about a larger group or population from a smaller group or sample observed. Inferential statistics finds its foundation in probability

Population is defined as all members of a specified group; a sample is a subset from the population.

There are different measurement scales that can be used in statistical methods and they have to be distinguished to choose the appropriate method for summarizing and analyzing data.

Nominal scales categorize data but do not rank them.

Ordinal scales sort data into categories that are order with respect of some characteristic, it may also involve numbers to identify categories.

Interval scales provide ranking but also assurance that the differences between scale values are equal.

Ratio scales have all the characteristics of interval scales as well as a zero point as the origin. Using this scale, ratios can be computed and amounts can be added or subtracted within the scale.

4.2 Frequency Distributions

Frequency distribution is one of the simplest ways to summarize data and can be defined as a tabular display of data summarized into a relatively small number of intervals. Frequency distributions help in the analysis of large amounts of statistical data and they work with all types of measurement scales.

The basic procedure to construct a frequency distribution can be stated as follow:

1. Sort the data in ascending order.
2. Calculate the range of the data, defined as Range = Maximum value – Minimum value.
3. Decide on the number of intervals in the frequency distribution, k.
4. Determine interval width as Range/k; round it rather down to ensure the final interval includes the maximum value of the data.
5. Determine the intervals by successively adding the interval width to the minimum value, to determine the ending points of the interval, stopping after reaching an interval that includes the maximum value.
6. Count the number of observations falling in each interval.
7. Construct a table of the intervals listed from smallest to largest that shows the number of observations falling in each interval.

An interval is a subset S of a totally ordered set T with the property that whenever x and y are in S and $x<z<y$ the z is in S. In our case, the interval can be defined as a set that contains every real number between two indicated numbers, and possibly the two numbers themselves, in this case, the interval is a set of values within an observation falls. Each observation falls into only one interval. The actual number of observations in a given interval is called the absolute frequency. Intervals are also called classes, ranges or bins. The relative frequency is the absolute frequency of each interval divided by the total number of observations.

Suppose we have twelve observations sorted in ascending order: -4.17, -2.96, -1.06, 0.28, 1.34, 2.1, 2.58, 3.05, 3.59, 4.21, 6.17 and 10.36. The minimum observation is -4.17 and the maximum observation is +10.36, so the range is +10.36-(-4.17) = 14.53. If we set *k*=4, the interval width is 14.53/4=3.63. Exhibit 4.1 illustrates step 5, the repeated addition of the interval width to determine the endpoints for the interval.

Endpoints of interval
-4.17+3.63=-0.54
-0.54+3.63=3.10
3.10+3.63=6.73
6.73+3.63=10.36

Exhibit 4.1

Thus the intervals are [-4.17 to -0.54), [-0.54 to 3.10), [3.10 to 6.73), [6.73 to 10.36). Exhibit 4.2 summarizes steps 5 to 7 from the basic procedure.

Interval		Absolute Frequency	Relative Frequency
1	$-4.17 \leq observation < -0.54$	3	25.00%
2	$-0.54 \leq observation < 3.10$	5	41.67%
3	$3.10 \leq observation < 6.73$	3	25.00%
4	$6.73 \leq observation < 10.36$	1	8.33%

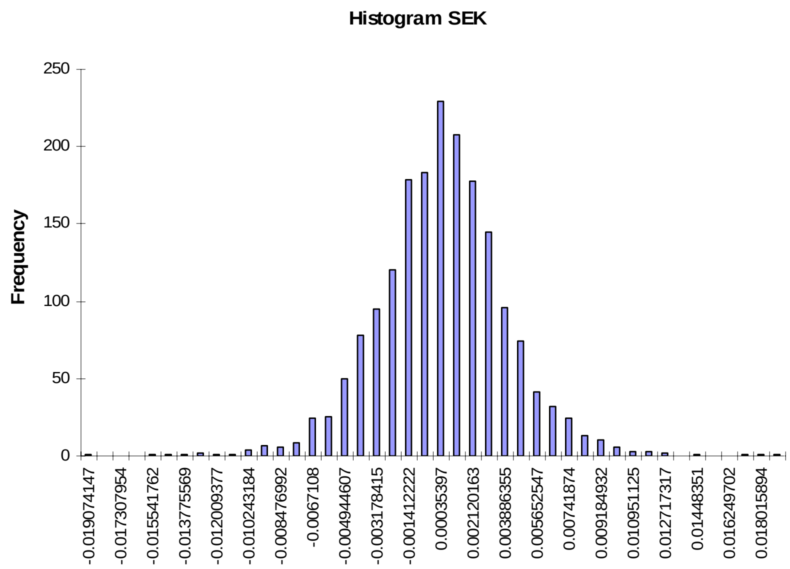
Frequency Distribution

Exhibit 4.2

The intervals do not overlap, so each observation can be placed just into one interval.

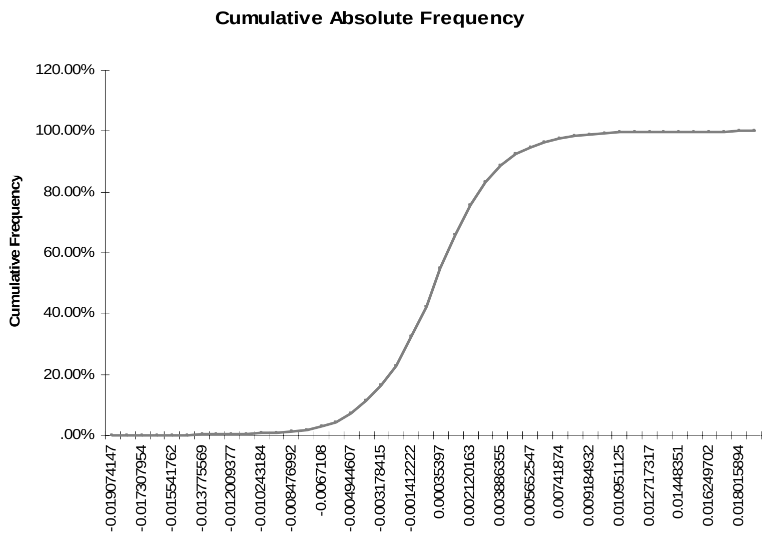
4.3 Graphic Representation

To visualize important characteristics of the data easily a graph should also be included in the analysis. One of the more typical graphs is the histogram, which is the graphical equivalent of a frequency distribution and can be defined as a bar chart of data that have been grouped into a frequency distribution. Exhibit 4.3 shows a histogram of the daily returns of the Swedish Crown (SEK) against the Euro in the past five years.



Histogram: daily return of the SEK Exhibit 4.3

Another form of graph is the cumulative frequency distribution. This graph can plot the relative or the absolute frequency against the upper interval limit. This kind of graph allows us to see how many or what percent of the observations lie in certain value or interval. Exhibit 4.4 presents a cumulative absolute frequency distribution from the data used to construct histogram 4.3



Cumulative absolute Frequency Distribution Exhibit 4.4

4.4 The Arithmetic Mean

The arithmetic mean is the sum of all observations divided by the number of observations. In statistics, the mean can be calculated for the population or just for a sample.

The population mean μ, is the arithmetic mean value of the entire population represented by

$$\mu = \frac{\sum_{i=1}^{N} X_i}{N} \tag{4.1}$$

Where N is the number of observations of the entire population and X_i is the value of each observation.

The sample mean or average $\overline{X}$ is the arithmetic value of a sample given by

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n} \tag{4.2}$$

Where n is the number of observations in the sample.

4.5 The Geometric Mean

If we were talking about the interpretations of numbers according to their product instead of their sum, like rates of growth, we were being talking about a geometric mean:

$$\overline{X}_G = \sqrt[n]{\prod_{i=1}^{n} X_i} \tag{4.3}$$

It is important to state that the geometric mean exists only if the product under the radical sign is non-negative.

$$\ln \overline{X}_G = \frac{1}{n} \ln\left(X_1, X_2, X_3 \ldots X_n\right)$$

Or

$$\ln \overline{X}_G = \frac{1}{n} \sum_{i=1}^{n} \ln X_i$$

4.6 The Harmonic Mean

The Harmonic mean is an average useful for sets of numbers which are defined in relations to some unit or amount per unit like speed (distance per unit of time). The harmonic mean of a set of observations $X_1, X_2, \ldots, X_n$ is

$$\overline{X}_H = \frac{n}{\sum_{i=1}^{n} \frac{1}{X_i}}$$

With $X_i > 0$ for $i = 1,2,\ldots,n$

The harmonic mean is the value obtained by summing the reciprocals of the observations, then averaging the sum by dividing the number of observations and finally, taking the reciprocal of the average.

4.7 The Median

The media is defined as the value of the middle item of a set of items that has been sorted in ascending or descending order. It can also be defined as a number dividing the higher half of a sample or a population from the lower half. At most, half the population have values less than the median and at most half have values greater than the median. If both groups contain less than the half the population, then some values of the population have are equal to the median.

In an odd-numbered sample of n items, the median occupies the $\frac{(n+1)}{2}$ position. In an even-numbered sample, the median can de defined as the mean of the values of items occupying the $\frac{n}{2}$ and $\frac{(n+2)}{2}$ positions.

4.8 The Mode

The mode is the most frequently occurring value in a distribution. The mode is a way of capturing important information about a population and is in general different from the mean and the median.

The mode is the value where the histogram reaches its peak; it will also only make sense when there is a linear order on possible values.

4.9 The Variance

The variance is one of the two most widely used measures of dispersion. Variance is defined as the arithmetic average of the squared deviations around the mean. The variance indicates how far the values from the mean are. If all the values within a population are known, the population variance can be computed and denoted by the symbol σ^2 and will be calculated with the following equation having the population $(X_1, X_2, X_3, ...X_N)$

$$\sigma^2 = \frac{\sum_{i=1}^{N}(X_i - \mu)^2}{N} \tag{4.4}$$

Where μ is the mean of the population and N is the size of the population.

The variance is the sum of the squared differences from the known population mean μ taking in consideration all N items; then find the mean squared difference dividing the sum by the size of the population.

Whether the differences are negative, all results will always be positive values after squaring.

In many cases, just a sample of the whole population can be seen. Therefore, the population variance cannot be computed.

When we deal with samples, the summary measures are called statistics.

The statistic that measures the dispersion in a sample $\left(X_1, X_2, X_3, \ldots X_n\right)$ is called the sample variance denoted by s^2 (instead of the population's mean μ), where $\overline{X}$ is the sample mean and n is the number of observations of the sample.

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \tag{4.5}$$

To calculate the sample variance first we have to calculate the sample mean $\overline{X}$, afterwards, we have to calculate each observation's squared deviation from the sample mean $\left(X_i - \overline{X}\right)^2$.

Having these results, we proceed to sum the squared deviations from the mean $\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$ and finally divided the result by $(n-1)$.

The equation for the sample variation is nearly the same as the one for the population variance, except for the mean used for the calculation and the different divisor. By dividing by $(n-1)$ in the sample variance equation (4.5) the statistical properties can be improved, being the equation an unbiased estimation of the population.

4.10 The Standard Deviation

As we reviewed in section 4.9, the variance is measured in squared unites. One way to return to the original units is by computing the square root of the variance, known as the standard deviation which is expressed in the same units as the observations. The standard deviation for a population is denoted by σ and calculated using the following equation

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(X_i - \mu)^2}{N}} \tag{4.6}$$

Where μ is the mean of the population and N is the size of the population.

As for the population standard deviation, the sample standard deviation will be computed by taking the positive square root of the sample variance

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2} \tag{4.7}$$

The standard deviation is a measure of the average distance of the values from their mean. If the value of the standard deviation tends to zero, it means that the values are close to the mean; on the other hand, the higher the standard deviation is, the longer the distance to the mean. If all the data values are equal, the standard deviation will be zero.

The standard deviation has no maximum value, although it is limited in most data sets following Chebyshev's inequality, which states the proportion of the observations within k standard deviations of the arithmetic mean is at least

$$1 - \frac{1}{k^2} \text{ for all } k > 1$$

Exhibit 4.5 denotes some proportions of this inequality.

Proportions from Chebyshev's Inequality

K	Interval Around the Sample Mean	Proportion
1.00	$\overline{X} \pm 1.00s$	0%
1.25	$\overline{X} \pm 1.25s$	36%
1.50	$\overline{X} \pm 1.50s$	56%
2.00	$\overline{X} \pm 2.00s$	75%
2.50	$\overline{X} \pm 2.50s$	84%
3.00	$\overline{X} \pm 3.00s$	89%
3.50	$\overline{X} \pm 3.50s$	92%
4.00	$\overline{X} \pm 4.00s$	94%

Exhibit 4.5

For a more precise calculation of the standard deviation for a sample, it would be necessary to take at least five hundred observations if the sample is big enough. As it can be seen in exhibit 4.6, the standard deviation has a bigger variance when is calculated with just few observations, the more observations taken for the calculation the less variance will be reflected on the standard deviation.

