2007:2

Monitoring the volatility in a process which reflects trading in the financial market

Alaba Femi Awomewe

Thesis for the degree Master of Mathematical Modelling and Simulation,
10 credits (15 ECTS credits)

October 2007

Blekinge Institute of Technology
School of Engineering
Department of Mathematics and Science
Supervisors: Assistant Professor Eric Järpe (University of Halmstad)
Assistant Professor Claes Jogréus (Blekinge Institute of Technology)

Monitoring the volatility in a process which reflects trading in the financial market

Alaba Femi AWOMEWE

Blekinge Institute of Technology
Thesis Report

Master's Thesis in Mathematical Modeling and Simulation, 15 ECTS credits

Supervisor: Ass. Prof. Eric Järpe
Supervisor: Ass. Prof. Claes Jogréus

October 10, 2007

Department of Mathematics and Science
School of Engineering
Blekinge Institute of Technology
Karlskrona, Sweden

Preface

I give praise, honour and adoration to Almighty God for giving me the privilege to be able to complete this master programme successfully.

My sincere appreciation goes to my supervisors Ass. Professor Eric Järpe and Ass. Professor Claes Jogréus for their assistance and advice through out the period of writing this thesis; I will forever be grateful to you. I can not but also thank Professor Nail Ibragimov, Professor Elisabeth Rakus-Andersson and other lecturers for their invaluable contribution towards the achievement of this academics excellence.

Finally, I thanks my family members most especially Benson Awomewe, Gabriel Awomewe and Tope Oluwajana for there moral and financial support. I appreciate the efforts of my friends, Awodola Joseph, Dele Ogundele, Kehinde Adenuga, Damian Erewu, Femi Adelere, Oluwatoba Aduke and others for all their assistance.

Thank you all.

Alaba Femi AWOMEWE

Abstract

Recently, the financial market has become an area of increased research interest for mathematicians and statisticians. The Black and Scholes breakthrough in this area triggered a lot of new research activity. Commonly the research concerns the log returns of assets (shares, bond, foreign exchange, option). The variation in the log returns is called volatility and it is widely studied and because of its relevance for applications in the financial world. The volatility is mostly used for measuring the risk and also for forecasting future prices. In this work a process of trading activities is considered. It is assumed that at a random time-point a parameter change in the law of the trading process occurs, indicating changed trading behaviour. For inferential matters about the process it is of vital importance to be able to state that such a change has occurred quickly and accurately. The methods used to this end are stopping rules which signal alarm as soon as some statistic based on the on-line observations goes beyond some boundary. The model considered for the process of log returns is the family of Autoregressive Conditional Heteroskedastic (ARCH) model. It is widely accepted that this well describes a lot of phenomena in the financial market. In this work statements about this process will be derived, the stopping rules will be defined, evaluated and their properties discussed.

Contents

Chapter 1

Introduction

Modern finance started in the 1950s and 1960s with major step of development, of application of mathematics into finance, through the breakthrough that Markowitz, Trenor, Sharpe, Lintner, and Mossin see [26, 38, 23, 35, 28]. This led to the Capital Asset Pricing Models in the 1960s which became a quantitative model for measuring risk. Also important influence of research in investment practices in the 1960s by Samuelson and Fama [32, 14] efficient market hypothesis, which roughly says that security prices reflect information fully and immediately.

In 1973, two important papers were published; Black and Scholes [5] and Merton [27] brought a special revolution with their pricing method. It would be difficult to name theoretical works in financial literature without referencing to these two papers. Their results were the start of an intensified development of the science of financial mathematics and the applications stemming from them. It became the source of inspiration for other studies of more complex options and other types of derivatives. There is also the theory of option price in Bachelier [1] which developed much of the mathematics underlying modern economics theories on efficient markets, the random-walk model and Browian motion five year before Einsten [12].

The development of models for volatility has widened the horizon of the research of the financial market. The work of Anderson and Bollerslev (1998) [7], Schwert (1989) [34], and Barndorff-Neilsen and Sherpherd (2002) [4] are some recent examples of development of the models for activities in the financial market.

The volatility of financial derivatives reflects the intensity of the fluctuations of prices. Bachelier defined volatility as a degree of nervousness or of instability of the price. The volatility development in price, and returns on assets play a special role in the sense of the stress factor and stress indicator. In the level of market it is very important to differentiate between the level of the volatility that reflects the fundamental justified price movement and the excess volatility. Good and bad news of the financial market creates changes in volatility. While bad is assumed to bring about a decline in the log price of the asset, good news are considered to imply an increase in the log price. Thus both good and bad news influence an increased level of the volatility.

The volatility of the main financial instruments price, i.e exchange rates, interest rate, stock indexes can be considered as the "degree of risk". High volatility signifies high risk and vice versa.

There are two kinds of indicators of volatility, namely: historical volatility and implied volatility. While historical volatility is "backward looking", implied volatility is "forward looking". Professionals prefer to quantify the volatility of the market price in a "forward looking" manner in which it is done by deriving or deducing volatility figures from its option price, stock index, or exchange rate.

Many researchers have done a lot of works in the area of volatilities by using family of Autoregressive Conditional Heteroscedastic (ARCH) model, e.g. Engle, Lilje and Robins (1987) [13], Bollerslev, Engle and Wooldridge (1988) [6],Chou (1988) [10] Campell, Henschel (1992) [8] ,Backus and Gregory (1993) [2] and others.

Volatility has also been useful to forecast stock prices. The more stationary the volatility is the better our model will be in forecasting the future price or return of the asset. In this research work changes in the log price of the financial instrument are studied. Time is considered to be discrete. The underlying assumption and properties of ARCH family are discussed. Properties of the ARCH family models are derived and the behaviour is analysed. Estimates of the volatility parameter by using maximum likelihood method is found. A stopping rule is created such that when the variation or a function of the historical variation values is moving beyond some boundary, an alarm is signalled.

Basic terms and terminology of financial market and financial instrument will be discussed below, so that when later in the thesis work such is mentioned it will not be strange.

1.1 Financial Markets

A financial market is a medium which allows people to trade financial instruments (stocks, currencies, bonds, etc). Basically investors or governments can not all the time have enough funds to execute their projects, so they need to reach wealthy individuals to raise funds. These funds are channeled through the financial market through the giver (saver) and the borrower. The giver give out the money by buying stock, bond, securities and the borrower in return issues stock or bond for the money collected. By this act the saver earns interest in his money which is much better than keeping the money and the borrower will have enough funds to execute its projects which will bring more profit or benefit. This process will amplify the improvement of the national economy.

More over, the financial market which dates back several hundred years has been experiencing improvement of the liquidity everyday. Previously the only instrument traded was precious metal, while nowadays many commodities are traded in the financial market. This improvement has changed the face of the market, from borrower and saver; it has become a typical issue of buyer and seller. That is a buyer may want to buy a stock to keep for a period when the price will increase, in order to sell and make high returns.

