Self-adjointness and conservation laws of nonlinear dispersive wave equations modelling elasto-plastic flows

Johan Ermstål

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Supervisor: Prof. Nail H. Ibragimov

Examiner: Dr. Raisa Khamitova

Department of mathematics and science

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Abstract:

Two nonlinear dispersive wave equations arising in elasto-plastic flow have been investigated for self-adjointness. For these equations their symmetries are calculated and conservation laws are constructed using two different methods: an old method based on Noether´s Theorem and a new one developed by Prof. Nail Ibragimov. The new method works for a larger number of equations than the old one. It is complementing the old one in the way that it gives some conservation laws that otherwise would have been impossible to obtain.

Keywords:

Conservation laws, Differential Algebra, Self-adjoint, Wave equation, Symmetry.

Contact Information

Author:

Johan Ermstål
E-mail: johan.ermstal@hotmail.com

Supervisor:

Professor Nail H. Ibragimov
E-mail: nailhib@gmail.com

Examiner:

Dr. Raisa Khamitova
E-mail: raisa.khamitova@bth.se

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1. Notations

I follow the notations used in Prof. Ibragimov´s textbook [2]. Our variables in the two equations are the dependent variable u and the two independent variables t and x where t is numbered as independent variable x^1 and x as x^2. The vector of independent variables is $x = (x^1, x^2) = (t, x)$.

For the derivatives of the dependent variable u the notations that will be used are as follows.

u_i is the first derivative of u with respect to x^i.

u_{ij} is the second order derivative of u with respect to x^i and x^j, and so on.

$u_{(1)}$ are all first order derivatives, $u_{(2)}$ are all second order derivatives etc.

$D_i(u) = u_i$,

$D_j\big(D_i(u)\big) = D_j(u_i)$

where the total derivative D_i with respect to an arbitrary independent variable x^i is

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots.$$

The variational derivative with respect to the dependent variable u is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i \left(\frac{\partial}{\partial u_i}\right) + D_i D_j \left(\frac{\partial}{\partial u_{ij}}\right) + \cdots. \qquad (1.1)$$

2. Introduction

2.1 Problem Statement

The following two equations

$$u_{tt} + u_{xxxx} + \lambda u = u_{xx}\sigma'(u_x), \qquad \lambda = \text{const.} > 0 \tag{2.1}$$

and

$$u_{tt} + u_{xxxx} = \sigma(u) \tag{2.2}$$

are investigated for self-adjointness and conservation laws. σ is an arbitrary function in both cases. In Equation (2.1) the function σ is dependent on the variable u_x, σ' is the first derivative of σ with respect to u_x. In (2.2) σ depends on u.

2.2 Background

The Equation (2.1) is a nonlinear wave equation arising in elasto-plastic flow [1]. It governs the longitudinal motion of an elasto-plastic bar and anti-plane shearing deformation in elasto-plastic-microstructure models. Equation (2.2) is another form of Equation (2.1) and it is briefly introduced in [1].

Conservation laws can be considered among the most fundamental principles of physics. They are symmetries or constants of nature that, according to modern physics, cannot be violated. In classical mechanics some standard conservation laws are those of energy, linear and angular momenta, etc. [2]. For example the equation of motion of a particle

$$m\dot{\boldsymbol{v}} = 0$$

has the conserved vector

$$E = m|\boldsymbol{v}|^2.$$

The energy of the particle is constant for the free motion of the particle. This is exactly what conservation laws are, quantities that are invariant for any specific equation. And in this case the physical significance of the conservation law is already known.

In my work conserved vectors will be constructed for Equations (2.1) and (2.2). The physical significance of the conservation laws is not given in this thesis, for that a physicist in the area would have to analyze the conservation laws. I hope that someone with knowledge in this specific field of physics will analyze the conservation laws in the future and hopefully explain their physical meaning.

I have used two methods for constructing conservation laws. The first method comes from Noether's Theorem which can be found in [2]. The Equations (2.1) and (2.2) will also be investigated for self-adjointness. If they are self-adjoint, another method of calculating conservation laws described in [3] will be used. If they are not self-adjoint, this method can still be used, but instead nonlocal conservation laws will be obtained. They will contain an additional dependent variable.

3. Calculations

3.1 Investigation of self-adjointness of Equations (2.1) and (2.2)

3.1.1 Theory and methods of calculation

This theory is mostly taken from my lecture notes from the course "Mathematical Modelling with Lie Group Analysis", but everything and even deeper theory can be found in [3].

My Equations (2.1) and (2.2) are of the fourth order. An arbitrary fourth order equation

$$F\left(\boldsymbol{x}, u, u_{(1)}, u_{(2)}, u_{(3)}, u_{(4)}\right)=0 \tag{3.1}$$

has the formal Lagrangian

$$\mathcal{L}=vF\left(\boldsymbol{x}, u, u_{(1)}, u_{(2)}, u_{(3)}, u_{(4)}\right),$$

where v is a new dependent variable, $v=v(\boldsymbol{x})$. The function F^{*} is calculated as the variational derivative of the formal Lagrangian

$$F^{*}=\frac{\delta \mathcal{L}}{\delta u}=\left(\frac{\partial}{\partial u}-D_{i} \frac{\partial}{\partial u_{i}}+D_{i} D_{j} \frac{\partial}{\partial u_{i j}}-D_{i} D_{j} D_{k} \frac{\partial}{\partial u_{i j k}}\right. \\ \left.+D_{i} D_{j} D_{k} D_{l} \frac{\partial}{\partial u_{i j k l}}\right)(\mathcal{L})$$

where the total derivative D_i acts on both v and u as follows:

$$D_{i}=\frac{\partial}{\partial x^{i}}+u_{i} \frac{\partial}{\partial u}+v_{i} \frac{\partial}{\partial v}+u_{i j} \frac{\partial}{\partial u_{j}}+v_{i j} \frac{\partial}{\partial v_{j}} \\ +u_{i j k} \frac{\partial}{\partial u_{j k}}+v_{i j k} \frac{\partial}{\partial v_{j k}}+u_{i j k l} \frac{\partial}{\partial u_{j k l}}+v_{i j k l} \frac{\partial}{\partial v_{j k l}} .$$

The Equation (3.1) is said to be self-adjoint if the equation

$$\left.F^{*}\right|_{v=u}=aF \tag{3.2}$$

holds for a certain variable coefficient a.

3.1.2 Investigation of self-adjointness of Equation (2.1)

Equation (2.1) has the function

$$F = u_{tt} + u_{xxxx} + \lambda u - u_{xx}\sigma'(u_x)$$

which gives the formal Lagrangian

$$\mathcal{L} = v\big(u_{tt} + u_{xxxx} + \lambda u - u_{xx}\sigma'(u_x)\big).$$

The function F^* is calculated according to (1.1) and has the form

$$F^* = \frac{\delta\mathcal{L}}{\delta u} = v_{tt} + v_{xxxx} + \lambda v - v_{xx}\sigma'(u_x) - v_x u_{xx}\sigma''(u_x).$$

Now it is checked whether Equation (2.1) is self-adjoint. Using Equation (3.2) we obtain

$$u_{tt} + u_{xxxx} + \lambda u - u_{xx}\sigma'(u_x) - u_x u_{xx}\sigma''(u_x)$$

$$= a\big(u_{tt} + u_{xxxx} + \lambda u - u_{xx}\sigma'(u_x)\big).$$

The left hand side of the equation will be identically equal to the right hand side of the equation only when $a = 1$ and the term with $u_x u_{xx}$ vanishes. Hence $\sigma''(u_x) = 0$ and

$$\sigma(u_x) = Au_x + B,$$

where A and B are arbitrary constants. Thus, I have obtained the following result.

Proposition 1

Equation (2.1) is not generally self-adjoint. However, there are two special cases.

Case 1. $A \neq 0$.

Equation (2.1) transforms to the following equation:

$$u_{tt} + u_{xxxx} + \lambda u = Au_{xx}. \qquad (3.3)$$

Case 2. $A = 0$.

Equation (2.1) changes to the form

$$u_{tt} + u_{xxxx} + \lambda u = 0. \qquad (3.4)$$

3.1.3 Investigation of self-adjointness of Equation (2.2)

Equation (2.2) has the function

$$F = u_{tt} + u_{xxxx} - \sigma(u)$$

and its formal Lagrangian has the form

$$\mathcal{L} = v(u_{tt} + u_{xxxx} - \sigma(u)).$$

The function F^* is calculated as

$$F^* = \frac{\delta \mathcal{L}}{\delta u} = v_{tt} + v_{xxxx} - v\sigma'(u).$$

Now it is checked whether the equation is self-adjoint. By using Equation (3.2) we obtain

$$u_{tt} + u_{xxxx} - u\sigma'(u) = a(u_{tt} + u_{xxxx} - \sigma(u)).$$

The left hand side will be identically equal to the right hand side only when the coefficient $a = 1$ and

$$\sigma(u) = u\sigma'(u).$$

Solving this equation we obtain that it holds only if the function $\sigma(u)$ is linear on u:

$$\sigma(u) = Au, \qquad A\text{=arbitrary constant.}$$

Proposition 2

Equation (2.2) is not generally self-adjoint. However, it has two special cases.