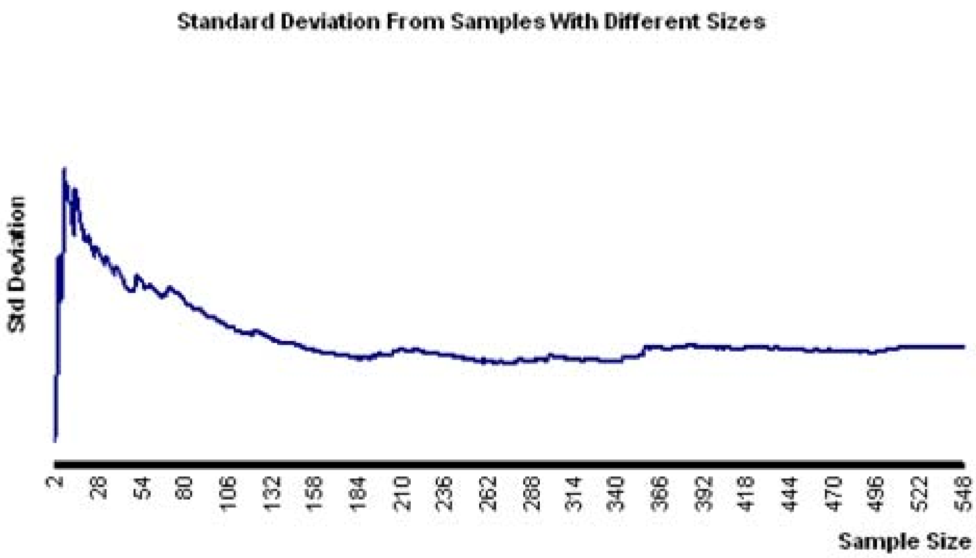


Exhibit 4.6

4.11 Skewness

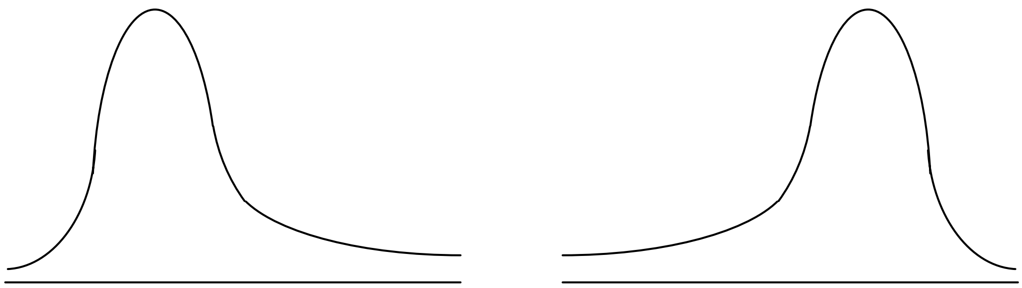
Mean and variance may not describe an investment's return adequately. In calculations of variance, the deviations around the mean are squared, so it could not be known whether large deviations are likely positive or negative; is important then to analyze other important characteristics like the degree of symmetry in the distribution.

If the distribution is symmetrical about its mean each side of the distribution will be a mirror image of the other. Talking about investments, the gain and losses intervals would exhibit the same frequencies. This analysis is also used extensively in Risk Management.

A normal distribution will have an equal mean and median, it is completely described by its mean and variance and roughly 68 percent of the observations lie between plus and minus one standard deviation; 95 percent between plus and minus two standard deviations and 99 percent lie between plus and minus three standard deviations from the mean.

A distribution that is not symmetrical is called skewed. A distribution with positive skewness (skewed to the right) has frequent small losses and a few extreme gains; a distribution with negative skewness (skewed to the left) has few extreme losses and frequent small gains.

Exhibit 4.7 shows positively and negatively skewed distribution, the positively skewed distribution has a long tail on its right side where the mode is less than the median; the negatively one has a long tail on its left side and the median is less than the mode.



Distribution Skewed to the Right (Positively Skewed)

Distribution Skewed to the Left (Negatively Skewed)

Skewness Exhibit 4.7

Skewness is computed using each observation's deviation from its mean as the average cubed deviation from the mean standardized by dividing by the standard deviation cube to make the measure free of scale. A symmetric distribution has skewness equal to zero; a positive or negative result will indicate if the skewness is positive or negative. Cubing the numerator will preserve the sign comparing the calculation with the standard deviation. The sample skewness or relative skewness S_K can be computed using equation (4.8)

$$S_K = \left[\frac{n}{(n-1)(n-2)}\right]\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^3}{s^3} \tag{4.8}$$

If the size of the population or the number of observations n is to large, equation 4.8 can be reduced to

$$S_K \approx \left(\frac{1}{n}\right)\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^3}{s^3} \tag{4.9}$$

4.12 Kurtosis

Kurtosis is the statistical measure that tells when a distribution is more or less peaked than a normal distribution. A distribution that is more peaked than normal is called leptokurtic (*lepto* from the Greek, slender); this distribution has fatter tails than the normal distribution. A distribution that is less peaked than the normal is called platykurtic (*platy* from the Greek word for broad), and a distribution identical to the normal is called mesokurtic (*messo* being the Greek word for middle). Exhibit 4.8 shows the three different types of kurtosis.

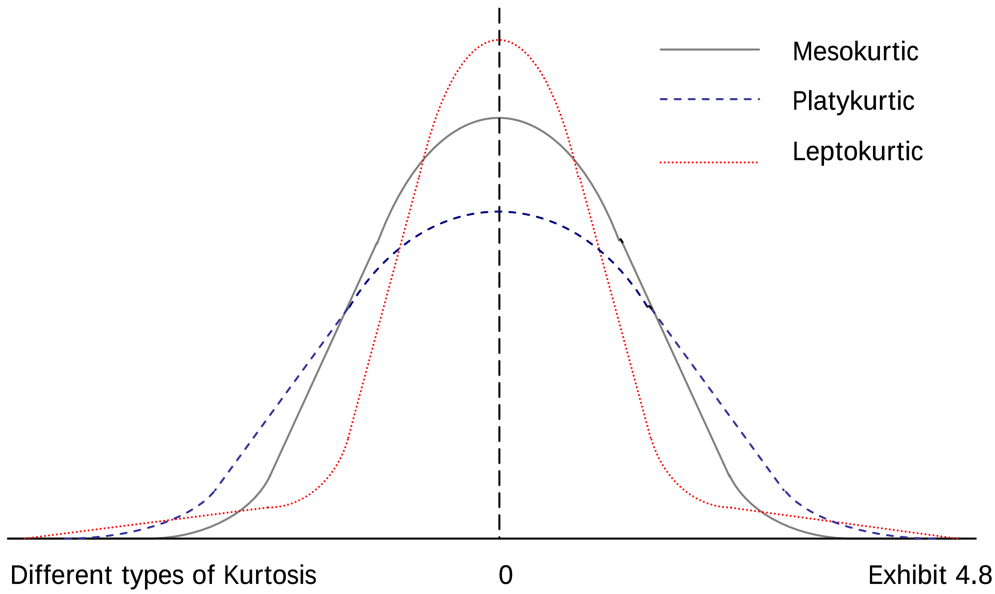
The equation to calculate the kurtosis involves finding the average of deviations from the mean raised to the power of four and then standardizing that average by dividing by the standard distribution raised to the fourth power. A normal or other mesokurtic distribution has a kurtosis equal to zero. A leptokurtic distribution has a kurtosis greater than zero and a platykurtic distribution less than zero. A kurtosis of 1.0 or larger would be considered unusually large. To calculate the kurtosis from a sample equation 4.10 is used.

$$K = \left(\frac{n(n+1)}{(n-1)(n-2)(n-3)} \frac{\sum_{i=1}^{n} (X_i - \overline{X})^4}{s^4} \right) - \frac{3(n-1)^2}{(n-2)(n-3)} \qquad (4.10)$$

If the number of observations within the sample becomes too large, equation 4.10 approximately equals

$$K \approx \frac{n^2}{n^3} \frac{\sum (X - \overline{X})^4}{s^4} - \frac{3n^2}{n^2} = \frac{1}{n} \frac{\sum (X - \overline{X})^4}{s^4} - 3 \qquad (4.11)$$

Mesokurtic, Platykurtic and Leptokurtic Distributions



Different types of Kurtosis

Exhibit 4.8

4.13 Covariance

Covariance measures how much two variables vary together. The covariance becomes more positive for each pair of values which differ from their mean in the same direction, and becomes more negative with each pair of values which differ from their mean in opposite directions. The more often they differ in the same direction, the more positive the covariance; the more often they differ in opposite directions, the more negative the covariance.

The covariance between tow real-numbered random variables X and Y, with expected values $E[\overline{X}] = \mu$ and $E[\overline{Y}] = \nu$ is defined by the following equation

$$Cov(X,Y) = E[(X - \mu)(Y - \nu)] \tag{4.12}$$

Alternative notations for covariance are $\sigma(X,Y)$ and σ_{XY}.

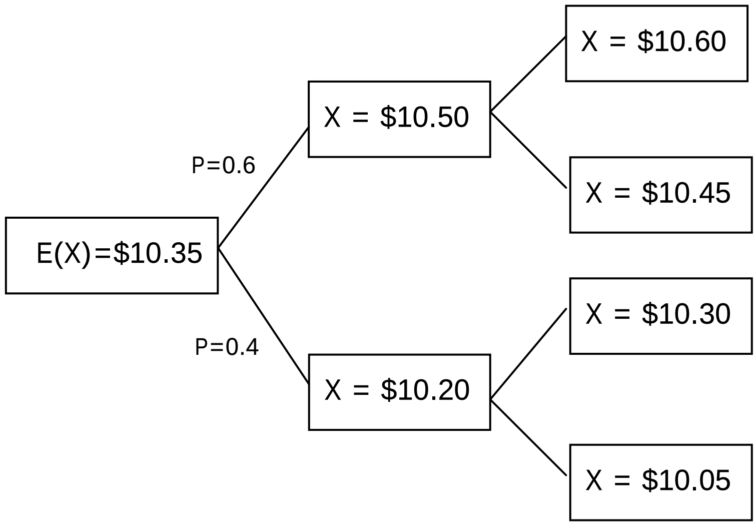
Equation 4.12 states that the covariance between two random variables is the probability-weighted average of the cross-products of each random variable's deviation from its own expected value.

4.14 Expected Value

The expected value is the probability-weighted average of the possible outcome of a random variable. The expected value of X equals the expected value of X given Scenario 1 times the probability of this Scenario, plus the expected value of X given Scenario 2 times its probability and so on. Equation 4.13 describes the expected value.

$$E(X) = E(X \mid S_1)P(S_1) + E(X \mid S_2)P(S_2) + \ldots + E(X \mid S_n)P(S_n) \tag{4.13}$$

The expected value of X can be shown graphically in a tree diagram with different Scenarios and the probability of those Scenarios.



Tree Diagram: Expected values and Scenarios for X Exhibit 4.9

Let w_i be any constant and R_i any random variable. The expected value of a constant times a random variable equals the constant times the expected value of the random variable

$$E(w_1 R_1) = w_1 E(R_1)$$

The expected value of a weighted sum of random variables equals the weighted sum of the expected values, using the same weights.

$$E(w_1 R_1 + w_2 R_2 + \ldots + w_n R_n) = w_1 E(R_1) + w_n E(R_n) + \ldots + w_n E(R_n) \qquad (4.14)$$

4.15 Correlation

The correlation indicates the strength and direction of a linear relationship between two random variables. Correlation refers to the departure of two variables from independence.

The correlation $\rho_{X,Y}$ between two random variables X and Y with expected values μ and ν, respectively, and standard deviations σ_X and σ_Y is given by

$$\rho_{X,Y} = \frac{E((X-\mu)(Y-\nu))}{\sigma_X \sigma_Y}$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \tag{4.15}$$

Equation 4.15 can be also written as

$$\rho_{X,Y} = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X)^2 - E^2(X)}\sqrt{E(Y)^2 - E^2(Y)}}$$

The correlation is defined only if both standard deviations are finite and both of them are nonzero. The correlation cannot exceed 1 in absolute value, meaning that the value for a correlation goes from -1 to 1.

Talking about a sample instead of a population, the correlation equation will have some differences being

$$r_{xy} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{(n-1)s_x s_y} \tag{4.16}$$

Where $i = 1,2,...,n$ and n in the number of observations.

As with the population correlation, the sample correlation can also be written as

$$r_{xy} = \frac{n\sum x_i y_i - \sum x_i y_i}{\sqrt{n\sum x_i^2 - \left(\sum x_i\right)^2}\sqrt{n\sum y_i^2 - \left(\sum y_i\right)^2}}$$

Chapter Five
The Stock Market

5.1 What are stocks?

A stock or share is a portion of the capital of a company owned by the holder. Shares are values whose profits cannot be determined by predefined calculations. Their profits are a function of the economic and financial development and the supply and demand relationship within the market. Stocks can be issued only by companies and can only be traded in established exchange markets also called bourses, word which derives from the Latin '*bursa*' meaning purse. There are several types of stocks available in the market, common stocks, preferred stocks and treasury stocks.

The first stock appeared in year 1602 issued by Dutch East India Company; this idea rose the economic growth in Europe in the 17th century.

The owners of a company or shareholders might want to raise the capital to invest in new projects, research or development. By issuing stocks they can sell a part or the whole company in many small portions to other companies or private investors interested in the organization. Each investor who holds even a single stock share the ownership of the organization and has the right to receive a fraction of the profits the company makes, these profits are known as dividends. Dividend amount and payment date have to be declared (announced) and can be paid to the investor in one of the following methods:

1. Cash Dividends: This is the most common way of companies to share the profits and are those dividends paid in real cash being a form of investment interest- income.

2. Stock dividends or scrip dividends: This profit sharing consists in giving each stockholder additional stocks of the issuing or other company, usually issued in portions of stocks owned.

