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Change point detection in an Ornstein-Uhlenbeck process (a reflection of trading in financial markets)

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In loving memory of my dear mother

Preface

A deep interest in the application of mathematics in solving real life problems dictated my choice of the master's programme in mathematical modelling and simulation. The decision to conduct my thesis in financial mathematics was informed by recent interest and advances in the application of mathematics in modern finance and global financial markets. I am eternally grateful to my supervisors Assistant professors Eric Järpe and Claes Jogréus for their guidiance all through the period of this project. Many thanks to the teachers and staff of the department of Mathematics and Science of the Blekinge Institute of Technology, Karlskrona especially Dr. Raisa Khamitova, Professor Nail Ibragimov, Professor Elisabeth Rakus-Anderson and Dr. Robert Nyqvist.

Words and space will not suffice to thank all my friends and loved ones who supported me in various ways all through the period of my study. I am thankful for the unquantifiable support I received from my aunt Iyabode Daodu, my dear friends Iyeyinka Olayanju, Taiwo and Sola Ajayi, Olumide Ajiboye, Yeonhee Oh, Akinyemi Akinyugha, Olujide Ogunle, Akingbade and Adebola Akintounde, Funso Somorin, Biodun Osunkunle, Odunayo Borisade, Ayobola Fadahunsi and many others too numerous to mention.

I am ultimately grateful to the Almighty God for His ever abiding love and grace towards me.

Fadahunsi, Adetokunbo Ibukun.

Abstract

The financial market has become an area of increasing research interest for mathematicians and statisticians in recent years. Mathematical models and methods are increasingly being applied to study various parameters of the market. One of the parameters that have attracted lots of interest is 'volatility'. It is the measure of variability of prices of instruments (e.g. stock, options etc.) traded in the market. It is used mainly to measure risk and to predict future prices of assets. In this paper, the volatility of financial price processes is studied using the Ornstein-Uhlenbeck process. The process is a mean reverting model which has good and well documented properties to serve as a model for financial price processes. At some random time point, a parameter change in the distribution of the price process occurs. In order to control the development of prices, it is important to detect this change as quickly as possible. The methods for detecting such changes are called 'stopping rules'. In this work, stopping rules will be derived and analysed. Using simulations and analytical methods, the properties of these stopping rules will be evaluated.

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Chapter 1

Introduction

Finance over the years has become one of the fastest developing areas in the modern banking and corporate world. This growth, coupled with the sophistication of modern financial products, provides a rapidly growing drive for new mathematical models and modern methods [46]. The genesis of the application of mathematics to finance can be traced back to March 29, 1900. On that day, a French postgraduate student, by the name Louis Bachelier, successfully defended his PhD thesis, *Théorie de la Spéculation*, at the Sorbonne. His pioneering analysis of the stock and options markets contains several ideas of enormous value in both finance and probability. Bachelier's thesis can be viewed as the origin of mathematical finance and of several important branches of stochastic calculus. His thesis was historically the first attempt to use advanced mathematics in finance and the introduction of Brownian motion [6], with his work on random walks, five years before Einstein's celebrated study of Brownian motion [9]. The thesis of Bachelier, together with his subsequent works, deeply influenced the whole development of stochastic calculus and mathematical finance.

Bachelier, in his thesis, had developed the theory of option pricing, a topic that had begun occupying Samuelson and other economists in the 1950s [8]. Since Bachelier, several researchers in the 1950s and 1960s like Lintner [23], Markowitz [26], Mossin [29], Sharpe [38], Treynor [42], to mention a few, have recorded astonishing breakthroughs in the development of the application of mathematics in finance. These paved way for the development of the Capital Asset Pricing Model, a quantitative model for pricing an individual security or a portfolio. The doyen of mathematical statisticians of the post second world war era, L.J. Savage pushed for the recognition of Bachelier's work among economists. His push prompted an intensive period of development in financial economics which would eventually culminate in the Nobel Prize-winning solution to the option pricing problem by Black, Scholes and Merton in 1973. The same year, the world first listed options exchange opened its doors in Chicago, see [8, 4, 28].

Bachelier had considered the option pricing problem and had come up with a formula extremely close to the Black-Scholes formula (derived 70 years later) using a method that would later be called stochastic analysis. Bachelier represented asset prices as stochastic processes and computed the quantities of interest by exploiting the connection between the processes and partial differential equations. As Bachelier viewed it and Black and Scholes conclusively established, the question of the relationship between the price of an underlying asset and the value of say, a call option, on that underlying asset, is best addressed by first describing the underlying asset as a 'stochastic process' [8]. Bachelier, in his memoir of 1906, defined stochastic processes which are now called processes with independent increments and Markov processes. He also derived the distribution of the Ornstein-Uhlenbeck process (also known as the Guass-Markov process), see Davis and Etheridge [8], Courtault, et al [6] and Finch [13]. Consequent upon the views of Bachelier and the assertions of Black and Scholes, the Ornstein-Uhlenbeck process describes to a high level of accuracy the behaviour of the prices of many quantities of interest in the financial market.

The emergence of various volatility models triggered off widespread research interest in the behaviour of prices of instruments traded in the financial market. Among published works on such models are Schwert [37], Officer [32], Bollerslev, et al [5], Abel [1], Mascaro and Meltzer [27], Lauter-bach [22], Barndorff-Nielsen and Shepherd [2].

In finance, volatility is simply the degree to which financial prices tend to fluctuate. Volatility represents uncertainty and the degree of risk faced by market players. Large volatility implies that returns fluctuate in a wide range. Volatility measures variability, or a dispersion about a central tendency - it is simply a measure of the price movement in a stock, futures, options, or any other market. Like most other market (random processes) parameters, volatility can not be directly observed but rather just estimated. Black and Scholes defined volatility as the amount of variability in the returns of an underlying asset [2, 5, 37]. Bachelier defined volatility as the degree of nervousness or instability of prices [8].

There are two indicators of volatility namely, historical volatility (or expost volatility) and implied volatility (or ex-ante volatility). The historical volatility is the volatility of a series of asset prices which is based on looking back over the historical price path of the particular asset while implied volatility is forward looking. Most market players prefer implied volatility to historical volatility.

A number of researchers have carried out studies on volatility using the Ornstein-Uhlenbeck process among whom are Griffin and Steel [17], Nicolato and Vanardos [31], Schoebel and Zhu [36], to name a few.

Volatility is an important phenomenom. Its usefulness in predicting future prices of underlying assets can not be overemphasized. The goal of this paper is to model financial asset prices using the Ornstein-Uhlenbeck process. The volatility of the process is monitored by deriving and applying monitoring methods for the model. Underlying assumptions and properties of the model are derived and discussed. Time is considered to be continuous. In this chapter, the background is given. In chapter 2, the model, problem and methods of detection, the stationarity conditions and the global distribution of the process are presented. Some optimality conditions of change-point detection methods are also defined in the second chapter. All results which comprise calculations of the optimal conditions of the detection methods are presented in Chapter 3. The results are discussed and interpreted in Chapter 4. Finally, data for charts and simulation program codes may be found in the Appendix.

1.1 Financial Markets

Financial markets comprise mainly *money markets* and *capital markets.* Money markets are also called *credit markets.* They are the markets for debt securities that mature in the short term (usually less than a year). Examples of such securities are treasury bills, commercial papers, bankers' acceptance, government agency securities. Money market securities are usually highly liquid and have a relatively low default risk. Capital markets on the other hand, are markets for long term (usally one or more years) securities issued by government and corporations. In contrast to the money markets, both debt instruments (bonds) and equity shares (common and preferred shares) are traded. Capital markets securities are characterized by greater default and market risk relative to money-market instruments but they yield relatively higher returns as compensations for the higher risk. Examples of capital markets include New York Stock Exchange (NYSE), London Stock Exchange, Stockholm Stock Exchange, Tokyo Stock Exchange, Nigeria Stock Exchange (NSE). These markets are organised markets. Securities are traded through thousands of dealers and brokers on the over-the-counter (or unlisted) market, a term used to denote the informal system of telephone contacts among brokers and dealers. Other types of markets include *the foreign exchange market*, which handles international financial transactions between different countries e.g. the US and other countries, *the commodity markets* which handle various commodity futures, *the mortgage markets* which involve various home loans, *the insurance, shipping and other markets* handling short-term credit accomodations in their operations. There are two sub-divisions of the *capital market*, namely *the primary market* and *the secondary market.* The *primary market* is the market for new issues while the *secondary market* is the market in which previously (second hand) securities are traded. The London Stock Exchange, New York Stock Exchange, Stockholm Stock Exchange, Frankfurt Stock Exchange are examples of secondary markets.

