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Choquet and Sugeno Integrals

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DEDICATION

To my Brother Ilyas Khan (more than my life), Parents and family, who is everything for me.

ABSTRACT

In real word many problems, most of criteria have interdependent or interactive characteristics, which cannot be evaluated by additive measures exactly .For the human subjective evaluation processes it will be more better to apply Choquet and Sugeno integrals model together with the definition of $\lambda -$fuzzy measure, in which the property of additivity is not necessary. My thesis presents the application of fuzzy integrals as tool for criteria aggregation in the decision problems.

Finally, this research gives the examples of evaluating medicine with illustrations of hierarchical structure of $\lambda -$Fuzzy measure for Choquet and Sugeno integrals model.

Keywords: Choquet and Sugeno integrals, $\lambda -$***Fuzzy measure***

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Introduction

Our tools for information of aggregation are the weighted average method for example, a linear integral or Lebesque integral (1966). These methods consider that the information sources involved are non-interactive or independent and, hence their weighted effects are viewed as additivity, but in real world in many problems ,this approach is not realistic.

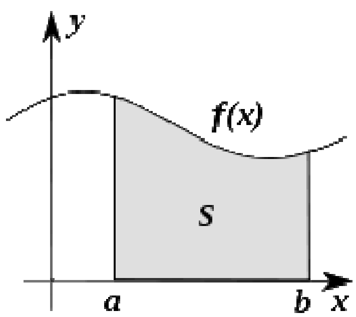
For the development of these problems, Sugeno (1974) introduced the concept of fuzzy measure and fuzzy integral, Sugeno replaced the additively requirement of normal (classical) measures with weaker requirement of monotonic (w.r.t set conclusion) and continuity. Sugeno gave $\lambda-$ *fuzzy measures* that satisfying the $\lambda-$*additive* axiom and it is the particular case of fuzzy measure. $\lambda-$ *fuzzy measures*has great importance in practical applications. $\lambda-$ *fuzzy measures* has many applications in artificial intelligence, neural network, image processing etc. A quite different definition was proposed by Sugeno and Murofushi (1989& 1991), using a functional defined by Choquet in capacity theory (Choquet 1953).The Choquet integral is based on $\lambda-$ *fuzzy measures* is provided the computational scheme for aggregation information according to Chen(1998).[4]The definition of Sugeno integral based on max and min ,the $\min-\max$ integral calculation can only determine some interval at which the measure values are possibly located, on the hand the unique solution is obtained if the Choquet integral is used.

Chapter 1: Riemann and Lebesque Integrals

Riemann Integral

Overview

Let $f(x)$ be a non-negative function of the interval $[a, b]$ and let s be the region under the function $f(x)$ .We want to find the area of s.

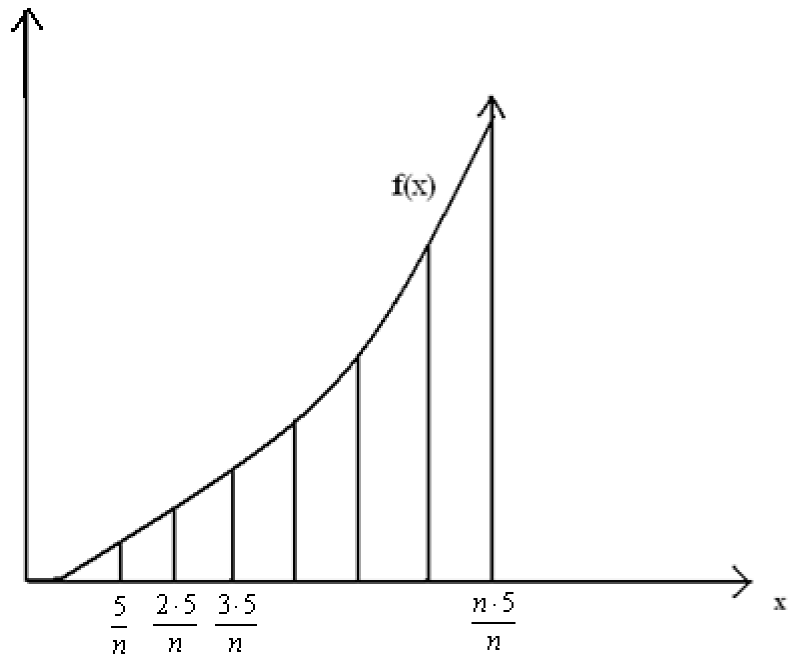


$$Area = \int_a^b f(x)dx$$

The basic idea of Riemann integral is to use approximations for area s. We will take better and better approximations.

Example 1.1: Find the Riemann sums and integral if $f(x) = x^2$ in the interval $[0,5]$

Solution: Suppose that we divide the interval $[0,5]$ into n equal subintervals



$$\Delta x = \Delta x_i = \frac{5-0}{n} = \frac{5}{n} \text{ , where } \quad i = 1,2,3,........,n$$

$$f(c_i) = i\Delta x_i \text{, Where } i = 1,2,3,........,n$$

Now we introduce the Riemann sum

$$Area = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

$$Area = \frac{5}{n}\left[\left(\frac{5}{n}\right)^2 + \left(\frac{2\cdot 5}{n}\right)^2 +,..................,+\left(\frac{n.5}{n}\right)^2\right]$$

$$Area = \frac{5}{n}\left(\frac{5}{n}\right)^2 \left(1^2 + 2^2 + 3^2 +,............,+n^2\right)$$

$$Area = \frac{5^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$Area = \frac{125}{6}\left[\frac{n(n+1)(2n+1)}{n^3}\right]$$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \frac{125}{6}\left[\frac{n(n+1)(2n+1)}{n^3}\right]$$

This is a Riemann sums

Now we introduce the Riemann integral

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i = \frac{125}{6}\lim_{n\to\infty}\left[\frac{n(n+1)(2n+1)}{n^3}\right]$$

$$\int_a^b f(x)dx = \frac{125}{3}$$

Example 1.2: $f(x) = x^2$ in the interval $[0,5]$

Suppose that we divide the interval $[0,5]$ into five rectangles

From the above example we know that

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \frac{125}{6}\left[\frac{n(n+1)(2n+1)}{n^3}\right]$$

So we take $n = 5$

$$Area = \sum_{i=1}^{5} f(c_i)\Delta x_i = 55$$

$$Area = \int_0^5 x^2 dx = 41.666$$

$$Error = (Actual - approximate)\cdot 100 \text{ Percent}$$

$$Error = 13.333 \text{Percent}$$

Now if we divide the interval $[0,5]$ into 10 rectangles then

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \frac{125}{6}\left[\frac{n(n+1)(2n+1)}{n^3}\right]$$

So we will take $n = 10$

$$Area = \sum_{i=1}^{10} f(c_i)\Delta x_i = 48.125$$

$$Error = (Actual - approximate) \cdot 100 \text{ Percent}$$

$$Error = 6.4584 \text{ Percent}$$

According to the above example which explains the idea of Riemann sums, we observe that when the area becomes smaller and smaller we will get better condition of approximation.

1.1 Definition of Partition of an Interval

A partition P of an interval [a, b] is made by means of a sequence $(x_0, x_1, x_2, ..., x_n)$ such that $a = x_0 \le x_2, ..., \le x_n = b$, the partition P divides the interval [a, b] into n subintervals $[x_{i-1}, x_i] = I_i$, where i = 1,2,3,……, n. We can call these the ith subintervals.

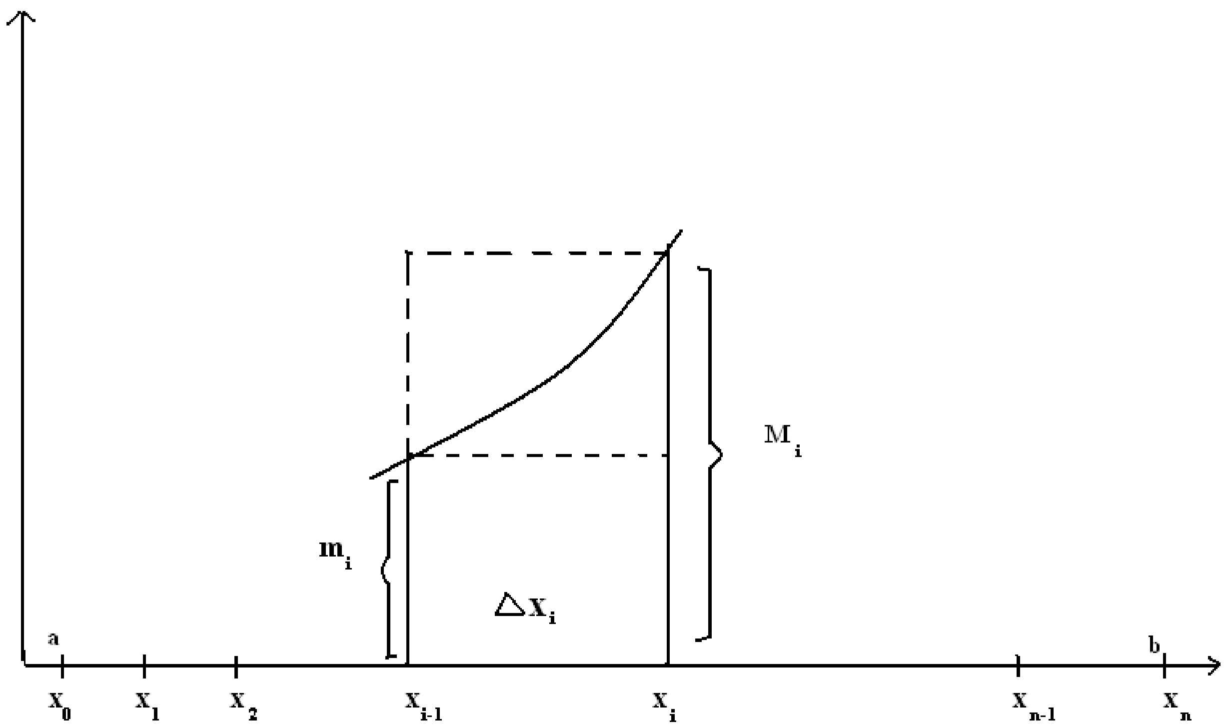
Let $f : [a, b] \to R$ be a bounded function and let P be the partition of [a, b]. Let $S \in P$ and suppose

$M_S(f) = \sup\{f(x) : x \in S\}$,

$m_S(f) = \inf\{f(x) : x \in S\}$.

If $S = I_i$ then we can write $M_i = M_S(f)$ and $m_i = m_S(f)$

Write $\Delta x_i = x_i - x_{i-1}$ the length of I_i



The lower sum of f for P is given by $L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} m_{I_i}(f) \Delta x_i$.

The upper sum of f for P is given by $U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=}^{n} M_{I_i}(f) \Delta x_i$

Since $m_i \le M_i$ for $1 \le i \le n$ so it will follow that

$L(f,P) \le U(f,P)$ for all bounded f and for all partitions P.

We approximate the area under the curve by choosing the partitions P progressively finer.

Definition

A partition P_1 is called refines of a partition P if every point of P is a point in P_1.

Example if $P_1 = (0,1,2,3,4,5)$ and $P = (0,2,3,5)$ then P_1 is called refines of P

Theorem1.1: If P_1 refines P then $L(f,P) \le L(f,P_1) \le U(f,P_1) \le U(f,P)$.

From this theorem follows that $\sup_p L(f,P) \le \inf_P U(f,P)$ (2)

Where the inf and sup are taken over all partitions P

Definition: A function $f:[a,b] \to R$ is a Riemann integrable if the inequality of equation (2) is valid for f. Now we are coming to the proper definition of Riemann sums and the integral.

1.2 Definition of Riemann Integral

Consider a continuous function f(x) between x=a and x=b , we want to divide the closed interval [a, b] into n intervals.

Let $x_0 = a$ and $x_n = b$ define $\Delta x_i = \dfrac{b-a}{n}$ and $\Delta x_i = x_{i+1} - x_i$ for $i = 1,2,3,.........,n,$ this is the partition of interval $[a,b]$ into n subintervals $[x_i, x_{i+1}]$ each with length $\Delta x.$

$$S_{n,upper} = \sum_{i=0}^{n} f(x_{i+1})\Delta x_{i+1}$$

$$S_{n,lower} = \sum_{i=0}^{n} f(x_i)\Delta x_i$$

$$S_{n,lower} \le \int_a^b f(x)dx \le S_{n,upper}$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\Delta x_i$$

1.3 Definition of Sigma-Algebra

Suppose X is a nonempty set and F is a the power set of a set X then F is called sigma-algebra if it has the following properties

1 F contains the set X as an element
2 If E is a subset of F then complement of E is also a subset of F.
3 The union of countable sets in F is also in F.

Example 1.3: $X = \{1,2,3\}$

$$F = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$
$$X \in F$$
$$\{1\} \in F \Rightarrow \{1,2\} \in F$$
$$\{1,2,3\} \cup \{2,3\} = \{1,2,3\} \in F$$

1.4 Definition of a Measureable Set

A measure m is a function defined on sigma-algebra F over a set X and taking values in the closed interval $[0, \infty]$ such that the following properties are satisfied.

1 $m(\phi) = 0$
2 If $E_1, E_2, E_3,$......... is a countable sequence of pair wise disjoint sets in F then
3 $m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$.

Then (X, F, m) is called measureable space and the members of F are called measureable sets.

1.5 Definition of Lebesque Integral

A new method was presented by Lebesque in 1902, as a new approach to the domain of the function in comparison the Riemann integral. He chose to make partition of the range. Thus, for each interval in the partition, rather than asking for the value of the function between the end points in the x-axis, we asked how much of the domain is mapped by the function to some value between two end points of the y-axis. In the Lebesque integral define we consider the closed interval [c, d] and the partition P of ranges such that $y_0 \le y_1 \le y_2 \le .... \le y_n$ .Let $y_0 = c$ and $y_n = d$ define as $\Delta y = \frac{d-c}{n}$ and $\Delta y_i = y_i - y_{i-1}$ so the upper and the lower sums will be defined as

$$L(f, P) = \sum_{i=1}^{n} \Delta y_i m(\{x : y_{i-1} \le f(x) < y_i\}$$

$$U(f,P)=\sum_{i=1}^{n}\Delta y_i m(\{x: y_{i-1}<f(x)\le y_i\}$$

$$L(f,P)\le U(f,P)$$

$$L(f,P)\le \int_E f(x)dx \le U(f,P)$$

$$\int_E f(x)dx=\lim_{n\to\infty}\sum_{i=1}^{n} y_i m(\{y_{i-i}\le f(x)\le y_i\})$$

1.6 Characteristic and Simple Function [15]

Let us consider any set A. The function

$$X_A(x)=\begin{cases}1 & \text{if } x\in A\\ 0 & \text{if } x\notin A\end{cases}$$

is called characteristic function of A .A finite linear combination of characteristic functions

$$s(x)=\sum a_i X_{E_i}(x)$$

is simple function if all sets E_i are measurable.

1.7 Definition of Lebesque Integral for a Simple Function [15]

If $s(x)=\sum a_n X_{A_n}(x)$ is simple function $m(A_n)$ and is finite for all n, then the Lebesque integral of s is defined as

$$\int s(x)dx=\sum a_n m(A_n)$$

If E is a measurable set then we define

$$\int_E s(x)dx=\int X_E(x)s(x)dx$$

Example1.4: If $f(x)=2$ over the closed interval $[2,3]$ the constant function can be written as a simple function

$$f(x)=2X_R(x).$$

Then the Lebesque integral over $[2,3]$ will be

$$\int_{[2,3]} f(x)dx = \int 2X_{[2,3]}(x)dx$$

$$= 2m([2,3])$$

$$= 2(3-2)$$

Theorem: If f is bounded on [a, b] such that f is Riemann integrable then it is Lebesque integrable on [a, b] but the reverse is not always true

Example1.5: Consider a Dirichlet function

$$g(x) = \begin{cases} 1, & \text{if } x = \text{rational} \\ 0, & \text{if } x = \text{irrational} \end{cases}$$

Then g (x) is Lebesque integrable but not Riemann integrable because

Let us consider a partition $P: 0 = x_0 < x_1 < x_2, ..., < x_n = 1$ of interval $[0,1]$ .Then it is easy to check that

SUP of $f(x) = 1 \quad x \in [x_{i-1}, x_i]$ and inf of $f(x) = 0 \quad x \in [x_{i-1} x_i]$

Hence the upper and lower Riemann sums of g(x) with respect to P satisfied

$U(P, f) = 1$ and $L(P, f) = 0$

then $g(x)$ is not Riemann integrable.

1.8 The General Definition of Lebesque Integral

Suppose that f(x) is a measureable function define on both positive and negative parts of $f(x)$. Then

$$f^+(x) = \max(f(x), 0).$$

$$f^-(x) = \max(-f(x), 0).$$

So we can write

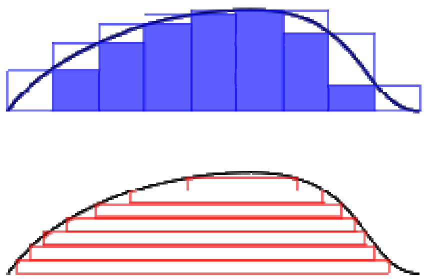
$$f = f^+ - f^-$$

Then the Lebesque integral will be

$$\int_E f(x)dx = \int_E f^+(x)dx - \int_E f^-(x)dx\ .$$

1.9 Riemann and Lebesque Approaches

[11]



Divide the x-axis into equal rectangles, measure the altitude, then take the area of each rectangle and then take the summation of all these rectangles in the blue figure.These indicate the Riemann integral,while in Lebesque integral we take the partition of the range as the red figure indicates.

Chapter 2: Fuzzy Set

2.1 Fuzzy Sets

If we use the expression ``a set`` then we can say that the collection of well defined distinct objects is called set. Define $A = \{a, b, c, d\}$, since we can count number of elements then A is called finite set .If we cannot count the number of elements then the set is called an infinite set for example $B = \{1, 3, 5, 7 \ldots\ldots\}$

Consider a function $\chi_A : X \to \{0,1\}$, which is called a characteristic function of the crisp set A if and only if $\forall\ x \in X$ we have

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

In fuzzy set theory the classical sets are called crisp sets for the reason that we can differentiate between fuzzy set and the classical set, in fuzzy set theory the above characteristic function is converted to a membership function if we assign every $x \in X$ a value from the unit interval $[0,1]$ instead of the set $\{0,1\}$.[2]

2.2 Definition of a Fuzzy Set

A set $A = \{(x, y) : (x, \mu_A(x)\}$, $x \in X$ is called a fuzzy set where $\mu_A(x)$ is the membership function of a fuzzy set A and is defined as $\mu_A : X \to [0,1]$, the value of $\mu_A(x)$ is called the membership degree of X .

Every element $x \in X$ has thus a membership degree $y = \mu_A(x) \in [0,1]$.

2.3 The Notation for Fuzzy Set

The discrete set is define as

$$A = \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} +, \ldots\ldots\ldots\ldots, + \frac{\mu_A(x_n)}{x_n}$$

where $x_1, x_2, \ldots\ldots\ldots, x_n$ are members of A

and $\mu_A(x_1), \mu_A(x_2), \ldots\ldots\ldots\ldots, \mu_A(x_n)$ are the membership degrees of $X_1, X_2, X_3, \ldots\ldots, X_n$.

The sign "+" has a symbolic character as a joint of the set's elements.