3. Property dividends or dividends in specie: theses dividends are the ones paid out in form of assets from the issuing or other corporation, commonly paid in form of goods or services provided by the company.

5.2 Common Stocks

Common Stocks or common shares are the ones which typically have voting rights in corporate decisions. This kind of stocks, as the name implies, are the more commonly held type of stocks in a corporation.

5.3 Preferred Stocks

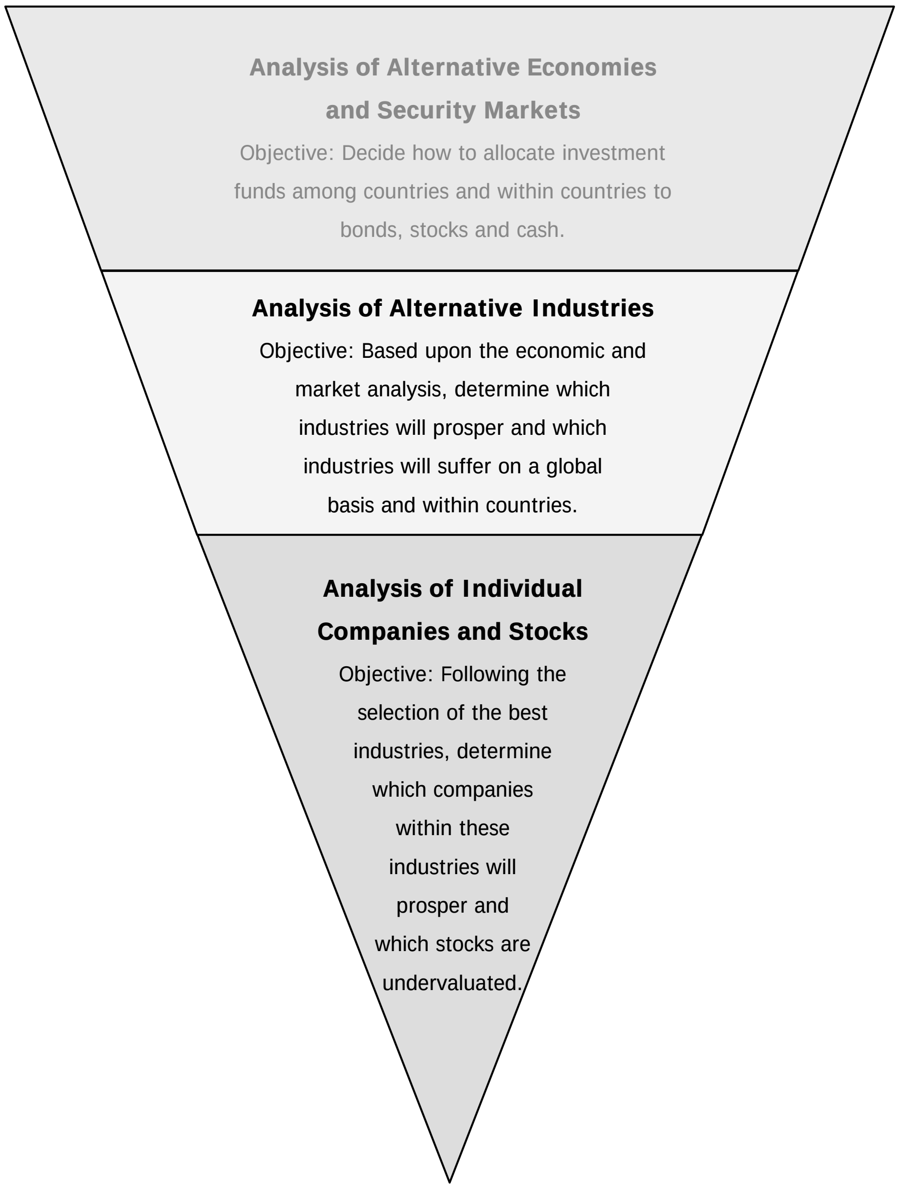
Preferred stocks have priority in the distribution of dividends and assets carrying also additional rights above the common stocks. There are issued to distinguish between the control of and the economic interest of the company.

5.4 Treasury Stocks

Treasury stocks are the shares which are bought back in the market by the issuer company. Organizations buy their own stocks in the market in order to decrease the number of stocks circulating or when they perceive the shares are undervalued for example. Treasury stocks does not pay dividends, have no voting rights and cannot exceed the 5% of total capitalization.

5.5 Security Valuation

The investment process as described by Frank K. Reilly and Keith C. Brown in the book *Investment Analysis & Portfolio Management* should begin with an analysis of the aggregate economies and overall securities markets. After a macroeconomic analysis, an examination of different industries in a global perspective should be done. Once the industry was globally studied, the investor will be in the position to properly evaluate the stocks issued by individual firms within the industry. This overview of the investment process is shown on Exhibit 5.1.



Overview of the Investment Process Exhibit 5.1

These three different types of analysis will be described further on in this chapter as a reference prior to discuss some stock valuation models. It's important to take them in consideration although we will not study them deeper.

5.5.1 Macroeconomic Analysis

The economic conditions within a country will affect the direct and foreign investments. Monetary and fiscal policies influence the aggregate economy of a country and these conditions have also a direct influence over the global industries and every single company within the economy.

The fiscal policy initiatives, such as taxing or tax cuts have a direct impact in spending, either encouraging or reducing it. Increases or decreases in government spending on unemployment insurance, education, defense, streets or buildings also influence the general economy. We can clearly explain this by analyzing one important part of the Gross Domestic Product (GDP) which is the public sector account, being this, the difference from money earned by taxes minus the government spending. Having a direct impact on GDP every monetary and fiscal policy will impact the economy as a whole, every investment sector and every company within a country.

From the investment perspective, if a country has a fiscal policy that high taxes investment profits, investors will not longer want to place their resources within the country because of the high cuts on their returns. Inflation causes differences in real and nominal interest rates.

The macroeconomic situation within a country will motivate the increase or decrease in foreign and domestic investment; prior to take an investment decision is crucial to study the country as a whole to eliminate uncertainty regarding possible market externalities within the economy that could present adverse scenarios.

It is difficult to conceive an industry or company that can avoid the impact of macroeconomic developments and changes, reason why macroeconomic factor should be analyzed before industries are studied.

5.5.2 Industry Analysis

After the completing a macroeconomic analysis the next step is to analyze the industry as a whole within a country. There are several conditions that can affect the industry like import or export taxes, mayor strikes, shortage or excess in supply of raw materials and government regulations among others.

Industries will react to economic changes at different points in the business cycle, but sooner or later they will be forced to adapt to the new industry conditions.

Companies operating in international markets will benefit or suffer from two or more industry situations, in the home country and in the foreign countries.

The global business environment will determine how well or how bad an individual company will perform. A great company in a poor industry or a poor firm in a great industry would not be good prospects for investment; the best prospect in conclusion will be a good company in a great industry, even if the company is not the best within the sector.

A good industry analysis should content the industry performance over time, the performance of the companies, the business cycle and industry sectors. Some other variables shall be analyzed, like demographics, consumer sentiment, interest rates and inflation, lifestyles, politics and regulations.

5.5.3 Company Analysis

After studying and determining how appropriate for investment the industry is investors shall continue with a company analysis comparing individual firms and studying their financial statements using financial ratios and cash flow values to determine the performance of each firm.

The aim of company analysis is to identify the best company in a promising industry by determining its value and determining the intrinsic price of its stock. The objective will be then to identify the best stock; however, the best stock is not necessarily issued by the best company.

5.6 Weight and Expected Returns in Portfolio Analysis

The expected value is the probability-weighted average of the possible outcome of a random variable as seen in section 4.14. A portfolio is an investment alternative where the money is placed in different securities at the same time. Given this portfolio with n securities, the expected return on the portfolio is a weighted average of the expected returns on the component securities (eq. 4.14):

$$E(R_P) = E(w_1 R_1 + w_2 R_2 + \ldots + w_n R_n)$$
$$= w_1 E(R_1) + w_2 E(R_2) + \ldots + w_n E(R_n)$$

Suppose we have estimated the returns on the assets in a portfolio composed by three securities:

Weights and Expected Returns

Asset Class	Weight	Expected Return (%)
OMX Stockholm 30	0.45	12
ABB Ltd	0.25	6
Ericsson A	0.30	15

Exhibit 5.2

We calculate the expected return:

$$E(R_P) = w_1 E(R_1) + w_2 E(R_2) + w_3 E(R_3)$$
$$E(R_P) = 0.45(12\%) + 0.25(6\%) + 0.30(15\%)$$
$$E(R_P) = 11.40\%$$

Letting R_P stand for the return of the portfolio, the portfolio variance is (from eq. 4.14)

$$\sigma^2(R_P) = E\left[(R_P - ER_P)^2\right]$$

$$
\begin{aligned}
&= E[(w_1R_1 + w_2R_2 + w_3R_3 - E(w_1R_1 + w_2R_2 + w_3R_3))^2] \\
&= E[(w_1R_1 + w_2R_2 + w_3R_3 - w_1E(R_1) - w_2E(R_2) - w_3E(R_3))^2] \\
&= E[(w_1(R_1 - E(R_1)) + w_2(R_2 - E(R_2)) + w_3(R_3 - E(R_3)))^2] \\
&= E[(w_1(R_1 - E(R_1)) + w_2(R_2 - E(R_2)) + w_3(R_3 - E(R_3))) \\
&\times (w_1(R_1 - E(R_1)) + w_2(R_2 - E(R_2)) + w_3(R_3 - E(R_3)))] \\
&= E[w_1^2(R_1 - E(R_1))(R_1 - E(R_1)) + w_1w_2(R_1 - E(R_1))(R_2 - E(R_2)) \\
&+ w_1w_3(R_1 - E(R_1))(R_3 - E(R_3)) + w_2^2(R_2 - E(R_2))(R_1 - E(R_1)) \\
&+ w_2w_2(R_2 - E(R_2))(R_2 - E(R_2)) + w_2w_3(R_2 - E(R_2))(R_3 - E(R_3)) \\
&+ w_3w_1(R_3 - E(R_3))(R_1 - E(R_1)) + w_3w_2(R_3 - E(R_3))(R_2 - E(R_2)) \\
&+ w_3^2(R_3 - E(R_3))(R_3 - E(R_3))]
\end{aligned}
$$

$$
\begin{aligned}
&= w_1^2\sigma^2(R_1) + 2w_1w_2Cov(R_1, R_2) + 2w_1w_3Cov(R_1, R_3) \\
&+ w_2^2\sigma^2(R_2) + 2w_2w_3Cov(R_2, R_3) + w_3^2\sigma^2(R_3)
\end{aligned} \tag{5.1}
$$

The variance terms $\sigma^2(R_1)$, $\sigma^2(R_2)$ and $\sigma^2(R_3)$ can be replaced by $Cov(R_1, R_1)$, $Cov(R_2, R_2)$ and $Cov(R_3, R_3)$ to rewrite equation 5.1 and having an equation to calculate the variance of a portfolio as follows:

$$
\sigma^2(R_P) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_i Cov(R_i, R_j) \tag{5.2}
$$

If the covariance in equation 5.2 is negative when the return of an asset is above its expected value and the return of the other asset tends to be below its expected value. The covariance is zero it the returns of the assets are unrelated and will be positive when the returns of both assets tend to be on the same side (above or below) their expected value at the same time.

5.7 The Capital Asset Price Model - CAPM

One of the more common models to valuate stocks which also involve risk is the Capital Asset Pricing Model (See [13]). For a better study of this model is important to change the focus towards an equilibrium analysis from a system in which the aggregate supply of financial instruments is equal to the aggregate demand of them, therefore different assumptions regarding the economy and investor’s behavior are necessary to announce:

1. The investors based their decisions in the expected rate of return over the investment and the variance of this returns over a specific period of time.

2. Every asset is infinitely and perfectly divisible.

3. Each investor can borrow and lend money at the same risk-free interest rate.

4. Transaction costs and taxes are irrelevant.

5. Information regarding expected returns, variance and covariance are available for everyone.

6. All investors have the same expectations.

The Capital Asset Price Model can be derived also using the following assumptions regarding the investment decisions:

1. All investors have rational expectations.

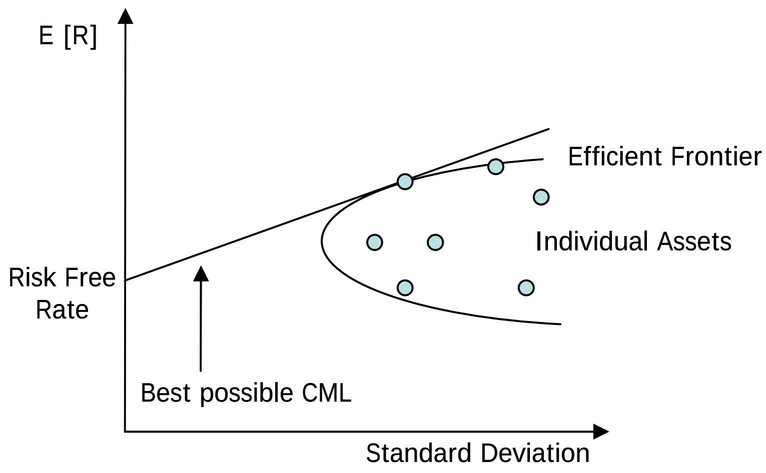
2. All investors are risk adverse.