1.2 Financial Instruments

These are instruments (assets) traded in financial markets. They are divided into two broad categories namely, *primary (underlying) assets* and *financial derivatives*. Underlying assets include but not limited to stocks, bonds, foreign currencies, commodities (gold, copper, oil,...). While derivatives include options, futures, forward contracts, swaps and hybrids.

1.2.1 Underlying Instruments

Primary or underlying assets are traded in the money and capital markets. Bonds, are examples of assets traded in the money market while equity shares, commodity (gold, copper, ...) are traded in the capital market.

- **Stocks**. The stock of a company or business entity represents the original capital paid or invested into the business by its founders. The stock of a business is divided into shares, the total of which must be stated at the time of the formation of the business. Shares represent a fraction of ownership in a business. The ownership of shares is documented by the issuance of a stock certificate. A stock certificate is a legal document that specifies the amount of shares owned by the shareholder, and other specifics of the shares. Stocks are largely classified into common stock and preferred stock. A major difference between the two classifications is that common stocks carry voting rights in corporate decisions while preferred stocks do not but are legally entitled to receive certain level of dividend payments before any dividend is issued to other shareholders. The owners of companies sell shares in order to raise additional capital to invest in new projects within the company. They may also simply wish to reduce their holding, freeing up capital for their own private use. Financing a company through the sale of stocks in a company is known as equity financing.

- **Bonds**. A bond is a debt security. It is a formal contract to repay borrowed money with interest in fixed intervals. It is like a loan in which the issuer is the borrower (the debtor), the holder is the lender (the creditor). Depending on the terms of the bond, the issuer is obliged to pay the holder interest (coupons) and/or to repay the principal at a later date (maturity). Bonds provide the borrower access to external funds to finance long-term investments, or in the case of government, they make available funds for current expenditure. A major difference between stocks and bonds is that stockholders have ownership (equity stake) in the company while bondholders hold creditor stake (i.e. they are lenders) in the company. Another difference is that a bond has a defined term or maturity after which the bond will be redeemed whereas a stock can be held in perpetuity. Examples of bonds are fixed rate bonds (the paid interest is constant through the life of the bond), floating rate bonds (the coupon rate varies over the life of the bond), zero-coupon bonds (no reguar interest is paid), inflation linked bonds (principal amount and the interest payment are indexed to inflation), asset-based bonds (the interest and principal payment are based on underlying cash flow from other assets).

1.2.2 Financial derivatives

A financial derivative is a transaction (or contract) whose value depends on or, as the name implies, is derived from the value of the underlying asset such as stocks, bonds, mortgages, markets indexes, or foreign currencies. A stock option for example is a derivative whose value depends on the price of a stock. Typically, one party with an unwanted risk passes some or all of that risk to a second party. The first party can assume a different risk from the second party, pay the second party to assume the risk, or as is often the case, create a combination. The participants of derivative activities are divided into two broad types namely, *dealers* and *end-users*. *Dealers* include investment banks, merchant banks, commercial banks and independent brokers. Due to the growth in the involvement of more organisations in international financial transactions, the number of *end-users* have become large in recent years. *End-users* include businesses, banks, securities firms, insurance companies, governmental units at the local, state and federal levels, "supernational" organisations such as the World bank, mutual funds, private and public pension funds. A common reason derivates are traded is so that the risk of financial operations can be conrolled. Uses of derivatives include the management of foreign exchange exposure especially unfavourable exchange rate movements, *hedging of position* (that is, to set up two financial assets so that any unfavourable price movement in one asset is offset by favourable price movements in the other asset).

- **Options**. An option in finance is a contract between a buyer and a seller which gives the buyer the right but not the obligation to buy or sell an underlying asset at an agreed predetermined price called the strike price or simply, strike, on a later date, called the exercise time. A call option gives the buyer the right to buy the underlying asset while a put option gives the buyer the right to sell the underlying asset. Financial derivatives such as American and European call and put options are referred to as plain vanilla products. Derivative securities which have certain features that make them more complex than the commonly traded plain vanilla products are called exotic options or simply, exotics. These products are traded in the over-the-counter (OTC) derivative market. Exotic options are important aspects of the portfolio of an investment bank because they are usually more profitable than plain vanilla products. Examples of exotic options are compound option, chooser option, barrier option, binary/digital option, lookback option, constant proportion portfolio insurance (CPPI), cliquet or ratchet option, variance swap, rainbow option and Bermudan option.

- **Forward Contracts**. A forward contract is a written agreement between two parties for the purchase or sale of a stipulated amount of a commodity, foreign currency or any other risky asset at a specified future time, known as the delivery date, usually 30, 90 or 180 days, for a price F, fixed at the present moment, called the forward price. An investor who agrees to buy the asset is said to enter into a long forward contract or to take a long forward position. If an investor agrees to sell the asset, we speak of a short forward contract or a short forward position. No money is paid at the time when a forward contract is exchanged.

- **Futures**. A future is a contract to purchase or sell a given amount of an item for a given price by a certain date in the future (hence, the name futures market). The seller of a futures contract agrees to deliver the item to the buyer of the contract, who agrees to purchase the item. The contract specifies the amount, valuation method, quality, month and means of delivery, and exchange to be traded in. The month of delivery is the expiration date, in other words, the date on which the commodity or financial instrument must be delivered. As in the case of a forward contract, it costs nothing to initiate a futures position. The difference lies in the cash flow during the lifetime of the contract. A long forward contract involves just a single payment at delivery while a futures contract involves a random cash flow, known as marking to market. Trading in futures is conducted by hedgers and speculators. Hedgers protect themselves with futures contracts in the commodity they produce or in the financial instrument they hold. Speculators use futures contracts to obtain capital gain on price rises of the commodity, currency, or financial instrument.

- **Swaps**. A swap is the exchange of assets or payments. It is a simultaneous purchase and sale of a given amount of securities, with the purchase being effected at once and the sale back to the same party to be carried out at a price agreed upon today but to be completed at a specified future date. Swaps are basically of two types namely, *interest rate swaps* and *currency swaps*. Interest rate swaps typically involve exchanging fixed interest payments for floating interest payments. Currency swaps are the exchange of one currency into another at an agreed rate, combining a spot and forward contract in one deal.

- **Hybrids**. A hybrid foreign currency option involves the purchase of a put option and the simultaneous sale of a call option or vice versa so that the overall cost is less than the cost of a straight option.

Chapter 2

Methods

2.1 The behaviour of prices in financial markets

Assume that time is measured in days $t = 0, 1, 2, ...$ and let

$$S = (S_t)_{t \geq 0} \tag{2.1}$$

be the price process of a financial instrument (e.g. stock, bond, or exchange rate of two currencies). The prices of financial instruments $S_t, t \geq 0$, assume a random pattern. This means that they vary in highly irregular manners. Hence, it can sometimes be a daunting task to predict a future price based on the present or past prices of the instrument without the right modelling tools. This assertion is supported by the random walk theory. Economists and statisticans have historically accepted the random walk theory as a more accurate description of the behaviour of prices than various "technical" or "chartist" procedures for predicting prices. They have conducted several tests and continue to believe that the behaviour of prices are random due to the efficiency of the markets. The most notable among papers which have been written on this stochastic behaviour of prices inculude Kendall and Bradford [20], Markiel [25] and Fama [11, 12]

As mentioned in the introduction, Bachelier was the first to describe prices of financial instruments using the concept and methods of probability theory. Taking this approach and A.N. Kolmogorov axiomatics of probability theory, we shall consider that all observations of $\{S_t : t \geq 0\}$ are carried out with respect to some probablity space

$$(\Omega, \mathcal{F}, P)$$

where, Ω is the sample space of the log returns, $\mathcal{F}$ is the σ-algebra of subsets of Ω and P is the probability measure on $\mathcal{F}$.

The integral role time and dynamics play in financial theory, necessitates the definition of the probablity space more specifically by assuming that the space has a flow $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ which is a sequence of σ-algebras such that

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_t \subseteq \cdots \subseteq \mathcal{F}.$$

The above non-decreasing σ-subalgebras of $\mathcal{F}$, which is also called a filtration, is the sequence of sets of observable events through t. In other words, it is the information on the market situation that is available to an observer up to time t. S_t is $\mathcal{F}_t$ measurable which implies that prices are formed on the basis of the developments observable on the market up to time t. Adding the flow, we define a filtered probablity space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$$

which is also called a stochastic basis.