Basically, the financial market is rather different from other markets because, intermediaries are required in transacting business. The intermediaries are called financial institutions; they are among others banks, insurance companies, pension funds, stock exchanges, investment companies and so on.

The financial market can be divided into primary and secondary. In the primary market such sales of government and corporate body equity can be done by private or public offer. An investment bank underwrites the equities of the issue bodies and then resells them to public. Although it is a risk for the investment bank, but it tries as much as possible putting all effort to reduce the risk of not being able to sell equities it underwrites. While secondary market, is the place where the investor can trade their stock that was purchased in the primary market. Major example of the secondary market is stock exchange; we have New York stock exchange, Nigeria stock exchange and many others.

1.2 Financial Instruments

These are the instruments that are traded in the financial market. They are broadly divided into two group; primary (underlying) instruments and secondary (derivative) instruments. The primary instruments comprises of stocks, bonds, bank accounts, foreign currencies and the derivatives consist of options, future contracts, warrants, swaps. For better understanding we will discuss on only few of the afore-mentioned instruments below.

1.3 Underlying Instruments

- **Bank Account**
  Bank account can be referred to as the fund that individual has entrusted in bank in which the bank is obliged to pay interest. It is considered primarily because it is a unit of measurement of various securities. The interest on bank account is normally calculated as a simple interest or a compound interest. For instance if we deposit an initial money called B_0, and we can measured the return on the amount in N years knowing the rate of the interest. It is an investment because it brings some returns.

- **Stock**
  This is fund raise by corporation through the issuance house by distributing of shares. Shares are divided into two categories, ordinary shares and preference shares. The ordinary shares are referred to equities or common stock; while the preference shares are referred to preferred stock. The major different between these two stocks are, the dividend received by ordinary shares holder is proportional to the profit or the success of the company; while the preference shares dividend is fixed. The risk borne by ordinary shares is more than preference shares holders. When buying the stock the investor is not mostly concerned about the dividend, but rather they are concerned about the future fluctuation of the stock which earns them more returns. The trading of the stock in secondary market (stock exchange) involves approaching a stock broking firm that will acts as intermediary. The area of the interest is the return on buying a stock to date and sell in the future. For instance let us consider a stock price S_0 today and sell at price S_1 in the future, the return in this will be $S_1 - S_0$, the analysis of the return will be discussed later in this work.

- **Bonds**
  These are promissory notes issued to raise fund by government or private enterprises. In more precisely it is debt security in which the issuer is bound to pays both the face value and the interest to the holder at date of maturity. Bond is similar to stock, but the major different is that the stock holder, owned part of the issuer company whereas the bond holder are just lender. The major attraction of the investor to bonds security is that, it is riskless security, especially if it is a government bond. Also the interest rate is fixed but there is guarantee that it will be paid at regular bases and the principal will be paid at specific time.
  Bonds can be classified according to the date of there maturities. They are: bills which have a maturity of less than one year, notes have a maturity of between one year and 10years while last bonds have a maturity that is more than 10years.
  Investor take a critically look at riskiness of corporation before purchasing their bond, that is, the ability of the company to pay the interest and also study the going concern concept of the company. The information concerning this analysis of the corporate company were obtained from their financial statement, audited account and other indicator from stock market. Also investor can trade the bond before the date of maturity to raise fund.

- **Foreign Currency**
  This is the currency of another country either in reserve, cross rates and so on. It is the major measurement of countries economic well-being and it is used in the payment of international trade. The agreement between different nations to agree upon regulation of exchange rate and fluctuation of the exchange rate has made foreign currency an investment instruments. Investors can buy a currency today and keep for later future when the exchange rate can be increase for better return. Or investor can now involve in a future contract today at a spot rate in order to benefit from the speculative increase in foreign exchange for tomorrow. The issue of forward contract will be discussed under the derivative.

1.3.1 Derivative

Derivative is a financial instrument in which, rather than trade or exchange the asset itself, the market participant enters into an agreement to exchange money or assets at future date based on the underlying assets. The agreement entered into by the participant may or may not depend on the performance of the underlying assets. The major financial instruments in these categories are swaps, option and future.

- **Option**
  An option is the security issued by a financial institution and giving the right but not obligation to buy (for call option) or sell (for put option) particular underlying assets on the specific terms at fixed period or during certain period. Each option has a buyer called the holder and a seller called the writer. When the option contract is exercised, it is the responsibility of the writer to fulfil the terms of the contract by delivering it to the appropriate party, when the nature of the security can not be delivered it is settle by cash. The holder of the option have a risk of loss limited to the amount paid to acquire the option. Both call option and put option from financial mathematics point of view work in opposite directions. As the gain from one increase the gain from other one decrease. Option is classified into two kinds in respect to the way there are exercise; which are American option and European Option. European option is the kind of option which the exercise is only at a fixed date called maturity date or expiry date, while American option is the type of option that can be exercise at time (arbitrary e.g. random). Most of the option is American option which gives the buyer freedom to choose of exercise time.

- **Future**
  Future contract is an agreement reached between a buyer and a seller on particular commodity of future time at certain price of today. It is normally been traded in the floor of exchange because the buyer or the seller needs not to see each other. The major reason for investor to involve in this contract is to minimise the risk of uncertainty of the future price. The buyer want to reduce the risk of price rising and seller want to minimised the risk of future down of price. The exchange bore the default risk hence a standard regulation is applied by them, which make it for investor to pay an initial deposit and the margin is adjusted daily to reflect the gain or losses since the future price is determine by demand and supply on the floor of exchange.

  Forward contract also is similar to future contract, but the major different is that contract is not traded in exchange. Usually the contract is between institutions, such like banks and one blue chip company. To reduce the risk one assume position of long position by agreeing to buy the asset while other assume a short position to sell the same asset. Basically since both parties will agree to honour the contract; one party will gain while the other party will lose the same amount.

Chapter 2

Methods

2.1 Characteristics of the Price of Financial Instruments

The price of stock or the exchange rate of two country currencies measured at different times can be expressed as follows. Let

$$S = (S_t) \tag{2.1}$$

when $t \geq 0$ and t is time of days t= 0, 1, 2,... Empirical studies have shown that uncertainty in financial data behave in such a way that sometimes it is very hard to use the past data to predict the future price of the financial instruments. Prominent among the papers which have been written on this kind of stochastic behaviour is A. Cowles 1933 [11]. Often it is more convenient to consider the log returns

$$H_t = \ln \frac{S_t}{S_{t-1}} \tag{2.2}$$

where S_t, $t \geq 1$ are the asset prices, rather than the sequence of prices itself.