3. Each investor will invest in just two portfolios: risk free asset and the tangent portfolio.

4. Perfect equilibrium: aggregate asset supply = aggregate asset demand.
The net loans must be equal to zero and all risky assets have to be in the investor’s portfolio.

5. The tangent portfolio is equal to the market portfolio.

These assumptions give us the Capital Market Line (CML) which describes the inverse relationship between portfolio's risk and return. The CAPM also assumes that the risk-return profile of a portfolio can be optimized reducing the risk to the lowest possible level for each return level. All such portfolios compromise the efficient frontier (Exhibit 5.3)



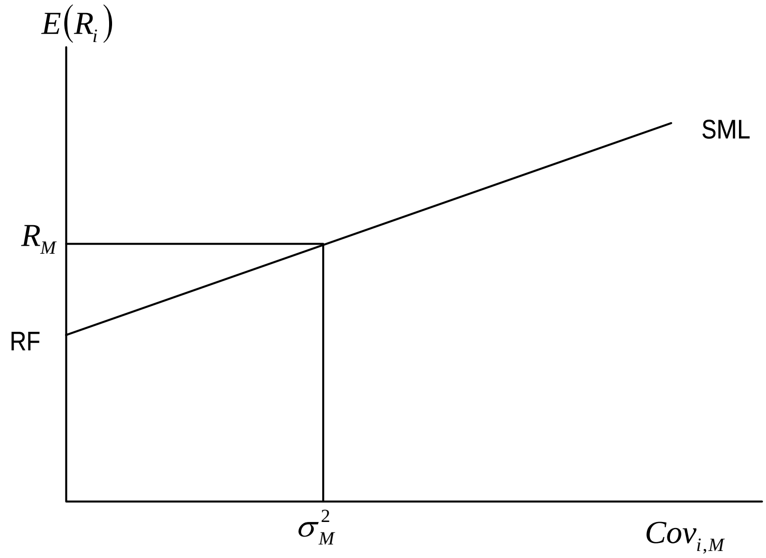
Efficient Frontier Exhibit 5.3

Having a portfolio composed by different securities, the relevant risk measure for an individual risky asset is its covariance with the market portfolio $\left(Cov_{i,M}\right)$. The return of the market portfolio should be consistent with its own risk, which is the covariance of the market with itself.

In section 5.6 we stated that the covariance of an asset with itself is its variance

$$Cov_{i,i} = \sigma_i^2$$

The covariance of the market with itself is the variance of the market rate of return. Exhibit 5.4 shows the risk-return relationship with the systematic variable $(Cov_{i,M})$ as the risk measure.



Security Market Line Exhibit 5.4

The return of the market portfolio (R_M) should be consistent with its own risk, which is the covariance of the market with itself. The equation for the Security Market Line in exhibit 5.4 is

$$E(R_i) = RFR + \frac{R_M - RFR}{\sigma_M^2}(Cov_{i,M})$$

$$= RFR + \frac{Cov_{i,M}}{\sigma_M^2}(R_M - RFR)$$

The term $Cov_{i,M} / \sigma_M^2$ can be defined as β_i obtaining equation 5.3 which describes the Capital Asset Pricing Model

$$E(R_i) = RFR + \beta_i (R_M - RFR) \tag{5.3}$$

Beta can be viewed as a standardized measure of systematic risk because it relates the covariance to the variance of the market portfolio. The market portfolio has a beta of 1.0; if an asset has a beta above 1.0 the asset has a higher normalized systematic risk than the market, meaning it is more volatile than the overall market portfolio and the other way around for a beta less than 1.0.

The market risk premium in the CAPM model is the difference between the market expected return and the risk-free rate of return. However, talking about a risk-free rate of return we cannot leave the concept of real interest rate a side. If we apply equation 3.1 in the CAPM model we would have a real expected return. If we calculate the expected return for short periods of time, inflation could be a factor that has no impact on the return, but if we analyze the return for a longer period of time, the inflation can play an important role, especially in countries with high inflation rates.

We can conclude that there is no “risk-free” interest rate even if it’s the rate for the government bonds; the inflation risk is always present. Under this assumption we could adjust equation 5.3 by inflation

$$E(R_i) = \frac{(1+RFR)}{(1+\pi)} + \beta_i\left[\left(\frac{1+R_M}{1+\pi}\right) - \left(\frac{1+RFR}{1+\pi}\right)\right] \qquad (5.6)$$

Chapter Six
Money Market and Bond Valuation

6.1 Bond Features

Bonds can be defined as long-term, fixed-obligation debt securities packaged in convenient, affordable denominations for sale to individuals and financial institutions. They are sold to the public and are considered fixed-income securities because they impose fixed financial obligations on the issuer who agrees to pay a fixed amount of interest periodically to the holder and repay a fixed amount of principal at the date of maturity.

Short term issues with maturities of one year or less are traded in the money market. Intermediate-term issues with maturities in excess of one year but less than ten years are instruments known as notes and the long-term obligations, with maturities longer than ten years are called bonds.

All bonds have different characteristics based on its intrinsic features, its type, its indenture provisions and the features that affect its cash flows and/or its maturity.

6.2 Intrinsic Features

There are some important intrinsic features in all bonds, this features are the following:

Coupon: the coupon of a bond indicates the income that the investor will receive over the life or holding period of the issue; this is known as interest income, coupon income or nominal yield.

Term of maturity: specifies the date or the number of years before the bond matures or expires. The maturity can be called a term bond, which has a single maturity date but can also be a serial obligation bond which has a series of maturities being each maturity a subset of the total issue.

Principal: The principal or par value represents the original value of the obligation. The principal is not the market value of the bond. The market price rises above or falls below the principal because of differences between the coupons and the prevailing market interest rate. If the market interest rate is above the coupon rate, the bond will sale at a discount par. If the market rate is below the bond's coupon, it will sell at a premium above par.

Ownership: bonds differ in terms of ownership. With a bearer bond, the holder or bearer is the owner, so the issuer keeps no record of ownership. Interest from a bearer bond is obtained by clipping coupons attached to the bond and sending them to the issuer for payment. In contrast, the issuers of registered bonds maintain records of owners and pay interest direct to them.

6.3 Bond Valuation

The price of any bond can be expressed as the sum of the present value of all cash flows plus the present value of the principal and can be computed with the following equation:

$$B = \sum_{t=1}^{n} \frac{CF_t}{\left(1+\frac{y}{m}\right)^{mn}} + \frac{P}{\left(1+\frac{y}{m}\right)^{mn}} \qquad (6.1)$$

Where,

B = the price of the bond

CF_t = Cash Flow of coupon payment in time t

P = Principal

y = yield or bond interest rate

The denominator equals equation 3.4 to compute the present value of every cash flow and the principal. The interest rate of a bond can be composed of a reference interest rate, for example T-Bills in the USA plus a overrate defined by the issuer.

6.4 Duration

The price of any bond is determined by its interest rate or yield making the price sensitive to every change in this rate. There is an inverse relationship between the yield and the price of the bond; if interest rates increases, the price will fall and if interest rates decline the price will rise.

A measure of the interest rate sensitivity of a bond is called duration. Duration indicates the price volatility of a bond in response to interest rate changes. The duration, expressed in years also measures the period of time in which the initial investment will be recovered. If the bond has no coupons (zero coupon bond) the duration equals the maturity.

To compute the duration, we begin the calculation with the bond price, this time we will calculate the price using interest rates with continuous compounding.

The price of the bond is given by B which is a function of the yield.

$$B(y) = \sum CF_t e^{-yt} + Pe^{-yt}$$

We will forget about the principal for this calculation since is irrelevant to calculate the duration. The equation results in

$$B(y) = \sum CF_t e^{-yt} \tag{6.2}$$

We proceed to calculate the first and the second derivative which will be used in further calculations

$$B'(y) = -\sum tCF_t e^{-yt}$$

$$B''(y) = \sum t^2 CF_t e^{-yt}$$

Since we are trying to measure the sensitivity of the bond to changes in the interest rates, our bond is now a function of the yield plus a change in the interest rates.

$$B(y+\Delta y)=\sum CF_t e^{-yt} \tag{6.3}$$

We can expand equation 6.3 by a Taylor series where just the first two derivatives are needed

$$f(x+\varepsilon)=f(x)+f'(x)\varepsilon+\frac{f''(x)}{2!}\varepsilon^2+\ldots+\frac{f^n(x)}{n!}n^2$$

Substitute formula 6.3

$$B(y+\Delta y)=B(y)+B'(y)\Delta y+\frac{B''(y)}{2}\Delta y^2$$

we change $B(y)$ just for B being this the price of the bond and then divide by B to eliminate the price and keep just the duration and convexity. Convexity will be discuss in section 6.5

$$B+\Delta B=B+B'\Delta y+\frac{1}{2}B''\Delta y^2$$

$$\frac{\Delta B}{B}=\frac{B'}{B}\Delta y+\frac{1}{2}\frac{B''}{B}\Delta y^2$$

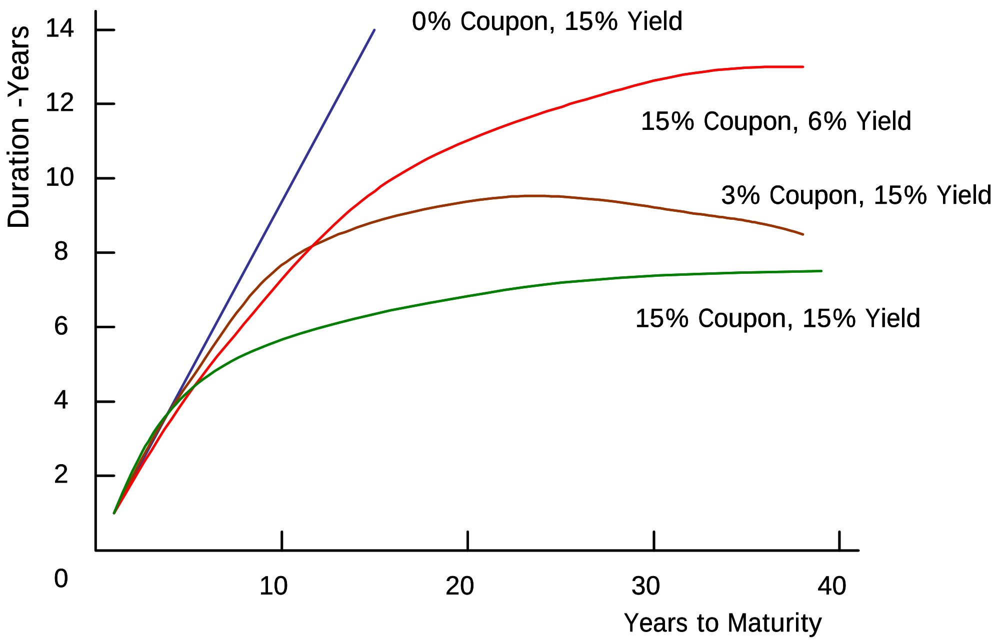
we obtain now the equation for the Duration and Convexity substituting B for equation 6.2.

$$B_{DC}=1-\left[\frac{\sum tCF_t e^{-yt}}{\sum CF_t e^{-yt}}\Delta y\right]+\left[\frac{1}{2}\frac{\sum t^2 CF_t e^{-yt}}{\sum CF_t e^{-yt}}\Delta y^2\right] \tag{6.4}$$

The part of equation 6.4 that gives us the equation to calculate the duration of a bond is

$$Duration = \frac{B'}{B} = \left[\frac{\sum tCF_t e^{-yt}}{\sum CF_t e^{-yt}} \Delta y \right] \tag{6.5}$$

The duration for a bond with coupon payment will be always less than its maturity; mathematically, the denominator will always be less than the numerator because it represents the sum of the first derivative from the price of the bond or denominator. There is an inverse relationship between coupon size and duration, a bond with longer coupon or a higher interest payment has shorter duration because more of the total cash flows come earlier as interest payment. A zero coupon bond will have always a duration equals to his maturity. There is a positive relationship between duration and maturity, if the maturity of a bond increases, the duration will also be higher. Exhibit 6.1 shows the relationship between duration, maturity and coupon size. The duration can be also defined as a linear approximation of the price of a bond which follows a curvilinear function.