An important function of prices is the logarithmic return or continuously compounded return (also known as the "force of interest"). It is a measure of the ratio of money gained or lost on an investment relative to the amount of money invested. It is defined as:

$$h_t = \ln \frac{S_t}{S_{t-1}} \tag{2.2}$$

2.2 The Ornstein-Uhlenbeck Process

In 1930, Leonard Ornstein and George Eugene Uhlenbeck introduced a random process now known as the Ornstein-Uhlenbeck process, see [13, 19, 43]. The Ornstein-Uhlenbeck process is the most widely used mean reverting stochastic process in financial modelling especially in interest rates and commodities. The Vasicek model of interest rates is an example of the Ornstein-Ulenbeck process [44]. To define this process, let us first consider some properties of a stochastic process.

A stochastic process $\{X_t : t \geq 0\}$ is said to be

- **Stationary** if, for all $t_1 < t_2 < \cdots < t_n$ and $h > 0$, the random n-vectors $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ and $(X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_n+h})$ are identically distributed; that is, their joint probabilities remain unchanged when shifted in time or space.

- **Gaussian** if, for all $t_1 < t_2 < \cdots < t_n$, the n-vector $(X_{t_1}, \ldots, X_{t_n})$ is multivariate normally distributed.

- **Markovian** if, for all $t_1 < t_2 < \cdots < t_n$, $P(X_{t_n} \leq x | X_{t_1}, \ldots, X_{t_{n-1}}) = P(X_{t_n} \leq x | X_{t_{n-1}})$; that is, the future value conditional on the present, is not dependent on the past.

- **continuous in probability** if, for all $u \in \mathbb{R}^+$ and $\epsilon > 0$, $P(|X_v - X_u| \geq \epsilon) \to 0$ as $v \to u$

Definition 2.2.1. A stochastic process $\{X_t : t \geq 0\}$ is said to be an Ornstein-Uhlenbeck process if it is stationary, Gaussian, Markovian and continuous in probability.

The Ornstein-Uhlenbeck process is also known as the mean-reverting process. It is the continuous time analog of the discrete-time autoregressive AR(1) process.

$\{X_t : t \geq 0\}$ satisfies the following stochastic differential equation

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t \tag{2.3}$$

where,
θ = mean reversion rate,
μ = mean,
σ = volatility
$\theta > 0$, $\mu, \sigma \in \mathbb{R}$ and W_t is the Wiener process.

2.2.1 Stationarity conditions

We seek to derive the stationarity conditions of the Ornstein-Uhlenbeck process.

A particular solution of equation (2.3) is $f(X_t, t) = X_t e^{\theta t}$. Apply Itô's lemma to $f(X_t, t)$, to obtain

$$df(X_t, t) = \theta X_t e^{\theta t} dt + e^{\theta t} dX_t \tag{2.4}$$

Substituting (2.3) into (2.4), we obtain

$$df(X_t, t) = \theta e^{\theta t} dt + e^{\theta t} \left[\theta(\mu - X_t)dt + \sigma dW_t\right] \tag{2.5}$$

Simplifying (2.5), we have

$$df(X_t, t) = \mu\theta e^{\theta t} + e^{\theta t}\sigma dW_t \tag{2.6}$$

Integrating (2.6) from 0 to t, we get

$$X_t e^{\theta t} = X_0 + \mu(e^{\theta t} - 1) + \int_0^t \sigma e^{\theta s}\, dW_s \tag{2.7}$$

By dividing (2.7) by $e^{\theta t}$, we achieve

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \int_0^t \sigma e^{\theta(s-t)}\, dW_s \tag{2.8}$$

From (2.8), we have that the first moment which is the conditional expectation is

$$E(X_t|X_0 = c) = ce^{-\theta t} + \mu(1 - e^{-\theta t}), \text{ for some constant } c. \tag{2.9}$$

The covariance function is obtained from (2.9)

$$\begin{aligned} C(X_s, X_t|X_0 = c) &= E[(X_s - E[X_s])(X_t - E[X_t])] \\ &= E\left[\int_0^s \sigma e^{u-s} dW_u \int_0^t \sigma e^{v-s} dW_v\right] \\ &= \sigma^2 e^{-\theta(s+t)} E\left[\int_0^s e^{\theta u} dW_u \int_0^t e^{\theta v} dW_v\right] \\ &= \frac{\sigma^2}{2\theta}\left(e^{-\theta|s-t|} - e^{-\theta(s+t)}\right) \end{aligned} \tag{2.10}$$

From (2.10), we obtain the conditional varaince function for $\{X_t : t \geq 0\}$ as

$$V(X_t|X_0 = c) = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t}) \tag{2.11}$$

From (2.9) and (2.11), we observe that

$$\lim_{t\to\infty} E(X_t|X_0 = c) = \mu \tag{2.12}$$

and

$$\lim_{t\to\infty} V(X_t|X_0 = c) = \frac{\sigma^2}{2\theta} \tag{2.13}$$

Consequently, $E_\Pi(X_t) = \mu$ and $V_\Pi(X_t) = \frac{\sigma^2}{2\theta}$ are the stationary moments of $\{X_t : t \geq 0\}$. The volatility of $\{X_t : t \geq 0\}$ is

$$\frac{\sigma}{\sqrt{2\theta}} \tag{2.14}$$

Also,

$$\begin{aligned} C_\Pi(X_s, X_t) &= \frac{\sigma^2}{2\theta}(e^{-\theta|s-t|}) \\ &= \frac{\sigma^2}{2\theta} e^{-\theta|h|} = C(h), \quad \text{where} \quad h = s - t \end{aligned} \tag{2.15}$$

The autocorrelation coefficient for $\{X_t : t \geq 0\}$ is

$$\begin{aligned} R_{X_t,X_{t+h}}(t, t+h) &= \frac{C(X_t, X_{t+h})}{V(X_t)} \\ &= e^{-\theta|h|} \\ R_{X_t,X_{t+h}}(t, t+h) &= e^{-\theta|h|} = R(h) \end{aligned} \tag{2.16}$$

From (2.12) and (2.16), and the fact that the process is Gaussian we note that $\{X_t : t \geq 0\}$ is strongly stationary.

2.2.2 Global distribution of the Ornstein-Uhlenbeck process

The global distribution of a process is equal to the product of conditional ditributions.

The Ornstein-Uhlenbeck process $\{X_t : t \in \mathbb{R}\}$ is a Guassian process. Hence $X_t = (X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ is a multivariate Guassian distributed n-vector conditional on $X_{t_0} = x_{t_0}$.

We seek to calulate the global density function of the above vector.

$$f_{X_{t_0},\ldots,X_{t_n}|X_{t_0}}(x_{t_0}, \cdots, x_{t_n}) = \pi(X_{t_0}) \prod_{i=1}^{n} f_{X_{t_i}|X_{t_{i-1}}}(x_{t_i}|x_{t_{i-1}}) \tag{2.17}$$

$$f(x_{t_i}) = \frac{1}{\sqrt{2\pi\sigma_{t_i}^2}} \exp\left\{-\frac{(x_{t_i} - \mu_{t_i})^2}{2\sigma_{t_i}^2}\right\} \tag{2.18}$$

$$\begin{aligned} f(x_{t_i}, x_{t_{i-1}}) = & \frac{1}{2\pi\sigma_{t_i}\sigma_{t_{i-1}}\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x_{t_i} - \mu_{t_i}}{\sigma_{t_i}}\right)^2\right.\right. \\ & \left.\left. -2\rho\left(\frac{x_{t_i} - \mu_{t_i}}{\sigma_{t_i}}\right)\left(\frac{x_{t_{i-1}} - \mu_{t_{i-1}}}{\sigma_{t_{i-1}}}\right) + \left(\frac{x_{t_{i-1}} - \mu_{X_{t_{i-1}}}}{\sigma_{t_{i-1}}}\right)^2\right]\right\} \end{aligned} \tag{2.19}$$

$\{X_t : t \in \mathbb{R}\}$ is a stationary process. Hence,

$$\sigma_{t_i} = \sigma_{t_{i-1}} = \frac{\sigma}{\sqrt{2\theta}} \quad \text{and} \quad \mu_{t_i} = \mu_{t_{i-1}} = \mu.$$

Also $\rho = e^{-\theta|h_i|}$, where $h_i = t_i - t_{i-1}$.