Using the probability approach and A.N. Kolmogorov axiomatics of probability theory, we assume the probability space $(\Omega, \mathcal{F}, P)$

where $\Omega = \mathbb{R}$ is the sample space of the log returns, $\mathcal{F}$ is the Borel sets on $\mathbb{R}$ and P is the probability measure on $(\Omega, \mathcal{F})$.

Further more, the probability space is equipped with a flow $(\mathcal{F}_t)$ $t \geq 0$ which is a sequence of σ-algebras such that $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F}$. Adding the flow we define our filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, P)$. S_t is $\mathcal{F}_t$-measurable where $\mathcal{F}_t$ is the information accessible from observations by time t.

2.2 The Autoregressive Conditional Heteroskedastic Model ARCH(p).

This is a non-linear model which catches some of the cluster properties of financial time series and has been useful for eliminating some of the shortcomings of solutions to Black-Scholes equation, by taking into consideration heavy tail and cluster of variance. Let us assume that $\varepsilon_t, t \geq 0$, is white noise i.e a sequence of independent and identically distributed random variables $\varepsilon_t \sim N(0,1)$ and set $\mathcal{F}_t = \sigma(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t)$.
Further let $\{H_t : t \in \mathbb{Z}^+\}$ be a weakly stationary random process such that $H_t = \sigma_t \varepsilon_t$, where $\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i H_{t-i}^2$. Then $E(H_t) = 0$ and $E(H_t^2|\mathcal{F}_{t-1}) = \alpha_0 + \sum_{i=1}^{p} \alpha_i H_{t-i}^2$. where $\alpha_0 > 0, \alpha_i \geq 0, i = 1, \ldots, p$. ${\sigma_t}^2$ is the conditional variance and μ_t is the conditional mean. Then H_t is called the ARCH(p) process.

In a special case when $p = 1$, then the process becomes ARCH(1) process which will be the case that we are considering in this thesis work.

2.2.1 Derivation of the volatility for ARCH(1)

By the definition of ARCH(1) we have that

$$H_t = \sigma_t \varepsilon_t \tag{2.3}$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 H_{t-1}^2 \tag{2.4}$$

so from Equations 2.3 and 2.4 we have

$$H_t^2 = (\alpha_0 + \alpha_1 H_{t-1}^2)\varepsilon_t^2 \tag{2.5}$$

and

$$H_t = \sqrt{\alpha_0 + \alpha_1 H_{t-1}^2}\varepsilon_t \tag{2.6}$$

Then back to Equation 2.5 and take the expectation

$$E(H_t^2) = E(\alpha_0 + \alpha_1 H_{t-1}^2)E(\varepsilon_t^2) \tag{2.7}$$

Since $\varepsilon_t \sim N(0,1)$ it follows that

$$E(H_t^2) = E(\alpha_0 + \alpha_1 H_{t-1}^2) = \alpha_0 + \alpha_1 E(H_{t-1}^2) \tag{2.8}$$

Apply a weakly stationary condition that $E(H_t^2) = E(H_{t-1}^2)$, then we arrived at

$$E(H_t^2) = \frac{\alpha_0}{1-\alpha_1} \tag{2.9}$$

Thus the volatility is $\sqrt{\frac{\alpha_0}{1-\alpha_1}}$

2.3 The Change Point Problem

The change point problem is the situation where one wants to know when a (parameter) change in the distribution of the process under consideration has occurred. The most commonly studied special case is when the process variables are considered to be independent and identically distributed before and after the change point. One observes the processes sequentially, with the aim to raise alarm as soon as it is clear that the distribution has changed. For some aspects one has to assume that the distribution of the change point is known but in some cases it is partial or not known, which makes it difficult to detect the change. The relevance of the studies had implications for many areas, like e.g. in health, environmental, and in financial sciences. In this study we consider parameter values such that the process is stationary, at least before change.

Let us consider the process $\{H_u : u = 1, 2 \ldots\}$ takes values. Also let us assume that H_u changes at time τ. The change most commonly considered in the literature is the change in mean or variance, see e.g. Chen and Gupta (1997)[9].

The change point for the conditional distribution of H_t given H_{t-1} at the time t can be a random time-point τ, which indicates that the change occurs at time τ. More formally,

$$H_t = \begin{cases} f_0(h_t|h_{t-1}) & \text{if } t < \tau\{\text{the change has not occured}\} \\ f_1(h_t|h_{t-1}) & \text{if } t \geq \tau\{\text{the change has occured}\} \end{cases}$$

The change in volatility in this research work is defined as

$$H_t = \begin{cases} \sqrt{\alpha_0 + \alpha_1 H_{t-1}^2}\varepsilon_t & \text{if } t < \tau\{\text{the change has not occured}\} \\ \sqrt{a(\alpha_0 + \alpha_1 H_{t-1}^2)}\varepsilon_t & \text{if } t \geq \tau\{\text{the change has occured}\} \end{cases}$$

Where $a > 0$ representing an increment of volatility

2.4 Likelihood Estimation and Likelihood Ratio

Likelihood estimation is to estimate a parameter in a distribution by the value which maximizes the likelihood function based on a sample from that distribution.

Consider $H_1, ...H_{t-1}, H_t$ be a sample of a random variable of log return. Let $f_H(h;\theta)$ be the density function of H, where $\theta \in \Theta$ is a parameter in the parameter space Θ and let $f_H(h_1, \ldots, h_t; \theta)$ be the joint density function of $H = (h_1, \ldots, h_t)$. Then the likelihood function is $L(\theta) = \prod_{i=1}^{p} f_h(h_t; \theta)$
The value, $\dot{\theta}$, of Θ which maximizes the likelihood function is called the maximum likelihood estimator of the parameter θ.

To perform this maximization one may equivalently maximize i.e find zeros of the derivative of the log likelihood function $\frac{\partial}{\partial\theta}\log(f_H(h_1, \ldots, h_t; \theta))$
The likelihood ratio is, $\frac{L(\theta_1)}{L(\theta_0)}$ (or equivalently the log of this fraction), is the rational expression formed by two likelihood functions. This is commonly used in hypothesis testing and sequential analysis for discriminating between alternative parameter values, θ_0 and θ_1.

2.4.1 Likelihood Ratio

Likelihood ratio is the ratio of of the likelihood fuctions. It can be expressed as follows

$$\text{Likelihood ratio} = \frac{\text{probability with condition having the test result}}{\text{probability without the condition having the test result}}$$

This can be expressed for the change point problem as the log likelihood ratio, when $\tau \geq 1$, and $0 \leq$ s $\leq$ t, as below

$$\begin{aligned} L(s,t) &= \log \frac{f(h_0, h_1, ...h_t|\tau = s)}{f(h_0, h_1, ...h_t|\tau > t)} \\ &= \log \frac{f_0(h_0)\prod_{u=1}^{s-1} f_0(h_u|h_{u-1})\prod_{u=s}^{t} f_1(h_u|h_{u-1})}{f_0(h_0)\prod_{u=1}^{t} f_0(h_u|h_{u-1})} \\ &= \sum_{u=s}^{t} \log \frac{f_1(h_u|h_{u-1})}{f_0(h_u|h_{u-1})} \end{aligned}$$

2.5 Stopping Time

Sometimes we face a situation of taking some kind of action whose nature is fixed. This usually occurs in financial markets when the investor needs to make a decision either to hold its portfolio or disposeof it in order to have more return. In order to take such actions one has to have available information to make this decision at any particular time. Usually these type of properties are expressed in mathematical terms as probability space that have been discussed above, i.e. let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_t)$ be a flow.