Duration vs. Maturity Exhibit 6.1

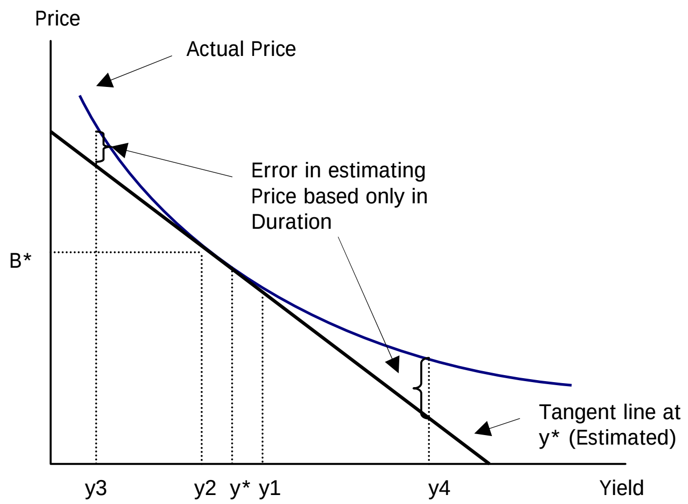
6.5 Convexity

Convexity is the measure of the curvature of the price-yield relationship shown in exhibit 6.2. To calculate the convexity we start from equation 6.2 and having equation 6.4 as a result, the convexity will be the calculation of the second derivative of the bond price divided by the bond price.

$$Convexity = \frac{\sum t^2 CF_t e^{-yt}}{\sum CF_t e^{-yt}} \tag{6.6}$$

There is an inverse relationship between convexity and coupon so as between yield and convexity; being a direct relationship the one between convexity and yield. The price of the bond can be modified by the duration and convexity when there is a change in the yield multiplying the price of the bond B times equation 6.4.

$$B_{DC} = B\left[-D\Delta y + \frac{1}{2}C\Delta y^2\right] \tag{6.7}$$



Price-Yield Relationship Exhibit 6.2

The duration line is a tangent line to the curve representing the price-yield relationship. This tangent line provides a good estimation of actual prices with small changes in the yield, for example from initial yield $y*$ to y_1 or y_2. If the yield presents bigger changes to y_3 or y_4 an error in the price calculation will be found if the price is calculated only based on duration. Never the less, the error can be calculated for the duration and for the convexity having this error expressed in basic points (0.01%) using the following equation

$$Error_{DC} = \left(\frac{B_{DC}}{B} - 1 \right) * 10{,}000 \qquad (6.8)$$

Adding the error amount to the bond calculation will provide a more accurate market price for the bond.

6.6 Market Price Equation

The price of a bond in day by day trading can be calculating using a non-conventional formula. There are important variables that change every day like the interest rate offered and asked by suppliers and investors willing to buy.

The days to go for the next interest payment change; so will also change the remaining maturity of the bond.

First of all, it is necessary to calculate the period of coupon payment

$$Coupon = \frac{C\% * CC}{360} \qquad (6.9)$$

Where,

$C\%$ = Coupon rate, interest payment from the principal value.

CC = Coupon cut, number of days between each interest payment.

Having the coupon interest rate expressed in one day we can proceed with the calculation of the bond price

$$B = NV \frac{\left[\dfrac{\left[1-\left(\dfrac{acr}{mkr-or}\right)\right]}{\left[1-\dfrac{(mkr+or)cp}{36{,}000}\right]^{(nc-1)}} + \left(\dfrac{acr}{mkr+st}\right) + \left(\dfrac{ccr*cp}{36{,}000}\right) \right]}{\left[1+\dfrac{(mkr+or)*cp}{36{,}000}\right]^{\left(1-\frac{dcc}{cp}\right)}} \qquad (6.10)$$

where

NV : Bond nominal value

acr : Actualized coupon rate

mkr : Market interest rate

or : Overrate

cp : Coupon period

nc : Number of coupons

ccr : Current coupon rate

dcc : Days passed from current coupon

Suppose a bond was issued 10 days ago with a nominal value of $100 a maturity of 728 days and will pay a coupon of 5.56% every 28 days. The current market yield is 10.42% and an overrate of 0.89%. The coupon rate has changed to 5.40% being this the actualized coupon rate.

We will have the following data

NV : $100

acr : 11.02%

mkr : 10.42%

or : 0.89%

cp : 28

$nc\ :\ \frac{728}{28} = 26$

ccr : 5.56%

dcc : 10

We now substitute all data in equation 6.10 to compute the bond's price:

$$B = \$100 * \frac{\left[\frac{\left[1 - \left(\frac{5.40\%}{10.42\% - 0.89\%}\right)\right]}{\left[1 - \frac{(10.42\% + 0.89\%)28}{36{,}000}\right]^{(26-1)}} + \left(\frac{5.40\%}{10.42\% + 0.89\%}\right) + \left(\frac{5.56\% * 28}{36{,}000}\right) \right]}{\left[1 + \frac{(10.42\% + 0.89\%) * 28}{36{,}000}\right]^{\left(1 - \frac{10}{28}\right)}}$$

$$B = \$89.65$$

The market price of the bond is $89.65.

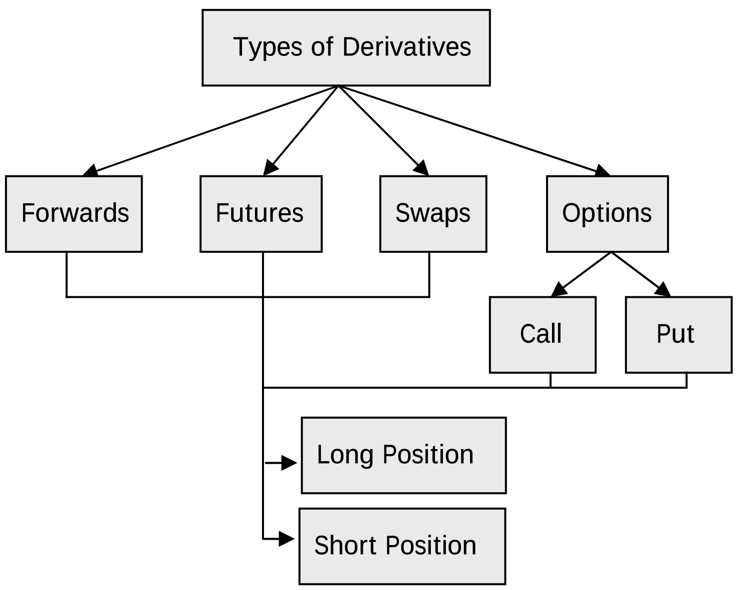
Chapter Seven
Derivative Markets and Securities

7.1 Overview of Derivatives

A derivative instrument is a security whose payoff is explicitly tied to the value of some other variable or some other financial security. The security that determines the value of a derivative instrument is called an underlying asset. However, derivatives may have payoffs that are functions of nonfinancial variables, such as the weather or the outcome from the agricultural season. The main point is that the payments derived from a derivative security are deterministic functions of some other variable whose value will be revealed before or at the time of payoff.

There are different types of derivative securities, the main are forwards, futures, swaps and options. There can also be found some derivatives whose underlying asset is another derivative.

In each type of derivative the holder and issuer can have two different positions, short position when selling and long when buying.



Types of Derivatives Exhibit 7.1

7.2 Forward Contracts

Future and forward contracts are closely related, but forwards are the simpler of the two. A forward contract on a commodity is a contract to buy or sell a specific amount of an underlying asset at a specific price at a specific time in the future.

A forward contract is specified by a legal document, the terms of which bind the two parties involved to a specific transaction in the future. However, a forward contract on a priced asset, such as rice, is also a financial instrument, since it has an intrinsic value determined by the market for the underlying asset. Forward contracts may have underlying assets other than physical commodities, such as interest rates or foreign currency.

There is no established market for forwards, they are traded in Over the Counter Markets and both, buyer and seller establish the terms and conditions for the contract.

Forward Contracts Exhibit 7.2

Suppose person *A* agrees today $(t = 0)$ to buy from person *B* ten tons of rice in two years $(t = 2)$ paying $10,500 for each ton, person *A*, the buyer, is said to be long 10 tons of rice and person *B*, the seller is said to be short. Person *B* might not have the underlying asset at time $t = 0$, the day the contract is pact, it would really not matter since he made the commitment to deliver the rice in two years.

Almost always, the initial payment associated with a forward contract is zero, the contract has no value when is initiated. Neither party pays any amount of money to obtain the contract, the payment will be made when the underlying asset is delivered and the purchaser pays the forward price or price agreed the day the contract was made. Sometimes a security deposit in cash is required by both parties.

Suppose we buy one commodity at price S on the spot market and at the same time we enter a forward contract in a short position at time T to deliver one unit at price F. The underlying asset will be stored until time T when it will be delivered receiving F against the delivery. Using the future value analysis, S and F should be equivalent, therefore, the forward price can be calculated with a continuous compounding with the following equation

$$F = Se^{rt} \tag{7.1}$$

Equation 7.1 supposes the interest rates for money and for the underlying asset are equal and no storing costs are involved.

If the interest rates are not equal and no other costs are involved, r will represent the interest rate for lending or borrowing money and q the interest rate or growth rate on the price of the underlying asset. Equation 7.1 can be written as

$$F = Se^{(r-q)t} \tag{7.2}$$

The rate for money should be bigger than the one for the underlying asset. If the interest rate for the commodity is larger than the interest rate for money the forward price would be less than the spot price and nobody will go short on forward contracts. If both rates are equal, the price will be the same today and in the future, but it would be better to borrow money and buy the underlying asset because one monetary unit today will not have the same value in the future.

Equations 7.1 and 7.2 will only give the theoretic forward price. The forward price of a contract will be given by the market supply and demand relationship based on this theoretic price.

The cost of storage and other costs associated with forward prices are called cost of carry. They are added to the forward price. Mathematically they can be added in the exponent making the forward price bigger

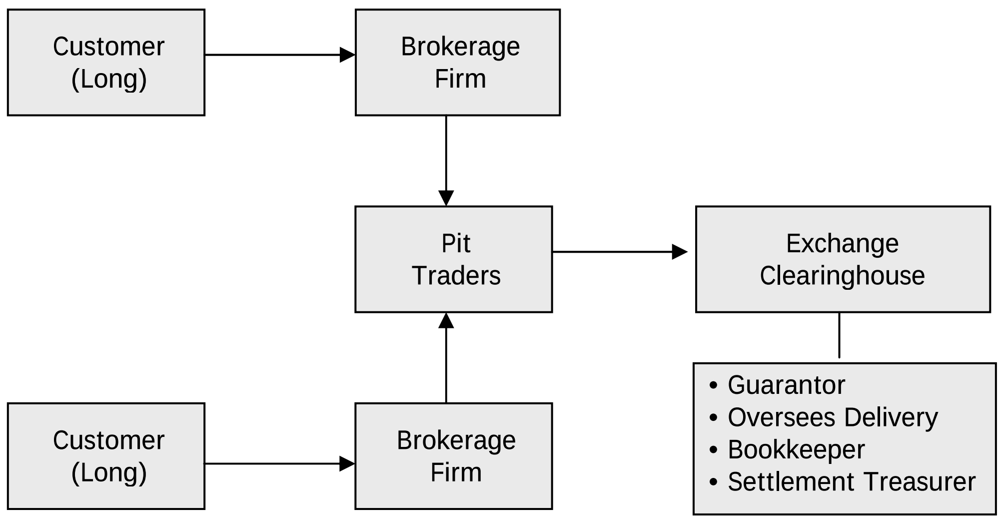
$$F = Se^{(r-q)t+c} \tag{7.3}$$

There is other concept related to forward contracts which is the measure of the benefit of holding the commodity known as the convenience yield. The convenience yield is a negative holding cost but it represents the interest of having the underlying asset in hand to be used when needed. Adding the convenience yield to the future contract gives us the final equation to valuate future contracts.

$$F = Se^{(r-q)t+c-y} \tag{7.4}$$

7.3 Future Contracts

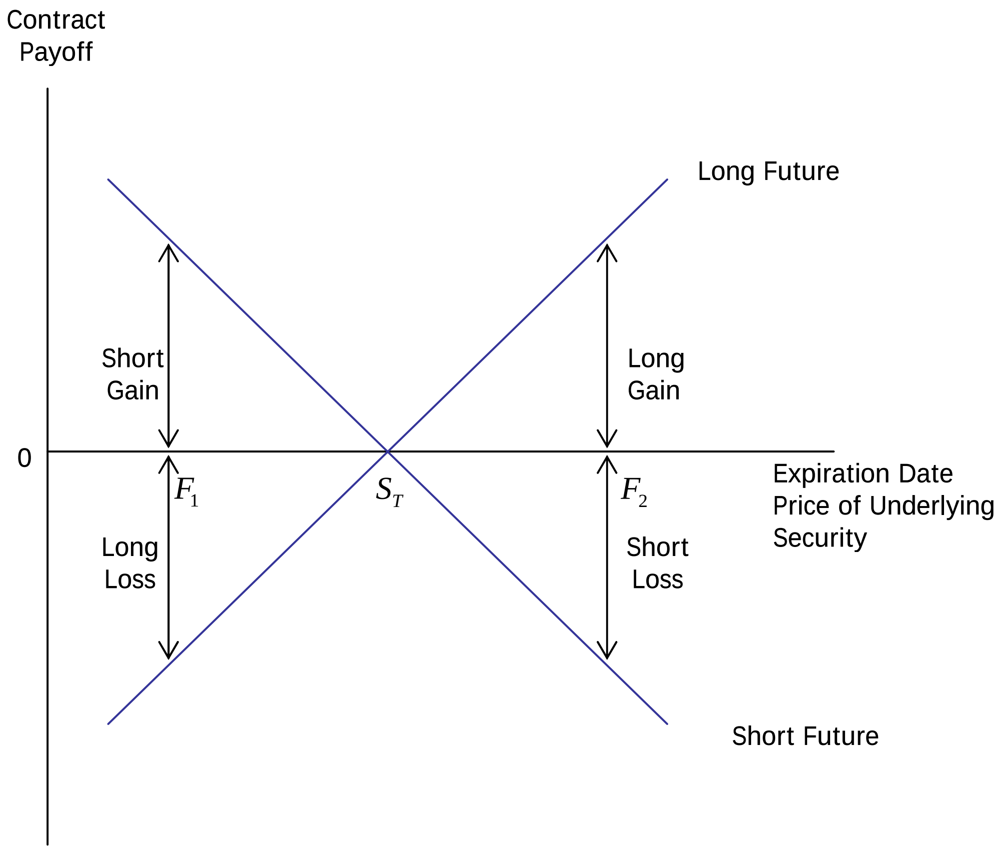
Futures and forwards are very likely, the main difference is that futures are standardized contracts traded in the Future Market. The valuation for these contracts are the same as for forwards and the size of the contracts is fixed; if a farmer needs to buy 2.5 tons of seed, he could buy this exact amount with a forward contract. If the farmer wants to buy a future on seed, and each contract represents just one ton, he has two buy either two or three contracts to cover his shortage. To have access to the future market, investors have to contract the services of a brokerage firm to buy or sell the required contracts. As in every market, there is a clearing house which takes care of all the settlement issues. Exhibit 7.3 shows the future market structure.