For continuous fuzzy set, a fuzzy set A can be defined as

$$A = \int_X \frac{\mu(x)}{x} \text{ or } \sum_{x \in X} \frac{\mu_A(x)}{x}$$

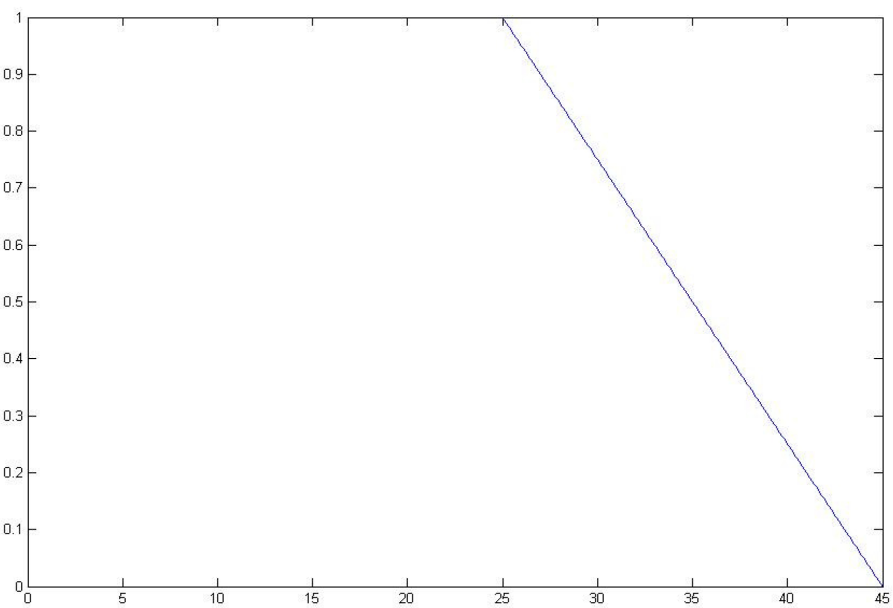
Example2.1: Consider the non-fuzzy finite set $\text{"young"} = \{15, 16, 18, 20, 25, 30, 35, 40, 45\}$

Let us decide the strength of the relationship between the set and each value belonging to its support.

$$\text{"young"} = \{(15,1),(16,1),(18,1),(20,1),(25,1),(30,0.7),(35,0.4),(40,0.1),(45,0)\}$$

We know that $A = \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \ldots\ldots\ldots\ldots + \frac{\mu_A(x_n)}{x_n}$ So

$$\text{"young"} = \frac{1}{15} + \frac{1}{16} + \frac{1}{18} + \frac{1}{20} + \frac{1}{25} + \frac{0.7}{30} + \frac{0.4}{35} + \frac{0.3}{40} + \frac{0}{45}$$



To find the membership function we will find the straight line equation from $(45,0)$ to $(25,1)$

Equation of straight line is $(y - y_1) = m(x - x_1)$

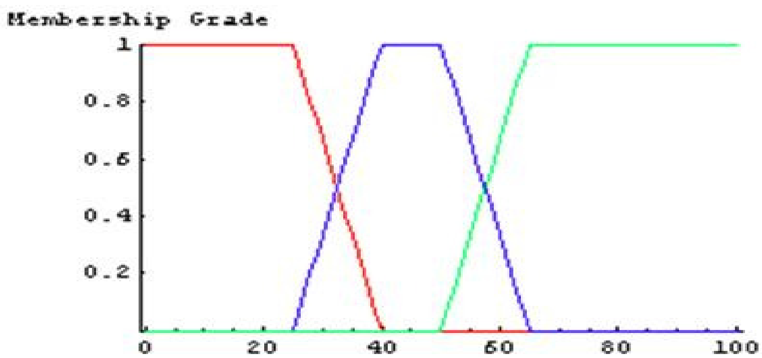
$$y = -\frac{1}{20}x + \frac{9}{4}$$

Membership Function

$$y = \mu_{"young"}(x) = 1 \text{ if } 0 \le x < 25 \quad \text{and} \quad y = \mu_{"young"}(x) = -\frac{1}{20}x + \frac{9}{4} \quad \text{if} \quad 25 \le x \le 45$$

By using the same way, we can also find the memberships for “middle-aged” and “old-aged” people.

Example 2.2: Let us consider three fuzzy sets of the universe X=[0, 100] in which we discern three groups of age ” young”, “middle-aged” and “old-aged” people.



The member ship function of “young”

The equation of a straight line from $(40,0)$ to $(25,1)$

Equation of straight line is $(y - y_1) = m(x - x_1)$

$$y = -\frac{1}{15}x + \frac{8}{3}$$

Membership function

$$y = \mu_{"young"}(x) = 1 \text{ if } 0 \le x < 25 \quad \text{and} \quad y = \mu_{"young"}(x) = -\frac{1}{15}x + \frac{8}{3} \quad \text{if} \quad 25 \le x \le 40$$

The membership function for “middle-aged”

The straight line equation from $(25,0)$ to $(40,1)$

$$y = \frac{1}{15}x - \frac{5}{3}$$

$y = \mu_{"middle-aged"}(x) = \frac{1}{15}x - \frac{5}{3}$ if $25 \le x \le 40$

Similarly we can find $y = \mu_{"middle-aged"}(x) = -\frac{1}{15}x + \frac{13}{3}$ if $50 \le x \le 65$

$y = \mu_{"middle-aged"}(x) = 1$ if $40 \le x \le 50$

The member ship function for "old-aged" people

The straight line equation from $(50,0)$ to $(65,1)$

$$y = \frac{1}{15}x - \frac{10}{3}$$

$y = \mu_{"old-aged"}(x) = \frac{1}{15}x - \frac{10}{3}$ if $50 \le x \le 65$ and $y = 1$ if $x \ge 65$

2.4 Special Continuous Membership Functions

Now we are introducing the formula for continuous functions, the s-class function $s(x, \alpha, \beta, \gamma)$ with parameter α, β and γ,

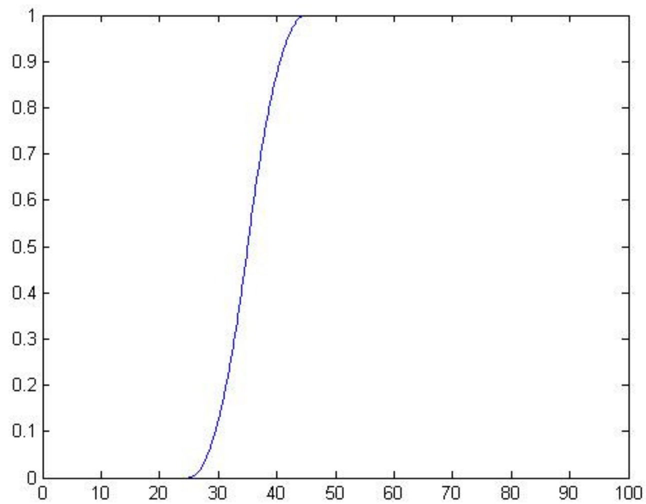
where $\beta = \frac{\alpha + \gamma}{2}$ and $\alpha < \beta < \gamma$

$$y = \mu_A(x) = s(x, \alpha, \beta, \gamma) = \begin{cases} 0 & \text{for } x \le \alpha, \\ 2\left(\frac{x-\alpha}{\gamma-\alpha}\right)^2 & \text{for } \alpha < x \le \beta, \\ 1 - 2\left(\frac{x-\gamma}{\gamma-\alpha}\right)^2 & \text{for } \beta < x \le \gamma, \\ 1 & \text{for } x > \gamma, \end{cases}$$

Example 2.3: Now we are using the above formula, draw the graph of the s-function

with $\alpha = 25$ and $\gamma = 45$

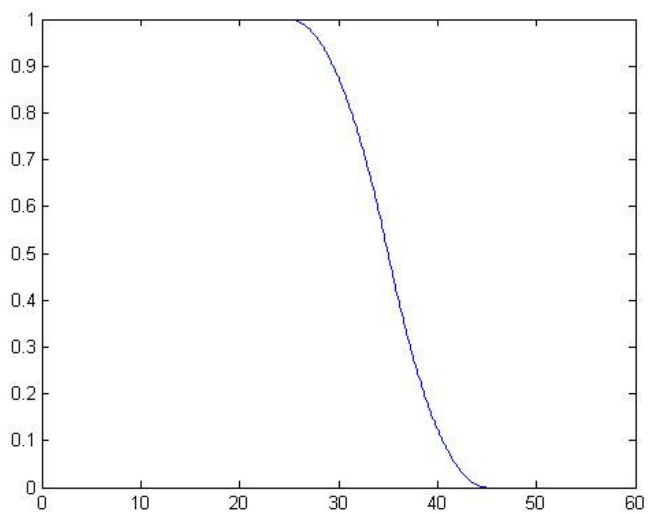
The graph of $s(X, 25, 35, 45)$ will be



Now we are drawing the membership function for "young" people, by utilizing the formula

$$\mu_{\text{"young"}}(x) = 1 - s\,(x, 25, 35, 45)$$

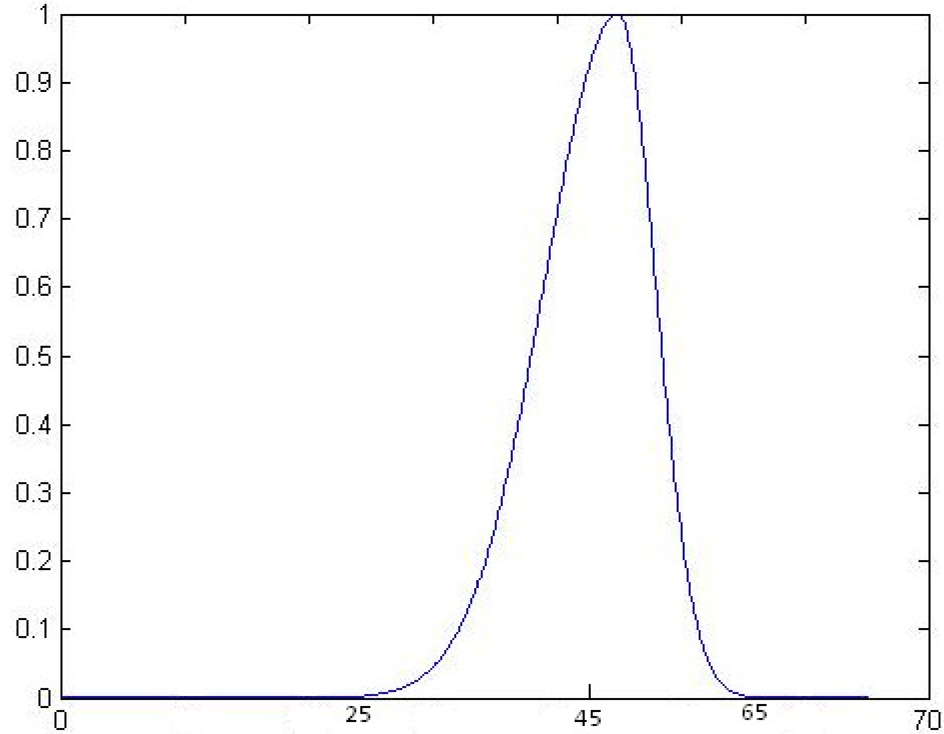
This means that the function will be constant up to 25 , then the function will be decreasing and at 45 reaches zero



Example 2.4: If we modify the above formula more, we will have the member ship function of "middle-aged" people

$$y = \pi(x,20,45) = \begin{cases} s(x,25,35,45) & \text{for} \quad x \le 45, \\ 1 - s(x,45,55,65) & \text{for} \quad x > 45. \end{cases}$$

The graph of the membership function of “middle-aged” people



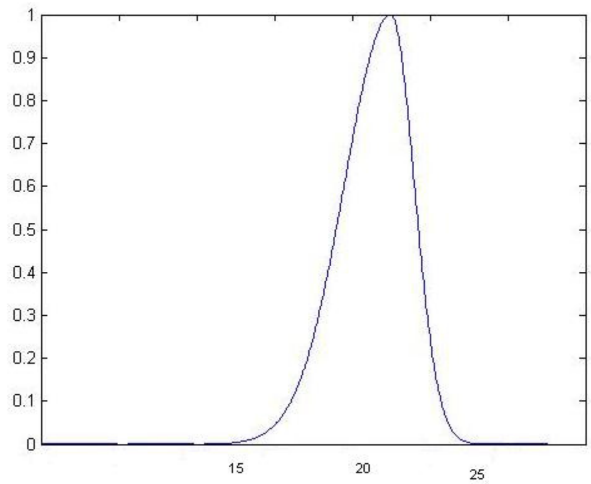
We can also construct the membership function for “old-aged”

Example2.5: A = "The real number closed to 20"

The member ship function will be $\mu_A(x) = \dfrac{1}{1+(x-20)^2}$

And the fuzzy set will be $A = \int_R \dfrac{\frac{1}{1+(x-20)^2}}{x}$ where "R" is a space of real numbers.

The graph will be



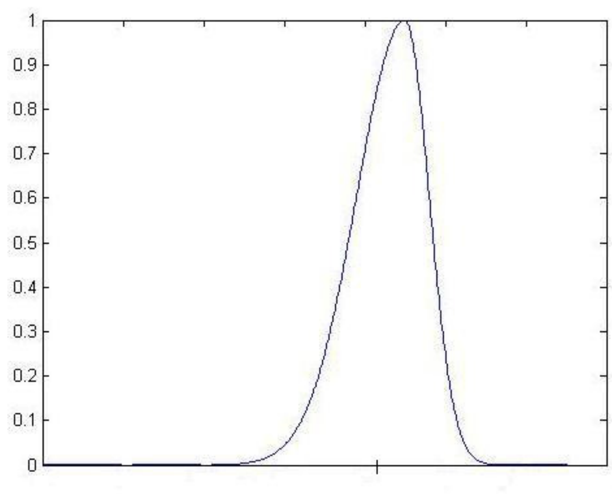
The general way of finding of the real number closed to "a"

Example2.6: "The real number closed to a"

The member ship function will be $\mu_A(x) = \dfrac{1}{1+(x-a)^2}$

And the fuzzy set will be $A = \int_R \dfrac{\overline{1+(x-a)^2}^{\,1}}{x}$ where "R" is a space of real numbers.

The grap of member ship function $\mu_A(X)$



Chapter 3: Fuzzy Measure Choquet and Sugeno Integrals

Fuzzy Measure

3.1 Measure

Measure is one of the most important concepts in mathematics, as well as the i-e,concept of integration w. r. t a given measure. In the classical definition of measure we use additive property. Additivity is very effective in many applications, but in many real world problems we do not require measure with respect to the additive feature, for example in fuzzy logic, artificial intelligence, data mining, decision making theory etc, a large amount of open problems for example the efficiency of a set of workers is being measured, we use the definition of non additive measure.

The fuzzy measure does not require additivity in most cases, in fuzzy measure we require monotonicity related to inclusion of sets.

3.2 Definition of Fuzzy Measure (Monotonic) [5]

Suppose that (X, F, μ) is a measureable space, a fuzzy measure is a function $\mu : F \rightarrow [0,\infty]$ such that the following properties are held

1 $\mu(\phi) = 0$

2 If $A, B \in F$ and $A \subseteq B$ then $\mu(A) \leq \mu(B)$

3 If $A_n \in F$ and $A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4$............ then $\lim_{n \to \infty} \mu(A_n) = \mu(\lim_{n \to \infty} A_n)$

3.3 Definition of the Additive Measure

Let us consider (X, F, m) to be a measure space .An additive measure m is a function $m : F \rightarrow [0,\infty]$ i-e defined on sigma-algebra F over a set X and taking values in the interval $[0,\infty]$ such that the following properties are satisfied

1 $m(\phi) = 0$

2 If $E_1, E_2, E_3, \ldots\ldots\ldots$ is a countable sequence of pair wise disjoint subsets of F then $m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$

Example of additive measure is a well known example of Lebesque measures, he generalize the concept of length of a segment.

3.4 Sugeno Fuzzy Measure [5]

A Sugeno fuzzy measure is a function $\mu : F \rightarrow [0,1]$ such that

1 $\mu(\phi)=0$

2 If $A \subseteq B$ then $\mu(A) \le \mu(B)$, If $A_n \in F$ and $A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4$............ then $\lim_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n)$

3.5 Definition of Possibility Measure (Zedeh, 1978)

Let (X, F, m) be a measureable space .A possibility measure is a function $m: F \to [0,1]$,satisfying the following condition

1 $m(\phi)=0$

2 $m(X)=1$

3 If $A \subseteq B$ then $m(A) \le m(B)$

4 $m(\bigcup_{i\in I} A_i) = \sup_{i\in I}\{m(A_i)\}$

Definition of $t-conorms$ **or** $s-norms$

$t-conorms$ or $s-norms$ are associative, commutative, and monotonic two-placed functions s that map from $[0,1]\times[0,1]$ into $[0,1]$.These properties are formulated with the following conditions:

1. $s(1,1)=1$; $s(\mu_{\bar{A}}(X),0) = s(0,\mu_{\bar{A}}(X)) = \mu_{\bar{A}}(X),\ x \in X$
2. $s(\mu_{\bar{A}}(X), s(\mu_{\bar{B}}(X)) \le s(\mu_{\bar{C}}(X), s(\mu_{\bar{D}}(X))$ if $s(\mu_{\bar{A}}(X) \le s(\mu_{\bar{C}}(X)$ and $s(\mu_{\bar{B}}(X) \le s(\mu_{\bar{D}}(X)$
3. $s(\mu_{\bar{A}}(X) \le s(\mu_{\bar{B}}(X)) = s(\mu_{\bar{B}}(X) \le s(\mu_{\bar{A}}(X))$
4. $s(\mu_{\bar{A}}(X), s((\mu_{\bar{B}}(X),(s(\mu_{\bar{C}}(X)) = (s(\mu_{\bar{A}}(X),(\mu_{\bar{B}}(X)),(\mu_{\bar{C}}(X))$

3.6 Definition of s-Decomposable Measure (Weber 1984)

Weber defined the s-Decomposable measure by giving a general concept *of* λ-fuzzy measure and the possibility measures.

Suppose that s is a t-conorm then for any measureable space (X, F, m), a s-decomposable measure is a function $m: F \to [0,1]$ such that

1 $m(\phi)=0$

2 $m(X)=1$

3 For all disjoint subsets $A, B \in F$, $m(AUB) = S(m(A), m(B))$

3.7 Definition of Sugeno λ-Fuzzy Measure

Let $X = \{x_1, x_2, x_3, \ldots\ldots\ldots x_n\}$ be a finite set and consider $\lambda \in (-1,\infty)$, an λ-measure is a function $g_\lambda : 2^X \to [0,1]$ such that it satisfied the following condition

1 $g_\lambda(X) = 1$

2 If $A, B \in 2^X$ then $g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B)$ with $A \cap B = \phi$

Moreover, let X be a finite set, $X = \{x_1, x_2, ..., x_n\}$ and $P(X)$ be the class of all subsets of X the fuzzy measure $g(X) = g_\lambda(\{x_1, x_2, ..., x_n\})$ can be formulated as (Leszczyński et al., 1985)

$$g_\lambda(\{x_1, x_2, ..., x_n\}) = \sum_{i=1}^{n} g_i + \lambda \sum_{i_1=1}^{n-1} \sum_{i_2=i_1+1}^{n} g_{i_1} . g_{i_2} + ... + \lambda^{n-1} g_1 . g_2 .... g_n$$

$$= \frac{1}{\lambda}\left[\prod_{i=1}^{n}(1+\lambda g_i) - 1\right] \text{ where } \lambda \in (-1, \infty)$$

$$g_\lambda(\{x_1, x_2, ..., x_n\})\lambda = \prod_{i=1}^{n}(1+\lambda g_i) - 1$$

$$g_\lambda(\{x_1, x_2, ..., x_n\})\lambda + 1 = \prod_{i=1}^{n} (1+\lambda g_i) \text{ by definition } g_\lambda(\{x_1, x_2, ..., x_n\}) = 1$$

And $\lambda + 1 = \prod_{i=1}^{n}(\lambda g_i + 1)$

Now we evaluate the value of λ

According to the fundamental theorem regarding the λ-fuzzy measure (Leszczyński et al., 1985), λ-value has three cases, as follows:

(i) If $\sum_{i=1}^{n} g_i > g(X)$ then $-1 < \lambda < 0$.