$$\begin{aligned} f(x_{t_i}|x_{t_{i-1}}) &= \frac{f(x_{t_2}, x_{t_1})}{f(x_{t_1})} \\ &= \frac{\frac{1}{2\pi\frac{\sigma^2}{2\theta}\sqrt{1-e^{-2\theta|h_i|}}}\exp\left\{-\frac{(x_{t_i}-\mu)^2 - 2e^{-\theta|h_i|}(x_{t_i}-\mu)(x_{t_{i-1}}-\mu) + (x_{t_{i-1}}-\mu)^2}{2\frac{\sigma^2}{2\theta}(1-e^{-2\theta|h_i|})}\right\}}{\frac{1}{\sqrt{2\pi\frac{\sigma^2}{2\theta}}}\exp\left\{-\frac{(x_{t_{i-1}}-\mu)^2}{2\frac{\sigma^2}{2\theta}}\right\}} \\ &= \frac{1}{\sqrt{2\pi\hat{\sigma}^2}}\exp\left\{-\frac{\left[x_{t_i} - x_{t_{i-1}}e^{-\theta|h_i|} - \mu(1 - e^{-\theta|h_i|})\right]^2}{2\hat{\sigma}^2}\right\} \end{aligned} \tag{2.20}$$

where

$$\hat{\sigma}^2 = \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta|h_i|}\right)$$

From (2.20), we have that the global density function (product of condtional distributions) of an n-dimensional Ornstein-Uhlenbeck process is

$$f_{X_{t_0},\dots,X_{t_n}}(x_{t_0},\dots,x_{t_n}) = \prod_{i=1}^{n} f_{X_{t_i}|X_{t_{i-1}}}(x_{t_i}|x_{t_{i-1}})$$
$$= \frac{1}{(\sqrt{2\pi\hat{\sigma}^2})^n}\exp\left\{-\sum_{i=1}^{n}\frac{\left[x_{t_i}-x_{t_{i-1}}e^{-\theta|h_i|}-\mu(1-e^{-\theta|h_i|})\right]^2}{\sigma^2(1-e^{2\theta|h_i|})/\theta}\right\} \quad (2.21)$$

where $h_i = t_i - t_{i-1}$.

2.3 The Ornstein-Uhlenbeck process as a model of financial markets

With properly chosen drift and diffusion parameters, an Ornstein-Uhlenbeck process can be used to mathematically model trading activities in a financial market. Below are graphical representations which demonstrate the semblance between the paths of an Ornstein-Uhlenbeck process with long term mean $\mu = 1$, mean reversion rate $\theta = 3$, time step $h = 0.01$ and noise term $\sigma = 0.5$ and the DAX (Deutscher Aktien Index (German Stock Index)) from January 2, 2008 to January 15, 2010.

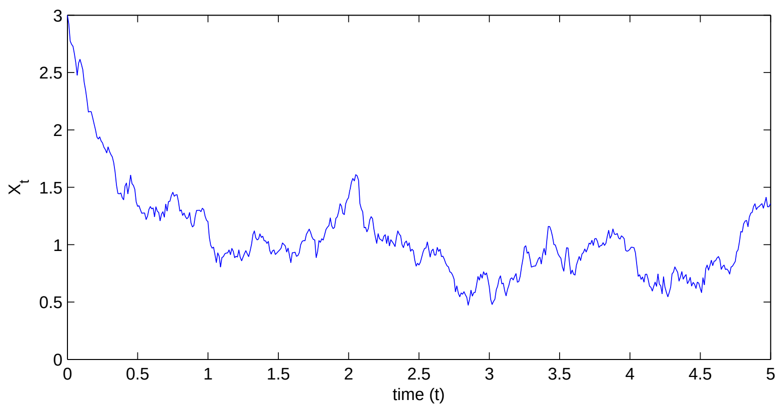

Figure 2.1: *An Ornstein-Uhlenbeck process with* $X_0 = 3$.

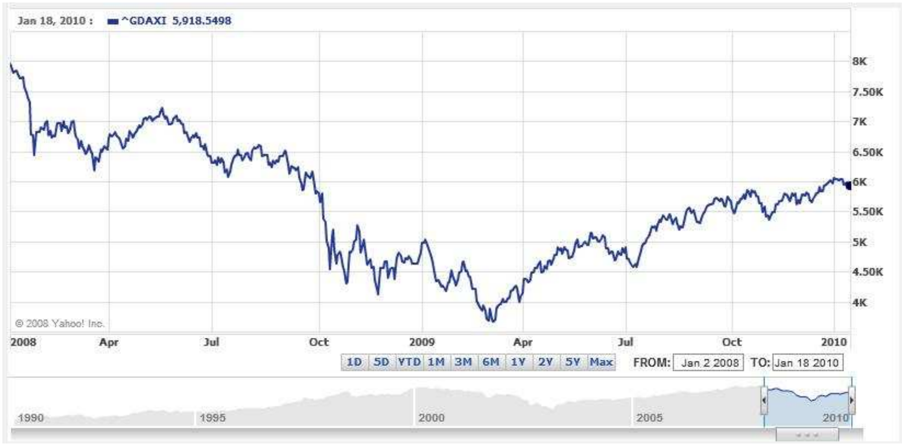

Figure 2.2: *The DAX index - German Stock Index.*

2.4 The Change-point problem

The change-point problem arises from the need for controlling a random process and the foundation for deciding whether or not to continue observing the process or to stop and to take counteractions to prevent developments in some undesired direction. To do this, continual observation of time series are made with the goal of detecting any important change in the parameters which define the underlying process as soon as possible after the change has occurred. Surveillance, statistical process control, monitoring and change-point detection are various names for methods with the goal of addressing the change-point problem. These methods are studied and applied in economics, medicine, environmental control, finance, engineering and many other areas. Worthy of note among papers on the study of these methods include Barnett and Turkman [3], Lao, et al [21], Williamson and Hudson [45], Yashchin [48]. The underlying process is sequentially observed with the goal of raising an alarm as soon as the change occurs. An application in which the whole process is stopped as soon as an alarm occurs is called active change-point detection. Observations are made sequentially and at each time $t \in \mathcal{T}$, where $\mathcal{T}$ is the time parameter space, a decision is made whether to go on making other observations or to stop immediately and not make any other observation. The opposite is called passive change-point detection in which case, actions at an earlier time point do not affect the distribution of the process under surveillance. An example is the case of flood warning system when an alarm does not affect the height of the flood wave. Change-point detection methods may be based on likelihood ratios of the process under investigation, see Frisén and de Maré [16].

We consider the change in the parameters of an Ornstein-Uhlenbeck process $\{X_t : t \geq 0\}$ (where X_t is the obervation made at time t) occurring at time,

$t = \tau$. The most common paramater change studied in most literature on change-point detection is a shift in the mean or volatility (standard deviation) of the process under consideration. A shift from a desired value σ_0 to an undesired value σ_1 is considered. The stationarity of the Ornstein-Uhlenbeck process makes it possible to assume that if a change in the volatility occurs, there will be a sharp movement to another constant level $\sigma_1 > \sigma_0$ (and the value remains at that level). This implies that the volatility is a function of t such that $\sigma(t)$ has a constant value σ_0 for $t = 1, 2, \ldots, \tau - 1$ and the constant value σ_1 for $t = \tau, \tau + 1, \ldots$, where τ is a random time-point. Let $C(s) = \{\tau \leq s\}$ denote the event that the change has occurred by time s and $D(s) = \{\tau > s\}$ denote that the change has not occurred by that time. We seek to construct an alarm $A(s)$ such that $\{X_s : t \leq s\}$ being a subset of $A(s)$ implies that $C(s)$ has occurred for each decision time s.

The Ornstein-Uhlenbeck process $\{X_t : t \in \mathbb{R}\}$ is defined on the probability space $(\Omega, \mathcal{F}, P)$ with respect to a filtration $\{\mathcal{F}_t\}$. A paramter change (say volatility), occurs in the process at time, $t = \tau$. Conditional on $\{\mathcal{F}_t\}$, the model including the change-point may be given by

$$f_X(x_t|x_{t-1}) = \begin{cases} f_0(x_t|x_{t-1}) & \text{if } t < \tau\{\text{change has not occurred}\} \\ f_1(x_t|x_{t-1}) & \text{if } t \geq \tau\{\text{change has occurred}\} \end{cases}$$

The change-point problem is to detect the change as accurately and as quickly as possible. A change-point detection method is a stopping rule, T, defined by

$$T = \inf\{t \geq 0 : A(\{X_s : 0 \leq s \leq t\}) > G\}$$

where $A(.)$ is called the alarm function and $G \in \mathbb{R}$ is called the threshold.