Future Contracts Exhibit 7.3

As forwards, futures are contracts that make the investors acquire the obligation to buy or sell a specific amount of underlying asset at a specific time and to a specific price in the future.

The payoff function is a difference of the future price and the spot price. If the spot price is lower than the future price, the seller would have a gain and the buyer a loss because he could have bought the commodity directly in the spot market cheaper. In the contrary, if the future price is lower than the spot price, the gain goes to the buyer who would have to pay more directly from the spot market.



Payoffs to Long and Short Future (Forward) Positions Exhibit 7.3

Forward and future contracts are use usually as hedge instruments, they are not used as speculative investments to make profits; buyers cover future needs and fixed the price today, they know exactly the amount they will have to pay and in some cases, if they expect the price of the commodity to grow, they will realize a gain.

If the position of the underlying asset is long, the position for the future has to be short to hedge the portfolio, meaning that investors always take the opposite position in future contracts from the one they have in the commodity. To calculate the optimal hedge ratio, we start with a portfolio composed of a commodity and the contrary position in future contracts multiplied by an optimal hedge size

$$P_t = S - hF$$

The portfolio at time t is a difference of spot and future prices

$$P_t = \left(S_t - S_0\right) - h\left(F_t - F_0\right) = \Delta S - h\Delta F$$

The variance value of the portfolio is then given by

$$\sigma_P^2 = \sigma_{\Delta S}^2 + h^2\sigma_{\Delta F}^2 - 2Cov_{\Delta S,\Delta F}$$

Minimizing this expression and solving for h leaves

$$h = \frac{Cov_{\Delta S,\Delta F}}{\sigma_{\Delta F}^2} = \rho\frac{\sigma_{\Delta S}}{\sigma_{\Delta F}} \tag{7.4}$$

Suppose a window maker is short of 417,200 kilos of aluminum. To cover this shortage, he wants to buy futures over aluminum in the future market. Each contract represents 10,000 kilos. The buyer needs to calculate the optimal hedge ratio to avoid buying excessive contracts. If $\sigma_{\Delta S} = 20\%$, $\sigma_{\Delta F} = 23\%$ and the correlation between spot and future price changes is $\rho = 0.87$ the optimal hedge ratio can be computed by

$$h = 0.87\frac{0.2}{0.23} = 0.7565$$

$$\frac{(0.7565)(417{,}200)}{10{,}000} = 31.56 \Rightarrow 32$$

The window maker should buy then just 32 future contracts on aluminum to cover his short position.

7.4 Forward Interest Rates

Forwards can also be used for interest rates in the future. There are known interest rates for different periods today; an investor knows the interest rate today if he invests for six, nine or twelve months today. If an investor wants to place some amount of money in an investment for a period of five starting in four months, he does not know now the interest rate for such a period in the future. To solve this need, investors use a security known as FRA or Forward Rate Agreement. The FRA can be calculated with following equation

$$R_T * T = R_t * t + R_{t \to T} * (T - t) \tag{7.5}$$

Using the information form the example above, the interest rate today for a four months period is $R_t = 3.00\%$, for a nine month period starting today, the interest rate is $R_T = 5.00\%$. Based on this information, the investor agrees to invest his money starting in four months, for a period of five months at a 6.60% interest rate

$$(5.00\%)(9) = (3.00\%)(4) + R_{t \to T}(9 - 4)$$

$$R_{t \to T} = \frac{(5.00\%)(9) - (3.00\%)(4)}{(9 - 4)}$$

$$R_{t \to T} = 6.60\%$$

7.5 Swaps

Swaps are derivative securities where two counterparties exchange one stream of cash flows against another stream. These streams are called the legs of the swap. The cash flows are calculated over a notional value which is not exchanged.

Swaps are used to cover different risks. A liability with a variable interest rate can be transformed into a fixed rate one or the other way around. If a company has a debt in a foreign currency it can cover this exchange rate risk by acquiring a swap over foreign currencies.

As an example, consider the most common swap called plain vanilla, an interest rate swap. Party *A* agrees to make a series of monthly payments to party *B* to a fixed rate of interest rate on a notional principal. In return, party *B* makes a series of monthly payments to party *A* based on a floating interest rate (such as LIBOR rate) and the same notional principal. The floating rate can also include an overrate, LIBOR+10 basic points for example. Usually, swaps are netted, meaning that only the difference of required payments is made by the party that owes this difference.

Swaps are derivative securities in which a market maker, such as a brokerage firm, finds the two counterparties and acts like an intermediary for these payments having a margin as a gain.

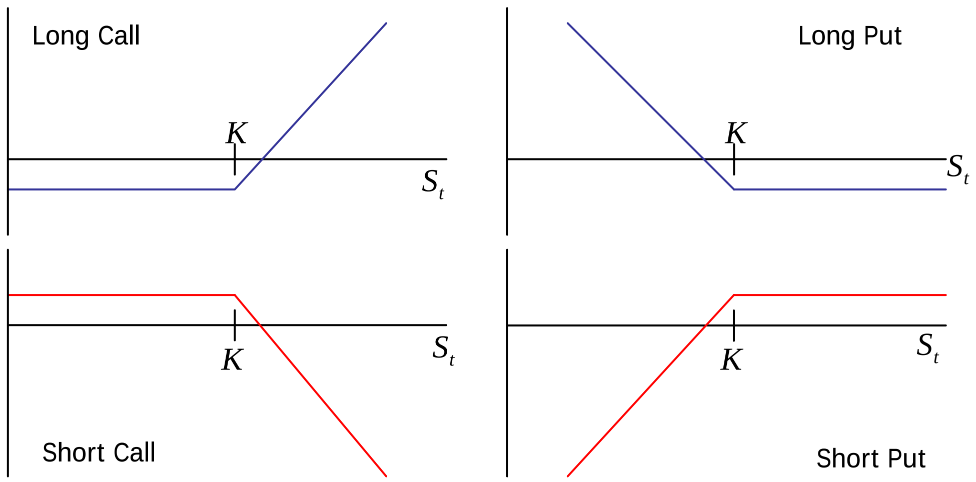
Interest Rate Swap Exhibit 7.4

7.6 Options

An option is a derivative financial instrument that gives the buyer the right but not the obligation to buy or sell an underlying asset at a specific exercise price K, known as strike at a specific time in the future. For the seller, the option gives the obligation to buy or sell the underlying asset if the buyer decides to exercise his/her right.

There are two different kinds of options, the Call that provides the buyer the right but not the obligation to buy an underlying asset and the seller has the obligation to sell the commodity if the buyer exercises the option. The Put provides the party holding the long position the right but not the obligation to sell an underlying asset and the party holding the short position is obligated to buy if the option is exercised. The buyer of the option pays a prime to the seller of this one, if the option is not exercised; the seller gets a profit by receiving this prime.

Options can be divided in two, American and European options. The European options can only be exercised at the end of the contract or expiration date; the American options can be exercised at any time during the life of the option.



Call and Put Options Exhibit 7.5

The long call and the long call have a limited loss equal to the prime and an unlimited gain and in the opposite side, short positions have unlimited losses and limited gains.

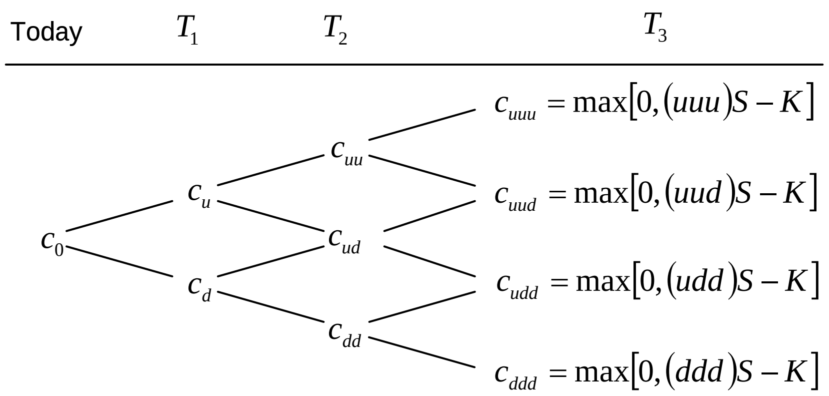
The payment function of a call can be expressed as the maximum value between the difference between the spot price and the strike and zero:

$$C(S_t) = \max(S_t - K, 0)$$

The payment function of a call is the maximum value between the difference between the strike and the spot price and zero:

$$P(S\ \ ) = \max(K - S_t, 0)$$

Any option has an intrinsic value determined by the payment functions and a time value. The time value reflects uncertainty, the longer the time to exercise the option the longer this value. The value of an option can be calculated using binomial and trinomial trees. The binomial tree considers the probability of the price going either up and down with time and the trinomial trees also includes the probability of the price remaining the same trough time.



Binomial Tree for a Call Exhibit 7.6

The binomial model is a discrete method for valuating options because it allows security price changes to occur in distinct upward or downward movements. It also can be assumed that the prices change continuously trough time. This was the approach taken by Fischer Black and Myron Scholes on their equation published in 1973. The key assumptions of the Black-Scholes model are:

- The price of the underlying instrument follows a geometric Brownian motion with constant drift and volatility.
- It is possible to short sell the underlying asset (sell without having the asset).
- There are no arbitrage opportunities.
- Trading the underlying asset is continuous.
- There are no transaction costs or taxes.
- All securities are perfectly divisible.
- The risk-free interest rate exists and is constant, and the same for all maturity dates.

The underlying asset price process assumed by Black and Scholes is

$$\frac{\Delta S}{S} = \mu[\Delta T] + \sigma\varepsilon\sqrt{\Delta T}$$

That is, an underlying asset's return $(\Delta S / S)$ from the present through any future period T has both an expected component $(\mu[\Delta T])$ and a "noise" component $(\sigma\varepsilon\sqrt{\Delta T})$, where μ is the mean return and ε is the standard normally distributed random error term.

Assuming the continuously compounded risk-free rate, and the asset's variance remain constant until the expiration date, the Black-Scholes equation for options is

$$X = \phi S_0 e^{-qt} N(\phi d_1) - \phi K e^{-rt} N(\phi d_2) \qquad (7.6)$$

Where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$$

And

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t}$$

The value for ϕ will be -1 for put and $+1$ for call. The formula uses the function $N(x)$, the cumulative normal probability distribution. This is the cumulative distribution of a normal random variable having mean 0 and variance 1 expressed as

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-y^2}{2}dy}$$

The Black-Scholes model can be expressed as a partial differential equation that follows a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t \,,$$

where W_t is Brownian.

Let V be some sort of option on S. Mathematically V is a function of S and t while $V(S,t)$ is the value of the option at time t if the price of the underlying asset at time t is S. The value of the option at the time that the option matures is known. To determine its value at an earlier time we need to know how the value evolves as we go backward in time. By Ito's lemma for two (see [8] and [18]) variables we have

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S \frac{\partial V}{\partial S} dW \qquad (7.7)$$

Now, consider a portfolio consisting of one unit of the option V and $-\partial V / \partial S$ units of underlying asset.

$$\Pi = V - S\frac{\partial V}{\partial S}$$

The composition of this portfolio, called delta hedged, will vary from time-step to time-step. We consider the change in return of the portfolio

$$dR = dV - \frac{\partial V}{\partial S} dS$$

By substituting equation 7.7 we get

$$dR = \left(\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \tag{7.8}$$

Equation 7.8 contains no dW which means that is entirely delta neutral or risk-less. By the assumption that there is no arbitrage and supply and demand are infinite, the rate of return of the portfolio must equal the rate of return on any other risk-free instrument. Assuming the risk-free return is r over the time period $[t, t+dt]$

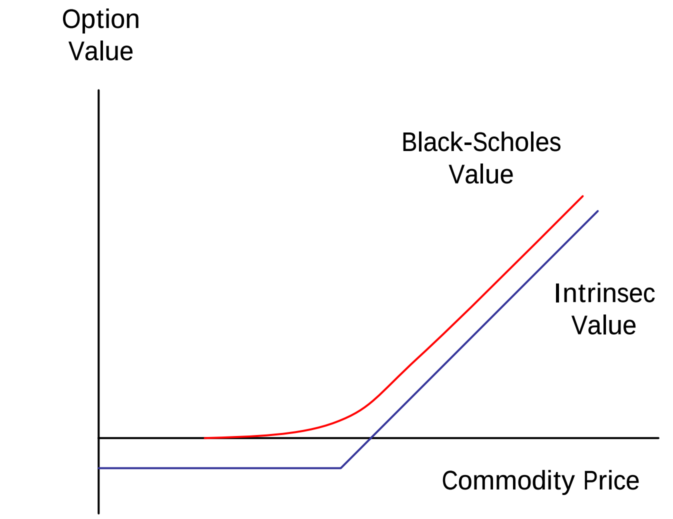
$$r\Pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

Substituting in for Π and dividing by dt the Black-Scholes partial differential equation is obtained.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{7.9}$$

This model is formulated in terms of stochastic differential equations, however, under certain simplifying assumptions it can be approximated by usual differential equations. The linear equation 7.9 can also be written as

$$u_t + \frac{1}{2} A^2 x^2 u_{xx} + Bxu_x - Cu = 0$$



Black-Scholes Value on Options Exhibit 7.7

There are different sensibility measures associated with the Black-Scholes model known as the Greeks. Greeks are derivatives of the Black-Scholes model with respect of different variables measuring changes in prices.