(ii) If $\sum_{i=1}^{n} g_i = g(X)$ then $\lambda = 0$.

(iii) If $\sum_{i=1}^{n} g_i < g(X)$ then $\lambda > 0$.

3.8 Definition (Discrete Case)

A fuzzy measure μ on finite set X is a function $\mu : 2^X \rightarrow [0,1]$ satisfying the following axioms

1 $\mu(\phi) = 0$

2 If $A \subseteq B$ then $\mu(A) \leq \mu(B)$

3 $\mu(X) = 1$

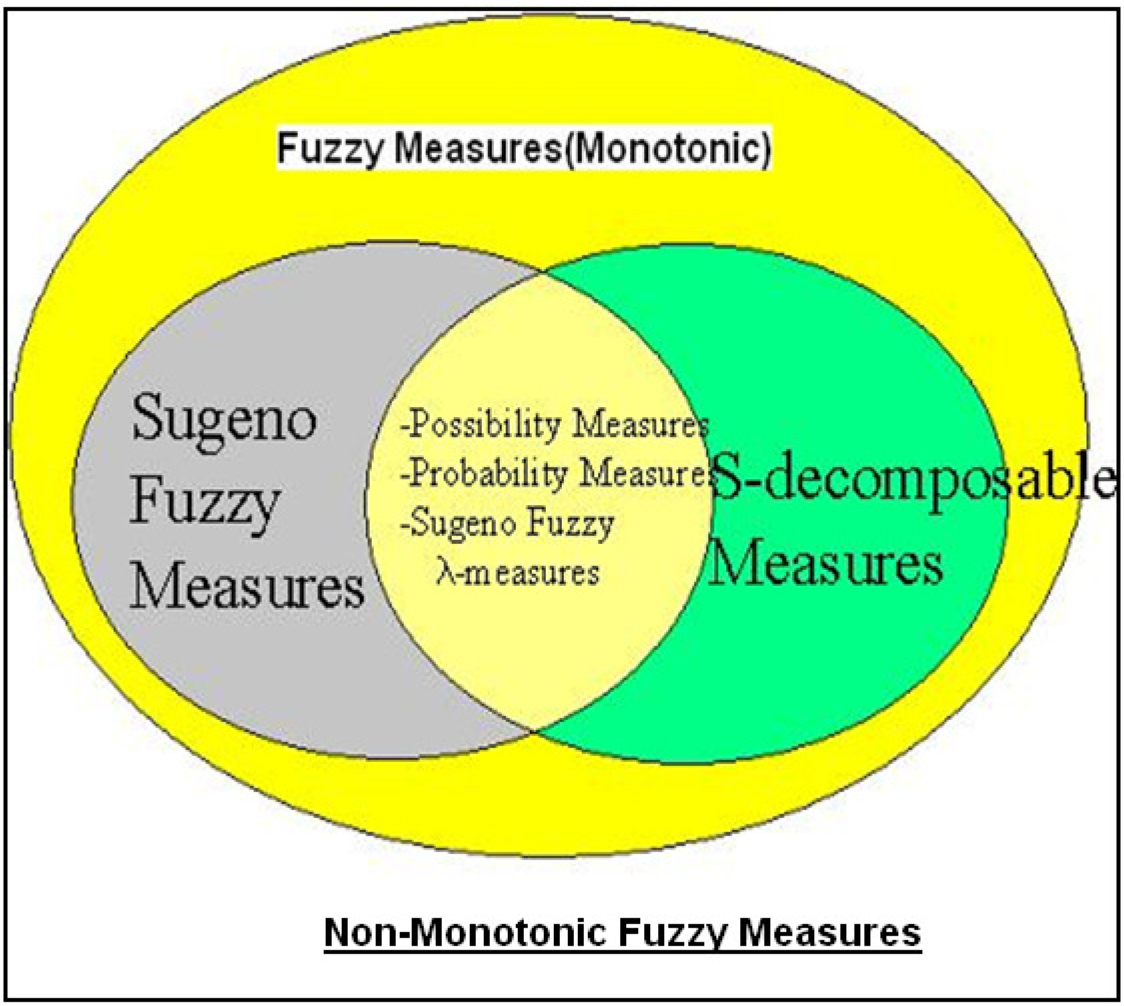
3.9 Discussion on Fuzzy Measure

In most of the cases we will consider the fuzzy measure as monotonic because it has applications in many fields for example in neural networks, data mining, image processing and so many other fields we have also a non monotonic fuzzy measure which is defined as:

Suppose that (X, F, μ) is measureable space, a fuzzy measure is a function $\mu : F \to [0,\infty]$ such that
$\mu(\phi) = 0$ is called non monotonic fuzzy measure.

Since the non-monotonic has the least number of conditions so every monotonic fuzzy measure is a non monotonic fuzzy measure.

We will consider a fuzzy monotonic measure in most of cases.



Example3.1: consider a set $X = \{a, b, c\}$, we introduce a function μ such that

Function	values
$\mu(\phi)$	0
$\mu(\{a\})$	0.2
$\mu(\{b\})$	0.4
$\mu(\{c\})$	0.6
$\mu(\{a, b\})$	0.8
$\mu(\{a, c\}$	0.6
$\mu(\{b, c\})$	0.9
$\mu(\{a, b, c\})$	1.2

This is a monotonic fuzzy measure since, e.g, $\{a\} \subset \{a, c\} \rightarrow \mu(\{a\}) \leq \mu(\{a, c\})$

Example 3.2: Consider a set $X = \{a, b, c\}$, we introduce a function μ such that

Function	values
$\mu(\phi)$	0
$\mu(\{a\})$	0.9
$\mu(\{b\})$	1
$\mu(\{c\})$	0.6
$\mu(\{a, b\})$	0.8
$\mu(\{a, c\}$	0.6
$\mu(\{b, c\})$	0.9
$\mu(\{a, b, c\})$	1.3

This is the example of non monotonic fuzzy measure since, e.g, $\{a\} \subset \{a, c\} \to \mu(\{a\}) \geq \mu(\{a, c\})$

Examples of Fuzzy Measure (for calculation):

λ - Fuzzy measure is the best example of fuzzy measure

Example 3.3: Consider the set $X = \{a, b, c\}$ the fuzzy density values as follow

$$g_\lambda(\{a\}) = 0.4$$
$$g_\lambda(\{b\}) = 0.3$$
$$g_\lambda(\{c\}) = 0.2$$

First we calculate the value of λ

Since $g_\lambda(X) = 1$ then

$$\lambda + 1 = \prod_{i=1}^{n} (\lambda g_i + 1)$$

$$\lambda + 1 = (0.4\lambda + 1)(0.3\lambda + 1)(0.2\lambda + 1)$$

$$0.024\lambda^3 + 0.26\lambda^2 - 0.1\lambda = 0$$

The roots of the above equation will be

$\lambda = \{0, -11.87, 0.3719\}$

But $\lambda \in (-1, \infty)$

So we will take $\lambda = 0.3719$

For $\lambda = 0$ g is additive measure

If $\lambda = 0.3719$ then

$$g_\lambda(\{a\}) = 0.4$$

$$g_\lambda(\{b\}) = 0.3$$

$$g_\lambda(\{c\}) = 0.2$$

$$g_\lambda(\{a,b\}) = g_\lambda(\{a\}) + g_\lambda(\{b\}) + \lambda g_\lambda(\{a\})\, g_\lambda(\{b\}) = 0.7446$$

$$g_\lambda(\{a,c\}) = g_\lambda(\{a\}) + g_\lambda(\{c\}) + \lambda g_\lambda(\{a\})\, g_\lambda(\{c\}) = 0.6298$$

$$g_\lambda(\{b,c\}) = g_\lambda(\{b\}) + g_\lambda(\{c\}) + \lambda g_\lambda(\{b\})\, g_\lambda(\{c\}) = 0.5223$$

$$g_\lambda(X) = 1$$

For $\lambda = 0.3719$ this is fuzzy measure because for example

If $\{a\} \subseteq \{a,b\}$ then $g\{a\} \le g\{a,b\}$

If $\{a,b\} \subseteq X$ then $g\{a,b\} \le g(X)$

Similarly we can take all other cases

Example3.4: Calculation of λ-fuzzy measure for set the students in bth of complex analysis. Let X= {a(those students who got 5 grade in complex analysis), b(those students who got 4 grade in complex analysis), c= (those students who got 3 grade in complex analysis)}

Solution Let the associated densities are

$$g_1 = g_\lambda(\{a\}) = 0.5$$

$$g_2 = g_\lambda(\{b\} = 0.4$$

$$g_3 = g_\lambda(\{c\}) = 0.3$$

First we calculate the value of λ

We know that

$$\lambda + 1 = \prod_{i=1}^{n} (\lambda g_i + 1)$$

$$\lambda + 1 = (0.5\lambda + 1)(0.4\lambda + 1)(0.3\lambda + 1)$$

$$0.06\lambda^3 + 0.47\lambda^2 + 0.2\lambda = 0$$

And the roots of the above equation will be

$\lambda = \{0,-23.0665, -0.4335\}$

But $\lambda \in (-1, \infty)$

So we will take $\lambda = -0.4335$

If $\lambda = 0$ then the measure is additive measure

If $\lambda = -0.4335$ then

$$g_\lambda(\{a\}) = 0.5$$

$$g_\lambda(\{b\}) = 0.4$$

$$g_\lambda(\{c\}) = 0.3$$

$$g_\lambda(\{a,b\}) = g_\lambda(\{a\}) + g_\lambda(\{b\}) + \lambda g_\lambda(\{a\})\, g_\lambda(\{b\}) = 0.8133$$

$$g_\lambda(\{a,c\}) = g_\lambda(\{a\}) + g_\lambda(\{c\}) + \lambda g_\lambda(\{a\})\, g_\lambda(\{c\}) = 0.7350$$

$$g_\lambda(\{b,c\}) = g_\lambda(\{b\}) + g_\lambda(\{c\}) + \lambda g_\lambda(\{b\})\, g_\lambda(\{c\}) = 0.6480$$

$$g_\lambda(X) = 1$$

For $\lambda = -0.4335$ the above problem represent a fuzzy measure

3.10 Definition of Choquet Integral

Let us suppose that g be a fuzzy measure on X, then Choquet integral of a function $f : X \to [0, \infty]$ w.r.t fuzzy measure g is defined

$$(c)\int f\ dg = \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right) g(A_i) \text{ where } A_i \subset X \text{ for } i = 1,2,3,\ldots\ldots,n$$

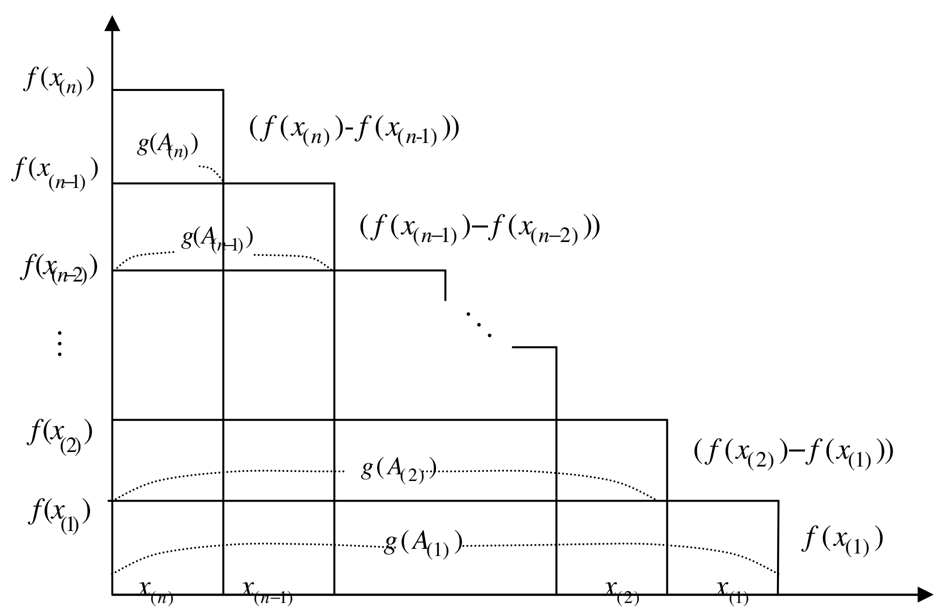
where $\{ f(x_1), f(x_2), f(x_3), \ldots\ldots, f(x_n)\}$ are the ranges and they are defined as

where $f(x_1) \le f(x_2) \le f(x_3) \le \ldots\ldots \le f(x_n)$ and $f(x_0) = 0$

$(c)\int f\ dg = \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right) g(A_i)$ is an equation is called Choquet integral.

The basic concept of Choquet integral can be illustrated from the figure below.

[4]



Explanation of Choquet Integral w.r.t figure above

Let $X = \{x_1, x_2, x_3, \ldots, x_n\}$ be n objects with ranges $\{f(x_1), f(x_2), f(x_3), \ldots, f(x_n)\}$ such that $f(x_1) \le f(x_2) \le f(x_3) \le \ldots \le f(x_n)$

Now the measurements of these objects are $g(A_1)\ g(A_2)\ \ldots\ldots\ldots\ldots\ g(A_n)$

The area of first rectangle $= f(x_1) \cdot g(A_1)$ where $g(A_1) = g(\{x_1, x_2, x_3, \ldots\ldots x_n\})$

The area of second rectangle $= (f(x_2) - f(x_1)) \cdot g(A_2)$ where $g(A_2) = g(\{x_2, x_3, \ldots\ldots x_n\})$

The area of third rectangle $= (f(x_3) - f(x_2)) \cdot g(A_3)$ where $g(A_3) = g(\{x_3, x_4, \ldots\ldots x_n\})$

The area of nth rectangle $= (f(x_n) - f(x_{n-1})) \cdot g(A_n)$ where $g(A_n) = g(\{x_n\})$

The sum of these areas will be

$$(c)\int fdg = f(x_1) \cdot g(A_1) + (f(x_2) - f(x_1)) \cdot g(A_2) + (f(x_2) - f(x_1)) \cdot g(A_2) + \ldots\ldots\ldots\ldots + (f(x_n) - f(x_{n-1})) \cdot g(A_n)$$

$$(c)\int f\ dg = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))g(A_i)$$

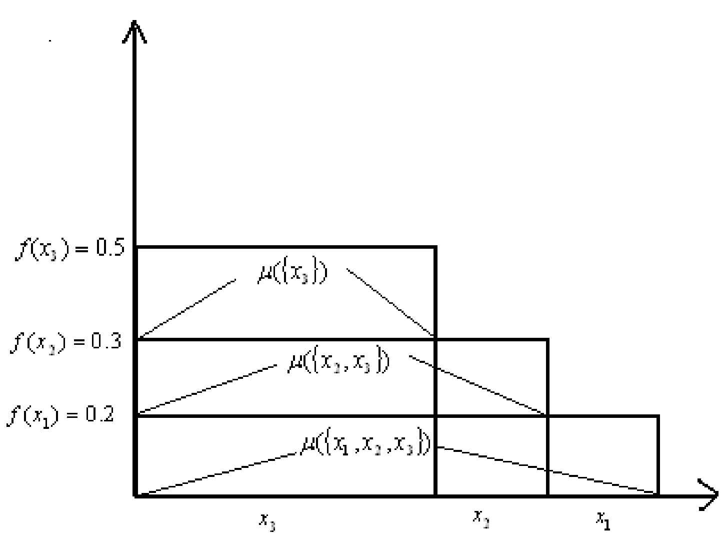
Example3.5: Consider a set $X = \{x_1, x_2, x_3\}$ where the ranges are defined as $f(x_1) = 0.2, f(x_2) = 0.3, f(x_2) = 0.5$, such that the fuzzy measure (monotonic) is defined as

Function	values
$\mu(\phi)$	0
$\mu(\{x_1\})$	0.2
$\mu(\{x_2\})$	0.4
$\mu(\{x_3\})$	0.5
$\mu(\{x_1, x_2\})$	0.8
$\mu(\{x_2, x_3\}$	0.6
$\mu(\{x_1, x_3\})$	0.9
$\mu(\{x_1, x_2, x_3\})$	1.2

The Choquet integral for this problem is defined as

$$(c)\int f\, d\mu = \sum_{i=1}^{3} \left(f(x_i) - f(x_{i-1})\right)\mu(A_i)$$

$$= f(x_1) \cdot \mu(\{x_1, x_2, x_3\}) + \left(f(x_2 - f(x_1)\right) \cdot \mu(\{x_2, x_3\}) + \left(f(x_3 - f(x_2)\right) \cdot \mu(\{x_3\})$$



$$= 0.2 \cdot 1.2 + (0.3 - 0.2) \cdot 0.6 + (0.5 - 0.3) \cdot 0.5$$

$$(c)\int f \ d\mu = 0.4$$

This is the Choquet integral for monotonic fuzzy measure

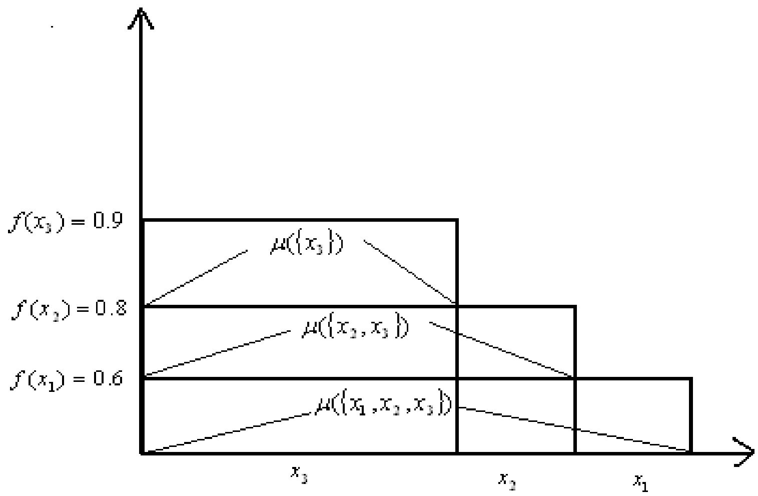
Example 3.6: Consider a set $X = \{x_1, x_2, x_3\}$, where the ranges are defined as $f\ (x_1) = 0.6, f\ (x_2) = 0.8, f\ (x_3) = 0.9$ such that the fuzzy measure (non monotonic) is defined as

Function	values
$\mu(\phi)$	0
$\mu(\{x_1\})$	0.9
$\mu(\{x_2\})$	1
$\mu(\{x_3\})$	0.6
$\mu(\{x_1, x_2\})$	0.8
$\mu(\{x_2, x_3\}$	0.9
$\mu(\{x_1, x_3\})$	0.6
$\mu(\{x_1, x_2, x_3\})$	1.2