Definition 1 *A random variable, T, taking value in $\{1, 2, \ldots\} \cup \{\infty\}$, is called a stopping time (with respect to the flow $\mathcal{F}$) if $\{T = t\} \in \mathcal{F}_t$ for all $t \geq 0$.*

Also note that the stopping time T satisfies $\{T > n\} = \{T \leq n\}^c \in \mathcal{F}_t$ for all t. Stopping time T is not required to be finite, because $\mathcal{F}$ is flow, so it can take infinite values. The information gained up to T can be denoted $\mathcal{F}_T$, and the collection of all events A such that $A \cap \{T \leq t\} \in \mathcal{F}_t$ for all t. It can be seen that $\mathcal{F}_T$ is a σ -field and we think of events whose occurrence or non-occurrence is known by time T, so it is impossible to consider what happens in the future after time T.
Stopping time is essential in change point problems because most situations require that a decision of whether to state that the change has occurred (and thus start taking potential counter actions) or to state that it has not yet occurred (and thus continue collecting observations) should be made immediately. Otherwise a life may be lost or in our particular case a fund can be lost. Basically, also as far as it is important in stopping our process as soon as change occurred, it may also be very important that a false alarm is not made, i.e. the stopping must be accurate. Specifically in most cases there is a cost for false alarm and also a penalty for not stopping quickly when a change in the process occured.

2.6 Method of Detecting a Change Point

Several methods have been derived and considered in the vast literature of change point detection (see e.g. Frisén (2003) [16]). In this study, the Shewhart method, the CUSUM method, and the Windows method will briefly be derived. Some of the performance measures of the Shewhart method will be derived and its properties will be discussed.

2.7 The Shewhart Method

In 1931 Shewhart presented the method which later became the Shewhart method [36]. This method has been very popular in the area of surveillance and it has been used in monitoring in several fields, for change point problem. For detailed properties of the method see e.g Frisén (2007)[17]. Let us denote a process by $X = X(t), t = 1, 2, \ldots.$ where $X(t) = \{x(1), \ldots, x(t)\}$ is the observation at time t. At each observable time s we like to decide whether the process is in control $C(s)$ or out of control $D(s)$. Let $X_s = \{X(t); t \leq s\}$. The alarm statistic is a function of the variables up to time t,while probability $p(X_t)$, is based on the observations and the alarm has threshold value C.

The alarm time can be given in likelihood ratio as

$$T = \min\{t \geq 1 : L(t,t) \geq C\} \tag{2.10}$$

where C is a real number called threshold value.

2.7.1 Derivation of Shewhart for ARCH(1)

The likelihood ratio has been defined in 2.4.1 the modefication for Shewart loglikehood ratio is given as

$$L(t,t) = \sum_{u=t}^{t} \log \frac{f_1(h_u|h_{u-1})}{f_0(h_u|h_{u-1})} \tag{2.11}$$

$$L(t,t) = \sum_{u=t}^{t} \log \frac{\frac{1}{\sigma_u\sqrt{2a\pi}} \exp(-\frac{H_u^2}{2a\sigma_u^2})}{\frac{1}{\sigma_u\sqrt{2\pi}} \exp(-\frac{H_u^2}{2\sigma_u^2})} \tag{2.12}$$

$$L(t,t) = \log \frac{\frac{1}{\sigma_t\sqrt{2a\pi}} \exp(-\frac{H_t^2}{2a\sigma_t^2})}{\frac{1}{\sigma_t\sqrt{2\pi}} \exp(-\frac{H_t^2}{2\sigma_t^2})} \tag{2.13}$$

$$L(t,t) = \log \frac{1}{\sqrt{a}} \exp((\frac{H_t^2}{2\sigma_t^2})(1 - \frac{1}{a})) \tag{2.14}$$

$$L(t,t) = (1 - \frac{1}{a})\frac{H_t^2}{2\sigma_t^2} - \frac{1}{2}\log a \tag{2.15}$$

2.8 The Cumulative Sum Method (CUSUM)

Similarly this method is one of the most prominent methods used in change point problems. The CUSUM method handles previous information differently depending on the position of in the time series, as it was suggested by Page (1954) [29] and its properties further investigated by Lorden (1971) [24].

The alarm function of the CUSUM method may preferably be based on a sufficient statistic such as the likelihood ratio. Yaschin (1993) [39] and Hawkins and Olwell (1998) [20] give more details about this.

The alarm function of the CUSUM method is expressed the likelihood of ratio as

$$T = \min\{t; \max(L(s,t); s = 1, 2, \ldots, t) \geq C\} \tag{2.16}$$

where C is a constant called threshold value.

2.8.1 Derivation of CUSUM method for ARCH(1)

This method can be derived in similar way as that of Shewhart method, as follows

$$L(s,t) = \sum_{u=s}^{t} \log \frac{f_1(h_u|h_{u-1})}{f_0(h_u|h_{u-1})} \tag{2.17}$$

$$L(s,t) = \sum_{u=s}^{t} \log \frac{\frac{1}{\sigma_u\sqrt{2a\pi}}\exp(-\frac{H_u^2}{2a\sigma_u^2})}{\frac{1}{\sigma_u\sqrt{2\pi}}\exp(-\frac{H_u^2}{2\sigma_u^2})} \tag{2.18}$$

$$L(s,t) = \sum_{u=s}^{t} \log \frac{1}{\sqrt{a}} \exp((\frac{H_u^2}{2\sigma_u^2})(1-\frac{1}{a})) \tag{2.19}$$

$$L(s,t) = \sum_{u=s}^{t} \log \frac{1}{\sqrt{a}} + \sum_{u=s}^{t}(\frac{H_u^2}{2\sigma_u^2})(1-\frac{1}{a}) \tag{2.20}$$

$$L(s,t) = \frac{s-t-1}{2}\log a + (1-\frac{1}{a})\sum_{u=s}^{t}\frac{H_u^2}{2\sigma_u^2} \tag{2.21}$$

2.9 The Windows Method

This method is one of the method in detecting change point. It's alarm time can be constructed in the likelihood ratio as

$$T = \min\{t \geq 1 : L(t-d,t) \geq C\} \tag{2.22}$$

Where d is defined as the parameter indicating the size of the window and C in the threshold value.