Case 1. $A \neq 0$

Equation (2.2) transforms to the following equation:

$$u_{tt} + u_{xxxx} = Au. \tag{3.5}$$

Case 2. $A = 0$

Equation (2.2) changes to the equation

$$u_{tt} + u_{xxxx} = 0. \tag{3.6}$$

Equation (3.5) can be simplified in the following way.

Let $\sigma = Au = \bar{u}, \quad x = b\bar{x}, \quad t = a\bar{t}.$

Then $\bar{u}_t = Au_t, \quad \bar{u}_x = Au_x, \quad \dfrac{\partial}{\partial t} = \dfrac{1}{a}\dfrac{\partial}{\partial \bar{t}}, \quad \dfrac{\partial}{\partial x} = \dfrac{1}{b}\dfrac{\partial}{\partial \bar{x}}$

where $a = \dfrac{1}{\sqrt{A}}, \; b = \dfrac{1}{\sqrt[4]{A}}$.

This gives $\dfrac{1}{A}\left(\dfrac{\bar{u}_{\bar{t}\bar{t}}}{a^2} + \dfrac{\bar{u}_{\bar{x}\bar{x}\bar{x}\bar{x}}}{b^4}\right) = \bar{u}$ which is equal to $\bar{u}_{\bar{t}\bar{t}} + \bar{u}_{\bar{x}\bar{x}\bar{x}\bar{x}} = \bar{u}$ and can be written as

$$u_{tt} + u_{xxxx} = u. \tag{3.7}$$

3.2 Calculation of symmetries for Equations (2.1) and (2.2)

3.2.1 Theory and methods of calculation

For calculating the conservation laws of the given equations their symmetries are needed. A symmetry will be found as an operator of the form

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}.$$

Equation (3.1) admits the symmetry X if the following equation is true

$$X\left(F\left(\boldsymbol{x},u,u_{(1)},u_{(2)},u_{(3)},u_{(4)}\right)\right)|_{(3.1)} = 0.$$

Since Equations (2.1) and (2.2) include derivatives, it is necessary to prolong the operator using prolongation formulas given in [4]. X prolonged to the fourth order derivatives has the form

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}$$
$$+\zeta_i^\alpha\frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha\frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \zeta_{i_1 i_2 i_3}^\alpha\frac{\partial}{\partial u_{i_1 i_2 i_3}^\alpha} + \zeta_{i_1 i_2 i_3 i_4}^\alpha\frac{\partial}{\partial u_{i_1 i_2 i_3 i_4}^\alpha}.$$

where the general prolongation formula is

$$\zeta_{i_1\ldots i_s}^\alpha = D_{i_s}\left(\zeta_{i_1\ldots i_{s-1}}^\alpha\right) - u_{j i_1\ldots i_{s-1}}^\alpha D_{i_s}\left(\xi^j\right) \tag{3.8}$$

and

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i\left(\xi^j\right). \tag{3.9}$$

A symmetry of a differential equation can, for example, be used for solving differential equations, constructing conservation laws etc. For deeper understanding of a symmetry of a differential equation and different ways of using it the reader is referred to [2], [3] and [4].

3.2.2 Symmetries of Equation (2.1)

In this case an arbitrary symmetry of Equation (2.1) has the form

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{11} \frac{\partial}{\partial u_{tt}}$$
$$+ \zeta_{22} \frac{\partial}{\partial u_{xx}} + \zeta_{2222} \frac{\partial}{\partial u_{xxxx}}.$$

The prolongation formulas (3.8) and (3.9) give the following:

$$\zeta_2 = D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2)$$
$$= D_x(\eta) - u_t \xi^1_x - u_t u_x \xi^1_u - u_x \xi^2_x - u_x^2 \xi^2_u,$$

$$\zeta_{11} = D_t(\zeta_1) - u_{tt} D_t(\xi^1) - u_{tx} D_t(\xi^2)$$
$$= D_t^2(\eta) - u_t D_t^2(\xi^1) - u_x D_t^2(\xi^2) - 2u_{tt} D_t(\xi^1) - \underline{2u_{tx} D_t(\xi^2)},$$

$$\zeta_{22} = D_x(\zeta_2) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2) =$$
$$D_x^2(\eta) - u_t D_x^2(\xi^1) - u_x D_x^2(\xi^2) - 2u_{tx} D_x(\xi^1) - 2u_{xx} D_x(\xi^2),$$

$$\zeta_{2222} = D_x(\zeta_{222}) - u_{txxx} D_x(\xi^1) - u_{xxxx} D_x(\xi^2)$$
$$= D_x^4(\eta) - u_t D_x^4(\xi^1) - u_x D_x^4(\xi^2) - 4u_{tx} D_x^3(\xi^1) - 4u_{xx} D_x^3(\xi^2)$$
$$-6u_{txx} D_x^2(\xi^1) - 6u_{xxx} D_x^2(\xi^2) - \underline{4u_{txxx} D_x(\xi^1)} - 4u_{xxxx} D_x(\xi^2).$$

The equation for calculating the symmetries of Equation (2.1), the so called determining equation, is written as

$$[\zeta_{11} + \zeta_{2222} + \lambda\eta - \zeta_{22}\sigma'(u_x) - u_{xx}\zeta_2\sigma''(u_x)]_{(2.1)} = 0. \qquad (3.10)$$

The substitution used in this case is

$$u_{xxxx} = u_{xx}\sigma'(u_x) - u_{tt} - \lambda u.$$

Equation (3.10) should be identically zero. All derivatives of u are considered as separate variables. They are singled out one by one and their coefficients are nullified.

The term with u_{txxx} is identically zero only if

$$D_x(\xi^1) = \xi^1_x + u_x\xi^1_u = 0.$$

The equation above holds if

$$\xi^1 = f(t).$$

Now all terms containing $D_x(\xi^1)$ disappear from the prolongation formulae.

The remaining term containing u_{tx} is identically zero only if

$$D_t(\xi^2) = \xi^2_t + u_t\xi^2_u = 0.$$

The equation above holds if

$$\xi^2 = g(x).$$

Now the prolongation formulae change to the following:

$$\zeta_2 = D_x(\eta) - u_x D_x(\xi^2) = D_x(\eta) - u_x\xi^2_x\,,$$

$$\zeta_{11} = D_t(\zeta_1) - u_{tt}D_t(\xi^1) = D^2_t(\eta) - u_t D^2_t(\xi^1) - \underline{2u_{tt}D_t(\xi^1)}\,,$$

where $D^2_t(\eta) = \eta_{tt} + 2u_t\eta_{tu} + u^2_t\eta_{uu} + \underline{u_{tt}\eta_u}\,,$

$$\zeta_{22} = D_x(\zeta_2) - u_{xx}D_x(\xi^2) = D_x^2(\eta) - u_x D_x^2(\xi^2) - 2u_{xx}D_x(\xi^2) \,,$$

where $D_x^2(\eta) = \eta_{xx} + 2u_x\eta_{xu} + u_x^2\eta_{uu} + u_{xx}\eta_u$,

$$\zeta_{2222} = D_x(\zeta_{222}) - u_{xxxx}D_x(\xi^2) = D_x^4(\eta) - u_x D_x^4(\xi^2)$$
$$-4u_{xx}D_x^3(\xi^2) - 6u_{xxx}D_x^2(\xi^2) - \underline{4u_{xxxx}D_x(\xi^2)} \,,$$

where $D_x^4(\eta) = \eta_{xxxx} + \cdots + \underline{u_{xxxx}\eta_u}$,

$$u_{xxxx} = u_{xx}\sigma'(u_x) - \underline{u_{tt}} - \lambda u \,.$$

The terms containing u_{tt} are in ζ_{11} and ζ_{2222} only and they give the coefficient

$$-2\xi_t^1 + 4\xi_x^2,$$

which is identically zero only when

$$\xi^1 = 2C_1t + C_2$$

and

$$\xi^2 = C_1x + C_3.$$

ζ_{2222} reduces to the form

$$\zeta_{2222} = D_x^4(\eta) - 4C_1(u_{xx}\sigma'(u_x) - \lambda u),$$

where

$$D_x^4(\eta) = \eta_{xxxx} + u_x^4\eta_{uuuu} + 4u_x^3\eta_{xuuu} + 6u_x^2\eta_{xxuu} + 4u_x\eta_{xxxu}$$
$$+\underline{4u_{xxx}\eta_{xu}} + 12u_{xx}u_x\eta_{xuu} + \underline{4u_{xxx}u_x\eta_{uu}} + 3u_{xx}^2\eta_{uu}$$
$$+6u_{xx}u_x^2\eta_{uuu} + \eta_u(u_{xx}\sigma'(u_x) - \lambda u) + 6u_{xx}\eta_{xxu} \,.$$