The Choquet integral for this problem is defined as

$$(c)\int f \ d\mu = \sum_{i=1}^{3} \left(f(x_i) - f(x_{i-1})\right)\mu(A_i)$$

$$= f(x_1) \cdot \mu(\{x_1, x_2, x_3\}) + \left(f(x_2 - f(x_1)\right) \cdot \mu(\{x_1, x_2\}) + \left(f(x_3 - f(x_2)\right) \cdot \mu(\{x_3\})$$



$$= 0.6 * 1.3 + (0.8 - 0.6) * 0.6 + (0.9 - 0.8) * 0.5$$

$$(c) \int f \, d\mu = 0.9$$

This is the Choquet integral for non monotonic fuzzy measure

Now the Choquet integral for λ - Fuzzy measure or sugeno λ - Fuzzy measure

Example 3.7: Consider the set $X = \{x_1, x_2, x_3\}$ where the ranges are defined as $f(x_1) = 0.4, f(x_2) = 0.6, f(x_2) = 0.8$ and fuzzy density values are

$$g_\lambda(\{x_1\}) = 0.4$$

$$g_\lambda(\{x_2\}) = 0.3$$

$$g_\lambda(\{x_3\}) = 0.2$$

First we calculate the value of λ

Since $\lambda + 1 = \prod_{i=1}^{n} (\lambda g_i + 1)$

$$\lambda + 1 = (0.4\lambda + 1)(0.3\lambda + 1)(0.2\lambda + 1)$$

$$0.024\lambda^3 + 0.26\lambda^2 - 0.1\lambda = 0$$

The roots of the above equation will be

$\lambda = \{0,-11.87,0.3719\}$

If $\lambda = 0$

$$g_\lambda(\{x_1, x_2\}) = g_\lambda(\{x_1\}) + g_\lambda(\{x_2\}) = 0.7$$

$$g_\lambda(\{x_3, x_2\}) = g_\lambda(\{x_3\}) + g_\lambda(\{x_2\}) = 0.5$$

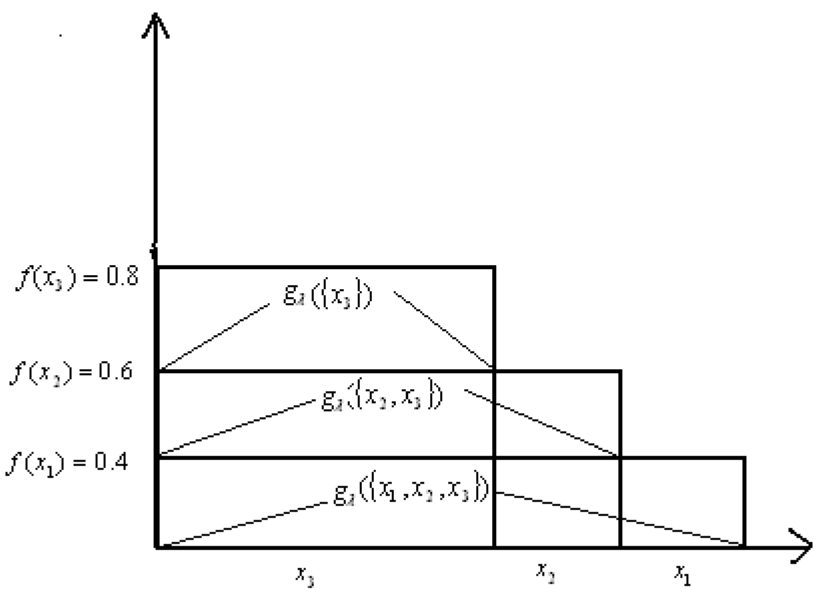
$$g_\lambda(\{x_1, x_3\}) = g_\lambda(\{x_1\}) + g_\lambda(\{x_3\}) = 0.6$$

$$g_\lambda(X) = 0.9$$

The Choquet integral

$$(c)\int f\, d\mu = \sum_{i=1}^{3}\left(f(x_i) - f(x_{i-1})\right)g_\lambda(A_i)$$

$$= f(x_1)\cdot g_\lambda(\{x_1, x_2, x_3\}) +$$
$$\left(f(x_2 - f(x_1)\right)\cdot g_\lambda(\{x_2, x_3\}) + \left(f(x_3 - f(x_2)\right)\cdot g_\lambda(\{x_3\})$$



$$= 0.4 * 0.9 + (0.6 - 0.4) * 0.5 + (0.8 - 0.6) * 0.2$$

$$(c)\int f\, d\mu = 0.5$$

If $\lambda = 0.3719$ then If

$$g_\lambda(\{x_1\}) = 0.4$$

$$g_\lambda(\{x_2\}) = 0.3$$

$$g_\lambda(\{x_1\}) = 0.4$$

$$g_\lambda(\{x_1, x_2\}) = g_\lambda(\{x_1\}) + g_\lambda(\{x_2\}) + \lambda g_\lambda(\{x_1\}) g_\lambda(\{x_2\}) = 0.7446$$

$$g_\lambda(\{x_2, x_3\}) = g_\lambda(\{x_3\}) + g_\lambda(\{x_2\}) + \lambda g_\lambda(\{x_3\}) g_\lambda(\{x_2\}) = 0.5223$$

$$g_\lambda(\{x_1, x_3\}) = g_\lambda(\{x_1\}) + g_\lambda(\{x_3\}) + \lambda g_\lambda(\{x_1\}) g_\lambda(\{x_3\}) = 0.7323$$

$$g_\lambda(X) = 1$$

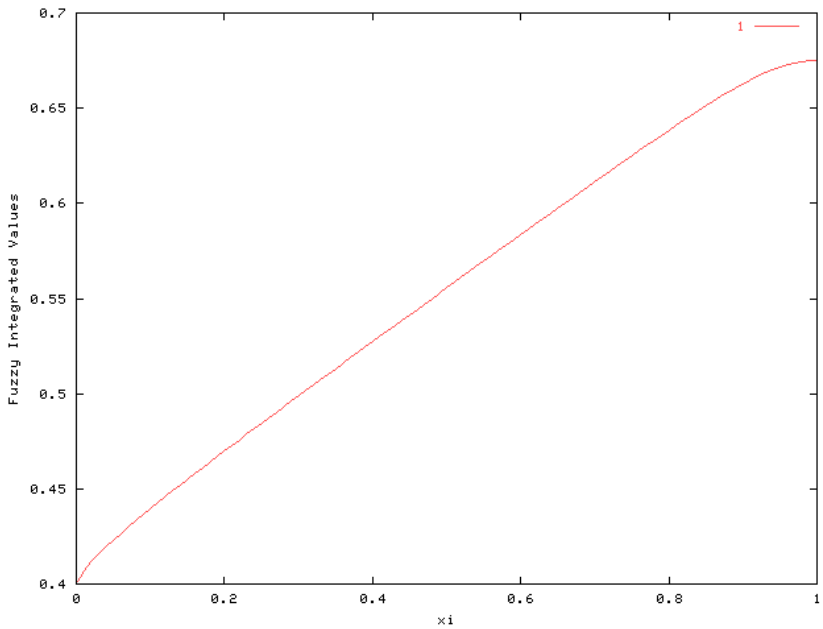
The Choquet integral will be

$$(c)\int f \, d\mu = \sum_{i=1}^{3} \left(f(x_i) - f(x_{i-1}) \right) g_\lambda(A_i)$$

$$= f(x_1) \cdot g_\lambda(\{x_1, x_2, x_3\}) + \left(f(x_2 - f(x_1)\right) g_\lambda(\{x_2, x_3\}) + \left(f(x_3 - f(x_2)\right) \cdot g_\lambda(\{x_3\})$$

$$= 0.4 * 1 + (0.6 - 0.4) * 0.5223 + (0.8 - 0.6) * 0.2$$

$$(c)\int f \, d\mu = 0.5445$$



Example 3.8: The teacher of mathematics has to evaluate her student according to their level in complex analysis, fuzzy logic and numerical analysis. She gives equal importance to complex analysis and fuzzy logic and less importance to numerical analysis, the student has got 45 points in complex analysis out of 60 points, 50 in fuzzy logic out of 60 and 53 in numerical analysis out of 60 analyses.

Solution: Suppose that the grad of importance

$$g_1 = g_\lambda(x_1) = g\,(\text{complex analysis}) = 0.45$$

$$g_2 = g_\lambda(x_2) = g\,(\text{fuzzy logic}) = 0.45$$

$$g_3 = g_\lambda(x_3) = g\,(\text{numerical analysis}) = 0.3$$

First we calculate the value of λ

Since

$$\lambda + 1 = \prod_{i=1}^{n} (\lambda g_i + 1)$$

$$\lambda + 1 = (0.45\lambda + 1)(0.45\lambda + 1)(0.3\lambda + 1)$$

$$0.06067\lambda^3 + 0.4725\lambda^2 + 0.2\lambda = 0$$

The roots of the above equation will be

$$\lambda = \{0, -7.3286, -0.4492\}$$

But $\lambda \in (-1, \infty)$

So we will take $\lambda = -0.4492$

If $\lambda = 0$ then the measure is additive measure

If $\quad \lambda = -0.4492$ then

$$g_\lambda(\{x_1\}) = 0.45$$

$$g_\lambda(\{x_2\}) = 0.45$$

$$g_\lambda(\{x_3\}) = 0.3$$

$$g_\lambda(\{x_1, x_2\}) = g_\lambda(\{x_1\}) + g_\lambda(\{x_1\}) + \lambda g_\lambda(\{x_1\})\, g_\lambda(\{x_2\}) = 0.8090$$

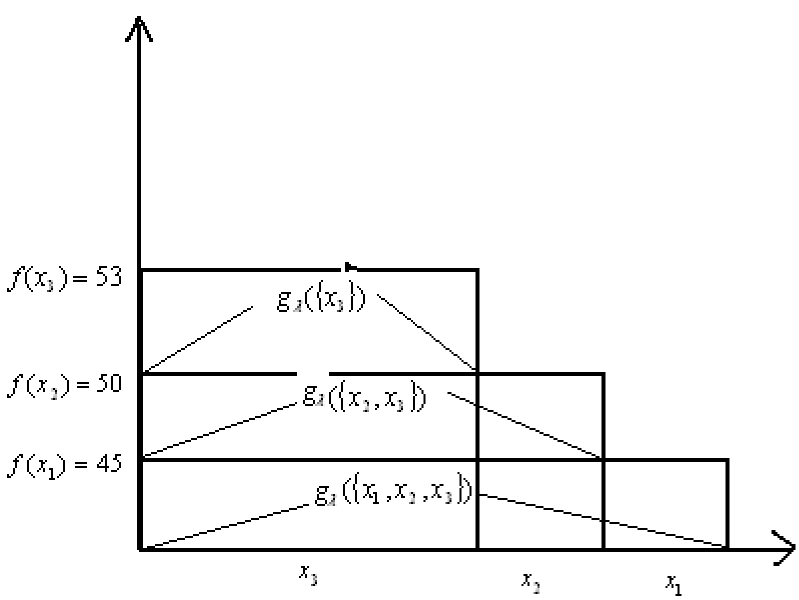
$$g_\lambda(\{x_1, x_3\}) = g_\lambda(\{x_1\}) + g_\lambda(\{x_3\}) + \lambda g_\lambda(\{x_1\})\, g_\lambda(\{x_3\}) = 0.6894$$

$$g_\lambda(\{x_2, x_3\}) = g_\lambda(\{x_2\}) + g_\lambda(\{x_3\}) + \lambda g_\lambda(\{x_2\})\, g_\lambda(\{x_3\}) = 0.6894$$

$$g_\lambda(X) = 1$$

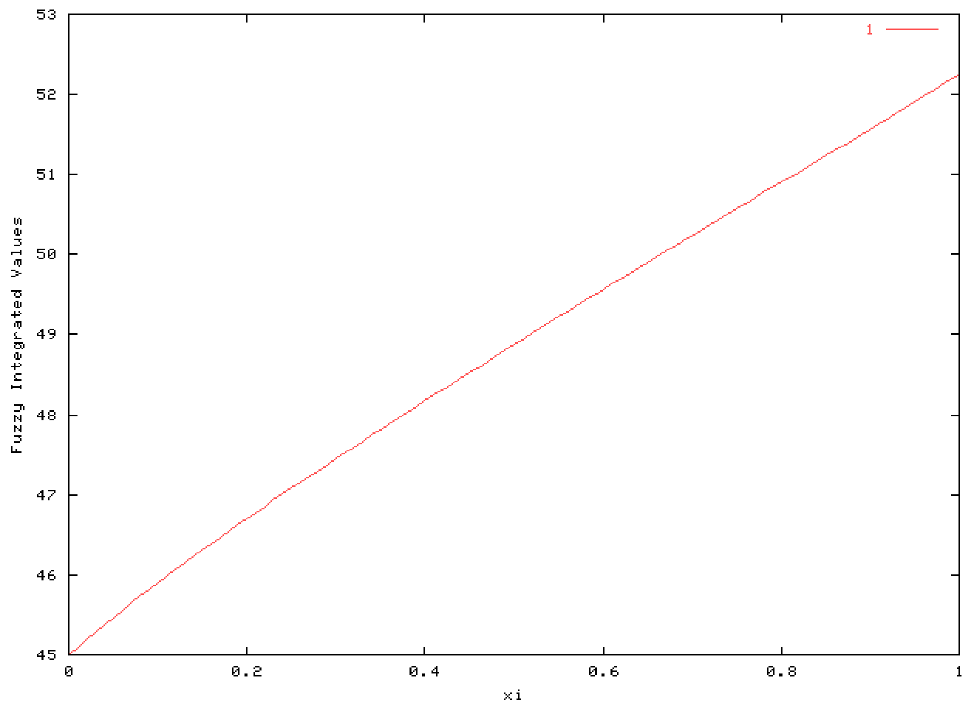
(c) $\int \mathrm{f}\ \mathrm{d}\mu = \sum_{i=1}^{3} \left(f(x_i) - f(x_{i-1})\right) g_\lambda(A_i)$

$= f(x_1) \cdot g_\lambda(\{x_1, x_2, x_3\}) + \left(f(x_2 - f(x_1)\right) \cdot g_\lambda(\{x_2, x_3\}) + \left(f(x_3 - f(x_2)\right) \cdot g_\lambda(\{x_3\})$



$(c)\int f \, d\mu = 45 * 1 + (50 - 45) * 0.8090 + (53 - 50) * .3$

$(c)\int f \, d\mu = 49.9450$



3.11 Sugeno Integral [Sugeno 1974]

Suppose that μ is a fuzzy normalized measure on X, the Sugeno integral of a function $f : X \rightarrow [0,1]$ w. r. t fuzzy measure μ is defined as

$$\int f(x) d\mu = \max_{1 \le i \le n} \left(\min \left(f(x_i), \mu(A_i) \right) \right)$$

where $\{ f(x_1), f(x_2), f(x_3), \ldots, f(x_n) \}$ are the ranges and they are defined as $f(x_1) \le f(x_2) \le f(x_3) \le \ldots \le f(x_n)$

Example3.9: Consider a set $X=\{x_1,x_2,x_3\}$ and the ranges are defined as $f(x_1)=0.2, f(x_2)=0.3, f(x_3)=0.5$, such that the fuzzy measure (monotonic) is defined as

Function	values
$\mu(\phi)$	0
$\mu(\{x_1\})$	0.2
$\mu(\{x_2\})$	0.4
$\mu(\{x_3\})$	0.5
$\mu(\{x_1,x_2\})$	0.8
$\mu(\{x_2,x_3\}$	0.6
$\mu(\{x_1,x_3\})$	0.9
$\mu(\{x_1,x_2,x_3\})$	1.2

To normalize the fuzzy measures, we are dividing all of them by the largest value in measure.

Function	values
$\mu(\phi)$	0
$\mu(\{x_1\})$	0.16667
$\mu(\{x_2\})$	0.333333
$\mu(\{x_3\})$	0.416666
$\mu(\{x_1,x_2\})$	0.6667
$\mu(\{x_2,x_3\}$	0.5
$\mu(\{x_1,x_3\})$	0.75
$\mu(\{x_1,x_2,x_3\})$	1

The Sugeno integral is defined as

$$\int f(x)d\mu = \max_{1\le i\le n}\left(\min\left(f(x_i),\mu(A_i)\right)\right)$$

$$= \max(\min(0.2,1), \min(0.3,0.5), \min(0.5,0.416666)$$

$$= \max(0.2,0.3,0.4166666)$$

$$\int f(x)d\mu = 0.416666$$

Example3.10: Consider a set $X = \{x_1, x_2, x_3\}$ and the ranges is defined as $f(x_1) = 0.6, f(x_2) = 0.8, f(x_3) = 0.9$ such that the fuzzy measure (non monotonic) is defined as

Function	values
$\mu(\phi)$	0
$\mu(\{x_1\})$	0.9
$\mu(\{x_2\})$	1
$\mu(\{x_3\})$	0.6
$\mu(\{x_1, x_2\})$	0.8
$\mu(\{x_2, x_3\}$	0.9
$\mu(\{x_1, x_3\})$	0.6
$\mu(\{x_1, x_2, x_3\})$	1.2

To normalize the fuzzy measures, we are dividing all of them by the largest value in measure.

Function	values
$\mu(\phi)$	0
$\mu(\{x_1\})$	0.75

$\mu(\{x_2\})$	.83333
$\mu(\{x_3\})$	0.5
$\mu(\{x_1, x_2\})$	0.66667
$\mu(\{x_2, x_3\}$	0.8333
$\mu(\{x_1, x_3\})$	0.5
$\mu(\{x_1, x_2, x_3\})$	1

The Sugeno integral

$$\int f(x)d\mu = \max_{1\le i\le n}\left(\min\left(f(x_i), \mu(A_i)\right)\right)$$

$$= \max(\min(0.6, 1), \min(0.8, 0.8333), \min(0.9, 0.5))$$

$$= \max(0.6, 0.8, 0.5)$$

$$\int f(x)d\mu = 0.8$$

Example 3.11: Consider the set $X = \{x_1, x_2, x_3\}$ with ranges $f(x_1) = 0.4, f(x_2) = 0.6, f(x_2) = 0.8$ and fuzzy density values are

$$g(\{x_1\}) = 0.4$$
$$g(\{x_2\}) = 0.3$$
$$g(\{x_3\}) = 0.2$$

First we calculate the value of λ

We know that $\lambda + 1 = \prod_{i=1}^{n}\left(\lambda g_i + 1\right)$

$$\lambda + 1 = (0.4\lambda + 1)(0.3\lambda + 1)(0.2\lambda + 1)$$

$$0.024\lambda^3 + 0.26\lambda^2 - 0.1\lambda = 0$$

And the roots of the above equation will be

$\lambda = \{0,-11.87,0.3719\}$

If $\lambda =$ 0.3719 then

$$g_\lambda(\{x_1\}) = 0.4$$

$$g_\lambda(\{x_2\}) = 0.3$$

$$g_\lambda(\{x_1\}) = 0.4$$

$$g_\lambda(\{x_1, x_2\}) = g_\lambda(\{x_1\}) + g_\lambda(\{x_2\}) + \lambda g_\lambda(\{x_1\}) g_\lambda(\{x_2\}) = 0.7446$$

$$g_\lambda(\{x_2, x_3\}) = g_\lambda(\{x_3\}) + g_\lambda(\{x_2\}) + \lambda g_\lambda(\{x_3\}) g_\lambda(\{x_2\}) = 0.5223$$

$$g_\lambda(\{x_1, x_3\}) = g_\lambda(\{x_1\}) + g_\lambda(\{x_3\}) + \lambda g_\lambda(\{x_1\}) g_\lambda(\{x_3\}) = 0.7323$$

$$g_\lambda(X) = 1$$

$$\int f(x) d\mu = \max\left(\min(f(x_1), g_\lambda(\{x_1, x_2, x_3\}), (\min(f(x_2), g_\lambda(\{x_2, x_3\}), \min(f(x_3), g_\lambda(\{x_3\}))\right)$$

$$= \max(\min(0.4, 1), \min(0.6, 0.5223), \min(0.8, 0.4))$$

$$= \max(0.4, 0.5223, 0.4)$$

$$= 0.5223$$

Chapter 4: The Difference between Choquet -Sugeno Integral and Classical Integrals

4.1 Measure

A measure on X is a non-negative additive set function define on 2^X i-e

$f : 2^X \to R_+$ a normalize measure is called probability measure.