2.9.1 Derivation of Windows Method for ARCH(1)

The windows is similar to CUSUM method , when s = d-t., which can be substitute into the result of CUSUM above to obtain

$$L(t-d,t) = \frac{-d-1}{2}\log a + (1-\frac{1}{a})\sum_{u=t-d}^{t}\frac{H_u^2}{2\sigma_u^2} \tag{2.23}$$

$$L(t-d,t) = (1-\frac{1}{a})\sum_{u=t-d}^{t}\frac{H_u^2}{2\sigma_u^2} - \frac{d+1}{2}\log a \tag{2.24}$$

2.10 Optimality of Stopping Time

To measure the quality of a stopping rule one may use different performance measures e.g. ARL, ED, PV has been discussed by some researchers see Järpe (2002) [21] and Frisén (2003) [16] which will be discussed in the subsection below. In general one wants a few false alarms as possible but at the same time a minimal expected delay of motivated alarm. Comparisons of e.g. expected delay of different method may be relevant after having calibrated according to e.g. false alarm probability.

2.10.1 Average Run Length (ARL)

This measures indicates the average number of observations that are made until the process is stopped, i.e alarm signalled. The ARL until alarm when actually there is no change is called ARL_0 while the ARL when there is a change in the parameter immediately as the process starts is called ARL_1. Although it has been suggested that the whole run length distribution should be reported instead of the average length but this is rarely done. Mathematically it can be denoted as below

$$ARL_0 = E(T|\tau = \infty) \tag{2.25}$$

$$ARL_1 = E(T|\tau = 1) \tag{2.26}$$

2.10.2 Expected Delay

This is a performance measure indicating the mean delay of a motivated alarm. The expected delay plays a significant role in the optimality of stopping rules, which minimized the rate of false alarms. Let the time of change be τ. The expected delay can be denoted by

$$ED = E(T - \tau|T \geq \tau) \tag{2.27}$$

So also the conditional expected delay is given as

$$CED(t) = E(T - \tau|T \geq \tau = t) \tag{2.28}$$

2.10.3 Predictive Value (PV)

The predictive value is the credibility of alarm at time see Frisén (2003) [16]. It let us know how probable a change anywhere in the past,when there is alarm. It can be expressed as

$$PV(t) = Pr(\tau \leq t|T = t) \tag{2.29}$$

Chapter 3

Results

3.1 Derivation of Stationary Condition

From Chapter 2 we got $E(H_t^2) = \frac{\alpha_0}{1-\alpha_1}$ and also $H_t = \sigma_t \varepsilon_t$ then we get

$$E(H_t^4) = E(\sigma_t^4)E(\varepsilon_t^4) \tag{3.1}$$

But since $E(\varepsilon_t^4) = 3$ Equation 3.1 now becomes

$$E(H_t^4) = 3E(\sigma_t^4) \tag{3.2}$$

Also from Chapter 2 it we have $\sigma_t^2 = \alpha_0 + \alpha_1 H_{t-1}^2$ and can now arrive at the equation

$$E(H_t^4) = 3E((\alpha_0 + \alpha_1 H_{t-1}^2)^2) \tag{3.3}$$

$$E(H_t^4) = 3E(\alpha_0^2 + 2\alpha_0\alpha_1 H_{t-1}^2 + \alpha_1^2 H_{t-1}^4) \tag{3.4}$$

Taking the expectation of all in the parenthesis gives

$$E(H_t^4) = 3\alpha_0^2 + 6\alpha_0\alpha_1 E(H_{t-1}^2) + 3\alpha_1^2 E(H_{t-1}^4) \tag{3.5}$$

It has been shown in Chapter 2 that $E(H_{t-1}^2) = E(H_t^2) = \frac{\alpha_0}{1-\alpha_1}$, which is weakly stationary condition. So also $E(H_t^4) = E(H_{t-1}^4)$ by the strong stationary condition. If these two conditions are applied then we get the following result and solving with respect to $E(H_t^4)$ renders

$$E(H_t^4) = 3\alpha_0^2 + 6\alpha_0\alpha_1 \frac{\alpha_0}{1-\alpha_1} + 3\alpha_1^2 E(H_t^4) \tag{3.6}$$

$$E(H_t^4) = \frac{3\alpha_0^2(1-\alpha_1) + 6\alpha_0^2\alpha_1}{(1-3\alpha_1^2)(1-\alpha_1)} \tag{3.7}$$

the above equation can be factorized to obtain

$$E(H_t^4) = \frac{3\alpha_0^2(1+\alpha_1)}{(1-3\alpha_1^2)(1-\alpha_1)} \tag{3.8}$$

By definition $\alpha_0 > 0, \alpha_1 \geq 0$. From $0 \leq E(H_t^2) = \frac{\alpha_0}{1-\alpha_1}$, then $1-\alpha_1 > 0$ and since $E(H_t^4) > 0$, $(1-3{\alpha_1}^2)(1-\alpha_1) > 0$

It also follows that $0 \leq \alpha_1 < 1$. Considered the above conditions it is easily to know that $1-3\alpha_1^2 > 0$ and which means that $0 \leq \alpha_1 < \frac{1}{\sqrt{3}}$.
The feasible parameter value for ARCH(1) are $(\alpha_0, \alpha_1) : \alpha_0 > 0, 0 \leq \alpha_1 < \frac{1}{\sqrt{3}}$

3.2 Deriving the Estimates Values of ARCH(1)

The MLE method can be used for deriving the parameters of ARCH(1)
Let us first consider a normal density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(x-\mu)^2}{2\sigma^2}) \tag{3.9}$$

Considering the case of ARCH(1) which is conditional normal distribution with conditional $\mu = 0$, and conditional variance $\sigma_t^2 = \alpha_0 + \alpha_1 H_{t-1}^2$ the distribution for the ARCH(1) now become

$$f_0(h_t|h_{t-1}) = \frac{1}{\sigma_t\sqrt{2\pi}} exp^{(-\frac{H_t^2}{2\sigma_t^2})} \tag{3.10}$$

the log-likelihood function can be written as function of σ_0 and σ_1

$$L(\alpha_0, \alpha_1) = \sum_{t=2}^{n} \log f_0(h_t|h_{t-1}) \tag{3.11}$$

then by substituting the equation (3.10) into (3.11) , the equation (3.11) will become

$$L(\alpha_0, \alpha_1) = \sum_{t=2}^{n} \log \frac{1}{\sigma_t\sqrt{2\pi}} exp^{(-\frac{H_t^2}{2\sigma_t^2})} \tag{3.12}$$

$$L(\alpha_0, \alpha_1) = \sum_{t=2}^{n} (\log{(2\pi)}^{-\frac{1}{2}} + \log(\sigma_t^2)^{-\frac{1}{2}} - \frac{H_t^2}{2\sigma_t^2}) \tag{3.13}$$

$$= -\frac{n-1}{2}\log(2\pi)^{-}\frac{1}{2}\sum_{t=2}^{n}\log(\sigma_t^2) - \frac{1}{2}\sum_{t=2}^{n}\frac{h_t^2}{\sigma_t^2} \quad (3.14)$$