The terms containing u_{xxx} are in $D_x^4(\eta)$ only and they give the coefficient

$$4\eta_{ux} + 4u_x\eta_{uu} \ ,$$

which is identically zero only when

$$\eta = h(t)u + k(t,x).$$

The prolongation formulae now change to the following:

$$\zeta_2 = hu_x + k_x - C_1u_x \ ,$$

$$\zeta_{11} = uh_{tt} + k_{tt} + \underline{2u_th_t} \ ,$$

$$\zeta_{22} = k_{xx} + hu_{xx} - 2C_1u_{xx} \ ,$$

$$\zeta_{2222} = h\big(u_{xx}\sigma'(u_x) - \underline{\lambda u}\big) + k_{xxxx} - 4C_1\big(u_{xx}\sigma'(u_x) - \underline{\lambda u}\big).$$

The term containing u_t has the coefficient $2h_t$, hence $h = C_4$,

$$\eta = C_4u + k(t,x)$$

and

$$\lambda\eta = \underline{\lambda C_4u} + \lambda k(t,x).$$

In the reduced determining equation terms containing u are in ζ_{2222} and $\lambda\eta$ and they give the coefficient

$$4C_1\lambda.$$

It is identically zero only when

$$C_1 = 0.$$

Now the prolongation formulae change to the following:

$$\zeta_2 = C_4 u_x + k_x \,,$$

$$\zeta_{11} = k_{tt} \,,$$

$$\zeta_{22} = k_{xx} + C_4 u_{xx} \,,$$

$$\zeta_{2222} = k_{xxxx} + C_4 u_{xx} \sigma'(u_x) \,.$$

The determining Equation (3.10) has reduced to the equation

$$k_{tt} + k_{xxxx} + \lambda k - k_{xx}\sigma'(u_x) - \underline{C_4 u_x u_{xx} \sigma''(u_x)} - k_x u_{xx} \sigma''(u_x) = 0.$$

In the general case we consider $\sigma''(u_x) \neq 0$. The term with $u_x u_{xx}$ has the coefficient

$$C_4 \sigma''(u_x).$$

It is identically zero only when

$$C_4 = 0.$$

Hence, the determining equation obtains the following form:

$$k_{tt} + k_{xxxx} + \lambda k - k_{xx}\sigma'(u_x) - \underline{u_{xx} k_x \sigma''(u_x)} = 0.$$

To nullify the term containing u_{xx} the derivative of k with respect to x has to be equal to zero. Hence, $k = k(t)$. The determining equation reduces to the form

$$k_{tt} + \lambda k = 0.$$

This equation is of the same form as the well-known linear equation for harmonic oscillations, see, for example, page 114 in [2]. It has the general solution

$$k = Acos(\sqrt{\lambda}t) + Bsin(\sqrt{\lambda}t),$$

where A and B are arbitrary constants. We set $A = C_5$ and $B = C_6$.
Thus, for Equation (2.1) the symmetry has

$$\xi^1 = C_2, \ \ \xi^2 = C_3, \ \ \eta = C_5 cos(\sqrt{\lambda}t) + C_6 sin(\sqrt{\lambda}t).$$

The symmetries of (2.1) are obtained by setting one constant at a time to a nonzero value and all others to zero. The symmetries admitted by Equation (2.1) are

$$X_1 = \frac{\partial}{\partial t}, \ X_2 = \frac{\partial}{\partial x}, \ X_3 = \cos(\sqrt{\lambda}t)\frac{\partial}{\partial u}, \ X_4 = sin(\sqrt{\lambda}t)\frac{\partial}{\partial u}.$$

3.2.3 Symmetries of Equations (3.3) and (3.4)

In the first special case, when Equation (2.1) is self-adjoint and has the form (3.3), the method of finding symmetries is exactly the same up to the point where we considered $\sigma''(u_x) \neq 0$. Now we choose $\sigma''(u_x) = 0$ and $\sigma(u_x) = Au_x$. Then the determining equation reduces to the equation

$$k_{tt} + k_{xxxx} + \lambda k - Ak_{xx} = 0. \qquad (3.11)$$

For Equation (3.4) the calculations are exactly the same and $\sigma(u_x) = B$ $=$ const. The determining equation reduces to the form

$$k_{tt} + k_{xxxx} + \lambda k = 0. \qquad (3.12)$$

Thus, $\xi^1 = C_2, \xi^2 = C_3, \eta = C_4 u + k(t,x)$, where $k(t,x)$ is any solution of Equation (3.11) or Equation (3.12) for Case 1 and Case 2, respectively.
When Equation (2.1) is self-adjoint it has the additional symmetries

$$X_5 = u\frac{\partial}{\partial u}\ ,\ X_6 = k(t,x)\frac{\partial}{\partial u}.$$

Notice that X_3 and X_4 are particular cases of X_6.

3.2.4 Symmetries of Equation (2.2)

For Equation (2.2) the determining equation is

$$[\zeta_{11} + \zeta_{2222} - \eta\sigma'(u)]_{(2.2)} = 0.$$

In this case Equation (2.2) is used to exclude u_{xxxx} from the determining equation,

$$u_{xxxx} = \sigma(u) - u_{tt}.$$

The terms containing u_{txxx} , u_{tx} , u_{txx}, u_{tt}, u_{xxx} and u_t give the same results as in Chapter 3.2.2. The determining equation reduces to the equation:

$$k_{tt} + k_{xxxx} + (C_4 - 4C_1)\sigma(u) - (C_4 u + k)\sigma'(u) = 0.$$

The equation splits into two equations

$$k_{tt} + k_{xxxx} = 0$$

and

$$(C_4 - 4C_1)\sigma(u) - (C_4 u + k)\sigma'(u) = 0.$$

The latter equation should be satisfied for any function σ. Hence,
$C_4 - 4C_1 = 0$ and $C_4 u + k = 0$, whence $C_4 = 0$, $k = 0$ and $C_1 = 0$. Thus, the symmetries admitted by Equation (2.2) are

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}.$$

3.2.5 Symmetries of Equation (3.7)

The calculations for Equation (3.7) are very similar. Equation (3.7) has the determining equation

$$[\zeta_{11} + \zeta_{2222} - \eta]_{(3.7)} = 0.$$

In this case $\sigma = u$. Hence, the determining equation reduces to the following equation:

$$k_{tt} + k_{xxxx} - k - 4C_1 u = 0$$

The equation splits into two equations:

$$C_1 = 0$$

and

$$k_{tt} + k_{xxxx} - k = 0.$$

This gives the additional symmetries

$$X_5 = u\frac{\partial}{\partial u}, \; X_6 = k(t,x)\frac{\partial}{\partial u},$$

where $k(t,x)$ is any solution of the equation

$$k_{tt} + k_{xxxx} - k = 0.$$

3.2.6 Symmetries of Equation (3.6)

The other self-adjoint equation, (3.6), has the determining equation

$$[\zeta_{11} + \zeta_{2222}]_{(3.6)} = 0.$$

In this case $\sigma = 0$. Most of the calculations are exactly the same. The determining equation reduces to the form

$$k_{tt} + k_{xxxx} = 0.$$

Since the constants C_1 and C_4 are not equal to zero, Equation (3.6) has the additional symmetry

$$X_7 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}.$$

Notice that $X_6 = k(t,x)\frac{\partial}{\partial u}$ has $k(t,x)$ satisfying the equation

$$k_{tt} + k_{xxxx} = 0.$$

3.3 Conservation laws

3.3.1 Theory and methods of calculation

The goal of this Master Thesis is to construct conservation laws for Equations (2.1), (2.2) and the special cases, when the equations are self-adjoint. The theory and methods of calculation are taken from [1] and [2]. For a deeper understanding the reader is referred to those books.

3.3.2 Definition of a conservation law

Consider the differential equation

$$F(\boldsymbol{x}, u, u_{(1)}, u_{(2)}) = 0. \qquad (3.13)$$

According to the definition [2], p.275, a vector field $C(\boldsymbol{x}, u, u_{(1)})$ with n components,

$$C = (C^1, \dots, C^n), \tag{3.14}$$

is called a conserved vector if it satisfies the equation

$$div\, C \equiv D_i(C^i) = 0 \tag{3.15}$$

on each solution $u = u(x)$ of Equation (3.13). Equation (3.15) is termed a conservation law for Equation (3.13). Trivial conserved vectors $C = (C^1, \dots, C^n)$ with C^i as a combination of F and its derivatives are excluded from consideration.