2.5 Stopping time

Financial market participants often face the dilemma of either holding on to their investments or disposing of them for returns. The decisions made at such instances are usually based on the information available to individual participants at any given time. The concept of stopping time which dictates a stopping rule that is characterized as a mechanism for deciding whether to retain or dispose of the investment on the basis of the present position and past events, plays a vital role in such decision making. With respect to random variables $X_1, X_2, \ldots$, a stopping time is a random variable T, with the property that for each t, the occurrence or non-occurrnce of the event $T = t$, depends solely on the values of $X_1, X_2, \ldots, X_t$. More generally, given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, a random variable $T : \Omega \to \mathbb{R}^+$, is defined as a stopping time if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. Simply, for

T to be a stopping time, we must be able to state whether or not $\{T \leq t\}$ has occurred based on the knowledge of the σ-algebra $\mathcal{F}_t$.

Stopping times are very important in change-point detection problem because they aid the decision whether to stop the observation of a process as soon as change occurs and take immediate actions in order to minimize the cost of further delay or to continue the observation in the case when the change has not occurred. It is important that the stopping time is accurate in order to minimize the cost of false alarms. An example of stopping time is the first-passage time or hitting time.

2.6 Optimality criteria

Various optimality criteria have been suggested in the vast literature on change-point theory. The Average Run Length *ARL*, Expected delay *ED*, conditional expected delay *CED*, probability of false alarm *PFA*, predictive value *PV*, are few of these criteria. The performance of a change-point detection method depends on so many things, for example the choice of the threshold value, the distribution of the change-point etc. Early detections and few false alarms are good properties that increase the quality of a change-point detection method, see Frisén [14, 15].

2.6.1 Average Run Length

The average run length until the first alarm is a measure that is widely used in quality control. The average run length until an alarm occurs, when the expected paramter change in the underlying process has not occurred is called the average in-control run length, denoted as ARL_0. The ARL_0 is not necessarily a good indicator of sound performance of a change-point detection method. The performance measure is of little or no help at all if no assumptions about when the change-point occurs are made. Even if these assumptions are given to be any value other than $\tau = 1$ (i.e. the change occurs at the beginning of the observation) or $\tau = \infty$ (which indicates no change occurs), it is not informative enough since it would have mixed false alarms and delayed true alarms. The average run length until a change in the underlying process actually occurs is referred to as the out-of-control average run length, denoted as ARL_1.

$$ARL_0 = E(T|\tau = \infty) \tag{2.22}$$

$$ARL_1 = E(T|\tau = 1) \tag{2.23}$$

2.6.2 Probability of false alarm

The probabilty that an alarm occurs when no change in the process under surveillance has taken place is called the probaility of false alarm.

$$PFA = P(T < \tau) \tag{2.24}$$

2.6.3 Expected Delay

Let τ be exponentially distributed with λ as the parameter of the distribution. The expected delay *(ED)* from the time of change τ, to the time of alarm T, is

$$ED(\lambda) = E(T - \tau | T > \tau) \tag{2.25}$$

$ED(\lambda) \to 0$ as $\lambda \to \infty$, where $\lambda = E(\tau)$. The conditional expected delay is given thus,

$$CED(t) = E(T - \tau | T > \tau = t) \tag{2.26}$$

2.6.4 Predictive Value

The predictive value gives credence to change-point detection methods. It is the probability that a change has actually occurred when the detection method indicates so. It provides information about the underlying process and predicts what will occur if no action is taken.

$$PV(t) = P(\tau \leq T | T = t) \tag{2.27}$$

2.7 Change-point detection methods

In the vast literature on change-point detection, several methods have been derived and studied, see e.g. Frisén [14], Srivastava and Wu [41]. The Likelihood ratio method, Shewhart method, Cumulative Sum method (CUSUM), the Exponential Weighted Moving Average method (EWMA) and a Window method will discussed in this paper. The performance measures and properties of the Shewhart method will be derived and discussed.

2.8 The Likelihood Ratio Method

While probability theory allows us to predict unknown outcomes based on known parameters, the likelihood function allows us to estimate unknown parameters based on known outcomes.

Let $X_1, \dots, X_t$ be a sample of log returns and $\psi \in \Psi$, an unknown constant parameter vector in the parameter space Ψ. Let $f(x_1, \dots, x_t; \psi)$ be the joint distribution function of $X = \{X_1, \dots, X_t\}$. The likelihood function is,

$$L_X(\psi) = f(X; \psi) \tag{2.28}$$

The value of $\psi \in \Psi$ which maximizes the likelihood function $L_X(\psi)$ of the unknown parameter vector is called the maximum likelihood estimator (MLE). The MLE may be obtained by finding the zeros of the derivative with respect to ψ of the natural logarithm of the likelihood function.

$$\frac{\partial}{\partial \psi} \ln f(X; \psi) \tag{2.29}$$

If ψ_0 and ψ_1 are two possible values of ψ, we say that ψ_0 is more likely than ψ_1 if

$$L_X(\psi_0) > L_X(\psi_1)$$

The likelihood ratio $\frac{L_X(\psi_0)}{L_X(\psi_1)}$ which is a ratio of the likelihood functions of ψ_0 and ψ_1 respectively is used in hypothesis testing for discriminating between the parameter values ψ_0 and ψ_1.

For the change-point problem defined in section 2.4, an alarm is triggered when the alarm function $A(.)$ which is the likelihood ratio exceeds a limit. For the online discrimination between events $C(s) = \{\tau \leq s\}$ and $D(s) = \{\tau > s\}$, the likelihood ratio method (see Frisén and de Maré [16]) means stopping as soon as the likelihood ratio exceeds a limit:

$$\sum_{t=1}^{s} w(s,t) L(s,t) > G \tag{2.30}$$

where $w(s,t) = P(\tau = t)/P(\tau \leq s)$ and the log likelihood ratio $L(s,t)$ is defined by

$$\begin{aligned} L(s,t) &= \ln \frac{f(X_0, \dots, X_t | \tau = s)}{f(X_0, \dots, X_t | \tau > s)} \\ &= \ln \frac{f_1(X_t|X_{t-1}) \cdots f_1(X_s|X_{s-1}) f_0(X_{s-1}|X_{s-2}) \cdots f_0(X_0)}{f_0(X_t|X_{t-1}) \cdots f_0(X_s|X_{s-1}) f_0(X_{s-1}|X_{s-2}) \cdots f_0(X_0)} \\ &= \sum_{r=t}^{s} \ln \frac{f_1(X_r|X_{r-1})}{f_0(X_r|X_{r-1})} \end{aligned}$$

2.9 The Shewhart method

In 1931, Walter A. Shewhart introduced the method that has since become the most commonly used method of surveillance. The Shewhart method is very simple and widely applied in industrial quality control, see Frisén [14, 15], Shewhart [39].

Assume we are given a process $X = \{X_t : t = 1, 2, \ldots\}$, where X_t is the variable whose value is observed at time t. Let $\sigma^2(t)$ be the shift function $V(X_t) = \sigma_0^2 I(t < \tau) + \sigma_1^2 I(t \geq \tau)$ random only through the randomness of τ and where $V(X_t|\tau) = \sigma^2(t)$ and $I(.)$ is the indicator function. We seek to descriminate between two states of the system namely, the in-control state $D(s)$ and the out-of-control state $C(s)$ for each decison time s. To do this, we can only consider all observations up to time s that is, $\{X_t : t \leq s\}$. The alarm statistic is a function $p(X_s)$ based on the observations up to time s. Given a control limit $G(s)$, the alarm time is given by:

$$T = \min\{s : p(\{X_t : t \leq s\}) > G\}$$

For our specific case, we consider $X = \{X_t : t \geq 0\}$ to be an Ornstein-Uhlenbeck process. We consider that at time $t = \tau$, there occurs a change in the volatility of the process from a desired value σ_0 to an undesired value σ_1. The in-control state in this case is $D(s) = \{\tau > s\}$, which denotes the event that the change is yet to occur while the out-of-control state is $C(s) = \{\tau \leq s\}$, which denotes the event that the change has occurred. As soon as the observation deviates too much from the target, an alarm is triggered. The alarm time which coincides with the stopping time is given by the likelihood ratio as

$$T = \min\{s \geq 1 : L(s, s) > G\} \tag{2.31}$$

where $G \in \mathbb{R}$ is called the control limit or the threshold.