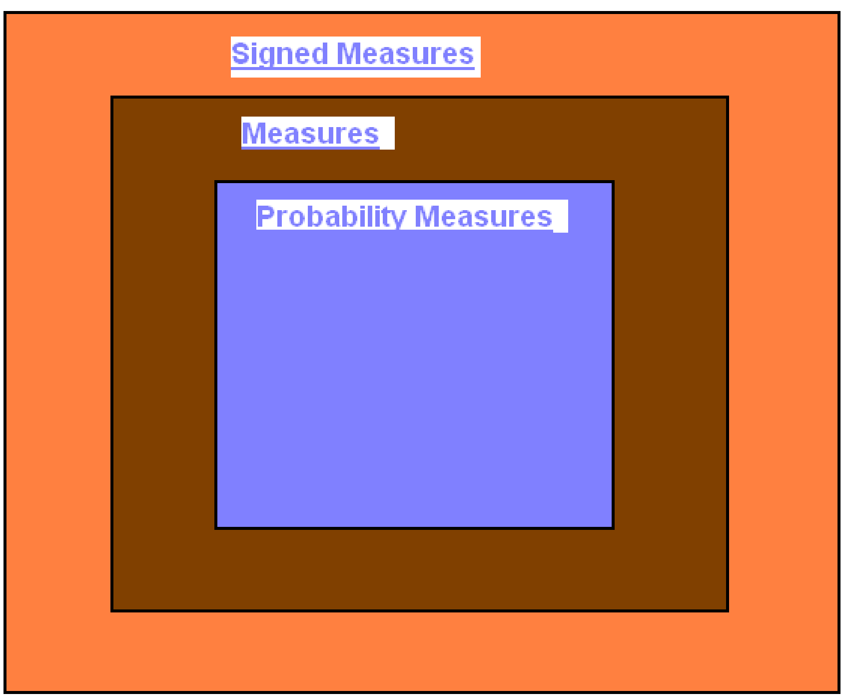
4.2 Signed Measure

A measureable space (X, F), that is, a set X with sigma algebra F on it, an extended signed measure is a function $g : F \to R \cup \{\infty, -\infty\}$ such that the following properties are satisfied

1. $g(\phi) = 0$
2. $g\left(\bigcup_{i=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} g(A_n)$ for any sequence of $A_1, A_2, A_3, ..., A_n, ...$ of disjoint sets in F.

A finite signed measure is defined in the same way, but it is only allowed to take $(finite)$ Real values.

A probability measure is a measure, and a measure is a signed measure



4.3 Recall the Definition of Fuzzy Measure [14]

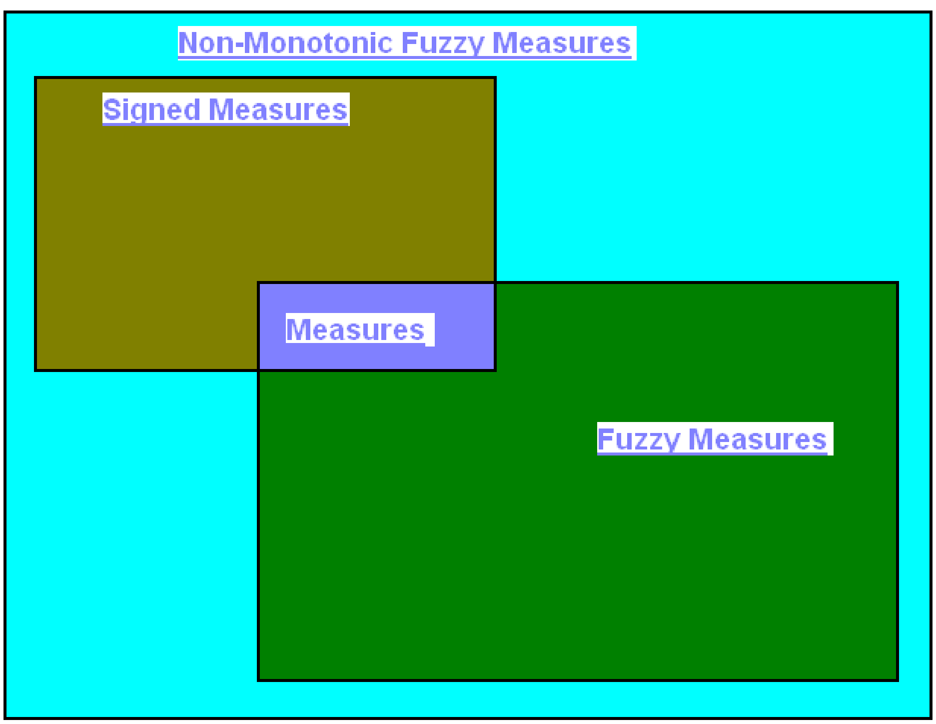
A monotonic fuzzy measure is a function $f : 2^X \rightarrow R_+$ such that it is monotonic and vanishes at empty set and non monotonic fuzzy measure is a function defines on 2^X and vanishes at empty set.

An additive fuzzy measure is a measure and an additive non monotonic fuzzy measure is signed measure.

The difference between fuzzy measure and a measure is that, the fuzzy measure is non additive and additive measures but the measure is additive only this means that every measure is fuzzy measure but it reverse is not true similarly every signed measure is non monotonic fuzzy measure but every non monotonic fuzzy measure is not necessarily signed measure.

The difference between fuzzy measure (monotonic or non monotonic fuzzy measure) and measure (measure or signed measure) is non additivity. Classical measures satisfy the additive condition. On the contrary the fuzzy measures can, but do not need fulfill the additive condition and it is the most important difference between these two classes of measures. And the most important difference: the classical measures are still made in measure units like meters, centimeters and so on, whereas fuzzy measure are abstract, e.g., effectiveness of work, effectiveness of medicines and the like. Therefore two workers who talk and smoke maybe produce less in comparison to their separate production. That is why

Effectiveness (A and B) < Effectiveness (A) + Effectiveness (B)



Example 4.1: Let $A \cap B = \phi$ i.e. A and B are two disjoint sets

Then the measure will be

$$m(A \cup B) = m(A) + m(B)$$

The fuzzy measure will be

$$\mu(A \cup B) < \mu(A) + \mu(B)$$

OR

$$\mu(A \cup B) > \mu(A) + \mu(B)$$

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

To compensate these inequality Sugeno define λ - fuzzy measure which is defined as

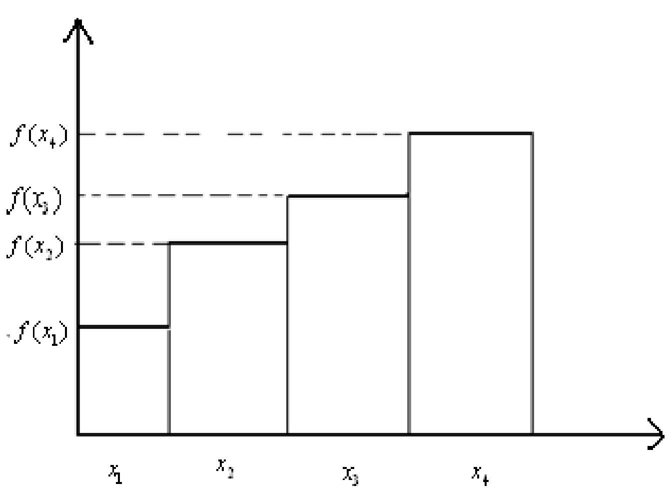
$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B) \text{ where } \lambda \in (-1, \infty)$$

4.4 Integration

Suppose that m be a signed measure on X and f is a function on X then the integration $\int f(x)\,dm$ w.r.t m is defined

$$\int f(x)\,dm = \sum_{x \varepsilon X} f(x) m(\{x\})$$

Let $X = \{x_1, x_2, x_3, x_4\}$.



Obviously $\int_X f(x)dm = \int \text{f(x) dm} = \sum_{x \epsilon X} f(x)m(\{x\}) =$

$f(x_1)m(\{x_1\}) + f(x_2)m(\{x_2\}) + f(x_3)m(\{x_3\}) + f(x_4)m(\{x_4\})$

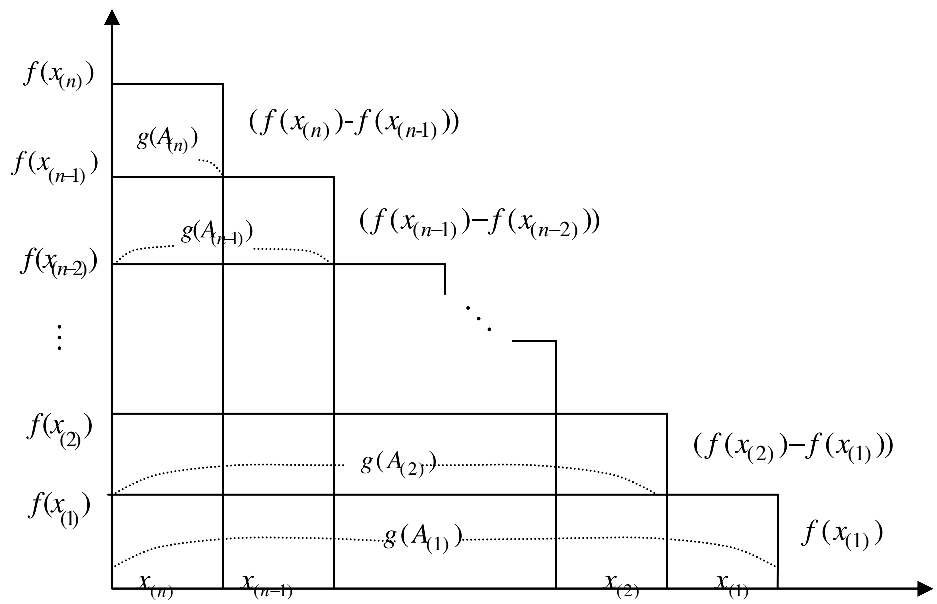
Fuzzy integrations (Recall the definition of Choquet and Sugeno integral)

4.5 Choquet Integration

Let g be a fuzzy measure on X. The Choquet integral of a function f: X→ $[0,\infty]$ with respect to fuzzy measure g is

$$(c)\int f\ dg = \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right)g(A_i)$$

[4]



Where $g(A_1) = g(\{x_1, x_2, x_3, \ldots\ldots x_n\})$

$g(A_2) = g(\{x_2, x_3, \ldots\ldots x_n\})$

$g(A_3) = g(\{x_3, x_4, \ldots\ldots x_n\})$

$g(A_n) = g(\{x_n\})$

4.6 Sugeno Integral [Sugeno 1974]

Suppose that μ is a fuzzy measure on X, the Sugeno integral of a function $\text{f} : \text{X} \rightarrow [0,1]$ w. r. t fuzzy measure μ is define as

$$\int f(x)d\mu = \max_{1 \le i \le n} \left(\min \left(f(x_i), \mu(A_i) \right) \right)$$

where $\{ f(x_1), f(x_2), f(x_3), \ldots, f(x_n) \}$ are the ranges and they are defined as

$$f(x_1) \le f(x_2) \le f(x_3) \le \ldots \le f(x_n)$$

4.7 Difference

The difference between classical integrals and fuzzy integrals, in classical integrals (integration w.r.t measure) we have measure or signed measure but in fuzzy integrals we have fuzzy measure (monotonic or non monotonic) the main difference between them is non-additively, in fuzzy integrals we have additive and non additive but in classic integral we have additive only.

Chapter 5: Practical Applications of Choquet and Sugeno Integrals

Practical applications of Choquet and Sugeno integrals in multicriteria decision making and human evaluation processes.

Some traditional aggregation operator

5.1 Arithmetic Mean

Consider we have the set of data $X = \{x_1, x_2, ..., x_n\}$ then the arithmetic mean is defined as $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$

5.2 Median

The median is another typical operator, this is not counting the values of objects x_i

themselves but only their ordering ,median is defined the middle value of the given objects

med($x_1, x_2, ..., x_n) = x_{(n+1)/2}$ where n is odd

med($x_1, x_2, ..., x_n) = \frac{1}{2}\left(x_{n/2} + x_{n/2+1}\right)$ where n is even

where $x_1 \le x_2, ... \le x_n$ we have arrange the elements in increasing order

5.3 Ordering Weighted Averaging Operator

This operator has been introduced by yager, which is defined as

$$OWA = F(a_1, a_2, ...a_n) = \sum_{i=1}^{n} w_i a_{(i)}$$

where $a_{(1)} \le a_{(2)} \le ....a_{(n)}$ and $\sum_{i=1}^{n} w_i = 1$

Properties of OWA

(1) monotonic i-e $F(a_1, a_2, ...a_n) \geq F(b_1, b_2, ...b_n)$ if $a_i \geq b_i \, for \; i = 1,2,3.....n$
(2) bounded i-e $Min(a_1, a_2, ...a_n) \leq F(a_1, a_2, ...a_n) \leq Max(a_1, a_2, ...a_n)$
(3) symmetric
(4) idempotent $F(a_1, a_2, ...a_n) = a \; if \;\; all \; a_i = a$
(5) If all weights equal to $\frac{1}{n}$, then OWA will becomes arithmetic mean.

5.4 Weighted Minimum and Maximum

They have been introduced by (Dubois, D., and Prade, H., 1985) in the frame work of possibility theory ,that are denoted by $\wedge \, and \vee$,and define as

$\min_{w_1, w_2, ..., w_n}(a_1, a_2, ...a_n) = \bigvee_{i=1}^{n}(1 - w_i) \vee a_i$ where a_i are the objects and w_i their weights of importance .

$\max_{w_1, w_2, ..., w_n}(a_1, a_2, ...a_n) = \bigwedge_{i=1}^{n} w_i \wedge a_i$

where the weights are normalized as $\bigvee_{I=1}^{n} w_i = 1$

All the above operators are very important, and we can present common solutions for the aggregation step, [8] all these operators are idempotent, continuous, and monotonically non-decreasing these are the basic operators for any problem of aggregation, we are calling them aggregation operators. But all these operators have some drawbacks because all large families do not possess all desirable properties, small families seem to be too restrictive, so that we are unable to model in some understandable way to interaction between criteria. We are introducing fuzzy integrals as new aggregation operators which are without these drawbacks.

Fuzzy integrals and $\lambda - fuzzy$ measure for human evaluation processes and decision making.

5.5 Fuzzy Measure

A fuzzy measure of the set X is a function: $g : 2^X \rightarrow [0,1]$ such that the following condition are satisfied

(1) $g(\phi) = 0$
(2) $g(X) = 1$ (1) and (2) is also called boundary condition
(3) If $A \subseteq B \in X$ then $g(A) \leq g(B)$ this property is called monotonicity

where $g(A)$ is indicates the weights of importance for a set A .A fuzzy measure is called additive if $g(A \cup B) = g(A) + g(B)$ whenever $A \cap B = \phi$,super additive if $g(A \cup B) \geq g(A) + g(B)$ whenever $A \cap B = \phi$ and sub additive if $g(A \cup B) \leq g(A) + g(B)$ whenever $A \cap B = \phi$.

Where X is finite, but however in practical applications it is enough to consider the universal set X finite. Let g_λ is a $\lambda - fuzzy$ measure, this is a special kind of measure define on 2^X of a finite set X (Sugeno, 1974) is satisfying the following additional condition.

$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B)$ for all $A, B \in X$ whenever $A \cap B = \phi$ where $\lambda \in (-1, \infty)$

Moreover, let X be a finite set, $X = \{x_1, x_2, ..., x_n\}$ and $P(X)$ be the class of all subsets of X the fuzzy measure $g(X) = g_\lambda(\{x_1, x_2, ..., x_n\})$ can be formulated as(Leszczyński et al., 1985)

$$g_\lambda(\{x_1, x_2, ..., x_n\}) = \sum_{i=1}^{n} g_i + \lambda \sum_{i_1=1}^{n-1} \sum_{i_2=i_1+1}^{n} g_{i_1} . g_{i_2} + ... + \lambda^{n-1} g_1 . g_2 .... g_n$$

$$= \frac{1}{\lambda}\left[\prod_{i=1}^{n}(1 + \lambda g_i) - 1\right] \text{ where } \lambda \in (-1, \infty)$$

$$g_\lambda(\{x_1, x_2, ..., x_n\})\lambda = \prod_{i=1}^{n}(1 + \lambda g_i) - 1$$

$$g_\lambda(\{x_1, x_2, ..., x_n\})\lambda + 1 = \prod_{i=1}^{n} (1 + \lambda g_i) \text{ by definition } g_\lambda(\{x_1, x_2, ..., x_n\}) = 1$$

$$\lambda + 1 = \prod_{i=1}^{n} (1 + \lambda g_i)$$

According to the fundamental theorem regarding the λ-fuzzy measure (Leszczyński et al., 1985), λ-value has three cases, as follows:

(i) If $\sum_{i=1}^{n} g_i > g_\lambda(X)$ then $-1 < \lambda < 0$.

(ii) If $\sum_{i=1}^{n} g_i = g_\lambda(X)$ then $\lambda = 0$.

(iii) If $\sum_{i=1}^{n} g_i < g_\lambda(X)$ then $\lambda > 0$.

5.6 Identification of $\lambda - fuzzy$ Measure

To identify the fuzzy measure uniquely we must specify the weights and the standard of identification, there are so many methods to identify the fuzzy measure but here we will use, singleton fuzzy measure ratio standard.

5.7 Singleton Fuzzy Measure Ratio Standard [9]

To identify the fuzzy measure such that

$$g_\lambda(\{1\}) : g_\lambda(\{2\}) : g_\lambda(\{2\}) : .... : g_\lambda(\{n\}) = w_1 : w_2 : w_3 : .... w_n$$

therefore, this standard marks point of each input`s single influence to the output.

5.8 Fuzzy Integration and its Properties

Definition: Let us suppose that g be a fuzzy measure on X, then Choquet integral of a function $f : X \to [0,1]$ w.r.t fuzzy measure g is defined as

$$(c)\int f \, dg = \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right) g(A_i) \text{ where } A_i \subset X \text{ for } i = 1,2,3,.....,n$$

where $\{f(x_1), f(x_2), f(x_3), ..., f(x_n)\}$ are the ranges and they are defined as

where $f(x_1) \le f(x_2) \le f(x_3) \le ..... \le f(x_n)$ and $f(x_0) = 0$

$(c)\int f \, dg = \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right) g(A_i)$ is an equation is called Choquet integral.

Definition of Sugeno Integral [Sugeno 1974]: Suppose that μ is a fuzzy measure on X, the Sugeno integral of a function $f : X \to [0,1]$ w. r. t fuzzy measure μ is defined as

$$\int f(x) d\mu = \max_{1 \le i \le n} \left(\min\left(f(x_i), \mu(A_i)\right)\right)$$

where $\{f(x_1), f(x_2), f(x_3), ..., f(x_n)\}$ are the ranges and they are defined as

$$f(x_1) \le f(x_2) \le f(x_3) \le ..... \le f(x_n)$$

Properties of Fuzzy Integrals [8]

(1) The Choquet and Sugeno integrals are monotonically non decreasing, idempotent and continuous operators this means that the fuzzy integral is always comprised between max and min.
(2) If the fuzzy measure is an additive then the Choquet integral is converted into weighted arithmetic mean, whose weights w_i are $g(\{x_i\})$
(3) The choquet integral is suitable for positive linear transformation. The Sugeno integral does not share this property but satisfies a similar property with min and max replacing product and sum. In this sense, it can be said that the Choquet integral is sutiable for

cardinal aggregation, where the number has real meaning, while the Sugeno integral is more suitable for ordinal aggregation, where only order make sense.