3.3.3 Methods of calculation

One way of constructing conservation laws is by using Noether's theorem described in detail in [2]. This method works only for equations that have a Lagrangian $\mathcal{L}$. In addition the following condition should be satisfied:

$$X(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = 0 \tag{3.16}$$

or

$$X(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = D_i(B^i). \tag{3.16´}$$

Let $X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}$ be a symmetry of the differential equation in question. The formula for components of a conserved vector is

$$C^i = \xi^i \mathcal{L} + W\left[\frac{\partial \mathcal{L}}{\partial u_i} - D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}}\right) + D_j D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}}\right) - D_j D_k D_l\left(\frac{\partial \mathcal{L}}{\partial u_{ijkl}}\right)\right] \tag{3.17}$$
$$+D_j(W)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}} - D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}}\right) + D_k D_l\left(\frac{\partial \mathcal{L}}{\partial u_{ijkl}}\right)\right]$$
$$+D_j D_k(W)\left[\frac{\partial \mathcal{L}}{\partial u_{ijk}} - D_l\left(\frac{\partial \mathcal{L}}{\partial u_{ijkl}}\right)\right] + D_j D_k D_l(W)\left[\frac{\partial \mathcal{L}}{\partial u_{ijkl}}\right], \qquad W = \eta - \xi^j u_j\,,$$

which is given for a Lagrangian containing derivatives of the fourth order, but it can be prolonged infinitely. Conservation laws obtained by the Noether theorem are termed local conservation laws. If an equation does not have a Lagrangian, but has symmetries, a nonlocal conservation law can be constructed using a formal Lagrangian. This is described in greater detail in [3]. A formal Lagrangian is created by multiplying the original equation by a new dependent variable v, as it was done in Section 3.1.1. However, a conservation law will be nonlocal, since it will include the new variable v. The formal Lagrangian for Equation (3.13) is

$$\mathcal{L} = v\, F\big(\boldsymbol{x}, u, u_{(1)}, u_{(2)}\big).$$

Then Formulae (3.17) are used to calculate a nonlocal conservation law. The first term in C^i that contains $\mathcal{L}$ can be neglected, because it gives a trivial conserved vector. If the equation in question is self-adjoint the found nonlocal conservation law can be transformed into a local conservation law by substituting u instead of v.

3.3.4 Conservation laws of Equation (2.1)

Calculations based on the Noether theorem

Conservation laws for Equation (2.1) are first constructed using the old method based on Noether's theorem. A Lagrangian of Equation (2.1) is therefore needed. The variational derivative of the Lagrangian is calculated according to

$$\frac{\delta \mathcal{L}}{\delta u} = \frac{\partial \mathcal{L}}{\partial u} - D_i \left(\frac{\partial \mathcal{L}}{\partial u_i}\right) + D_i D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}}\right) + \cdots$$

and it should give Equation (2.1).

It is easily seen that Equation (2.1) has a Lagrangian of the form

$$\mathcal{L} = -u_t^2 + u_{xx}^2 + \lambda u^2 + 2\int \sigma(u_x) du_x.$$

The operators X_1 and X_2 satisfy the condition (3.16). For the symmetry $X_1 = \frac{\partial}{\partial t}$ it follows from Equation (3.17) that

$$C^1 = u_{xx}^2 - u_t^2 + \lambda u^2 + 2\int \sigma(u_x) du_x + 2u_t^2$$

and

$$C^2 = -u_t[2\sigma(u_x) - D_x(2u_{xx})] - 2u_{tx}u_{xx}.$$

The conserved vector (C^1, C^2) is

$$\begin{cases} C^1 = u_{xx}^2 + u_t^2 + \lambda u^2 + 2\int \sigma(u_x) du_x \\ C^2 = 2u_t u_{xxx} - 2u_t \sigma(u_x) - 2u_{tx}u_{xx}. \end{cases} \tag{3.18}$$

We check whether the conserved vector satisfies Equation (3.15) and obtain the following:

$$\left[2u_t\left(u_{tt} + u_{xxxx} + \lambda u - u_{xx}\sigma'(u_x)\right)\right]_{(2.1)} = 0.$$

For the symmetry $X_2 = \frac{\partial}{\partial x}$ the formulae (3.17) yield:

$$C^1 = 2u_t u_x$$

and

$$C^2 = u_{xx}^2 - u_t^2 + \lambda u^2 + 2\int \sigma(u_x) du_x$$
$$-u_x[2\sigma(u_x) - D_x(2u_{xx})] - 2u_{xx}^2.$$

Hence, the conserved vector (C^1, C^2) has the components

$$\begin{cases} C^1 = 2u_x u_t \\ C^2 = -u_{xx}^2 - u_t^2 + \lambda u^2 + 2u_x u_{xxx} \\ \quad -2u_x\sigma(u_x) + 2\int \sigma(u_x) du_x \ . \end{cases} \tag{3.19}$$

After substitution in (3.15) we obtain

$$\left[2u_x\left(u_{tt} + u_{xxxx} + \lambda u - u_{xx}\sigma'(u_x)\right)\right]_{(2.1)} = 0.$$

The Noether theorem gives no conserved vectors for the symmetries X_3 and X_4 since the condition (3.16) or (3.16´) are not satisfied. Indeed,

$$X_3(\mathcal{L}) = 2\lambda u \cos\left(\sqrt{\lambda} t\right) + 2u_t\sqrt{\lambda} \sin\left(\sqrt{\lambda} t\right) \neq 0$$

and

$$X_4(\mathcal{L}) = 2\lambda u \sin\left(\sqrt{\lambda} t\right) - 2u_t\sqrt{\lambda} \cos\left(\sqrt{\lambda} t\right) \neq 0$$

The condition (3.16´) is also not satisfied.

Calculations based on Ibragimov´s theorem

In this case the adjoint equation inherits symmetries of the given equation. No extra conditions should be satisfied. Consider the formal Lagrangian

$$\mathcal{L} = v(u_{tt} + u_{xxxx} + \lambda u - u_{xx}\sigma'(u_x)).$$

The adjoint equation is

$$F^* \equiv v_{tt} + v_{xxxx} + \lambda v - \left(v_x \sigma'(u_x)\right)_x = 0.$$

Using the formal Lagrangian we obtain from the formulae (3.17) that

$$C^1 = \xi^1 \mathcal{L} - W D_t\left(\frac{\partial \mathcal{L}}{\partial u_{tt}}\right) + D_t(W)\frac{\partial \mathcal{L}}{\partial u_{tt}}$$
$$= \xi^1 \mathcal{L} - W D_t(v) + v D_t(W),$$

$$C^2 = \xi^2 \mathcal{L} + W\left[\frac{\partial \mathcal{L}}{\partial u_x} - D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right) - D_x^3\left(\frac{\partial \mathcal{L}}{\partial u_{xxxx}}\right)\right]$$
$$+D_x(W)\left[\frac{\partial \mathcal{L}}{\partial u_{xx}} + D_x^2\left(\frac{\partial \mathcal{L}}{\partial u_{xxxx}}\right)\right] - D_x^2(W) D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xxxx}}\right)$$
$$+D_x^3(W)\frac{\partial \mathcal{L}}{\partial u_{xxxx}}$$
$$= \xi^2 \mathcal{L} + W\left[-v u_{xx}\sigma''(u_x) - D_x\left(-v\sigma'(u_x)\right) - D_x^3(v)\right]$$
$$+D_x(W)\left[-v\sigma'(u_x) + D_x^2(v)\right] - D_x^2(W) D_x(v) + v D_x^3(W).$$

Thus,

$$C^1 = \xi^1 \mathcal{L} - W v_t + v D_t(W) \qquad (3.17')$$

$$C^2 = \xi^2 \mathcal{L} + W\left[v_x \sigma'(u_x) - v_{xxx}\right]$$
$$+D_x(W)\left[-v\sigma'(u_x) + v_{xx}\right] - v_x D_x^2(W) + v D_x^3(W).$$

In the general case Equation (2.1) is not self-adjoint, therefore only nonlocal conservation laws can be found with the new method. We also can ignore terms $\xi^1 \mathcal{L}$, $\xi^2 \mathcal{L}$, since they give a trivial conserved vector.
For the operator $X_1 = \frac{\partial}{\partial t}$ we have $W = -u_t$. Then from (3.17′) it follows that

$$C^2 = -u_t\left[v_x \sigma'(u_x) - v_{xxx}\right] - u_{tx}\left[-v\sigma'(u_x) + v_{xx}\right]$$
$$+u_{txx} v_x - u_{txxx} v.$$

Hence, the nonlocal conserved vector has the components

$$\begin{cases} C^1 = u_t v_t - u_{tt} v \\ C^2 = (v u_{tx} - u_t v_x)\sigma'(u_x) + u_t v_{xxx} \\ \quad -u_{tx} v_{xx} + u_{txx} v_x - u_{txxx} v. \end{cases} \tag{3.20}$$