2.9.1 The alarm function for the Shewhart method

For an n-dimensional Ornstein-Uhlenbeck process $X_t = \{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\}$, with parameters $(\mu_{t_1}, \sigma_{t_1}^2), (\mu_{t_2}, \sigma_{t_2}^2), \ldots, (\mu_{t_n}, \sigma_{t_n}^2)$ where

$$\mu_{t_i} = E(X_{t_i}) \quad \text{and} \quad \sigma_{t_i}^2 = V(X_{t_i}), \quad 1 \leq i \leq n \tag{2.32}$$

We seek to sequentially test the hypothesis

$$H_0 : \mu_{t_1} = \mu_{t_2} = \cdots = \mu_{t_n} \quad \text{and} \quad \sigma_{t_1} = \sigma_{t_2} = \cdots = \sigma_{t_n} \tag{2.33}$$

against the stronger alternative

$$\begin{aligned} H_1 : \mu_{t_1} = \mu_{t_2} = \cdots = \mu_{t_n} \quad & \text{and} \\ \sigma_0 = \sigma_{t_1} = \sigma_{t_2} = \cdots = \sigma_{t_{\beta-1}} \neq \sigma_{t_\beta} &= \sigma_{t_{\beta+1}} = \cdots = \sigma_{t_n} = \sigma_1 \\ & \text{where } \beta = [n\alpha] \quad \text{for some } \alpha \in (0, 1). \end{aligned} \tag{2.34}$$

Under H_1, we have that the mean of the process remains unchanged but the volatility of the process changes at the random time point $\{t = \tau\}$ from σ_0 to σ_1 while H_0 represents the process at the origin with the mean, $\mu = 0$. Let $\sigma_1^2 = \delta\sigma^2$, where $\delta > 0$ is the size of the shift. The alarm function (which indicates the change) expressed in terms of Likelihood ratio by the Shewhart method is given as follows

$$\begin{aligned} L(s,s) &= \sum_{r=s}^{s} \ln \frac{f_1(X_r|X_{r-1})}{f_0(X_r|X_{r-1})} \\ &= \ln \frac{\frac{1}{\sqrt{2\pi\hat{\sigma}^2\delta}} \exp\left\{ -\frac{\left[X_s - X_{s-1}e^{-\theta|h|}\right]^2}{2\hat{\sigma}^2\delta} \right\}}{\frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \exp\left\{ -\frac{\left[X_s - X_{s-1}e^{-\theta|h|}\right]^2}{2\hat{\sigma}^2} \right\}} \\ &= -\frac{1}{2}\ln\delta - \left[\frac{\left(X_s - X_{s-1}e^{-\theta|h|}\right)^2 (1-\delta)}{2\hat{\sigma}^2\delta} \right] \\ &= \frac{1}{2}\left[\left(\frac{X_s - X_{s-1}e^{-\theta|h|}}{\hat{\sigma}} \right)^2 \left(1 - \frac{1}{\delta}\right) - \ln\delta \right] \end{aligned} \qquad (2.35)$$

2.10 The Cumulative Sum method

The Cumulative Sum, CUSUM, was suggested by Page [33]. The CUSUM method has become one of the widely employed change-point detection methods. Yashchin [47], Hawkins and Olwell [18] give more detailed reviews of the CUSUM method.

The alarm condition of the method may be expressed by the partial likelihood ratio as

$$T = \min\{s; \max(L(s,t); t = 1, 2, \ldots, s) > G\} \qquad (2.36)$$

where $G \in \mathbb{R}$ is called the control limit or threshold.

The CUSUM method is a combination of various methods. Each of the methods is optimal with respect to the expected delay, to detect a change that occurs at a specific time.

2.10.1 The alarm function for the CUSUM method

We shall generate the alarm function for the CUSUM method by considering the shift in variance as stated in section (2.9.1)

$$
\begin{aligned}
L(s,t) &= \sum_{r=t}^{s} \ln \frac{f_1(X_r|X_{r-1})}{f_0(X_r|X_{r-1})} \\
&= \sum_{r=t}^{s} \ln \frac{\frac{1}{\sqrt{2\pi\hat{\sigma}^2\delta}} \exp\left\{ \frac{\left[X_r - X_{r-1}e^{-\theta|h|}\right]^2}{2\hat{\sigma}^2\delta} \right\}}{\frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \exp\left\{ \frac{\left[X_r - X_{r-1}e^{-\theta|h|}\right]^2}{2\hat{\sigma}^2} \right\}} \\
&= -\sum_{r=t}^{s} \ln 1\sqrt{\delta} + \frac{1}{2}\sum_{r=t}^{s} \left(\frac{X_r - X_{r-1}e^{-\theta|h|}}{\hat{\sigma}^2} \right) \left(1 - \frac{1}{\delta}\right) \\
&= \frac{1}{2}\left[\frac{1}{\hat{\sigma}^2}\left(1 - \frac{1}{\delta}\right) \sum_{r=t}^{s} \left(X_r - X_{r-1}e^{-\theta|h|}\right)^2 - (s+t)\ln\delta \right] \qquad (2.37)
\end{aligned}
$$

2.11 A Window method

The alarm condition for a window method is constructed in the likelihood ratio as

$$T = \min\{s \geq 0 : L(s, s-d) > G\} \qquad (2.38)$$

where G is the threshold and d is the fixed window width.

The most extreme window method is the Shewhart method with a window size of 1. The alarm function is given as follows

$$L(s, s-d) = \frac{1}{2}\left[\frac{1}{\hat{\sigma}^2}\left(1 - \frac{1}{\delta}\right) \sum_{r=s-d}^{s} \left(X_r - X_{r-1}e^{-\theta|h|}\right)^2 - (2s-d)\ln\delta \right] \qquad (2.39)$$

2.12 Exponentially Weighted Moving Average

Roberts [34] in 1959, described a change-point detection method based on exponentially weighted moving averages, EWMA. Robinson & Ho [35], Srivastava & Wu [41], Ng & Case [30], Crowder [7], Lucas & Saccucci [24], and Domangue & Patch [10] are among many in the literature who have positively commented on the quality and effectiveness of the method. The alarm statistic is an EWMA:

$$X_s = (1-\lambda)X_{s-1} + \lambda x(s) \quad s = 1, 2, \ldots \qquad (2.40)$$

where $0 < \lambda \leq 1$ and X_0 is the expected value μ_0 which is normalized to zero. If λ is near zero, all observations have approximately the same weight. If λ is equal to one, only the last observation is considered and the method reduces to the Shewhart method. The asymptotic variant (EWMAa) gives an alarm if X_s exceeds an alarm limit usually denoted as $G\sigma_X$, where G is a constant and σ_X is the asymptotic standard deviation of X_s. The alarm function is given as

$$T = \inf\{s > 0 : X_s > G\sigma_Z\} \tag{2.41}$$

Among many suggested variants of the EWMA with low alarm limits for early time points is the EWMAe. The exact variance of the process under observation is considered when applying the EWMAe instead of the asymptotic. For a normally distributed process as the Ornstein-Uhlenbeck process, the EWMAa is preferred to the EWMAe. This is beacause EWMAe gives more frequent alarms at the first time point for the same average in-control run length ARL_0. The conditional expected delay $CED(1)$ (and thus the average out-of-control run length ARL_1) is best for EWMAe. However, since many of the alarms at time 1 are false for EWMAe, the predictive value PV of an alarm at $T = 1$ is low. For larger values of t, EWMAa has better $CED(t)$ than EWMAe, see Sonesson [40], Frisén [15].

The choice of the λ is very important to the performance of the EWMA method and the choice for the optimal values has garnered great interest in the literature. Smaller values of λ result in good ability to detect early changes in a process while larger values are most useful for detecting changes that occur later in a process.

Chapter 3

Results

3.1 Parameter estimation

The Ornstein-Uhlenbeck process is defined by parameters, μ, θ and σ which represent the mean, the mean reversion rate and the volatility as stated in section (2.2). We shall use the Maximum Likelihood Estimation method to estimate the values of these paramteres.