(4) Any OWA operator with weights $w_1, w_2, w_3, \ldots\ldots, w_n$ is a Choquet integral whose fuzzy measure g is defined as

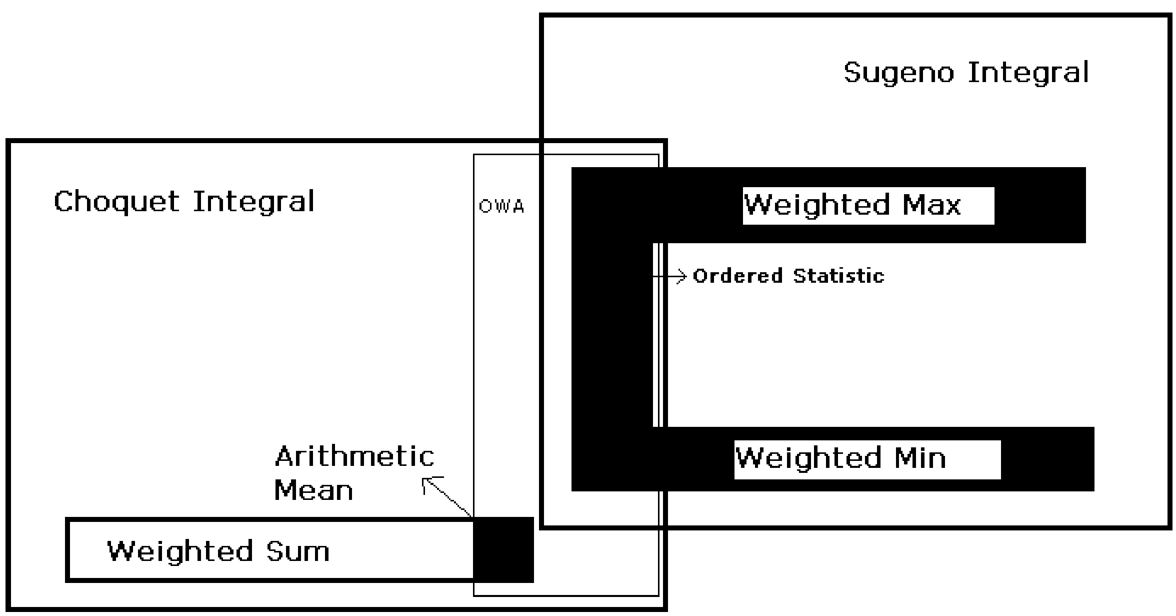
$g(A) = \sum_{j=0}^{i-1} w_{n-j},$ for all A such that $|A| = i$, where $|A|$ is the number of elements in the set A.

$\Rightarrow$ The Choquet integral encompasses both properties the weight arithmetic sums and OWA operators.

Which is said to be "*orthogonal*" ,this implies that ,shows its strong expressive power, since we can mix arbitrary the kinds of operators.

(5) The Choquet and Sugeno integrals contain all statistics order, in particular max, min and the median.

(6) Weighted minimum and maximum are the particular cases of Sugeno integral. This figure indicates a summary of all set relations between various aggregation operators and Sugeno-Choquet integrals.



5.9 Application of Choquet Integral

Example 5.1: The teacher of mathematics has to evaluate her students according to their level in complex analysis, fuzzy logic and numerical analysis. She gives equal importance to complex analysis and fuzzy logic and less importance to numerical analyses.
The points are given on a scale from 0 to 60.

Students	Complex analysis	Fuzzy logic	Numerical analysis
C_1	45	50	40
C_2	56	35	50
C_3	39	58	55
C_4	58	38	57

Suppose that the grades of importance is given by

$$g_\lambda(\{x_1\}) = g(\{\text{complex analysis}\}) = 0.45$$
$$g_\lambda(\{x_2\}) = g(\{\text{fuzzy logic}\}) = 0.45$$
$$g_\lambda(\{x_3\}) = g(\{\text{numerical analysis}\}) = 0.3$$

By the example 3.8

$$\lambda = -0.4492$$

$$g_\lambda(\{x_1, x_2\}) = 0.8090$$

$$g_\lambda(\{x_1, x_3\}) = 0.6894$$

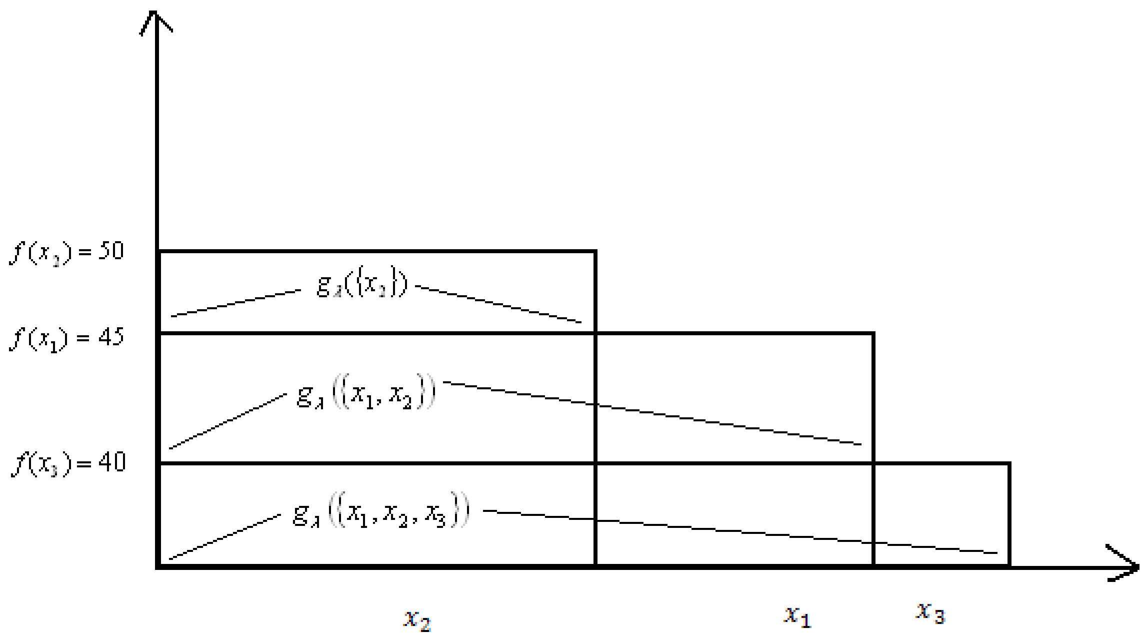
$$g_\lambda(\{x_2, x_3\}) = 0.6894$$

and

$$g_\lambda(X) = g_\lambda(\{x_1, x_2, x_3\}) = 1$$

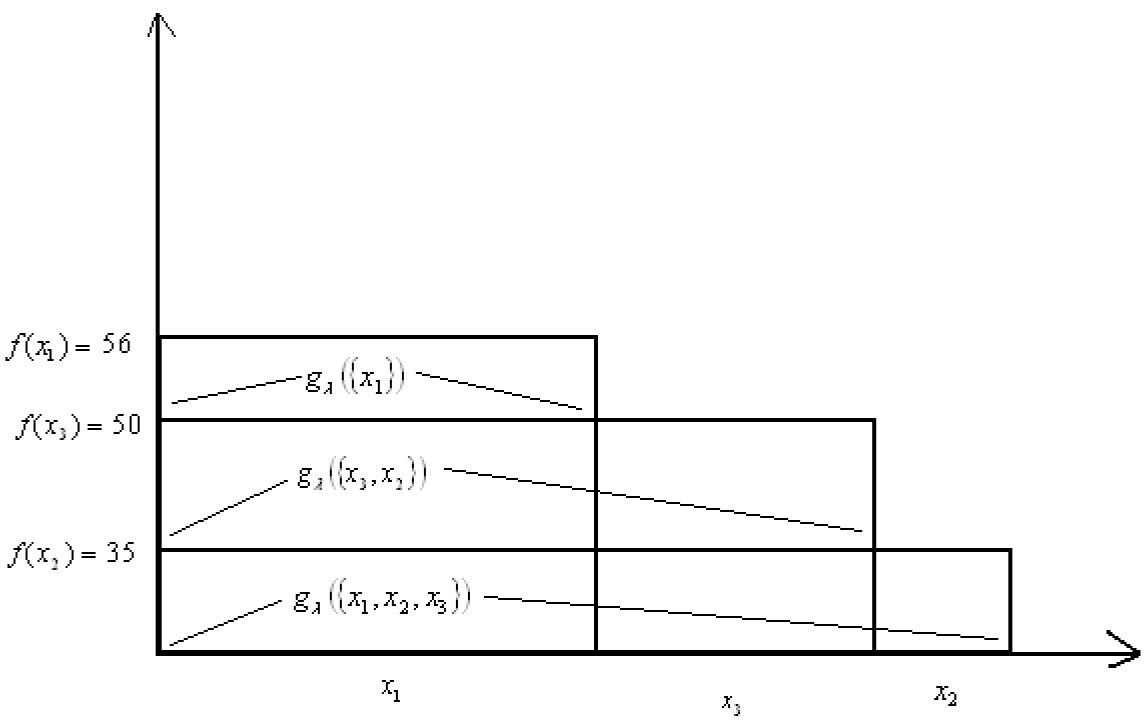
Construction of Choquet Integral

$$C_1 = (c)\int f dg_\lambda = f(x_3)\cdot g_\lambda(\{x_1,x_2,x_3\}) + (f(x_1) - f(x_3))\cdot g_\lambda(\{x_1,x_2\})$$
$$+ (f(x_2) - f(x_1))\cdot g_\lambda(\{x_2\})$$



$$C_1 = (c)\int fdg = 40*1+(45-40)*0.8090+(50-45)*0.45$$
$$C_1 = (c)\int fdg = 46.295$$

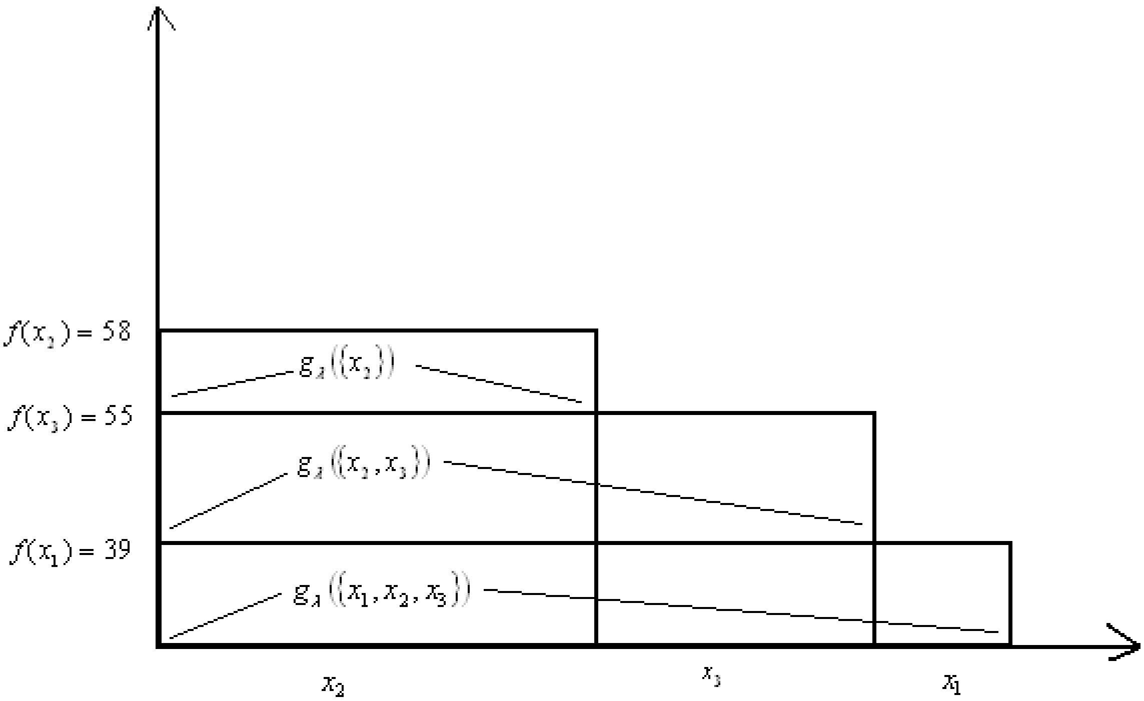
$$C_2 = (c)\int fdg$$
$$= f(x_2)\cdot g_\lambda(\{x_1,x_2,x_3\}) + \left(f(x_3)-f(x_2)\right)\cdot g_\lambda(\{x_3,x_2\}) + \left(f(x_1)-f(x_2)\right)\cdot g_\lambda(\{x_1\})$$



$$C_2 = (c)\int fdg = 35*1 + (50-35)*0.6894 + (56-50)*45$$

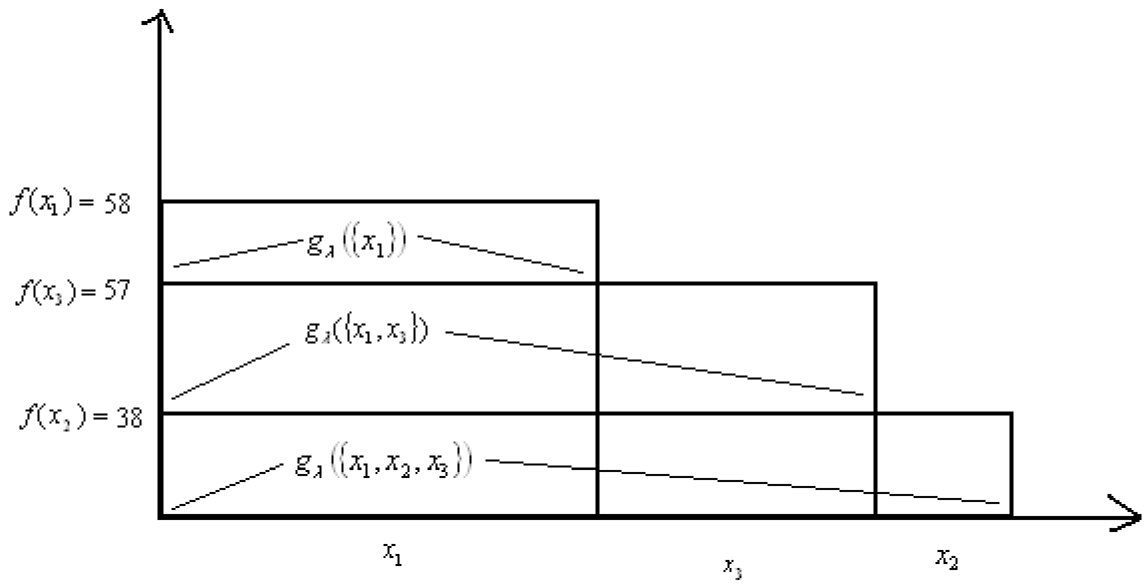
$$C_2 = (c)\int fdg = 48.041$$

$$C_3 = (c)\int fdg$$
$$= f(x_1)\cdot g_\lambda(\{x_1, x_2, x_3\}) + \left(f(x_3) - f(x_1)\right)\cdot g_\lambda(\{x_2, x_3\}) \quad + \left(f(x_2) - f(x_3)\right)\cdot g_\lambda(\{x_2\})$$



$$C_3 = (c)\int fdg = 39*1 + (55-39)*0.6894 + (58-55)*0.3$$
$$C_3 = (c)\int fdg = 51.3804$$

$$C_4 = (c)\int fdg$$
$$= f(x_2)\cdot g_\lambda(\{x_1, x_2, x_3\}) + \left(f(x_3) - f(x_2)\right)\cdot g_\lambda(\{x_3, x_2\}) + \left(f(x_1) - f(x_2)\right)\cdot g_\lambda(\{x_1\})$$



$$C_4 = (c)\int fdg = 38 * 1 + (57 - 38) * 0.6894 + (58 - 57) * 0.45$$

$$C_4 = (c)\int fdg = 51.5481$$

The ranking of students $C_4 \succ C_3 \succ C_2 \succ C_1$

5.10 Fuzzy Integral versus Traditional Method

Example 5.2: The teacher of mathematics has to evaluate her students according to their level in complex analysis and fuzzy logic. She gives equally important to complex analysis and fuzzy logic

The student c_1 has got 50 points in complex analysis and 30 in fuzzy logic out of 60 points. The student c_2 has got 45 points in complex analysis and 45 in fuzzy logic out of 60 points.

The student c_3 has got 20 points in complex analysis and 60 in fuzzy logic out of 60 points.

Suppose that $x_1 =$ "$complexanalysis$" and $x_2 =$ "$fuzzy\log ic$"

$g(\{x_1\}) = 0.7$ and $g(\{x_2\}) = 0.7$

First we will find the degree of interaction λ

According to mathematical reasoning if $\sum_{i=1}^{2} g_{\lambda}(x_i) > 1$ then -1< λ <0, According to mathematical reasoning if $\sum_{i=1}^{2} g_{\lambda}(x_i) > 1$ then -1< λ <0, λ is also called the degree of interaction.

$$\lambda + 1 = \prod_{i=1}^{n} (1 + \lambda g_i)$$

$$\lambda + 1 = (0.7\lambda + 1)^2$$

$$\lambda = -0.81632$$

And so $g_{\lambda}(\{x_1, x_2\}) = 1$

Choquet Integral

$$C_1 : \int f(x)dg = 1*30 + (50-30)*0.7 = 44$$

$$C_2 : \int f(x)dg = 1*45 = 45$$

$$C_3 : \int f(x)dg = 1*20 + (60-20)*0.7 = 48$$

The ranking of the above model

$C_3 \succ C_2 \succ C_1$.

Now we are introducing weight sum or additive model

$$C_1 : 30*\frac{0.7}{(0.7+0.7)} + 50*\frac{0.7}{(0.7+0.7)} = 40$$

$$C_2 : \quad 45*\frac{0.7}{(0.7+0.7)} + 45*\frac{0.7}{(0.7+0.7)} = 45$$

$$C_3 : \quad 20*\frac{0.7}{(0.7+0.7)} + 60*\frac{0.7}{(0.7+0.7)} = 40$$

$C_2 \succ C_3 = C_1$

According to mathematical reasoning if $g(\{x_1\}) + g(\{x_1\}) > 1$ then $\lambda < 0 \Rightarrow$ overestimation in the grades of importance if we use additive model, then $\frac{g(\{x_1\})}{g(\{x_1\}) + g(\{x_2\})} < \frac{g(\{x_1\})}{g(\{x_1, x_2\})}$ and

$\frac{g(\{x_2\})}{g(\{x_1\})+g(\{x_2\})} < \frac{g(\{x_2\})}{g(\{x_1, x_2\})}$, and we get an underestimation over evaluation if we use additive or weight sum model.

Thus if we use $\lambda < 0$,then the criteria relation have the substitutive effect ,impulse we can enhance the criteria, if we choose the professional skills then the Choquet integral is far more better than a traditional evaluation method.

5.11 The Application of Choquet and Sugeno Integral in Medical Diagnosis

Construction of Objectives [2]

We introduce the notions of a space of states $X = \{x_1, x_2, ..., x_n\}$ and a decision space (a space of alternatives) $A = \{a_1, a_2, ..., a_n\}$. We consider a decision model in which n alternatives $a_1, a_2, ..., a_n \in A$ act as drugs used to treat patients who suffer from a disease. The medicines should influence m states $a_1, a_2, ..., a_n \in A$, which are identified with m symptoms typical of the morbid unit under consideration.

Table 1: The representatives of effectiveness

Effectiveness	Representing z-value	$\mu(z)$
none	0	0
almost none	10	0.1
very little	20	0.2
little	30	0.3
rather little	40	0.4
medium	50	0.5
rather large	60	0.6
large	70	0.7
very large	80	0.8
almost complete	90	0.9
complete	100	1

(1) $b_{lj} = \frac{1}{b_{jl}}$

(2) If symptom j is more important than symptom l then b_{jl} gets assigned one of the numbers 1, 3, 5, 7 or 9 due to the difference of importance being equal, week, strong, demonstrated or absolute, respectively. If symptom l is more important than symptom j,

we will assign the value of b_{jl} .Having obtained the above judgments an $m\times m$ matrix $B=(b_{jl})_{j,l=1}^{m}$ is constructed.