Equation (3.15) gives

$$[u_t(v_{tt} + v_{xxxx} + \lambda v - [v_x \sigma'(u_x)]_x)]_{F=0,F^*=0} = 0.$$

Similarly, the nonlocal conserved vector for the symmetry $X_2 = \frac{\partial}{\partial x}$ with $W = -u_x$ is constructed as

$$C^1 = u_x v_t - u_{xt} v$$

$$C^2 = -u_x[v_x \sigma'(u_x) - v_{xxx}] - u_{xx}[-v\sigma'(u_x) + v_{xx}]$$
$$+u_{xxx} v_x - u_{xxxx} v.$$

Thus, the nonlocal conserved vector is

$$\begin{cases} C^1 = u_x v_t - u_{xt} v \\ C^2 = (u_{xx} v - u_x v_x)\sigma'(u_x) + u_x v_{xxx} \\ \quad +u_{xxx} v_x - u_{xx} v_{xx} - u_{xxxx} v. \end{cases} \tag{3.21}$$

Equation (3.15) gives

$$[u_x(v_{tt} + v_{xxxx} + \lambda v - [v_x \sigma'(u_x)]_x)]_{F=0,F^*=0} = 0.$$

For $X_3 = \cos(\sqrt{\lambda} t)\frac{\partial}{\partial u}$ we have $W = \cos(\sqrt{\lambda} t)$. From (3.17´) it follows that the nonlocal conserved vector has the components

$$\begin{cases} C^1 = -v_t \cos(\sqrt{\lambda} t) - v\sqrt{\lambda}\sin(\sqrt{\lambda} t) \\ C^2 = [v_x \sigma'(u_x) - v_{xxx}]\cos(\sqrt{\lambda} t). \end{cases} \tag{3.22}$$

Equation (3.15) gives

$$\left[-(v_{tt} + v_{xxxx} + \lambda v - [v_x \sigma'(u_x)]_x)\cos\left(\sqrt{\lambda}t\right)\right]_{F=0,F^*=0} = 0.$$

The operator $X_4 = \sin\left(\sqrt{\lambda}t\right)\frac{\partial}{\partial u}$ has $W = \sin\left(\sqrt{\lambda}t\right)$. Hence, the conserved vector has the components

$$\begin{cases} C^1 = -v_t \sin\left(\sqrt{\lambda}t\right) + \sqrt{\lambda}v\cos\left(\sqrt{\lambda}t\right) \\ C^2 = [v_x \sigma'(u_x) - v_{xxx}]\sin\left(\sqrt{\lambda}t\right). \end{cases} \quad (3.23)$$

Equation (3.15) gives

$$\left[-(v_{tt} + v_{xxxx} + \lambda v - [v_x \sigma'(u_x)]_x)\cos\left(\sqrt{\lambda}t\right)\right]_{F=0,F^*=0} = 0.$$

3.3.5 Conservation laws of Equation (3.3)

Calculations based on Ibragimov´s theorem

In this case $\sigma''(u_x) = 0, \sigma(u_x) = Au_x$. For the self-adjoint equation (3.3) the nonlocal conserved vectors are found by replacing $\sigma'(u_x)$ by the constant A in (3.20)-(3.21). The formal Lagrangian has the form
$\mathcal{L} = v(u_{tt} + u_{xxxx} + \lambda u - Au_{xx})$.
For the symmetry X_1 the nonlocal conserved vector is

$$\begin{cases} C^1 = u_t v_t - u_{tt} v \\ C^2 = u_t v_{xxx} - u_{tx} v_{xx} + u_{txx} v_x - u_{txxx} v + A(v u_{tx} - u_t v_x). \end{cases}$$

Since the equation is self-adjoint, a local conserved vector can be found by replacing v and its derivatives by u and the corresponding derivatives of u. This time we include $\mathcal{L}$ in the components C^1, C^2. Hence,

$$\begin{cases} C^1 = u_t^2 - u_{tt} u + \mathcal{L} = u_t^2 + u u_{xxxx} + \lambda u^2 - A u u_{xx} \\ C^2 = u_t u_{xxx} - u_{tx} u_{xx} + u_{txx} u_x - u_{txxx} u + A(u u_{tx} - u_t u_x). \end{cases}$$

We rewrite C^1 to the form $C^1 = \tilde{C}^1 + D_x(H)$, then $C^2 = C^2 + D_t(H)$. The conserved vector will have the components $\left(\tilde{C}^1, \tilde{C}^2\right)$. Indeed,

$$D_t(C_1) + D_x(C_2) = D_t\left(\tilde{C}^1 + D_t(H)\right) + D_x(C^2) =$$
$$= D_t\left(\tilde{C}^1\right) + D_x\left(C^2 + D_t(H)\right) = D_t\left(\tilde{C}^1\right) + D_x\left(\tilde{C}^2\right).$$

We have

$$uu_{xxxx} = D_x(uu_{xxx}) - u_x u_{xxx} = D_x(uu_{xxx} - u_x u_{xx}) + u_{xx}^2$$

$$Auu_{xx} = D_x(-Auu_x) + Au_x^2 .$$

Hence,

$$C^1 = u_t^2 + u_{xx}^2 + Au_x^2 + \lambda u^2 + D_x(uu_{xxx} - u_x u_{xx} - Auu_x)$$

$$\tilde{C}^2 = C^2 + D_t(uu_{xxx} - u_x u_{xx} - Auu_x) = 2u_t u_{xxx} - 2u_{tx} u_{xx} - 2Au_t u_x.$$

Thus, the conserved vector has the components

$$\begin{cases} C^1 = u_t^2 + u_{xx}^2 + Au_x^2 + \lambda u^2 \\ C^2 = 2u_t u_{xxx} - 2Au_t u_x - 2u_{xx} u_{tx}. \end{cases} \qquad (3.24)$$

Equation (3.15) gives

$$[2u_t(u_{tt} + u_{xxxx} + \lambda u - Au_{xx})]_{(3.3)} = 0.$$

For X_2 the nonlocal conserved vector is

$$\begin{cases} C^1 = u_x v_t - u_{xt} v \\ C^2 = Au_{xx} v - Au_x v_x + u_x v_{xxx} \\ \quad + u_{xxx} v_x - u_{xx} v_{xx} - u_{xxxx} v. \end{cases}$$

The corresponding local conserved vector is

$$\begin{cases} C^1 = u_x u_t - u_{xt} u \\ C^2 = Au_{xx}u - Au_x^2 + 2u_x u_{xxx} - u_{xx}^2 - u_{xxxx}u + \mathcal{L} \\ \quad = Auu_{xx} - Au_x^2 + 2u_x u_{xxx} - u_{xx}^2 - u_{xxxx}u \\ \quad +uu_{tt} + uu_{xxxx} + \lambda u^2 - Auu_{xx} \\ \quad = -Au_x^2 + 2u_x u_{xxx} - u_{xx}^2 + uu_{tt} + \lambda u^2. \end{cases}$$

It is possible to simplify it. We have

$$\begin{cases} C^1 = 2u_x u_t - D_x(u_t u) \\ C^2 = -Au_x^2 + 2u_x u_{xxx} - u_{xx}^2 + uu_{tt} + \lambda u^2 \end{cases}$$

then

$$\begin{cases} \tilde{C}^1 = 2u_x u_t \\ \tilde{C}^2 = -Au_x^2 + 2u_x u_{xxx} - u_{xx}^2 + uu_{tt} + \lambda u^2 - D_t(u_t u)\,. \end{cases}$$

Hence,

$$\begin{cases} C^1 = 2u_x u_t \\ C^2 = -u_{xx}^2 - u_t^2 - Au_x^2 + 2u_x u_{xxx} + \lambda u^2. \end{cases} \qquad (3.25)$$

Equation (3.15) gives

$$[2u_x(u_{tt} + u_{xxxx} + \lambda u - Au_{xx})]_{(3.3)} = 0.$$

The symmetry X_6 has $W = k(t,x)$. Hence, from (3.17´) the nonlocal conserved vector has the components

$$\begin{cases} C^1 = k_t v - k v_t \\ C^2 = k(Av_x - v_{xxx}) - k_x(Av - v_{xx}) - k_{xx}v_x + k_{xxx}v. \end{cases}$$

The corresponding local conserved vector is obtained by using $v = u$.

Then we have

$$\begin{cases} C^1 = k_t u - k u_t \\ C^2 = k(Au_x - u_{xxx}) - k_x(Au - u_{xx}) - k_{xx}u_x + k_{xxx}u. \end{cases} \tag{3.26}$$

Equation (3.15) gives

$$[u(k_{tt} + k_{xxxx} + \lambda k - Ak_{xx}) - k(u_{tt} + u_{xxxx} + \lambda u - Au_{xx})]_{(3.3),(3.11)} = 0$$

It is interesting to notice, that if k is replaced by u, $C^1 = 0, C^2 = 0$. Hence, for X_5 the component $C^1 = 0$, therefore X_5 does not give a conserved vector.