From the global distribution of the Ornstein-Uhlenbeck process defined by equation (2.21), the log likelihood function can be written in terms of μ, θ and σ as

$$\begin{aligned} L(\mu, \theta, \sigma) &= \sum_{i=1}^{n} \ln f(X_{t_i} | X_{t_{i-1}}; \mu, \theta, \hat{\sigma}) \\ &= -\frac{n}{2} - n \ln(\hat{\sigma}) - \frac{1}{2\hat{\sigma^2}} \sum_{i=1}^{n} \left[X_{t_i} - X_{t_{i-1}} e^{-\theta h_i} - \mu(1 - e^{-\theta h_i}) \right]^2 \end{aligned} \tag{3.1}$$

The estimate values of μ, θ and $\hat{\sigma}$, are calculated from (3.1), by taking the partial derivatives of the log likelihood function with respect to μ, θ and $\hat{\sigma}$ respectively.

$$\frac{\partial L(\mu, \theta, \hat{\sigma})}{\partial \mu} = \frac{1 - e^{\theta h_i}}{\hat{\sigma}^2} \sum_{i=1}^{n} \left[X_{t_i} - X_{t_{i-1}} e^{-\theta h_i} - \mu(1 - e^{-\theta h_i}) \right] \tag{3.2}$$

We set

$$\frac{\partial L(\mu, \theta, \hat{\sigma})}{\partial \mu} = 0$$

to get

$$\mu = \frac{\sum_{i=1}^{n} \left[X_{t_i} - X_{t_{i-1}} e^{-\theta h_i} \right]}{n(1 - e^{-\theta h_i})} \tag{3.3}$$

The partial derivative of the log-likelihood function with respect to θ is

$$\begin{aligned}\frac{\partial L(\mu,\theta,\hat{\sigma})}{\partial \theta} &= -\frac{h_i e^{-\theta h_i}}{\hat{\sigma}^2}\sum_{i=1}^{n}\left[X_{t_i} - X_{t_{i-1}}e^{-\theta h_i} - \mu(1-e^{-\theta h_i})\right](X_{t_{i-1}}-\mu) \\ &= -\frac{h_i e^{-\theta h_i}}{\hat{\sigma}^2}\sum_{i=1}^{n}\left[(X_{t_i}-\mu)(X_{t_{i-1}}-\mu) - e^{-\theta h_i}(X_{t_{i-1}}-\mu)^2\right]\end{aligned} \tag{3.4}$$

We set

$$\frac{\partial L(\mu,\theta,\hat{\sigma})}{\partial \theta} = 0$$

to get

$$\theta = -\frac{1}{h_i}\ln\frac{\sum_{i=1}^{n}\left[(X_{t_i}-\mu)(X_{t_{i-1}}-\mu)\right]}{\sum_{i=1}^{n}(X_{t_{i-1}}-\mu)^2} \tag{3.5}$$

The partial derivative of the log likelihood function with respect to $\hat{\sigma}$ is

$$\frac{\partial L(\mu,\theta,\hat{\sigma})}{\partial \hat{\sigma}} = -\frac{n}{\hat{\sigma}^2} + \frac{1}{\hat{\sigma}^3}\sum_{i=1}^{n}\left[X_{t_i} - X_{t_{i-1}}e^{\theta h_i} - \mu(1-e^{\theta h_i})\right]^2 \tag{3.6}$$

We set

$$\frac{\partial L(\mu,\theta,\hat{\sigma})}{\partial \hat{\sigma}} = 0$$

to get

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\left[X_{t_i} - X_{t_{i-1}}e^{\theta h_i} - \mu(1-e^{\theta h_i})\right]^2 \tag{3.7}$$

Recall,

$$\hat{\sigma}^2 = \frac{\sigma^2}{2\theta}\left(1-e^{-2\theta h_i}\right)$$

Hence,

$$\sigma^2 = \frac{2\theta}{n(1-e^{-2\theta h_i})}\sum_{i=1}^{n}\left[X_{t_i} - X_{t_{i-1}}e^{\theta h_i} - \mu(1-e^{\theta \alpha})\right]^2 \tag{3.8}$$

3.2 Some performance measures of the Shewhart method

In this section, we shall calculate some performance measures of the Shewhart method for the Ornstein-Uhlenbeck process. In section 2.6.1, we defined the average in-control run length, ARL_0 and the average out-of-control run length, ARL_1. Srivastava and Wu [41] generated expressions for the ARL_0 and ARL_1 for the Ornstein-Uhlenbeck process.

3.2.1 Average Run length

Theorem 3.2.1.

$$ARL_0 = \frac{1}{1 - \chi_1^2\left(\frac{G-c}{a\sigma^2}\right)} \quad and \quad ARL_1 = \frac{1}{1 - \chi_1^2\left(\frac{G-c}{a\delta\sigma^2}\right)} \tag{3.9}$$

where,

$$a = \frac{1}{\hat{\sigma}^2}\left(1 - \frac{1}{\delta}\right) \quad \text{and} \quad c = -\frac{1}{2}\ln\delta$$

Proof. Recall,

$$ARL_0 = E(T|\tau = \infty) = E_\infty(T) = \sum_{t=1}^{\infty} tP_\infty(T) \tag{3.10}$$

where,

$$T = \min\{t \geq 1 : L(t,t) > G\} \tag{3.11}$$

From equation (2.35), we have that

$$L(t,t) = a(X_t - bX_{t-1})^2 + c \tag{3.12}$$

where

$$b = e^{-\theta|h|}$$

Applying the Markov chain property, equation (3.10) becomes

$$\begin{aligned} ARL_0 &= \sum_{t=1}^{\infty} tP_\infty\left(L(1,1) \leq G, \ldots, L(t-1,t-1) \leq G, L(t,t) > G\right) \\ &= \sum_{t=1}^{\infty} tP_\infty\left(a(X_1 - bX_0)^2 + c \leq G, \ldots, a(X_{t-1} - bX_{t-2})^2 + c \leq G, a(X_t - X_{t-1})^2 + c > G\right) \\ &= \sum_{t=1}^{\infty} tP_\infty\left((X_1 - bX_0)^2 \leq \frac{G-c}{a}\right) \cdots P_\infty\left((X_{t-1} - bX_{t-2})^2 \leq \frac{G-c}{a}\right)\left(1 - P_\infty\left((X_t - bX_{t-1})^2 \leq G\right)\right) \end{aligned} \tag{3.13}$$

Now, $X_t \in N(0,\sigma^2)$ and $X_{t-1} \in N(0,\sigma^2)$. This implies that $(X_t - bX_{t-1}) \in N(0,\sigma^2)$. Hence equation (3.13) becomes

$$= \sum_{t=1}^{\infty} P_\infty\left(\left(\frac{X_1 - bX_0}{\sigma}\right)^2 \leq \frac{G-c}{a\sigma^2}\right) \cdots P_\infty\left(\left(\frac{X_{t-1} - bX_{t-2}}{\sigma}\right)^2 \leq \frac{G-c}{a\sigma^2}\right)\left(1 - P_\infty\left(\left(\frac{X_t - bX_{t-1}}{\sigma}\right)^2 \leq \frac{G-c}{a\sigma^2}\right)\right)$$

$\frac{X_t - bX_{t-1}}{\sigma} \in N(0,1)$ implies that $\frac{(X_t - bX_{t-1})^2}{\sigma^2} \in \chi_1^2$. Hence, equation (3.2.1) becomes

$$ARL_0 = \sum_{t=1}^{\infty} t\chi_1^2\left(\frac{G-c}{a\sigma^2}\right)^{t-1}\left(1 - \chi_1^2\left(\frac{G-c}{a\sigma^2}\right)\right) \tag{3.14}$$

We set

$$\chi_1^2\left(\frac{G-c}{a\sigma^2}\right) = Y_0$$

to get

$$ARL_0 = \sum_{t=1}^{\infty} tY_0^{t-1}(1 - Y_0)$$
$$= 1 - Y_0 \sum_{t=1}^{\infty} tY_0^{t-1} \tag{3.15}$$

We observe that

$$tY_0^{t-1} = \frac{d}{dY_0} Y_0^t \tag{3.16}$$

We substitute (3.16) into (3.15) to get

$$ARL_0 = 1 - Y_0 \sum_{t=1}^{\infty} \frac{d}{dY_0} Y_0^t$$
$$= 1 - Y_0 \frac{d}{dY_0} \sum_{t=1}^{\infty} Y_0^t$$
$$= 1 - Y_0 \frac{d}{dY_0} \left(\frac{1}{1 - Y_0} \right)$$
$$= \frac{1}{1 - Y_0} \tag{3.17}$$