(3) For example 5.4 we will construct the above judgments of $(m+2)\times(m+2)$ matrix $B=(b_{jl})_{j,l=1}^{m+2}$

The weights $w_1, w_2, ..., w_n \in W$ are decided as components of the eigenvector corresponding to the largest in magnitude eigen value of the matrix *B*.

Example 5.3 [2]: The following clinical data concerns the diagnosis "coronary heart disease". we consider the most substantial symptoms $x_1 =$ "pain in chest", $x_2 =$"changes in ECG" and

$x_3 =$ "increased level of LDL - cholesterol" . The medicines improving the patient's state are recommended as $a_1 = nitroglycerin$, $a_2 = beta - adrenergic\ blockade$ and

$a_3 = statineLDL - reductor$.

The physician has judged the relationship among efficiency of the drugs and retreat of the symptoms. We express the connection in the following table.

Table 2: The relationship between medicine action and retreat of symptom

$a_i \backslash x_j$	x_1	x_2	x_3
a_1	complete, $u_{11} = 1$	very large, $u_{12} = 0.8$	almost none, $u_{13} = 0.1$
a_2	medium, $u_{21} = 0.5$	rather large, $u_{22} = 0.6$	little, $u_{23} = 0.3$
a_3	little, $u_{31} = 0.3$	little, $u_{32} = 0.3$	very large, $u_{33} = 0.8$

Further, we concluded that the physician status of a patient is subjectively better if $x_1 =$"pain in chest" disappears. The next is assigned to $x_2 =$"changes in ECG" .And our last priority is assigned to $x_3 =$"increased level of LDL-cholesterol". We thus construct the matrix B .

$$B = \begin{matrix} & x_1 & x_2 & x_3 \\ x_1 & 1 & 3 & 5 \\ x_2 & \frac{1}{3} & 1 & 3 \\ x_3 & \frac{1}{5} & \frac{1}{3} & 1 \end{matrix} ..$$

The largest eigen value of B i – e $\lambda_{\text{"largest eigen value"}} = 3.0385$ and the correspondence eigen vector $V = (0.93295, 0.303787, 0.18659)$. V is composed of coordinates that are interpreted as the weights $w_1, w_2 and\ w_3$.

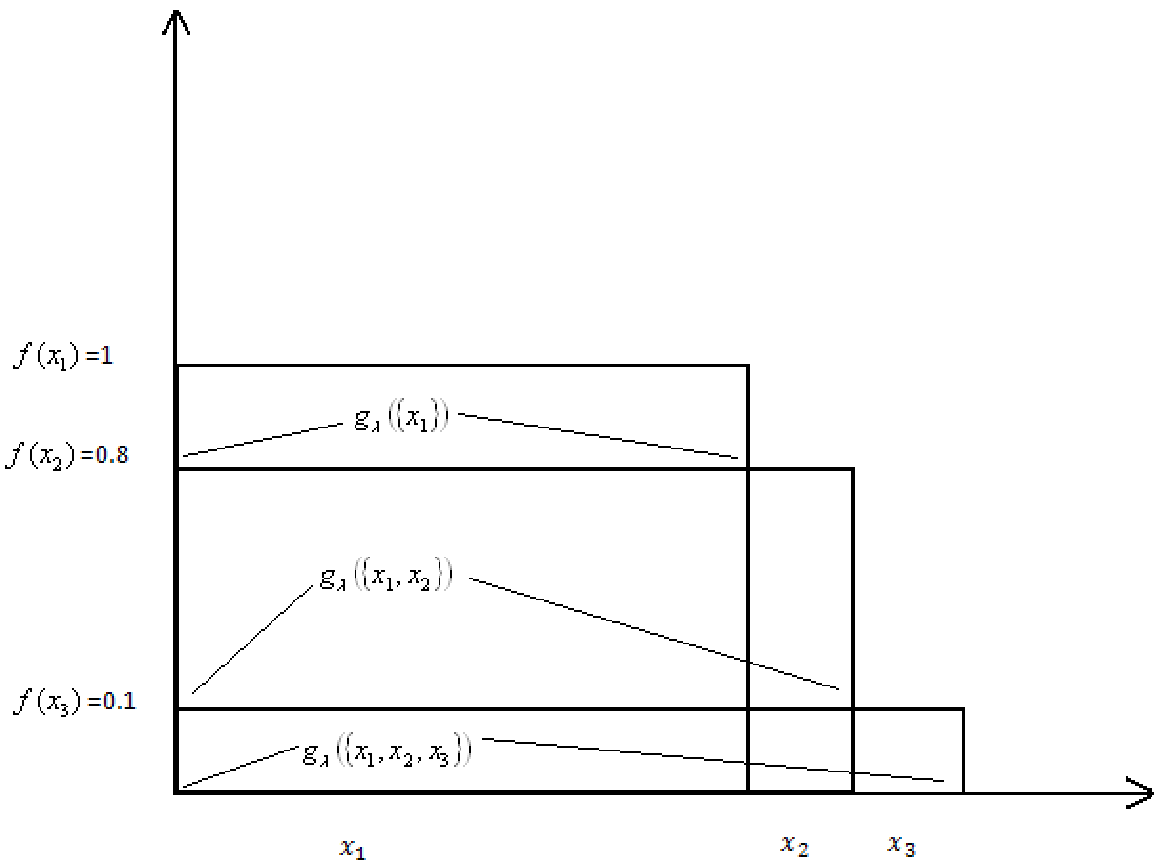
Construction of Choquet Integral

The weights $w_1, w_2, w_3, \ldots\ldots, w_n \in W$ act as the ranges of the function $g_\lambda : X \rightarrow W = [0,1]$ $w_1 = g_\lambda(x_1)$, $w_2 = g_\lambda(x_2)$ $w_3 = g_\lambda(x_3), \ldots\ldots\ldots, w_n = g_\lambda(x_n)$.

The Choquet integral for the ranges $\mathrm{f}(\mathrm{x}_3) \leq \mathrm{f}(\mathrm{x}_2) \leq \mathrm{f}(\mathrm{x}_1)$ is defined as

$$\mathrm{a}_1 = (\mathrm{c})\int fdg$$

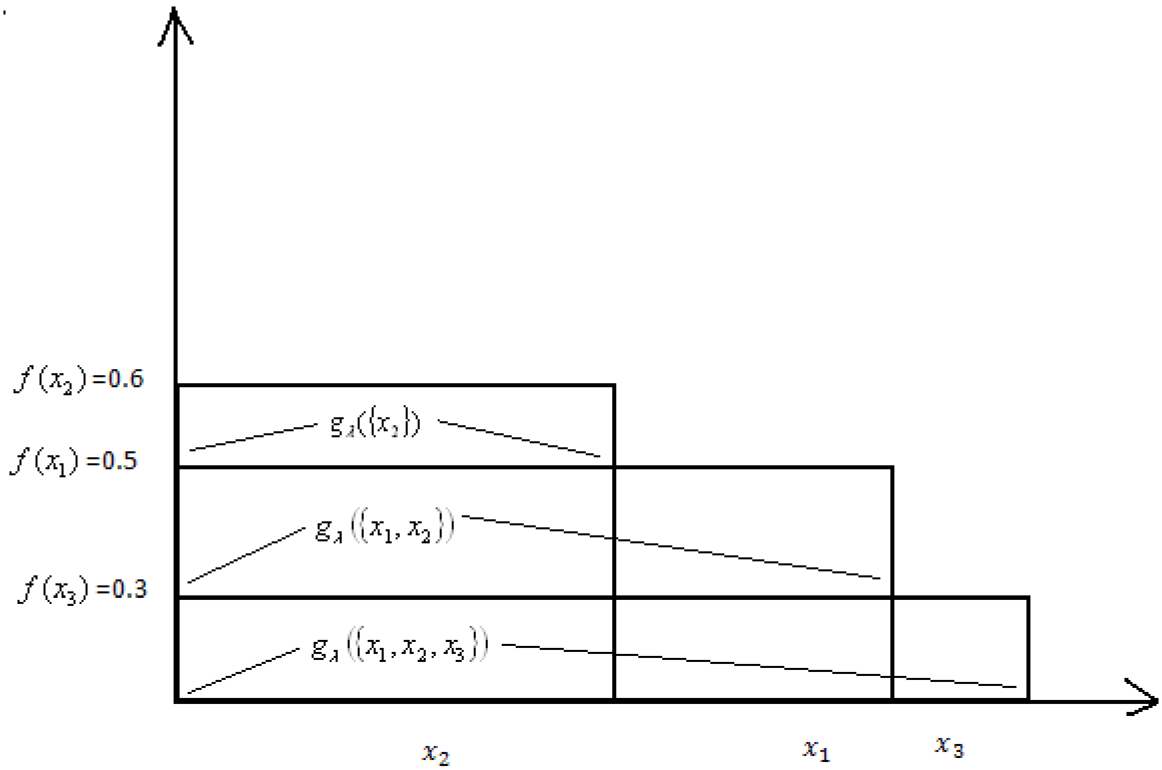
$$= f(x_3) \cdot g_\lambda(\{x_1, x_2, x_3\}) + (f(x_2) - f(x_3)) \cdot g_\lambda(\{x_1, x_2\}) + (f(x_1) - f(x_2)) \cdot g_\lambda(\{x_1\})$$



The Choquet for the ranges $\mathrm{f}(\mathrm{x}_3) \leq \mathrm{f}(\mathrm{x}_1) \leq \mathrm{f}(\mathrm{x}_2)$ is defined as

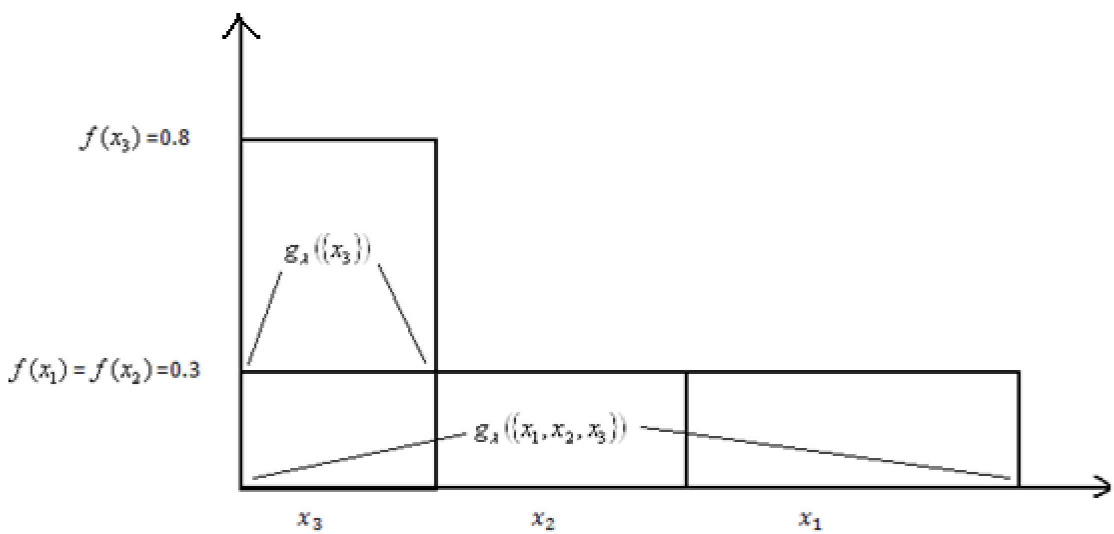
$$\mathrm{a}_2 = (\mathrm{c})\int fdg$$

$$= f(x_3) \cdot g_\lambda(\{x_1, x_2, x_3\}) + (f(x_1) - f(x_3)) \cdot g_\lambda(\{x_1, x_2\}) + (f(x_2) - f(x_1)) \cdot g_\lambda(\{x_2\})$$



The Choquet integral for the ranges $f(x_1) = f(x_2) \le f(x_3)$ is defined as

$$a_3 = (c)\int fdg$$
$$= f(x_1)\cdot g_\lambda(\{x_1, x_2, x_3\}) + (f(x_2) - f(x_1))\cdot g_\lambda(\{x_2, x_3\}) + (f(x_3) - f(x_2))\cdot g_\lambda(\{x_3\})$$



Consider

$$g_\lambda(\{x_1\}) = 0.93295$$
$$g_\lambda(\{x_2\}) = 0.303787$$
$$g_\lambda(\{x_3\}) = 0.18659$$

First we calculate the value of λ (degree of interaction)

Since $\lambda + 1 = \prod_{i=1}^{n} (\lambda g_i + 1)$

$$\lambda + 1 = (0.93295\lambda + 1)(0.303787\lambda + 1)(0.18659\lambda + 1)$$

$$0.05287\lambda^3 + 0.5141\lambda^2 + 0.42329\lambda = 0$$

And the roots of the above equation will be

$\lambda = \{0, -8.81, -0.9082\}$

But $\lambda \in (-1, \infty)$

We will take $\lambda = -0.9082$ only, because $\lambda = 0$ is additively.

If $\lambda = -0.9082$ then

$$g_\lambda(\{x_1, x_2\}) = g_\lambda(\{x_1\}) + g_\lambda(\{x_2\}) + \lambda g_\lambda(\{x_1\}) g_\lambda(\{x_2\}) = 0.979386$$

$$g_\lambda(\{x_2, x_3\}) = g_\lambda(\{x_3\}) + g_\lambda(\{x_2\}) + \lambda g_\lambda(\{x_3\}) g_\lambda(\{x_2\}) = 0.438924$$

$$g_\lambda(\{x_1, x_3\}) = g_\lambda(\{x_1\}) + g_\lambda(\{x_3\}) + \lambda g_\lambda(\{x_1\}) g_\lambda(\{x_3\}) = 0.961496$$

$$g_\lambda(X) = 1$$

$$a_1 = (c)\int f dg$$
$$= f(x_3) \cdot g_\lambda(\{x_1, x_2, x_3\}) + (f(x_2) - f(x_3)) \cdot g_\lambda(\{x_1, x_2\}) + (f(x_1) - f(x_2)) \cdot g_\lambda(\{x_1\})$$

$$= 0.1 * 1 + (.8 - 0.1) * 0.9793 + (1 - 0.8) * 0.93295$$

$$= 0.97$$

$$a_2 = (c)\int f dg$$
$$= f(x_3) \cdot g_\lambda(\{x_1, x_2, x_3\}) + (f(x_1) - f(x_3)) \cdot g_\lambda(\{x_1, x_2\}) + (f(x_2) - f(x_1)) \cdot g_\lambda(\{x_2\})$$

$$= 0.3*1 + (0.5 - 0.3)*0.96149 + (.6 - 0.5)*0.303$$

$$= 0.52$$

$$a_3 = (c)\int fdg$$

$$= f(x_1) \cdot g_\lambda(\{x_1, x_2, x_3\}) + (f(x_2) - f(x_1)) \cdot g_\lambda(\{x_2, x_3\}) + (f(x_3) - f(x_2)) \cdot g_\lambda(\{x_3\})$$

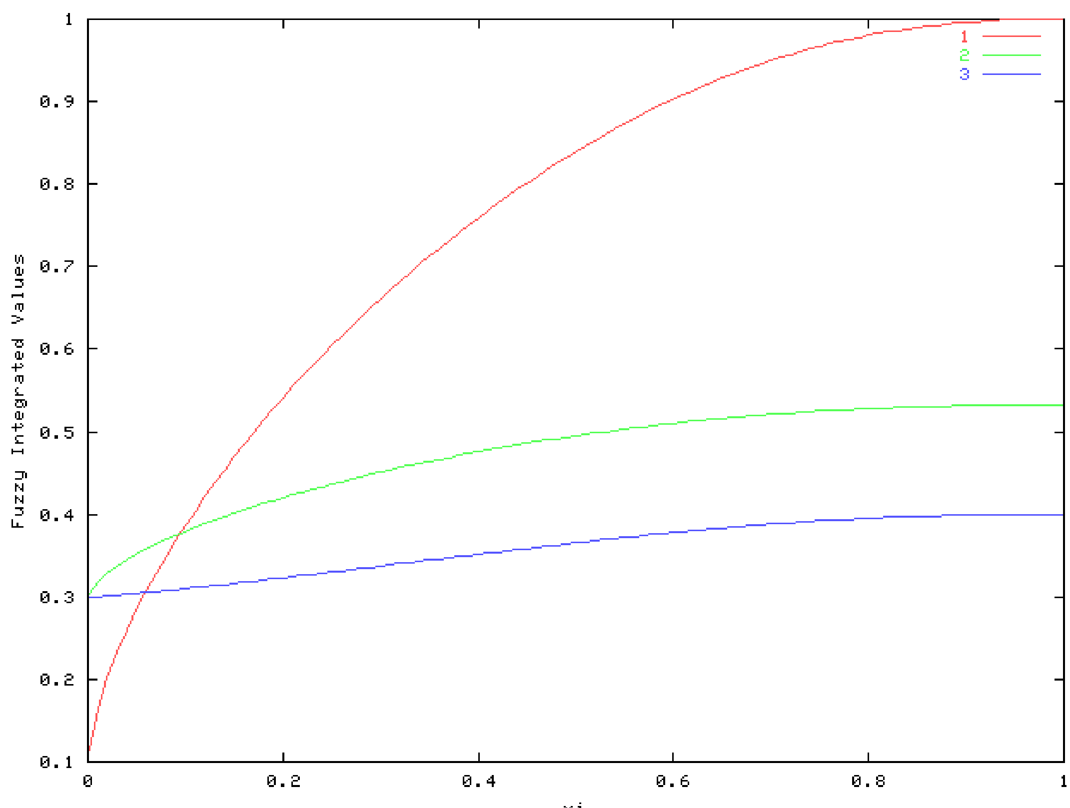
$$= 0.3*1 + 0 + (.8 - 0.3)*0.1866$$

$$= 0.39$$

The interpretation of Choquet integral in the drug ranking $a_1 \succ a_2 \succ a_3$

Choquet Integrated Values

No.	x_1	x_2	x_3	Choquet Integrated Values
1	1	0.8	0.1	0.972174
2	0.5	0.6	0.3	0.526261
3	0.3	0.3	0.8	0.393305



The Sugeno integral in the drug order

Construction of Sugeno integral

$$a_1 = \int fdg_\lambda = \max\left(\min\left(f(x_3), g_\lambda(\{x_1, x_2, x_3\})\right), \min\left(f(x_2), g_\lambda(\{x_1, x_2\})\right), \min\left(f(x_1), g_\lambda(\{x_1\})\right)\right)$$

$$a_2 = \int fdg_\lambda = \max\left(\min\left(f(x_3), g_\lambda(\{x_1, x_2, x_3\})\right), \min\left(f(x_1), g_\lambda(\{x_1, x_2\})\right), \min\left(f(x_2), g_\lambda(\{x_2\})\right)\right)$$

$$a_3 = \int fdg_\lambda = \max\left(\min\left(f(x_1), g_\lambda(\{x_1, x_2, x_3\})\right), \min\left(f(x_2), g_\lambda(\{x_3, x_2\})\right), \min\left(f(x_3), g_\lambda(\{x_3\})\right)\right)$$

$$a_1 = \max(\min(0.1,1), \min(0.8,0.979386), \min(1,.93295))$$

$$a_1 = \max(0.1,0.8,0.93295)$$

$$a_1 = 0.93295$$

$$a_2 = \max(\min(0.3,1), \min(0.5,0.979386), \min(0.6,0.93295))$$

$$a_2 = \max(0.3,0.5,0.303787)$$

$$a_2 = 0.5$$

$$a_3 = \max(\min(0.3,1), \min(0.3,0.438942), \min(0.8,0.18659))$$

$$a_3 = \max(0.3,0.3,0.18659)$$

$$a_3 = 0.3$$

The interpretation Sugeno integral in the drug ranking $a_1 \succ a_2 \succ a_3$.