Calculations based on the Noether theorem

The conserved vectors constructed for X_1 and X_2 using Noether's theorem are exactly the same as the ones calculated with the new method. The calculations are not given here. In this case the Lagrangian is

$$\mathcal{L} = -u_t^2 + u_{xx}^2 + \lambda u^2 + Au_x^2.$$

The symmetry $X_6 = k(t,x)\frac{\partial}{\partial u}$ does not satisfy Equation (3.16) or (3.16´). Indeed,

$$X_6(\mathcal{L}) = -2k_t u_t + 2k_{xx}u_{xx} + 2\lambda k u + 2Ak_x u_x \neq 0$$

or $X_6(\mathcal{L}) \neq D_i(B^i)$.

Hence, for X_6 it is impossible to construct a conserved vector by using Noether´s theorem. Similarly, if k is replaced by u, we obtain

$$X_5(\mathcal{L}) = -2u_t^2 + 2u_{xx}^2 + 2\lambda u^2 + 2Au_x^2 \neq 0$$

or $X_5(\mathcal{L}) \neq D_i(B^i)$.

It is not possible to find a conserved vector for X_5 with the methods used in this thesis.

3.3.6 Conservation laws of Equation (3.4)

Calculations based on the Noether theorem

Equation (3.4), $u_{tt} + u_{xxxx} + \lambda u = 0$, is self-adjoint. In this case the Lagrangian is

$$\mathcal{L} = -u_t^2 + u_{xx}^2 + \lambda u^2, \quad \sigma'(u_x) = 0, \quad A = 0.$$

For the operator $X_1 = \frac{\partial}{\partial t}$ the components of the conserved vector are obtained from (3.24) by setting $A = 0$. Hence,

$$\begin{cases} C^1 = u_t^2 + u_{xx}^2 + \lambda u^2 \\ C^2 = 2u_t u_{xxx} - 2u_{tx} u_{xx}. \end{cases} \tag{3.27}$$

Equation (3.15) gives

$$[2u_t(u_{tt} + u_{xxxx} + \lambda u)]_{(3.4)} = 0.$$

Similarly, for the second symmetry $X_2 = \frac{\partial}{\partial x}$, the conserved vector is constructed by using (3.25):

$$\begin{cases} C^1 = 2u_t u_x \\ C^2 = -u_{xx}^2 - u_t^2 + 2u_x u_{xxx} + \lambda u^2. \end{cases} \tag{3.28}$$

Equation (3.15) gives

$$[2u_x(u_{tt} + u_{xxxx} + \lambda u)]_{(3.4)} = 0.$$

For the symmetries X_5 and X_6 the results are the same as for the previous equation. The conditions (3.16) or (3.16´) are not satisfied, no conserved vectors can be constructed.

Calculations based on Ibragimov´s theorem.

For the new method of constructing conservation laws presented in [3] the result is exactly the same for X_1 and X_2, therefore those calculations will not be repeated. But for the symmetry $X_6 = k(t,x)\frac{\partial}{\partial u}$ another local conserved vector is found by using (3.26) with $A = 0$:

$$\begin{cases} C^1 = k_t u - k u_t \\ C^2 = -k u_{xxx} + k_x u_{xx} - k_{xx} u_x + k_{xxx} u. \end{cases} \tag{3.29}$$

Equation (3.15) gives

$$[(u-k)(u_{tt} + u_{xxxx})]_{(3.4)} = 0.$$

Even here it is interesting to note that after replacing k by u the conserved vector is $(C^1, C^2) = (0{,}0)$. Hence, X_5 provides no conservation law.

3.3.7 Conservation laws of Equation (2.2)

Nonlocal conserved vectors

Equation (2.2), $u_{tt} + u_{xxxx} = \sigma(u)$, is not self-adjoint, therefore only the new method described in [3] can be used for finding nonlocal conservation laws. Recall that the formal Lagrangian is

$$\mathcal{L} = v\big(u_{tt} + u_{xxxx} - \sigma(u)\big)$$

and the adjoint equation is

$$F^* = v_{tt} + v_{xxxx} - v\sigma'(u) = 0.$$

For X_1 we obtain from (3.17) that

$$C^1 = -u_t[-D_t(v)] - u_{tt}(v),$$

$$C^2 = -u_t[-D_x^3(v)] - u_{tx}[D_x^2(v)] - u_{txx}[-D_x(v)] - u_{txxx}v.$$

Hence, the conserved vector is

$$\begin{cases} C^1 = u_t v_t - u_{tt} v, \\ C^2 = u_t v_{xxx} - u_{tx} v_{xx} + u_{txx} v_x - u_{txxx} v. \end{cases} \tag{3.30}$$

Equation (3.15) gives

$$[u_t(v_{tt} + v_{xxxx}) - v(u_{tt} + u_{xxxx})_t]_{F=0,\, F^*=0} = 0.$$

For the symmetry X_2 the components are

$$C^1 = -u_x[-D_t(v)] - u_{xt}v,$$

$$C^2 = -u_x[-D_x^3(v)] - u_{xx}[D_x^2(v)] - u_{xxx}[-D_x(v)] - u_{xxxx}v.$$

Thus, the conserved vector is

$$\begin{cases} C^1 = u_x v_t - u_{xt} v \\ C^2 = u_x v_{xxx} - u_{xx} v_{xx} + u_{xxx} v_x - u_{xxxx} v. \end{cases} \tag{3.31}$$

Equation (3.15) gives

$$[u_x(v_{tt} + v_{xxxx}) - v(u_{tt} + u_{xxxx})_x]_{F=0, F^*=0} = 0.$$

Local conserved vectors (The Noether theorem)

Conservation laws can also be constructed by using Noether's theorem. In this case the Lagrangian has the form

$$\mathcal{L} = u_{xx}^2 - u_t^2 - 2\int \sigma(u)du.$$

Then for X_1 the formulae (3.17) give

$$C^1 = u_{xx}^2 - u_t^2 - 2\int \sigma(u)du - u_t[-2u_t],$$

$$C^2 = -u_t[-D_x(2u_{xx})] - u_{tx}[2u_{xx}],$$

whence the conserved vector is

$$\begin{cases} C^1 = u_{xx}^2 + u_t^2 - 2\int \sigma(u)du \\ C^2 = 2u_t u_{xxx} - 2u_{tx}u_{xx}. \end{cases} \tag{3.32}$$

Equation (3.15) gives

$$\left[u_t\left(2u_{tt} + 2u_{xxxx} - 2\sigma(u)\right)\right]_{(2.2)} = 0.$$

For the symmetry X_2 the calculations result in the following:

$$C^1 = 2u_t u_x\ ,$$

$$C^2 = u_{xx}^2 - u_t^2 - 2\int \sigma(u)du - u_x[-D_x(2u_{xx})] - 2u_{xx}^2.$$

Hence, the conserved vector is

$$\begin{cases} C^1 = 2u_x u_t \\ C^2 = 2u_x u_{xxx} - u_{xx}^2 - u_t^2 - 2\int \sigma(u)du\,. \end{cases} \tag{3.33}$$

Equation (3.15) gives

$$[u_x(2u_{tt} + 2u_{xxxx} - 2\sigma(u))]_{(2.2)} = 0 .$$

3.3.8 Conservation laws of Equation (3.7)

The Equation (3.7), $u_{tt} + u_{xxxx} = u$, is self-adjoint. To construct the conserved vectors for the symmetries X_1 and X_2, $2\int \sigma(u)du$ is simply replaced with u^2 in the calculations made for Equation (2.2). The new method gives the same conservation laws. The conserved vector for X_1 is

$$\begin{cases} C^1 = u_{xx}^2 + u_t^2 - u^2 \\ C^2 = 2u_{xxx}u_t - 2u_{xx}u_{tx}. \end{cases} \tag{3.34}$$

Equation (3.15) gives

$$[2u_t(u_{tt} + u_{xxxx} - u)]_{(3.7)} = 0.$$

And for X_2

$$\begin{cases} C^1 = 2u_x u_t \\ C^2 = 2u_x u_{xxx} - u_{xx}^2 - u_t^2 - u^2. \end{cases} \tag{3.35}$$

Equation (3.15) gives

$$[2u_x(u_{tt} + u_{xxxx} - u)]_{(3.7)} = 0.$$

For X_5 the new method gives $(C^1, C^2) = (0{,}0)$. But for $X_6 = k(t,x)\frac{\partial}{\partial(u)}$, where $k(t,x)$ is any solution of the equation

$$k_{tt} + k_{xxx}u - \lambda u = 0 \tag{3.36}$$

we derive

$$C^1 = k[-D_t(v)] + k_t v$$

$$C^2 = k[-D_x^3(v)] + k_x[D_x^2(v)] + k_{xx}[-D_x(v)] + k_{xxx}v.$$

The conserved vector is

$$\begin{cases} C^1 = k_t u - k u_t \\ C^2 = -k u_{xxx} + k_x u_{xx} - k_{xx} u_x + k_{xxx} u. \end{cases} \quad (3.37)$$

Equation (3.15) gives

$$[-k(u_{tt} + u_{xxxx}) + u(k_{tt} + k_{xxxx})]_{(3.7),(3.36)} = 0.$$

With the method based on Noether's theorem it is not possible to construct a conserved vector for X_5 and X_6, because they do not satisfy Equation (3.16) or (3.16´).