The Greeks are the following:

$Delta = \frac{\partial f}{\partial S}$: Delta is a measure of the sensibility of the option price with respect of small changes in the underlying asset's price.

$Gamma = \frac{\partial^2 f}{\partial S^2}$: Defines the curvature of the derivative price curve. Gamma is the second derivative of the model with respect of the spot price. Gamma will be calculated exactly the same for call and put.

$Vega = \frac{\partial f}{\partial \sigma}$: Measures the sensibility of the price with respect of changes in volatility.

$Theta = \frac{\partial f}{\partial t}$: Is the first derivative of the model with respect of time and measures the changes in price with respect of changes in time.

$Rho = \frac{\partial f}{\partial r}$: Measures the sensibility of the option price with respect of the interest rate.

Black-Scholes Greeks

	Call	Put
Delta	$N(d_1)$	$N(d_1)-1$
Gamma	$\dfrac{\phi(d_1)}{S\sigma\sqrt{t}}$	$\dfrac{\phi(d_1)}{S\sigma\sqrt{t}}$
Vega	$S\phi(d_1)\sqrt{t}$	$S\phi(d_1)\sqrt{t}$
Theta	$-\dfrac{S\phi(d_1)\sigma}{2\sqrt{t}}-rK^{-rt}N(d_2)$	$-\dfrac{S\phi(d_1)\sigma}{2\sqrt{t}}-rK^{-rt}N(-d_2)$
Rho	$Kte^{-rt}N(d_2)$	$-Kte^{-rt}N(-d_2)$

Exhibit 7.8

Chapter Eight
Price Behavior in the Financial Markets and The Stochastic Market Price Estimator Model

8.1 Technical Analysis

Talking about price behavior and the analysis of prices, we could immediately think about technical analysis as one very useful tool to analyze the tendency and estimate future ranges of prices. Technical analyst developed trading rules from observations of past price movements within the market. Analyzing historical changes in prices provides important information for future movements.

The technical analysis differs with the theory of efficient markets referred on chapter two because this type of analysis uses numerical series generated by the market activity historical data to predict future price trends instead of the availability of new information provided by the market. (Exhibit 8.1)

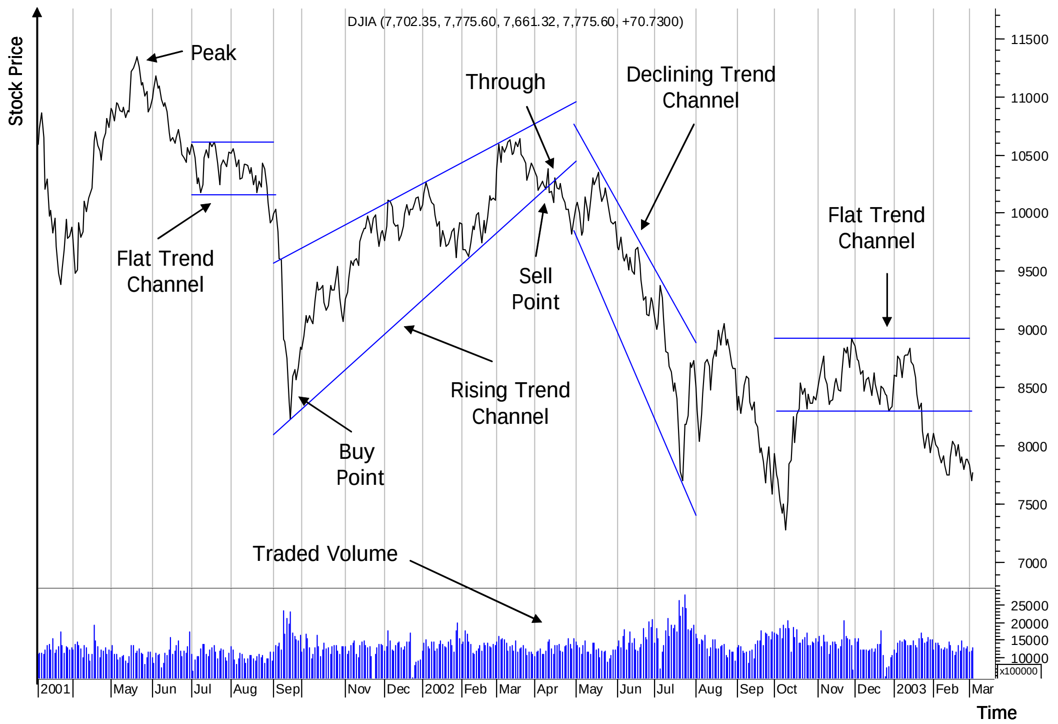
Historical price activities within the market Exhibit 8.1

Technical analysis does not try to analyze financial information about the company and is not completely accurate. Technical analysis is viewed by many practitioners as an art more than a science; however, different kind of mathematics, like differential equations or statistics can help us put technical analysis in a different perspective.

Technical analysis, as said before, places trading decisions on examinations of the historical data regarding prices and volume traded predicting a future behavior for the whole and for every single security traded. There are some assumptions regarding technical analysis that have to be considered:

- The market value of any security is set by supply and demand relation.
- Supply and demand are based on numerous rational and irrational factors.
- The trends in which prices move tend to persist for a long time.
- Trends will change reacting to shifts in supply and demand relationships.

There are some indicators within the technical analysis as shown in exhibit 8.2.



Technical analysis indicators Exhibit 8.2

First, we can identify three different trend channels, the bull or rising trend channel, the bear or declining trend channel and the flat trend channel. At the point where the trend changes, we can identify sell or buy points depending if the trend changes to a rising or to a declining one. The peak is the highest point on a trend and can only be identify once the trend has changed.

Prices are supposed to move in cycles that repeat over the time. The technical analysis tries to determine these cycles to estimate future range of prices.

8.2 External Variables

We have to keep always in mind that this kind of analysis, like the technical analysis, will just estimate price tendencies and ranges but will never provide us with an exact price and will never predict what will happen. It is impossible to put all variables in the market together and create an indicator because of externalities that can make any model collapse.

A sudden decision from the board of director of a company could change drastically the trend of a stock price; an unexpected change in interest rates from a country can make the whole market move and also politic decisions can affect the “normal” behavior in the market’s price levels.

Politicians can make a market either collapse or turn into a boom with the decisions they make. If the government of a nation decides to change the external politic for example, this could affect directly the foreign investment within the country and/or global industry provoking sudden changes in prices and market trends.

One good example of how politic decisions affect the global markets is the war in Iraq a couple of years ago. When the USA decided to invade the Iraqi territory prices of crude oil went sky high, including future oil contracts.

Macroeconomic and politic decisions change the investment expectations sometimes increasing them and increasing trading volume; but some other times decrease investments. Uncertainty is a variable that changes according to externalities having a direct impact in prices, having a positive correlation with them. If uncertainty grows prices change more drastically than with a small one.

8.3 Tendency Indicator

The first step to estimate future prices is to determine the tendency they have, either if is bullish (rising) or bearish (declining) one. Prices does not stay at the same level. They will always climb or decrease with time affected primarily by the supply and demand relationship. By reviewing historical prices the trend could be estimated even by a simple look at the graph.

It is possible to determine three different kind of tendencies which differed in terms of time, either short or long run, know as primary or secondary tendency. To calculate the different tendencies the values of i and j have to be different. To estimate a short run tendency the historical data used should be between 9 and 30 values, for example 12 and 26. Values like 50 and 200 can also be used but the tendency reflecting more historical data will give a long run trend result.

To estimate the tendency we can use three different equations based on exponential moving averages having X as a tendency result in the time n.

$$X_n = \bar{x}_i - \bar{x}_j \text{ where } \bar{x}_i < \bar{x}_j \text{ and } n = 1,2,\ldots,m$$

$$X_n = \left[\frac{p_1 + fp_2 + f^2 p_3 + \ldots + f^{i-1} p_i}{1 + f + f^2 + f^3 + \ldots + f^{i-1}}\right] - \left[\frac{p_1 + fp_2 + f^2 p_3 + \ldots + f^{j-1} p_j}{1 + f + f^2 + f^3 + \ldots + f^{j-1}}\right] \quad (8.1)$$

$$X_n = \left[(1-\alpha)^i p_0 + \alpha \sum_{k=1}^{i} (1-\alpha)^{i-k} p_k\right] - \left[(1-\alpha)^j p_0 + \alpha \sum_{k=1}^{j} (1-\alpha)^{j-k} p_k\right] \quad (8.2)$$

$$X_n = \left[\beta^{-i} p_0 + (\beta - 1)\beta^{-i-1} \sum_{k=1}^{i} \beta^k p_k\right] - \left[\beta^{-j} p_0 + (\beta - 1)\beta^{-j-1} \sum_{k=1}^{j} \beta^k p_k\right] \quad (8.3)$$

Where

$$f = 1 - \frac{2}{n+1} \text{ , } \beta = \frac{1}{1-\alpha}$$

Having X_n is easy to calculate X_{n+1} using equation 8.4, so it will be possible to calculate a daily tendency.

$$X_{n+1} = \frac{p_1 + f + X_n}{1 + f} \tag{8.4}$$

8.4 The Stochastic Market Price Estimator Model - SMPE

Once the tendency is calculated we proceed to calculate the estimate price from the security. The equation below can be applied to stock, money and foreign exchange markets just by substituting one of the variables. The model begins with the market price F in time *t*.

$$F_t = E[F] \text{ on time t}$$

The change in price with respect of time is given by:

$$\frac{dF}{dt} = \delta(t)F_t \text{ where } F(0) = F_0 = const.$$

$$\delta(t) = [r(t) - q(t)] + \text{"}noise\text{"}$$

$$r = r(t) = \text{ domestic interest rate}$$

$$q = q(t) = \text{ foreign interest rate}$$

The change in time can be expressed as the difference between the domestic interest and the foreign interest rate, both of them being a function of time. The noise is a random movement of the price F expressed by a constant α.

$$\frac{dF_t}{dt} = \delta_t F_t$$

$$\delta_t = [r_t - q_t] + \alpha W_t$$

$W_t =$ White noise, $\alpha =$ constant.

The difference between interest rates will be substituted by a new variable γ, so the model will have gamma as a difference in interest rates for the exchange markets or a mean value for stock and money market calculations.

$$[r_t - q_t] = \gamma_t \text{ , } \gamma_t = \gamma = const.$$

By the Ito interpretation X_t satisfies the following stochastic integral equation:

$$x_t = x_0 + \int_0^t b(s, x_s)ds + \int_0^t \sigma(s, x_s)dB_s$$

Or the differential form:

$$dx_t = b(t, x_t)dt + \sigma(t, x_t)dB_t \tag{8.5}$$

$$\sigma(t, x) = \alpha x \text{ , } \alpha = const.$$

Let $\gamma_t = \gamma = const$. By the Ito interpretation (8.5) the equation is equivalent to:

$$dF_t = \gamma F_t dt + \alpha F_t dB_t \text{ , here } \sigma(t.x) = \alpha x$$

Or

$$\frac{dF_t}{F_t} = \gamma dt + \alpha dB_t$$

Hence:

$$\int \frac{dF_t}{F_t} = \gamma t + \alpha B_t \ \ , \ B_0 = 0$$

To integrate the function the Ito equation: $g(t,x) = \ln x$, $x > 0$ can be used and as a result we will get

$$d(\ln F_t) = \frac{1}{F_t} dF_t + \frac{1}{2}\left(-\frac{1}{F_t^2}\right)(dF_t)^2$$
$$= \frac{dF_t}{F_t} - \frac{1}{2F_t^2}\alpha^2 F_t^2 dt$$
$$= \frac{dF_t}{F_t} - \frac{1}{2}\alpha^2 dt$$

Hence:

$$\frac{dF_t}{F_t} = d(\ln F_t) + \frac{1}{2}\alpha^2 dt$$

$$\ln\frac{F_t}{F_0} = \left(\gamma - \frac{1}{2}\alpha^2\right)t + \alpha B_t$$

$$\frac{F_t}{F_0} = e^{\left(\left(\gamma - \frac{1}{2}\alpha^2\right)t + \alpha B_t\right)}$$

$$F_t = F_0 e^{\left(\left(\gamma - \frac{1}{2}\alpha^2\right)t + \alpha B_t\right)}$$

The model also can be solved with the Stratonovich integral, giving us a different result and extra variables on the exponent which are used for a more approximated estimation of the price in time *t*:

$$x_t = x_0 \int_0^t b(s, x_s) ds + \frac{1}{2}\int_0^t \sigma'(s, x_s)\sigma(s, x_s) ds + \int_0^t \sigma(s, x_s) dB_s$$

$$dF_t = \gamma F_t dt + \alpha F_t dB_t$$

$$d\overline{F}_t = \gamma\overline{F}_t dt + \alpha\overline{F}_t dB_t \,,$$

$$\overline{F}_t = F_0 e^{(\gamma t + \alpha B_t)} \,.$$

The solution is a process of the type $x_t = x_0 e^{(\mu t + \alpha B_t)}$ where $\mu, \alpha = const$.