Let us substitute equation (2.4) of the form $\sigma_t^2 = \alpha_0 + \alpha_1 H_{t-1}^2$. equation 3.14 becomes

$$= -\frac{n-1}{2}\log(2\pi) - \frac{1}{2}\sum_{t=2}^{n}\log(\alpha_0 + \alpha_1 H_{t-1}^2) - \frac{1}{2}\sum_{t=2}^{n}\frac{H_t^2}{\alpha_0 + \alpha_1 H_{t-1}^2} (3.15)$$

The estimates of the value of α_0, α_1 can be done by taking the partial derivative. By simple partial differentiation we arrived at the following

$$\frac{\partial L(\alpha_0, \alpha_1)}{\partial \alpha_0} = -\frac{1}{2}\sum_{t=2}^{n}\frac{1}{\alpha_0 + \alpha_1 H_{t-1}^2} + \frac{1}{2}\sum_{t=2}^{n}\frac{H_t^2}{(\alpha_0 + \alpha_1 H_{t-1}^2)^2} \quad (3.16)$$

$$\frac{\partial L(\alpha_0, \alpha_1)}{\partial \alpha_0} = \sum_{t=2}^{n}\frac{H_t^2 - \alpha_0 - \alpha_1 H_{t-1}^2}{2(\alpha_0 + \alpha_1 H_{t-1}^2)^2} \quad (3.17)$$

If equation 3.17 is equal to zero we have

$$\sum_{t=2}^{n} H_t^2 = (n-1)\alpha_0 + \alpha_1 \sum_{t=2}^{n} H_{t-1}^2 \quad (3.18)$$

The partial derivative of the log-likelihood with respect to α_1 is

$$\frac{\partial L(\alpha_0, \alpha_1)}{\partial \alpha_1} = -\frac{1}{2}\sum_{t=2}^{n}\frac{H_{t-1}^2}{\alpha_0 + \alpha_1 H_{t-1}^2} + \frac{1}{2}\sum_{t=2}^{n}\frac{H_t^2 H_{t-1}^2}{(\alpha_0 + \alpha_1 H_{t-1}^2)^2} \quad (3.19)$$

$$\frac{\partial L(\alpha_0, \alpha_1)}{\partial \alpha_1} = \sum_{t=2}^{n}\frac{H_t^2 H_{t-1}^2 - \alpha_0 H_{t-1}^2 - \alpha_1 H_{t-1}^4}{2(\alpha_0 + \alpha_1 H_{t-1}^2)^2} \quad (3.20)$$

If equation 3.20 is equal to zero we have

$$\sum_{t=2}^{n} {H_t}^2 H_{t-1}^2 = \alpha_0 \sum_{t=2}^{n} H_{t-1}^2 + \alpha_1 \sum_{t=2}^{n} H_{t-1}^4 \quad (3.21)$$

The value of α_0 and α_1 can be obtained by solving equation (3.18) and (3.21) will gives the following expression

$$\alpha_0 = \frac{(\sum_{t=2}^{n} H_t^2)(\sum_{t=2}^{n} H_{t-1}^4) - (\sum_{t=2}^{n} H_{t-1}^2)(\sum_{t=2}^{n} H_t^2 H_{t-1}^2)}{(n-1)\sum_{t=2}^{n} H_{t-1}^4 - (\sum_{t=2}^{n} H_{t-1}^2)^2} \quad (3.22)$$

$$\alpha_1 = \frac{(n-1)(\sum_{t=2}^{n} H_t^2)(\sum_{t=2}^{n} H_{t-1}^2) - (\sum_{t=2}^{n} H_t^2)(\sum_{t=2}^{n} H_{t-1}^2)}{(n-1)\sum_{t=2}^{n} H_{t-1}^4 - (\sum_{t=2}^{n} H_{t-1}^2)^2} \tag{3.23}$$

3.3 Calculating the Performance ARL_0 and ARL_1 for Shewhart

The Shewart alarm time is given as $T = \min\{t \geq 1 : L(t,t) \geq C\}$ then we can derive the ARL_0 and ARL_1 as follows

$$ARL_0 = E(T|\tau = \infty) \tag{3.24}$$

$$ARL_0 = \sum_{t=1}^{\infty} tP(\min\{t : \frac{H_t^2}{\alpha_0 + \alpha_1 H_{t-1}^2} \geq (\frac{2a}{1-a})(C + \frac{1}{2}\log a)\} = t|\tau = \infty) \tag{3.25}$$

but if the $X \in N(0,1)$ it implies that $X^2 \in \chi_1^2$. Thus

$$\begin{aligned} P(L(t,t) \leq C|H_{t-1} = h_{t-1}, \tau > t &= P(\frac{H_t^2}{\alpha_0 + \alpha_1 H_{t-1}^2} \leq (\frac{2a}{a-1})(C + \frac{1}{2}\log a) \\ &= \chi_1^2(\frac{2a}{a-1})(C + \frac{1}{2}\log a) \end{aligned}$$

which is independent of H_{t-1} for simplicity purpose let $K = (\frac{2a}{a-1})(C + \frac{1}{2}\log a)$ Then will call further express it like this

$$ARL_0 = \sum_{t=1}^{\infty} tP(\frac{H_t^2}{\sigma_t^2} \geq K|\frac{H_{t-1}^2}{\sigma_{t-1}^2} < K)P(\frac{H_{t-1}^2}{\sigma_{t-1}^2} < K|\frac{H_{t-2}^2}{\sigma_{t-2}^2} < K)......P(\frac{H_1^2}{\sigma_1^2} < K) \tag{3.26}$$