We shall now calculate the ARL_1 in similar ways as we did the ARL_0

$$ARL_1 = E(T|\tau = 1) = \sum_{t=1}^{\infty} tP_0(T = t|\tau = 1)$$

$$= \sum_{t=1}^{\infty} tP_0 \left(\left(\frac{X_1 - bX_0}{\sigma} \right)^2 \leq \frac{G - c}{a\delta\sigma^2} \right) \dots P_0 \left(\left(\frac{X_{t-1} - bX_{t-2}}{\sigma} \right)^2 \leq \frac{G - c}{a\delta\sigma^2} \right) \left(1 - P_0 \left(\left(\frac{X_t - bX_{t-1}}{\sigma} \right)^2 \leq \frac{G - c}{a\delta\sigma^2} \right) \right)$$

$$= \sum_{t=1}^{\infty} t\chi_1^2 \left(\frac{G - c}{a\delta\sigma^2} \right)^{t-1} \left(1 - \chi_1^2 \left(\frac{G - c}{a\delta\sigma^2} \right) \right) \tag{3.18}$$

Let

$$Y_1 = \chi_1^2 \left(\frac{G - c}{a\delta\sigma^2} \right)$$

$$
\begin{aligned}
\text{ARL}_1 &= \sum_{t=1}^{\infty} tY_1^t(1 - Y_1) \\
&= 1 - Y_1 \sum_{t=1}^{\infty} tY_1^{t-1} \\
&= 1 - Y_1 \frac{d}{dY_1} \sum_{t=1}^{\infty} Y_1^t \\
&= 1 - Y_1 \frac{d}{dY_1} \left(\frac{1}{1 - Y_1} \right) \\
&= \frac{1}{1 - Y_1} \qquad (3.19)
\end{aligned}
$$

□

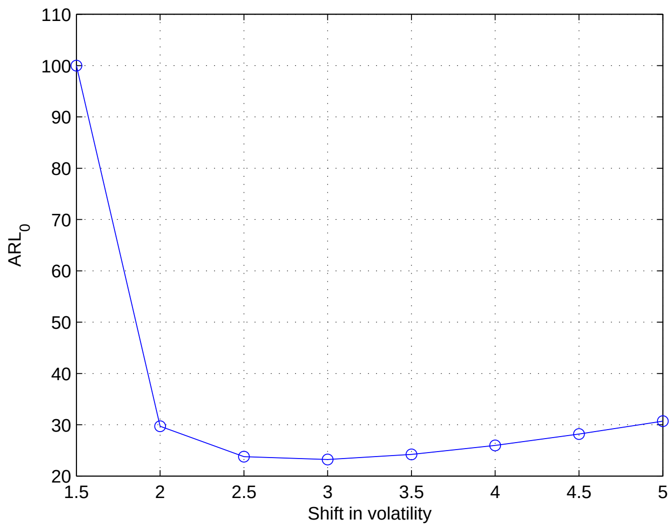

Figure 3.1: ARL_0 for the Shewhart method, when $G = 0.4044$, $\hat{\sigma}^2 = 0.4337$ and $\theta = 3$.

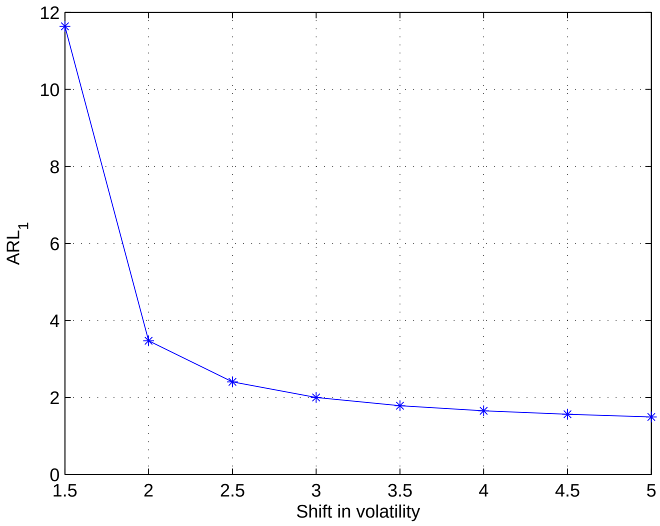

Figure 3.2: ARL_1 for the Shewhart method when $G = 0.4044$, $\hat{\sigma}^2 = 0.4337$ and $\theta = 3$.

The ARL_0 in this project is the expected number of times we had to run the process before an alarm is triggered off without a parameter change in the distribution of the process i.e. the run length until a false alarm. The threshold value which is used to compare various shift sizes is chosen such that the ARL_0 is equal to 100. Hence, at every 100 runs, we expect a false alarm. The ARL_1 measures the time until first alarm when the change occurs right at the start. This is very important as it is a measure of the expected delay to the motivated alarm assuming that the change occurs immediately as the monitoring is started.

Chapter 4

Conclusions

It is a common saying that "the only thing permanent in life is change". This statement is true even for global financial markets. The volatile nature of prices of financial instruments makes the monitoring of market parameters for changes a very salient task if market participants are to generate any yield from their investments. Hence, it is of utmost importance that properly designed change-point detection methods are constructed to detect these changes as soon as they occur.

We have shown that the Ornstein-Uhlenbeck process is a good mathematical reflection of activities in financial markets. We have also derived some change-point detection methods. The performance measures of these methods have in recent times garnered lots of attention in research and application. The average run length criteria (ARL) which are the most commonly used performance measure have been considered in this work. We observed however, that the average (in-control) run length ARL_0, is dependent on the shift size. This observation calls into question the integrity of the ARL_0 as a viable performance measure. While the concept of optimality is often difficult to specify, it is seldom possible to achieve uniform optimality of surveillance methods.

Using the ARL criteria, we examined the performance of the Shewhart method. The beauty of the Shewhart method is the fact that it keeps calculations simple. Hence, it is more easily applicable for market participants who have little or no mathematical background. From figures 3.1 and 3.2, we observed that the Shewhart method is very effective for detecting reasonably large changes. It can be seen that the value of the ARL_1 decreases as the size of shift increases. The implication of this is that when the anticipated shift is relatively small or moderate, the method might fail to timely detect the change. Hence, despite the criticism of the Shewhart method (for allotting so much weight to the last observation before the change), it remains a valuable tool for detecting large and possibly catastrophic changes when used with the average run length criteria.

The need to make timely decisions about whether to sell or to hold on to a financial instrument has made the application of change-point detection a very important tool of the modern financial market. This research will find usefulness among mathematicians and non-mathematicians, active and passive participants of the market. Possible future research could be the construction of more sophisticated detection methods using more advanced mathematical models which will better mirror the activities in financial markets. Performance measures like the minimax critieria, probability of false alarm, predictive value, expected delay of an alarm, to mention a few, should also be considered for the evaluation of these methods.

Appendix

The table below shows a simulation scenario of the ARL_0 and ARL_1 of the Shewhart method with threshold G=0.4044. See figures 3.1 and 3.2.

δ	Y_0	ARL_0	Y_1	ARL_1
1.5	0.9900	100.0046	0.9141	11.6364
2.0	0.9663	29.7054	0.7118	3.4696
2.5	0.9580	23.7820	0.5839	2.4033
3.0	0.9569	23.2228	0.4999	1.9997
3.5	0.9587	24.2328	0.4402	1.7863
4.0	0.9615	25.9780	0.3951	1.6532
4.5	0.9645	28.1770	0.3597	1.5618
5.0	0.9674	30.7105	0.3310	1.4947

δ = size of the shift in the volatility of the Ornstein-Uhlenbeck process

$Y_0 = \chi_1^2\left(\frac{G-c}{a\sigma^2}\right)$

$a = \frac{1}{\hat{\sigma}^2}(1-\frac{1}{\delta})$

$c = -\frac{1}{2}\ln\delta$

$\hat{\sigma}^2 = \sigma^2\left(\frac{1-e^{-2\theta h}}{2\theta}\right)$

$ARL_0 = \frac{1}{1-Y_0}$

$Y_1 = \chi_1^2\left(\frac{G-c}{a\delta\sigma^2}\right)$

$ARL_1 = \frac{1}{1-Y_1}$

The following are Matlab codes for the simulation of an Ornstein-Uhlenbeck process with long term mean $\mu = 1$, mean reversion rate $\theta = 3$, time step $h = 0.01$ and noise (diffusion) term $\sigma = 0.5$

```
close all; clear all;
x0 = 3;
theta = 3;
mu = 1;
h =0.01;
```

```
sigma = 0.5;
x_prv = x0;
x_total =x0;
rand_sq = randn(1,500);
for t=1:500
    x_new = x_prv*exp(-theta*h) + mu*(1-exp(-theta*h))
     + sigma*sqrt((1-exp(-2*theta*h))/(2*theta))*rand_sq(1,t);
    x_total = [x_total,x_new];
    x_prv=x_new;
end

plot(0:0.01:5,x_total);
ylabel('X_t');
xlabel('time (t)');
```

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