The process B_t is independent of F_0 and as a result we have:

$$E[F_t] = E[F_0]e^{\gamma t}$$

The same result when there is no noise in $\delta(t)$

$$Y_t = e^{\alpha B_t}$$

The process B_t could be replaced by

$$B_t = \mu + \frac{\sigma^2}{2} \,,$$

which represent the expected growth of prices under a geometric Brownian motion.

Applying the Ito Equation we obtain the following result:

$$dY_t = \alpha e^{\alpha B_t} dB_t + \frac{1}{2}\alpha^2 e^{\alpha B_t} dt$$

or

$$Y_t = Y_0 + \alpha \int_0^t e^{\alpha B_s} dB_s + \frac{1}{2}\alpha^2 \int_0^t e^{\alpha B_s} ds$$

Theorem: Let $f,g \in \nu(0,T)$ and let $0 \le S < u < T$ then

i) $\int_S^T f dB_t = \int_S^u f dB_t + \int_u^T f dB_t$

ii) $\int_S^T (cf + g) dB_t = c\int_S^T f dB_t + \int_S^T g dB_t$

iii) $E\left[\int_S^T f dB_t\right] = 0$

iv) $\int_S^T f dB_t$ is F_t – measurable

Since $E\left[\int_0^t e^{\alpha B_s} dB_s\right] = 0$, type iii) from the Theorem, we obtain:

$$E[Y_t] = E[Y_0] + \frac{1}{2}\alpha^2 \int_0^t E[Y_s]ds$$
$$E[Y_t] = e^{\frac{1}{2}\alpha^2 t}$$

The Ito result of the equation gives:

$$E[F_t] = E[F_0]e^{rt},$$

Which can be expressed as:

$$F_t = F_0 e^{rt}, \tag{8.6}$$

Which gives us the equation to calculate the forward price (see also 7.1) where

F_t = Future exchange rate in time t

F_0 = Exchange rate in time 0

r = Interest rate

t = Time

This equation is used in finance but will give just a theoretic exchange rate which will always be greater in time t as in time 0 since the exponent will probably not be negative.

If we modify the model by applying the Stratonovich integral the result will look like:

$$E[F_t] = E[F_0]e^{(\gamma + \frac{1}{2}\alpha^2)t},$$

where $\gamma = (r_t - q_t)$ for the exchange rate or the mean of the prices or interest rates for the stock and money market and α is the standard deviation of the relative change in prices.

Having this equation we introduce the tendency indicator as a new variable in the exponent represented by X (section 8.3) which will take values from 1 or -1 depending on the tendency, 1 if is bullish (rising tendency) and -1 if is bearish (declining tendency).

$$E[F_t] = E[F_0]e^{X((\gamma + \frac{1}{2}\alpha^2)t)}$$

The new model called the Stochastic Market Price Estimator (SMPE) can be described then by the equation:

$$P_t = P_0 e^{X\left(\left(\gamma + \frac{\alpha^2}{2}\right)t\right)} \tag{8.7}$$

With the adding of the tendency indicator X the expected price will not just grow like the Future Exchange Rate model 8.6. We can estimate then a more realistic price since it could also decline in the time *t*.

For the purposes of the model, α will be substituted for the standard deviation of the price growth given by:

$$\text{price growth} = R_t = \left(\frac{F_t}{F_{t-1}} - 1\right) * 100$$

and the equation 4.7 from chapter 4 for the standard deviation:

$$\text{standard deviation} = \sigma = \alpha = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}$$

For a more accurate result it will be necessary to calculate the standard deviation with at least 500 data, this will allow to have a standard deviation with a small variance as it was reviewed in exhibit 4.6 in section 4.10 in chapter four.

Prove 8.1:

The exchange rates from Euro against other currencies in the last two weeks of March 2006 are given in the following table:

Date	USD	JPY	DKK	GBP	SEK	CHF	NOK
14/03/2006	1.1948	141.31	7.4609	0.68855	9.382	1.5671	7.96
15/03/2006	1.2026	141.31	7.4618	0.6892	9.3875	1.5653	7.968
16/03/2006	1.2069	142.06	7.4616	0.69085	9.3637	1.5674	7.9995
17/03/2006	1.2185	141.47	7.4616	0.6945	9.3396	1.5735	7.9765
20/03/2006	1.2174	141.07	7.462	0.6937	9.334	1.5717	7.9535
21/03/2006	1.2144	141.61	7.4628	0.6942	9.373	1.5738	7.953
22/03/2006	1.2069	141.34	7.4617	0.6914	9.3775	1.5736	7.9435
23/03/2006	1.2055	141.12	7.4614	0.69175	9.3525	1.577	7.967
24/03/2006	1.1969	141.52	7.461	0.6908	9.373	1.5763	7.969
27/03/2006	1.2025	140.15	7.4617	0.6883	9.355	1.5733	7.967
28/03/2006	1.2084	141.09	7.4616	0.6907	9.3677	1.5715	7.9385

Source: European Central Bank Exhibit 8.3

The Future Exchange Rate will be calculated first using the equation derived by the Ito integral for the Euro (EUR) versus the Swedish Crown (SEK) having $F_0 = 9.382$ in time t_0. The calculation will be made for t_1 = 10 days.

$$F_t = F_0 e^{rt}$$

the interest rate from Euro is 2.5% and the interest rate in SEK is 1.5%, time will be expressed as $\frac{10}{360}$;

$$r = \%EUR - \%SEK = 0.025 - 0.015 = 0.01$$

$$F_t = 9.382 e^{((0.025 - 0.015) * \frac{10}{260})}$$

$$F_t = 9.382 * 1.002778$$

$$F_t = 9.3846$$

The market price ten days after F_0, that means March 28th was 9.3667, so the equation for the Future Exchange Rate assumes the prices will change depending just on the interest rates and not in the market history and the supply and demand relationship.

We will now analyze the same situation but with the SMPE model (equation 8.7)

$$P_t = P_0 e^{X\left(\left(\gamma + \frac{\alpha^2}{2}\right)t\right)}$$

Applying equation 8.1 with $i = 12$ and $j = 26$ we get:

$$f_{12} = 1 - \frac{2}{12 + 1} = 0.846154$$

$$f_{26} = 1 - \frac{2}{26+1} = 0.925926$$

and substituted in the tendency indicator with the historic prices from Euro (EUR) versus the Swedish Crown (SEK) from table 8.2 we will obtain the price's trend:

$$X_n = \left[\frac{9.444 + 0.846154 * 9.449 + 0.846154^2 * 9.4475 + ... + 0.846154^{11} * 9.382}{1 + 0.846154 + 0.846154^2 + 0.846154^3 + ... + 0.846154^{11}} \right]$$

$$- \left[\frac{9.2972 + 0.925926 * 9.307 + 0.925926^2 * 9.2713 + ... + 0.925926^{25} * 9.382}{1 + 0.925926 + 0.925926^2 + 0.925926^3 + ... + 0.925926^{25}} \right]$$

$$X_n = -0.0011137$$

since the result is a negative trend X will be substituted by -1 in the model.

Date	USD	JPY	DKK	GBP	SEK	CHF	NOK
06/02/2006	1.1981	142.28	7.4667	0.6833	9.2972	1.5558	8.047
07/02/2006	1.1973	141.24	7.4674	0.6859	9.307	1.5549	8.0465
08/02/2006	1.1948	141.51	7.4667	0.6861	9.2713	1.5529	8.0165
09/02/2006	1.1972	141.91	7.4653	0.6881	9.2617	1.5563	8.0155
10/02/2006	1.197	140.77	7.4649	0.6839	9.2623	1.5551	8.0755
13/02/2006	1.1888	140.35	7.4649	0.6837	9.3379	1.5559	8.102
14/02/2006	1.1898	139.57	7.4649	0.6858	9.335	1.5574	8.136
15/02/2006	1.1904	139.96	7.4654	0.6835	9.3325	1.5577	8.1375
16/02/2006	1.1858	140.11	7.4663	0.68425	9.394	1.5588	8.107
17/02/2006	1.1863	140.77	7.4653	0.6839	9.381	1.5621	8.06
20/02/2006	1.1932	141.11	7.4628	0.6848	9.3635	1.5617	8.0555
21/02/2006	1.1906	141.61	7.4625	0.6821	9.3887	1.5598	8.0545
22/02/2006	1.1875	140.91	7.462	0.6828	9.3762	1.5593	8.0575
23/02/2006	1.1957	139.88	7.461	0.68185	9.4103	1.5587	8.046
24/02/2006	1.1896	139.11	7.4604	0.6803	9.42	1.5602	8.0465
27/02/2006	1.1852	137.77	7.4598	0.68105	9.444	1.5637	8.0265
28/02/2006	1.1875	138.18	7.461	0.6796	9.449	1.5661	8.024
01/03/2006	1.1954	138.56	7.4604	0.6806	9.4475	1.5614	8.029
02/03/2006	1.1921	138.73	7.4603	0.6831	9.4623	1.5646	8.0095
03/03/2006	1.202	139.96	7.4618	0.685	9.4603	1.5628	8.012
06/03/2006	1.2017	141.04	7.4622	0.6855	9.4543	1.5607	7.979
07/03/2006	1.1913	140.34	7.4598	0.6858	9.4733	1.5616	7.9875

08/03/2006	1.1914	140.35	7.4586	0.68605	9.472	1.5598	8.0075
09/03/2006	1.192	140.1	7.4596	0.6858	9.4299	1.5635	8.0235
10/03/2006	1.1919	141.11	7.4601	0.6862	9.3925	1.5664	7.9735
13/03/2006	1.1922	141.94	7.46	0.69015	9.3786	1.5686	7.9755
14/03/2006	1.1948	141.31	7.4609	0.68855	9.382	1.5671	7.96

Source: European Central Bank — Exhibit 8.4

We now proceed to calculate the variables γ and α.

$$\gamma = (r-q) = \%EUR - \%SEK = 0.025 - 0.015 = 0.01$$

After calculating the standard deviation we get:

$$\alpha = 0.3156082$$

Now all variables are calculated and can be introduced in the model to calculate the expected exchange rate:

$$P_t = P_0 e^{X\left(\left(\gamma + \frac{\alpha^2}{2}\right)t\right)},$$

$$P_t = 9.382 e^{-1\left(\left((0.025-0.015)+\frac{0.3156082^2}{2}\right)\frac{10}{360}\right)},$$

$$P_t = 9.382 * 0.998340,$$

$$P_t = 9.3643.$$

The market price as can be seen in exhibit 8.4 with date March 28th 2006 is 9.3677 so the estimation has an error of 0.014% which is quite acceptable.

It is important to remark that this model only provides an estimation of the future price, never the less there can be either external factors or unusual market movements that can make this estimation not to be precise.

A confidence interval can be also calculated within the model to have a range of prices in which the market value will be found after time *t*. The half size of the confidence interval will be given by:

$$CI = z\frac{\sigma}{\sqrt{n}},$$

this half size will be add and subtract from the estimated future price given by the model to have the confidence interval.

The value of z is given by the normal standard distribution and will depend on the confidence level that wants to be applied to the model. Depending on the confidence lever the value of *z* will deferred, the values for different confidence levels can be seen in exhibit 8.5.

Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Normal Standard Distribution Exhibit 8.5

The finished model will be given by the following equation:

$$P_t = P_0 e^{X\left(\left(\gamma+\frac{\alpha^2}{2}\right)t\right)} \pm z\frac{\alpha}{\sqrt{n}}$$

Is important to keep in mind that the model will work better in the short run for the foreign exchange market; even better to estimate the prices within a month due to the change in interest rates for the domestic and for the foreign currency. If interest rates wouldn't change, the SMPE model would work more precisely also in the long run.

Chapter Nine
Conclusions

Different financial concepts have been analyzed in this work from a mathematical point of view, at the same time; mathematical concepts have been studied from a financial perspective which has brought both concepts together for a better understanding for both, mathematicians and financiers.

In the first chapter of this work the reader can find an overview of how financial markets are divided and how they work, including an important theory such as market efficiency.

Time Value of money, elementary mathematical finance, statistic and probability has been studied in this work to provide the reader with the basic knowledge and understanding of other related topics studied further in this work.

Different valuation models for money, stock and derivative markets have been presented in this work so the reader can obtain the basic knowledge from the different valuation methods, including the Black-Scholes Model for the option valuation, a model which was derived in two different forms.

The main contribution to financial mathematics has been a brand new model to estimate prices in the foreign exchange and the stock market. This model has been created by the author of this work and is known as the Stochastic Market Price Estimator – SMPE - which equation is presented again:

$$P_t = P_0 e^{X\left(\left(\gamma + \frac{\alpha^2}{2}\right)t\right)} \pm z\frac{\alpha}{\sqrt{n}}.$$

Hopefully, this model will be important for users who want to estimate prices in real markets directly by the users.

I hope you enjoyed reading this work as much as I enjoy making it.

Aldo Fabricio Gutiérrez Gómez

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