Since H_{t-1} independent, it become the derivative of the geometric sum as below

$$ARL_0 = \sum_{t=1}^{\infty} t(1 - B_0)B_0^{t-1} \tag{3.27}$$

When $B_0 = \chi_1^2((\frac{2a}{a-1})(C + \frac{1}{2}\log a)$ so we can solve the equation. It can be easily seen that

$$tB_0^{t-1} = \frac{d}{dB_0}B_0^t \tag{3.28}$$

Then we can substitute it into the equation

$$ARL_0 = \sum_{t=1}^{\infty}(1-B_0)\frac{d}{dB_0}B_0^t \tag{3.29}$$

$$ARL_0 = (1-B_0)\frac{d}{dB_0}\sum_{t=1}^{\infty}B_0^t \tag{3.30}$$

then

$$\sum_{t=1}^{\infty}B_0^t = \frac{B_0}{1-B_0} \tag{3.31}$$

$$ARL_0 = (1-B_0)\frac{d}{dB_0}(\frac{B_0}{1-B_0}) \tag{3.32}$$

$$ARL_0 = \frac{1}{1-B_0} \tag{3.33}$$

We now calculate the ARL_1

$$ARL_1 = E(T|\tau = 1) \tag{3.34}$$

but from change point problem defined in Chapter 2

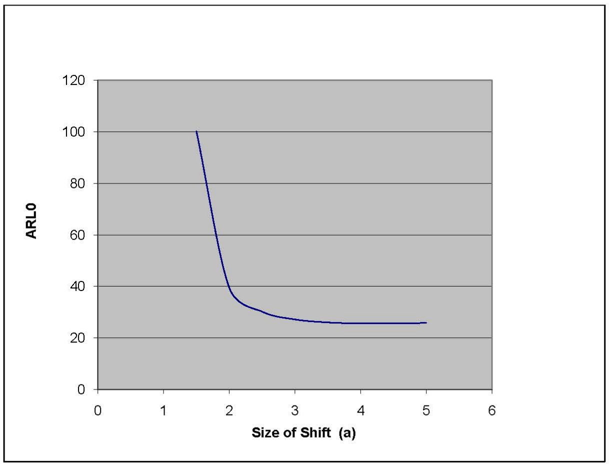
$$\begin{aligned} P(L(t,t)) \le C|H_t = H_{t-1}, \tau \le t) &= P(a\frac{H_t}{\alpha_0 + \alpha_1 H_{t-1}^2} \le (\frac{2a}{a-1})(C + \frac{1}{2}\log a) \\ &= P(\frac{H_t}{\alpha_0 + \alpha_1 H_{t-1}^2} \le (\frac{2}{a-1})(C + \frac{1}{2}\log a) \\ &= \chi_1^2(\frac{2}{a-1})(C + \frac{1}{2}\log a) \end{aligned}$$

From the same procedure used in ARL_0 above we can obtain

$$ARL_1 = \sum_{t=1}^{\infty}t(1-B_1)B_1^{t-1} \tag{3.35}$$

$$ARL_1 = \frac{1}{1-B_1} \tag{3.36}$$

where $B_1 = \chi_1^2(\frac{2}{a-1}(C + \frac{1}{2}\log a))$



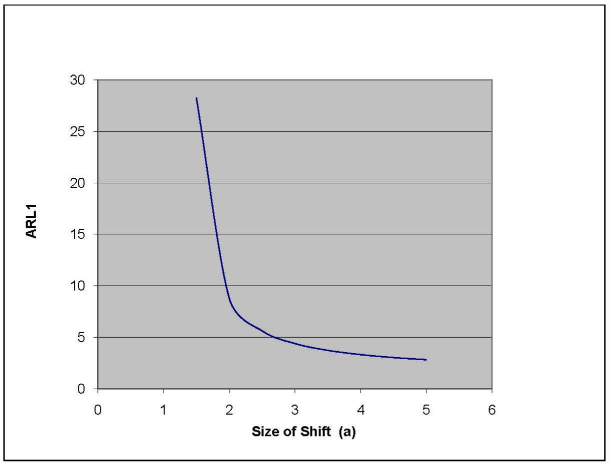


Figure 3.1: Graph of ARL_0 and ARL_1 against different volatility shift size (a) for Shewhart

The parameter ARL_0, which is the expected time for first alarm. This is firstly fixed to 100 runs for the first shift, to calculate the threshold value to be used for comparing different shifts. It indicates that for every 100 runs on average there is a false alarm. The most important is the expected delay to motivated alarm (ARL_1) which indicates delay to alarm assumming the process change immdediate monitoring start.

Specifically, when considering the ARL_1 in the plotted graph above, it can be seen that Shewhart is more effective for detecting reasonable large shift. The implication of this is that, when considering a shift that is very small there is tendecy that changes could have been occurred long time before alarm is signal. In financial markets, shift that will lead to crash in market or result in catashrophes is not so small, which will make this surveillance to detect it as soon as possible. As it can been seen that as the shift is increasing, ARL_1 is decreasing, which make it more effective for reasonable large shift.

Chapter 4

Conclusions

The importance of properly designed monitoring method can not be overemphasized and many researchers have pointed for the need in many areas e.g in health and finance. The performance measure for the surveillance methods are becoming very important and the effectiveness ARL_0 and ARL_1 have gained wider acceptance both in reseach area and application, that is why till today hardly anybody could do without them.

Basically, some of the methods for surveillance were derived in this thesis but the perforance measures was calculated on the Shewhart method. The beauty of the Shewhart method which includes keeping calculation simple has made it more applicable in practical areas. In spite of the criticism of Shewhart method it is still a valuable tool for surveillance when used with average run lenghts. From the plotted graph it is seen that Shewhart works most effectively for the detecting a reasonable large shift.

The urgent attention given to making timely decision based on the avaliable timely data has widened the horizon of surveillance majorly in financial market, i.e making decision as whether to sell or buy financial instruments at any particular time. This research will be more useful to practitioners in financial sector, where many would be scared by the rigorousness of constructing of surveillance for measuring the volatilty of log return of financial instruments which can be properly explained by the process ARCH(1).

The future research area is to construct a surveillance for higher order ARCH process, GARCH and other financial processes. Also to calculate more performance measures (Expected Delay, Conditional Expected Delay, Predictive Measure) for the process; analytical if possible for the Shewhart method and other methods like CUSUM and Windows method.

Notation

$\mathbb{Z}^+$	The set of positive integers.
$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$	filtered probability space with set of outcomes Ω, sigma algebra $\mathcal{F}$, flow or filtration $\{\mathcal{F}_n\}$ and probability measure P.
$E(H_t)$, $V(H_t)$	Expectation and variance of the random variable H_t .
S_t	Stock price or price of financial instrument
H_t	Log of return of financial instruments
σ_t	volatility
$\Phi(x)$,	Normal distribution fuction

Bibliography

[1] Bachelier, L. (1900)
The Theory of Speculation, *PhD Thesis, Universite Paris Sorbonne, France*

[2] Backus, D. K. and Gregory, A. W. (1993)
Theoretical relations between risk premiums and conditional variances, *Journal of Business and Economic Statistics 11*, (177–185).

[3] Barkat, M. (2005)
Signal detection and Estimation, *Artech House Incorporated*

[4] Barndorff-Nielsen, O. E. and Shepherd N. (2002)
Econometric analysis of realized volatility and its use in estimating stochastic volatility model, *Journal of the Royal Statistical scociety series B 64*, (253–280)

[5] Black, F. and Scholes, M. (1973)
The pricing of options and corporate liabilities, *Journal of Political Economy 81 no 3*, (637–659).