Choquet and Sugeno integrals of fuzzy decision process in a choice of optimal medicines.

Construction of Objectives [3]

We introduce the notions of a space of states $X = \{x_1, x_2, ..., x_n\}$ and a decision space (a space of alternatives) $A = \{a_1, a_2, ..., a_n\}$. We consider a decision model in which n alternatives $a_1, a_2, ..., a_n \in A$ act as drugs used to treat patients who suffer from a disease. The medicines should influence m states $a_1, a_2, ..., a_n \in A$, which are identified with m symptoms typical of the morbid unit under consideration.

The drugs-decisions constitute n elements in supports of fuzzy sets K_K, $\mathrm{k} = 1, \ldots, \mathrm{m}, \mathrm{m}+1, \mathrm{m}+2$, determined as some criteria-objectives, which restrict the set A [13].

Hence, we can treat each set K_K as a fuzzy subset of A, i.e., $K_K : A \to [0,1]$, $k = 1, \ldots, m, m+1, m+2$. [3](More explanation)

In the model of accepting the most optimal medicine a_i $i = 1, \ldots, n$, we assume that the first *m* restriction sets K_j, $j = 1, \ldots, m$, are defined by

$$K_j = \text{" influence of } a_1, \ldots\ldots, a_n \text{ on symptom } x_j\text{"} = \frac{a_1\text{'s effect concerning}}{a_1} + \ldots\ldots + \frac{a_n\text{'s effect concerning}}{a_n} \quad (1)$$

In spite of drug effectiveness, which definitely is the most important factor in the appreciation of drug action, we can introduce other substantial elements assisting drug decision making like side effects of medicines or their prices. We thus form the next fuzzy set.

$$K_{m+1} = \text{"side effect of } a_1, \ldots\ldots, a_n \text{ supporting the decision positively"} = \frac{1 - \text{side effect of } a_1}{a_1} + \ldots\ldots + \frac{1 - \text{side effect of } a_n}{a_n} \quad (2)$$

in which a physician estimates the strength of all side effects of the drugs. The side effects of drugs a_i, $i = 1, \ldots, n$, are rather unfavorable occurrences; therefore their lack in a_i should be emphasized by the larger membership value as-signed to a_i as an indication of a safe medicine consumption. For the purpose of enlarging membership values of these medicines that have not extensive side effects we use the complement operation 1–estimation of side effects.

The last constraint

$$K_{m+2} = \text{"estimation of price availability for } a_1, \ldots\ldots, a_n\text{"} =$$

$$\frac{\text{price availability of } a_1}{a_1} + \ldots\ldots + \frac{\text{price availability of } a_n}{a_n} \quad (3)$$

is added in order to enlarge a number of decisive indications.

Not all symptoms retreat after the cure has been carried out. One can only sometimes soothe their negative effects by, for example, the lowering of an excessive level of the indicator, the relief of pain, and the like.

Let us find a practical way of determining effectiveness of drugs as mathematical expressions, which should take place in the first *m* objectives. To simplify the symbols we assume that each symptom , where *X* is a space of symptoms (states), is understood as the result of the treatment of the symptom after the cure with the drugs $a_1, \ldots\ldots, a_n$ has been carried out.

Example 5.4 [3]: We have obtained the clinical data, which concerns the diagnosis "*coronary heart disease*". We consider the most substantial symptoms x_1 = "*pain in chest*", x_2 = "*changes in EKG*" and x_3 = "*increased level of LDL-cholesterol*". The recommended medicines that can improve the patient's state are listed as a_1 = *nitroglycerin*, a_2 = *beta-adrenergic blockade*, a_3 = *acetylsalicylic acid* (*aspirin*) and a_4 = *statine LDL-reductor*.

The physician has judged the relationship between efficiency of drugs and retreat of symptoms. We express the connections in Table 3

Table 3: The relationship between medicine action and retreat of symptom.

Drug action \Symptoms	x_1	x_2	x_3
a_1	complete	very large	almost none
a_2	medium	medium	little
a_3	little	little	very little
a_4	little	little	very large

$$K_1 = \text{" influence of } a_1, a_2, a_3, a_4 \text{ on } x_1\text{"} = \frac{1}{a_1} + \frac{0.5}{a_2} + \frac{0.3}{a_3} + \frac{0.3}{a_4}$$

$$K_2 = \text{" influence of } a_1, a_2, a_3, a_4 \text{ on } x_2\text{"} = \frac{0.8}{a_1} + \frac{0.5}{a_2} + \frac{0.3}{a_3} + \frac{0.3}{a_4}$$

and

$$K_3 = \text{" influence of } a_1, a_2, a_3, a_4 \text{ on } x_3\text{"} = \frac{0.1}{a_1} + \frac{0.3}{a_2} + \frac{0.2}{a_3} + \frac{0.8}{a_4}$$

The physician has evaluated side effects of the drugs in the set K_4 by assimilating the words from the first column of Table 1. We have already mentioned that the side effects of $a_i, i = 1, \ldots, 4$, are negative appearances and their lack in a_i, e.g., "*side effects* of a_i" = "*almost none*", should be expressed by the larger membership value assigned to a_i. We thus adopt the complement operation 1–estimation of side effects. The set K_4 is established in accordance with (2) as

$$K_4 = \text{"side effects of } a_1, a_2, a_3, a_4\text{"} = \frac{1 - \text{very little}}{a_1} + \frac{1 - \text{little}}{a_2} + \frac{1 - \text{rather large}}{a_3} + \frac{1 - \text{very little}}{a_4}$$

$$\mathrm{K}_4 = \text{"side effects of } \mathrm{a}_1, a_2, a_3, a_4\text{"} = \frac{1-0.2}{a_1} + \frac{1-0.3}{a_2} + \frac{1-0.6}{a_3} + \frac{1-0.2}{a_4}$$

$$\mathrm{K}_4 = \text{"side effects of } \mathrm{a}_1, a_2, a_3, a_4\text{"} = \frac{0.8}{a_1} + \frac{0.7}{a_2} + \frac{0.4}{a_3} + \frac{0.8}{a_4}$$

The prices of all medicines are not at the least inconvenient for patients to purchase them. If we note that the large value of a membership degree corresponds to a rather cheap and available medicine we can state the set K_5 by examining (3) as

$$\mathrm{K}_5 = \text{"price availability } \mathrm{a}_1, a_2, a_3, a_4\text{"} = \frac{0.8}{a_1} + \frac{0.8}{a_2} + \frac{0.9}{a_3} + \frac{0.8}{a_4}$$

The sets K_1–K_5 found in now utilized as columns of the matrix C, determined by a table

$$C = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 0.8 & 0.1 & 0.8 & 0.8 \\ 0.5 & 0.5 & 0.3 & 0.7 & 0.8 \\ 0.3 & 0.3 & 0.2 & 0.4 & 0.9 \\ 0.3 & 0.3 & 0.8 & 0.8 & 0.8 \end{bmatrix} \end{array}$$

The physical status of the patient as in example 5.2

The costruction of matrix B

$$B = \begin{array}{c} \\ K_1 \\ K_2 \\ K_3 \\ K_4 \\ K_5 \end{array} \begin{array}{c} \begin{array}{ccccc} K_1 & K_2 & K_3 & K_4 & K_5 \end{array} \\ \begin{bmatrix} 1 & 3 & 5 & 7 & 7 \\ \frac{1}{3} & 1 & 3 & 7 & 7 \\ \frac{1}{5} & \frac{1}{3} & 1 & 7 & 7 \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 1 & 3 \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{3} & 1 \end{bmatrix} \end{array}.$$

The largest eigen value of B i – e $\lambda_{\text{"largest eigen valu e"}} = 5.5805$ and corresponding eigen vector $\mathrm{V} = (0.83215, 0.46393, 0.26609, 0.08575, 0.055586)$. V is composed of coordinates that are interpreted as the weights $w_1, w_2, w_3, w_4 and w_5$.

Consider $g_\lambda(x_1) = .83215$

$g_\lambda(x_2) = .46393$

$g_\lambda(x_3) = .26609$

$$g_\lambda(x_4) = .08575$$

$$g_\lambda(x_5) = .055586$$

Since $\lambda + 1 = \prod_{i=1}^{n} (1 + \lambda g_i)$

so $\lambda + 1 = (0.83215\lambda + 1)(0.46393\lambda + 1)(0.26609\lambda + 1)(.08575\lambda + 1)(.055586\lambda + 1)$

$$.00494\lambda^5 + 0.018085\lambda^4 + 0.2139099\lambda^3 + 0.957524\lambda^2 + 0.704166\lambda = 0$$

$$\lambda = \{0, -18.9410, -8.3819 + 3.5974i, -8.3819 - 3.5974i, -0.9046\}$$

Since $\lambda \in (-1, \infty)$

We will take only $\lambda = -0.9046$

If $\lambda = -0.9046$ then

$$g_\lambda(\{x_1, x_2\}) = 0.946897$$

$$g_\lambda(\{x_1, x_3\}) = 0.897991$$

$$g_\lambda(\{x_2, x_3\}) = .618389$$

$$g_\lambda(\{x_1, x_4\}) = 0.853412$$

$$g_\lambda(\{x_2, x_4\}) = 0.51373$$

$$g_\lambda(\{x_3, x_4\}) = 0.331224$$

$$g_\lambda(\{x_1, x_5\}) = 0.845955$$

$$g_\lambda(\{x_2, x_5\}) = 0.496225$$

$$g_\lambda(\{x_3, x_5\}) = 0.308319$$

$$g_\lambda(\{x_4, x_5\}) = 0.137034$$

$$g_\lambda(\{x_1, x_2, x_3\}) = .0.985067$$

$$g_\lambda(\{x_1, x_2, x_4\}) = 0.959198$$

$$g_\lambda(\{x_1, x_3, x_4\}) = 0.914086$$

$g_{\lambda}(\{x_2, x_3, x_4\}) = 0.656174$

$g_{\lambda}(\{x_1, x_2, x_5\}) = 0.954871$

$g_{\lambda}(\{x_1, x_3, x_5\}) = 0.908424$

$g_{\lambda}(\{x_2, x_3, x_5\}) = 0.642882$

$g_{\lambda}(\{x_1, x_4, x_5\}) = 0.866086$

$g_{\lambda}(\{x_2, x_4, x_5\}) = 0.543487$

$g_{\lambda}(\{x_3, x_4, x_5\}) = 0.370158$

$g_{\lambda}(\{x_1, x_2, x_3, x_4\}) = 0.994407$

$g_{\lambda}(\{x_1, x_2, x_3, x_5\}) = 0.991122$

$g_{\lambda}(\{x_1, x_2, x_4, x_5\}) = 0.966553$

$g_{\lambda}(\{x_1, x_3, x_4, x_5\}) = 0.92371$

$g_{\lambda}(\{x_2, x_3, x_4, x_5\}) = 0.678767$

$g_{\lambda}(\{x_1, x_2, x_3, x_4, x_5\}) = 1$

Now construction of Choquet integral

$a_1 = (c)\int fdg$

$= f(x_3) \cdot g_{\lambda}(\{x_1, x_2, x_3, x_4, x_5\}) + (f(x_4) - f(x_3)) \cdot g_{\lambda}(\{x_1, x_2, x_4, x_5\}) + (f(x_1) - f(x_4)) \cdot g_{\lambda}(\{x_1\})$

$a_1 = (c)\int fdg = 0.1 * 1 + 0.7 * 0.966553 + 0.2 * 0.83215$

$a_1 = (c)\int fdg = 0.943$

$a_2 = (c)\int fdg = f(x_3) \cdot g_{\lambda}(\{x_1, x_2, x_3, x_4, x_5\}) + (f(x_2) - f(x_3)) \cdot g_{\lambda}(\{x_1, x_2, x_4, x_5\})$
$+ (f(x_4) - f(x_2)) \cdot g_{\lambda}(\{x_5, x_4\}) + (f(x_5) - f(x_4)) g_{\lambda}(\{x_5\})$

$a_2 = (c)\int fdg = .3 + .2 * .966553 + .2 * .137034 + .1 * 0.055586$

$a_2 = 0.5263$

$a_3 = (c)\int fdg$

$$= f(x_3) \cdot g_\lambda(\{x_1, x_2, x_3, x_4, x_5\}) + \left(f(x_2) - f(x_3)\right) \cdot g_\lambda(\{x_1, x_2, x_4, x_5\})$$
$$+ \left(f(x_4) - f(x_2)\right) \cdot g_\lambda(\{x_5, x_4\}) + (f(x_5) - f(x_4)) g_\lambda(\{x_5\})$$

$a_3 = (c)\int fdg \; = .2*1 + .1*.966553 + .1*.137034 + .5*0.055586$

$a_3 = 0.3382$

$a_4 = (c)\int fdg$

$$= f(x_1) \cdot g_\lambda(\{x_1, x_2, x_3, x_4, x_5\}) + \left(f(x_4) - f(x_1)\right) \cdot g_\lambda(\{x_3, x_4, x_5\})$$

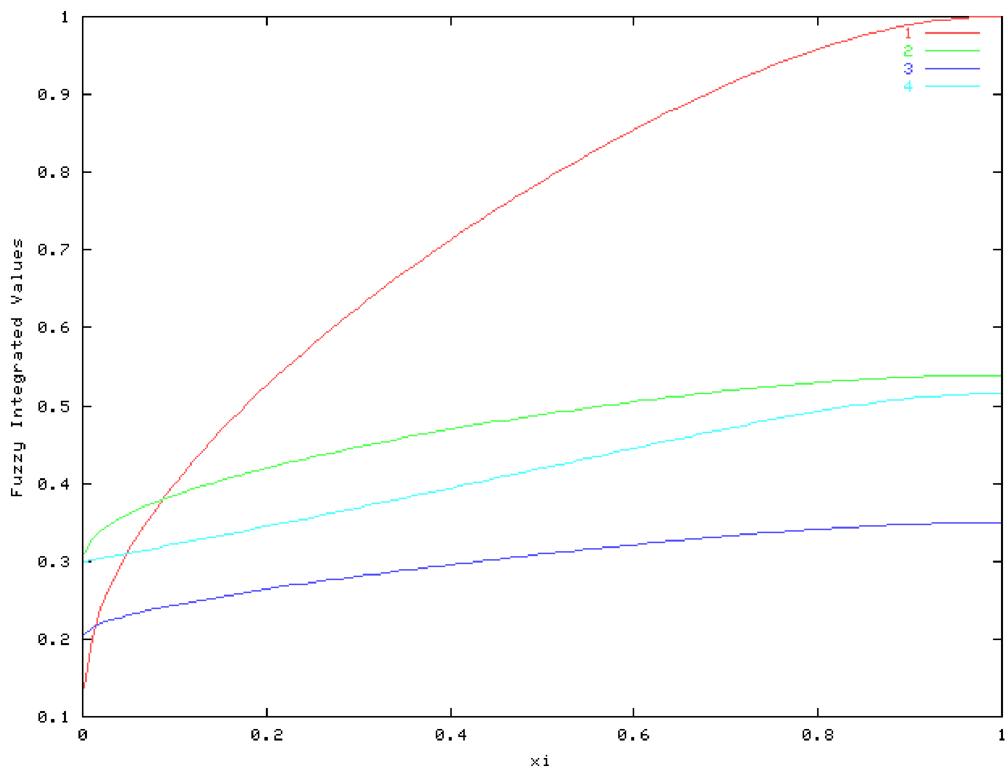
$a_4 = (c)\int fdg \; = 0.3*1 + 0.5*0.370158$

$a_4 = 0.4851$

The interpretation of Choquet integral in the drug ranking $a_1 \succ a_2 \succ a_4 \succ a_3$

Choquet Integrated Values

No.	x_1	x_2	x_3	x_4	x_5	Choquet Integrated Values
1	1	0.8	0.1	0.8	0.8	0.943029
2	0.5	0.5	0.3	0.7	0.8	0.526274
3	0.3	0.3	0.2	0.4	0.9	0.338152
4	0.3	0.3	0.8	0.8	0.8	0.485077



Construction of Sugeno Integral

$$a_1 = \max\begin{pmatrix} \min(f(x_3), g_\lambda(\{x_1, x_2, x_3, x_4, x_5\})), \min((f(x_2), g_\lambda(\{x_1, x_2, x_4, x_5\})), \min((f(x_4), g_\lambda(\{x_1, x_4, x_5\})), \\ \min((f(x_5), g_\lambda(\{x_1, x_5\})), \min((f(x_1), g_\lambda(\{x_1\})) \end{pmatrix}$$

$$a_2 = \max\begin{pmatrix} \min(f(x_3), g_\lambda(\{x_1, x_2, x_3, x_4, x_5\})), \min((f(x_1), g_\lambda(\{x_1, x_2, x_4, x_5\})), \min((f(x_2), g_\lambda(\{x_2, x_4, x_5\})), \\ \min((f(x_4), g_\lambda(\{x_4, x_5\})), \min((f(x_5), g_\lambda(\{x_5\})) \end{pmatrix}$$

$$a_3 = \max\begin{pmatrix} \min(f(x_3), g_\lambda(\{x_1, x_2, x_3, x_4, x_5\})), \min((f(x_1), g_\lambda(\{x_1, x_2, x_4, x_5\})), \min((f(x_2), g_\lambda(\{x_2, x_4, x_5\})), \\ \min((f(x_4), g_\lambda(\{x_4, x_5\})), \min((f(x_5), g_\lambda(\{x_5\})) \end{pmatrix}$$

$$a_4 = \max\begin{pmatrix} \min(f(x_1), g_\lambda(\{x_1, x_2, x_3, x_4, x_5\})), \min((f(x_2), g_\lambda(\{x_2, x_3, x_4, x_5\})), \min((f(x_3), g_\lambda(\{x_3, x_4, x_5\})), \\ \min((f(x_4), g_\lambda(\{x_4, x_5\})), \min((f(x_5), g_\lambda(\{x_5\})) \end{pmatrix}$$

$$a_1 = \max(\min(0.1,1).\min(0.8,0.966), \min(0.8,0.866), \min(0.8,0.84), \min(1,0.83215))$$

$a_1 = \max(0.1, 0.8, 0.8, 0.8, 0.83215)$

$a_1 = 0.83215$

$a_2 = \max(\min(0.3, 1), \min(0.5, 0.966), \min(0.5, 0.453), \min(0.7, 0.137034), \min(0.8, 0.05586)$

$a_2 = \max(0.3, 0.5, 0.453, 0.1347034, 0.05586)$

$a_2 = 0.5$

$a_3 = \max(\min(0.2, 1), \min(0.3, 0.966), \min(0.3, 0.453), \min(0.4, 0.137034), \min(0.9, 0.05586)$

$a_3 = \max(0.2, 0.3, 0.3, 0.137034, 0.05586)$

$a_3 = 0.3$

$a_4 = \max(\min(0.3, 1), \min(0.3, 0.678), \min(0.8, 0.370158), \min(0.8, 0.137034), \min(0.8, 0.05586)$

$a_4 = \max(0.3, 0.3, 0.370158, 0.137034, 0.05586)$

$a_4 = 0.370158$

The interpretation of Sugeno integral in the drug ranking $a_1 \succ a_2 \succ a_4 \succ a_3$

Conclusion

In typical multi-attribute evaluation process, each attribute must be independent from each other. Therefore the characteristics that have interaction among the criteria in real system cannot be solved by the concept of tradition additives measure alone. This review has shown the richness of these new tools for aggregation, the Sugeno-Choquet integrals. Consequently the hierarchical structure evaluation system of human subjective decision making by using $\lambda - fuzzy$ measures and Choquet-Sugeno integrals. I hope that this will encourage mathematician to use this technique in multi-criteria decision making.

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