3.3.9 Conservation laws of Equation (3.6)

Local conserved vectors (The Noether theorem)

For Equation (3.6), $u_{tt} + u_{xxxx} = 0$, the conserved vectors are also found by replacing $\int \sigma(u)du$, but this time it is just replaced by zero. In this case the Lagrangian is

$$\mathcal{L} = u_{xx}^2 - u_t^2.$$

For X_1 and X_2 the conserved vectors are the same for both methods. For X_1 the result is

$$\begin{cases} C^1 = u_{xx}^2 + u_t^2 \\ C^2 = 2u_{xxx}u_t - 2u_{xx}u_{tx}. \end{cases} \quad (3.38)$$

Equation (3.15) gives

$$[2u_t(u_{tt} + u_{xxxx})]_{(3.6)} = 0.$$

For X_2 the conserved vector has the components

$$\begin{cases} C^1 = 2u_x u_t \\ C^2 = 2u_{xxx}u_x - u_{xx}^2 - u_t^2. \end{cases} \tag{3.39}$$

Equation (3.15) gives

$$[2u_x(u_{tt} + u_{xxxx})]_{(3.6)} = 0.$$

The results for $X_5 = u\frac{\partial}{\partial u}$ and $X_6 = k(t,x)\frac{\partial}{\partial u}$ are exactly the same as for (3.7), X_5 provides no conservation law, the conserved vector for X_6 has the components (3.37), however, $k(t,x)$ is any solution of the equation

$$k_{tt} + k_{xxxx} = 0. \tag{3.40}$$

Equation (3.6) has the additional symmetry X_7. The conditions (3.16) or (3.16´) are not satisfied for this operator, the Noether theorem cannot be applied.

X_7 has the form

$$X_7 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u_t\frac{\partial}{\partial u_t} - 2u_{xx}\frac{\partial}{\partial u_{xx}}.$$

Invoking (3.16) we obtain

$$X(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = -u_{xx}^2 + u_t^2,$$

hence, neither the condition (3.16), nor (3.16´) are satisfied. Thus, X_7 gives no conservation law.

Local conserved vectors (Ibragimov´s theorem)

Let us use the new method now. In this case we have

$$W = -2tu_t - xu_x\ ,$$

the formal Lagrangian is

$$\mathcal{L} = v(u_{tt} + u_{xxxx}).$$

Then it follows from (3.17) that

$$C^1 = 2t\mathcal{L} + (-2tu_t - xu_x)[-D_t(v)] + (-2u_t - 2tu_{tt} - xu_{xt})v,$$

$$\begin{aligned}C^2 &= x\mathcal{L} + (-2tu_t - xu_x)[-D_x^3(v)] + (-2tu_{tx} - u_x - xu_{xx})[D_x^2(v)] \\ &+(-2tu_{txx} - 2u_{xx} - xu_{xxx})[-D_x(v)] + (-2tu_{txxx} - 3u_{xxx} - xu_{xxxx})v.\end{aligned}$$

Since the equation is self-adjoint, we can replace v by u. Hence,

$$\begin{cases} C^1 = 2t\mathcal{L} + (2tu_t + xu_x)u_t - (2u_t + 2tu_{tt} + xu_{tx})u \\ C^2 = x\mathcal{L} + (2tu_t + xu_x)u_{xxx} - (2tu_{tx} + u_x + xu_{xx})u_{xx} \\ +(2tu_{txx} + 2u_{xx} + xu_{xxx})u_x - (2tu_{txxx} + 3u_{xxx} + xu_{xxxx})u. \end{cases}$$

It is possible to simplify the conserved vector.

We have,

$$\begin{cases} C^1 = 2tu_t^2 + xu_xu_t - 2u_tu + 2tu_{xxxx}u - xu_{xt}u \\ C^2 = 2tu_tu_{xxx} + 2xu_xu_{xxx} - 2tu_{tx}u_{xx} + u_xu_{xx} + 2tu_{txx}u_x \\ -2tu_{txxx}u - 3u_{xxx}u + xuu_{tt} - xu_{xx}^2. \end{cases}$$

Then,

$$\begin{cases} C^1 = 2tu_{xx}^2 + 2tu_t^2 + 2xu_xu_t - u_tu + D_x[2t(uu_{xxx} - u_xu_{xx}) - xu_tu] \\ C^2 = 2tu_tu_{xxx} + 2xu_xu_{xxx} - 2tu_{tx}u_{xx} + u_xu_{xx} + 2tu_{txx}u_x \\ -2tu_{txxx}u - 3u_{xxx}u + xuu_{tt} - xu_{xx}^2. \end{cases}$$

Hence,

$$\begin{cases} \tilde{C}^1 = 2tu_{xx}^2 + 2tu_t^2 + 2xu_xu_t - u_tu \\ \tilde{C}^2 = 2tu_tu_{xxx} + 2xu_xu_{xxx} - 2tu_{tx}u_{xx} + u_xu_{xx} + 2tu_{txx}u_x \\ -2tu_{txxx}u - 3u_{xxx}u + xuu_{tt} - xu_{xx}^2 + D_t[2t(uu_{xxx} - u_xu_{xx}) - xu_tu]. \end{cases}$$

Thus,

$$\begin{cases} C^1 = 2t(u_t^2 + u_{xx}^2) + 2xu_xu_t - u_tu \\ C^2 = 4tu_tu_{xxx} + 2xu_xu_{xxx} - 4tu_{tx}u_{xx} \\ -u_xu_{xx} - u_{xxx}u - x(u_{xx}^2 + u_t^2). \end{cases} \tag{3.41}$$

Equation (3.15) gives

$$[(4tu_t + 2xu_x - u)(u_{tt} + u_{xxxx})]_{(3.6)} = 0.$$

4. Summary

4.1 Results for Equation (2.1)

Self-adjointness of Equation (2.1)

Equation (2.1),

$$u_{tt} + u_{xxxx} + \lambda u = u_{xx}\sigma'(u_x), \quad \lambda = \text{const.} > 0,$$

is generally not self-adjoint. It is self-adjoint only in two special cases, namely,

$$u_{tt} + u_{xxxx} + \lambda u = Au_{xx} \qquad (3.3)$$

and

$$u_{tt} + u_{xxxx} + \lambda u = 0. \qquad (3.4)$$

Symmetries of Equation (2.1)

Equation (2.1) admits the symmetries

$$X_1 = \frac{\partial}{\partial t}\ , \quad X_2 = \frac{\partial}{\partial x}\ , \quad X_3 = \cos\left(\sqrt{\lambda}t\right)\frac{\partial}{\partial u}, \quad X_4 = sin\left(\sqrt{\lambda}t\right)\frac{\partial}{\partial u}.$$

Equations (3.3) and (3.4) admit two additional symmetries

$$X_5 = u\frac{\partial}{\partial u}\ , \quad X_6 = k(t,x)\frac{\partial}{\partial u},$$

where $k(t,x)$ satisfies the equation

$$k_{tt} + k_{xxxx} + \lambda k - Ak_{xx} = 0 \qquad (3.11)$$

or

$$k_{tt} + k_{xxxx} + \lambda k = 0 \qquad (3.12)$$

for Equations (3.3) and (3.4), respectively. X_3 and X_4 are particular cases of X_6.