[6] Bollerslev, T. V., Engle, R. F., and Wooldridge, J. M. (1988)
A capital asset pricing model with time-varying covariances, *Journal of political Economy 98*, (116–131).

[7] Bollerslev, T. V. and Andersen,T. G. (1988)
Answering the skeptics: Yes standard volatility model do provide accurate forecasts, *International Economics Review 39*, (885–905).

[8] Campbell, J.Y. and Hentschel, L. (1992)
No news is good news: An asymmetric model of changing volatility in stock return, *Journal of Financial Economics 31*, (281–318).

[9] Chen, J. and Gupta, A. K. (1997)
Testing and locating variance change-points with application to stock price, *Journal of the American Statistics Association 92*, (739–747).

[10] Chou, R. Y. (1988)
Volatility persistence and stock valuation: Some empirical evidence using GARCH, *Journal of Applied Econometrics 3*, (279–294).

[11] Cowles A. (1993)
Can stock markets forecaster forecast, *Econometrica 1*, (309–324).

[12] Dimand, R. W. (1993)
The case of Brownian Motion: a Note of Bachelier's Contribution, *British Journal of History of Science, 26, part 2 No 89*, (233–234).

[13] Engle, R. F., Lilien, D. M. and Robins, R. P. (1987)
Estimating time-varying risk premia in the term structure: The ARCH-M model, *Econometrica 55*, (391–407).

[14] Fama, E. (1965)
The behavior of stock market prices, *Journal of Political Economy 96*, (246–273).

[15] Feller, W. (1968)
An introduction to probability theory and its application, Vol. 1 3rd edition *New York: Wiley*

[16] Frisén, M. (2003)
Statistical Surveillance. Optimality and Methods, *International Statistical review 71 vol. 2*, (403–434).

[17] Frisén, M. (2007)
Properties and use of the Shewart method and its followers, *Sequential Analysis 26*, (171–193).

[18] Grimmett, G. and Stirzaker, D. (2001)
Probability and Random Processes, 3rd edition *Oxford University Press.*

[19] Gupta, A. K. and Tanj, J. (1987)
On testing homogeneity of variance of Guassian model, *Journal of statistical computation and simulation 27*, (155–173).

[20] Hawkins, D. M and Olwel, D. H. (1998)
Cumulative sum chart and charting for quality improvement, *Springer-Verlag, New York.*

[21] Järpe, E. (2002)
Surveillance, environmental *Encyclopedia of Environmetrics (ISBN*

047189976) Edited by Abdel H. El-Shaarawi and Walter W.Piegorsch, John Wiley and sons Ltd Chichester. Vol. 4 , (2150–2155).

[22] Lavielle, M. (2005)
Using penalized contrast for change-point, *Signal Processing Vol 85 Issue 8*, (1501–1510).

[23] Linter, J. (1965)
The valuation of risk asset and the selection of risky investments in stock portfolios and capital budgets, *Review of Economics and Statistics 47(1)*, (13–37).

[24] Lorden, G. (1971)
Procedure for reacting to a change in distribution, *Annals of Mathematical Statistics, 41*, (1897–1908).

[25] Lyuu, Y. (2001)
Financial Engineering and Computational Mathematics and Algorithm, *Cambridge University Press.*

[26] Markowitz, H. M. (1952)
Portfolio Selection, *Journal of Finance 7*, (77–99).

[27] Merton, R. C. (1973)
The theory of rational pricing, *Bell Journal of economics and management science Vol 4 no 1*, (141–183).

[28] Mossin, J. (1966)
Equilibrium in capital asset market, *Econometrica 34(4)* , (7683–783).

[29] Page, E. S. (1954)
Continuous inspection schemes, *Biometrika, Vol 41 No 1/2* , (100–115).

[30] Robert, J. E. and Ekkehard, P. K. (1998)
Mathematics of Financial Markets, 2nd edition, *Springer-Verlag, New York*

[31] Saaty, T. L. (1981)
Modern Nonlinear Equations, *Dover, New York*

[32] Samuelson, P. A. (1965)
proof that properly anticipated prices fluctuate randomly, *Industrial Management Review 6*, (41–50).

[33] Samuelson, P.A. (1980)
Economics, 11th edition, *McGraw/Hill, New York.*

[34] Schwert, G. W. (1989)
Why does stock market volatility change over time?, *Journal of Finance 44*, (1115–1153).

[35] Sharpe, W.F. (1964)
Capital asset Prices: A theory of market equilibrium under conditions of risk, *Journal of Finance 19 (3)* , (425–442).

[36] Shewhart, W. A. (1931)
Economics Control of Quality of Manufactured Product, *Macmillan London.*

[37] Shiryaev, A. N. (1999)
Essentials of Stochastic Finance: Facts, Model, Theory, Vol. 3 *World scientific Publishing.*

[38] Trenor, J.L. (1962)
Toward a theory of market value of risky asset., *Unpublished manuscripts. A final version was published in 1999 in Asset Pricing and Portfolio Performance Metrics. Robert A. Korajczk (edition) London Books*, (15–222).

[39] Yaschin, E. (1993)
statistical control scheme, method, application and generalization. *International Statistical Reiew 61*, (41–66).

Appendix

Data for the graph of ARL_0 and ARL_1 against shift in the size of the volatility (a)
$C = 0.9035$. where C is the threshold value

a	K_0	B_0	ARL_0	K_1	B_1	ARL_1
1.5	6.6374	0.9900	100.1403	4.4260	0.9646	28.2345
2	5.0003	0.9746	39.4586	2.5002	0.8862	8.7846
2.5	4.5388	0.9669	30.1801	1.8155	0.8222	5.6228
3	4.3584	0.9632	27.1544	1.4528	0.7720	4.3844
3.5	4.2836	0.9615	25.9874	1.2239	0.7314	3.7231
4	4.2577	0.9609	25.5937	1.0644	0.6977	3.3089
4.5	4.2571	0.9609	25.5842	0.9460	0.6693	3.0236
5	4.2705	0.9612	25.7876	0.8541	0.6446	2.8137

a = shift size of volatility
C = threshold value, which is chosen to get first ARL_0 for first shift to be 100 and as standard for every other shift.
$K_0 = \frac{2a}{a-1}(C + \frac{1}{2}\log a)$
$B_0 = \chi_1^2(K_0)$
$ARL_0 = \frac{1}{1-B_0}$
$K_1 = \frac{2}{a-1}(C + \frac{1}{2}\log a)$
$B_1 = \chi_1^2(K_1)$
$ARL_1 = \frac{1}{1-B_1}$
$if \chi_1^2 = X^2, X \in N(0,1)$ then $P(X^2 \leq x) = P(-\sqrt{x} \leq X \leq \sqrt{x}) = 2\Phi(\sqrt{x}) - 1$
For expalanations about ARL_0 and ARL_1 see 2.10.1