Conservation laws of equation (2.1)

Calculations based on the Noether theorem

Equation (2.1) has the Lagrangian

$$\mathcal{L} = -u_t^2 + u_{xx}^2 + \lambda u^2 + 2\int \sigma(u_x)du_x.$$

The following conserved vectors are constructed:

$$X_1: \begin{cases} C^1 = u_{xx}^2 + u_t^2 + \lambda u^2 + 2\int \sigma(u_x)du_x \\ C^2 = 2u_t u_{xxx} - 2u_t\sigma(u_x) - 2u_{tx}u_{xx}\ ; \end{cases} \tag{3.18}$$

$$X_2: \begin{cases} C^1 = 2u_x u_t \\ C^2 = -u_{xx}^2 - u_t^2 + \lambda u^2 + 2u_x u_{xxx} \\ \quad -2u_x\sigma(u_x) + 2\int \sigma(u_x)du_x\ . \end{cases} \tag{3.19}$$

Calculations based on Ibragimov´s theorem

The formal Lagrangian is

$$\mathcal{L} = v(u_{tt} + u_{xxxx} + \lambda u - u_{xx}\sigma'(u_x)).$$

The adjoint equation is

$$F^* \equiv v_{tt} + v_{xxxx} + \lambda v - \left(v_x\sigma'(u_x)\right)_x = 0.$$

In the general case it is only possible to obtain nonlocal conserved vectors for Equation (2.1). They are

$$X_1:\begin{cases} C^1 = u_t v_t - u_{tt} v \\ C^2 = (v u_{tx} - u_t v_x)\sigma'(u_x) + u_t v_{xxx} \\ \quad -u_{tx} v_{xx} + u_{txx} v_x - u_{txxx} v \,; \end{cases} \tag{3.20}$$

$$X_2:\begin{cases} C^1 = u_x v_t - u_{xt} v \\ C^2 = (u_{xx} v - u_x v_x)\sigma'(u_x) + u_x v_{xxx} \\ \quad +u_{xxx} v_x - u_{xx} v_{xx} - u_{xxxx} v \,; \end{cases} \tag{3.21}$$

$$X_3:\begin{cases} C^1 = -v_t \cos\left(\sqrt{\lambda} t\right) - v\sqrt{\lambda} \sin\left(\sqrt{\lambda} t\right) \\ C^2 = [v_x \sigma'(u_x) - v_{xxx}] \cos\left(\sqrt{\lambda} t\right) \,; \end{cases} \tag{3.22}$$

$$X_4:\begin{cases} C^1 = -v_t \sin\left(\sqrt{\lambda} t\right) + \sqrt{\lambda} v \cos\left(\sqrt{\lambda} t\right) \\ C^2 = [v_x \sigma'(u_x) - v_{xxx}] \sin\left(\sqrt{\lambda} t\right). \end{cases} \tag{3.23}$$

For the self-adjoint version (3.3) of Equation (2.1) it is possible to obtain local conserved vectors for all its symmetries except X_5. They are

$$X_1:\begin{cases} C^1 = u_t^2 + u_{xx}^2 + A u_x^2 + \lambda u^2 \\ C^2 = 2u_t u_{xxx} - 2A u_t u_x - 2u_{xx} u_{tx} \,; \end{cases} \tag{3.24}$$

$$X_2:\begin{cases} C^1 = 2u_x u_t \\ C^2 = -u_{xx}^2 - u_t^2 - A u_x^2 + 2u_x u_{xxx} + \lambda u^2 \,; \end{cases} \tag{3.25}$$

$$X_6:\begin{cases} C^1 = k_t u - k u_t \\ C^2 = k A u_x - k u_{xxx} - k_x A u \\ \quad +k_x u_{xx} - k_{xx} u_x + k_{xxx} u, \end{cases} \tag{3.26}$$

where $k(t,x)$ satisfies the equation $k_{tt} + k_{xxxx} + \lambda k - A k_{xx} = 0$.
For the other self-adjoint version, (3.4), of the equation (2.1) it is also possible to obtain local conserved vectors for all its symmetries except X_5. They are

$$X_1:\begin{cases} C^1 = u_t^2 + u_{xx}^2 + \lambda u^2 \\ C^2 = 2u_t u_{xxx} - 2u_{tx} u_{xx} \,; \end{cases} \tag{3.27}$$

$$X_2 : \begin{cases} C^1 = 2u_t u_x \\ C^2 = -u_{xx}^2 - u_t^2 + 2u_x u_{xxx} + \lambda u^2 \ ; \end{cases} \tag{3.28}$$

$$X_6 : \begin{cases} C^1 = k_t u - k u_t \\ C^2 = -k u_{xxx} + k_x u_{xx} - k_{xx} u_x + k_{xxx} u \end{cases} \tag{3.29}$$

where $k(t, x)$ satisfies the equation $k_{tt} + k_{xxxx} + \lambda k = 0$.

4.2 Results for Equation (2.2)

Self-adjointness of Equation (2.2)

Equation (2.2),

$$u_{tt} + u_{xxxx} = \sigma(u),$$

is generally not self-adjoint. It is self-adjoint only in two special cases, namely,

$$u_{tt} + u_{xxxx} = u \tag{3.7}$$

and

$$u_{tt} + u_{xxxx} = 0. \tag{3.8}$$

Symmetries of Equation (2.2)

Equation (2.2) admits the symmetries

$$X_1 = \frac{\partial}{\partial t}\ , \quad X_2 = \frac{\partial}{\partial x}\,.$$

Equation (3.7) has the additional symmetries

$$X_5 = u\frac{\partial}{\partial u}\,, \quad X_6 = k(t,x)\frac{\partial}{\partial u}\,,$$

where $k(t,x)$ is any solution of the equation $k_{tt} + k_{xxxx} - k = 0$. Equation (3.6) has the additional symmetries X_5, X_6, where $k(t,x)$ is any solution of the equation $k_{tt} + k_{xxxx} = 0$, and

$$X_7 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\,.$$

Conservation laws of Equation (2.2)

Calculations based on the Noether theorem

Equation (2.2) has the Lagrangian

$$\mathcal{L} = u_{xx}^2 - u_t^2 - 2\int \sigma(u)du.$$

The following conserved vectors are constructed:

$$X_1\colon \begin{cases} C^1 = u_{xx}^2 + u_t^2 - 2\int \sigma(u)du \\ C^2 = 2u_t u_{xxx} - 2u_{tx}u_{xx}\,; \end{cases} \tag{3.32}$$

$$X_2\colon \begin{cases} C^1 = 2u_x u_t \\ C^2 = 2u_x u_{xxx} - u_{xx}^2 - u_t^2 - 2\int \sigma(u)du\,. \end{cases} \tag{3.33}$$

Calculations based on Ibragimov´s theorem

The formal Lagrangian is

$$\mathcal{L} = v(u_{tt} + u_{xxxx} - \sigma(u)).$$

The adjoint equation is

$$F^* \equiv v_{tt} + v_{xxxx} - v\sigma'(u) = 0.$$

In the general case it is only possible to obtain nonlocal conserved vectors for Equation (2.2). They are

$$X_1 : \begin{cases} C^1 = u_t v_t - u_{tt} v \\ C^2 = u_t v_{xxx} - u_{tx} v_{xx} + u_{txx} v_x - u_{txxx} v \, ; \end{cases} \quad (3.30)$$

$$X_2 : \begin{cases} C^1 = u_x v_t - u_{xt} v \\ C^2 = u_x v_{xxx} - u_{xx} v_{xx} + u_{xxx} v_x - u_{xxxx} v. \end{cases} \quad (3.31)$$

For the self-adjoint version (3.7) of Equation (2.2) it is possible to obtain local conserved vectors for all its symmetries except X_5. They are

$$X_1 : \begin{cases} C^1 = u_{xx}^2 + u_t^2 - u^2 \\ C^2 = 2u_{xxx} u_t - 2u_{xx} u_{tx} \, ; \end{cases} \quad (3.34)$$

$$X_2 : \begin{cases} C^1 = 2u_x u_t \\ C^2 = 2u_x u_{xxx} - u_{xx}^2 - u_t^2 - u^2 \, ; \end{cases} \quad (3.35)$$

$$X_6 : \begin{cases} C^1 = k_t u - k u_t \\ C^2 = -k u_{xxx} + k_x u_{xx} - k_{xx} u_x + k_{xxx} u, \end{cases} \quad (3.37)$$

where $k(t, x)$ satisfies the equation $k_{tt} + k_{xxxx} - k = 0$. For the other self-adjoint version, (3.6), it is also possible to find local conserved vectors for all its symmetries except X_5. They are

$$X_1 : \begin{cases} C^1 = u_{xx}^2 + u_t^2 \\ C^2 = 2u_{xxx} u_t - 2u_{xx} u_{tx} \, ; \end{cases} \quad (3.38)$$

$$X_2 : \begin{cases} C^1 = 2u_x u_t \\ C^2 = 2u_{xxx} u_x - u_{xx}^2 - u_t^2 \, ; \end{cases} \quad (3.39)$$

$$X_6 : \begin{cases} C^1 = k_t u - k u_t \\ C^2 = -k u_{xxx} + k_x u_{xx} - k_{xx} u_x + k_{xxx} u \,, \end{cases} \tag{3.37}$$

where $k(t, x)$ satisfies the equation

$$k_{tt} + k_{xxxx} = 0. \tag{3.40}$$

$$X_7 : \begin{cases} C^1 = 2t(u_t^2 + u_{xx}^2) + 2x u_x u_t - u_t u \\ C^2 = 4t u_t u_{xxx} + 2x u_x u_{xxx} - 4t u_{tx} u_{xx} \\ \quad -u_x u_{xx} - u_{xxx} u - x(u_{xx}^2 + u_t^2). \end{cases} \tag{3.41}$$

It is interesting to notice that the self-adjointness of the equations allows to derive more conservation laws than it is possible to calculate using only Noether´s theorem.

5. References

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