The multi-frequency solution for periodic nonlinear and dissipative waves

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Summary

The solution for multi-frequency plane waves propagating through a dissipative and nonlinear medium is shown for some examples of periodic conditions. The expression may for any given condition be expressed analytically as a ratio of Fourier series with Bessel function coefficients. In the examples are shown how the final appearance of any initial wave always is a pure periodic wave in the lowest frequency existing in the problem - the period of the condition.

For a single frequency source solutions for nonlinear evolution of plane waves through a dissipative medium have been known for a long time. The exact result of Mendousse (1953) [1] is in a form of a ratio between Fourier series with Bessel function coefficients. The generalization for multiple frequencies in the boundary condition of this solution has been presented by Hedberg [2].

For non-dissipative propagation the final form of the signals has one, two or even several shocks, and high frequencies are always present e.g. [3]. The presence of dissipation counteracts the shock creation of strong nonlinearity and the wave forms will be smoother.

The nonlinear equation for plane waves in a homogeneous dissipative medium is Burgers' equation, given in dimensionless variables as

$$\frac{\partial V}{\partial \sigma} - V\frac{\partial V}{\partial \theta} - \epsilon\frac{\partial^2 V}{\partial \theta^2} = 0 \tag{1}$$

The definition of the dimensionless variables - using: a characteristic velocity v_0 of the medium, b the effect of viscosity and heat conduction, $\tau = t - \frac{x}{c_0}$ the retarded time, ρ_0 the undisturbed density, c_0 the undisturbed sound velocity, $\beta = \frac{\gamma+1}{2}$ the nonlinearity for a fluid and a characteristic time $\frac{1}{\omega}$ - are

$$V = \frac{v}{v_0} \qquad \theta = \omega\tau \tag{2}$$

$$\sigma = \frac{\beta}{c_0^2}\omega v_0 x \qquad \epsilon = \frac{1}{2\beta}\frac{b\omega}{c_0 v_0 \rho_0} \tag{3}$$

Letting the boundary condition be a simple sinus wave:

$$V(\sigma = 0, \theta) = \sin\theta \tag{4}$$

, the solution to equation (1) is obtained as [4]

[1]:

$$V(\sigma,\theta) = -4\epsilon \frac{\sum_{n=1}^{\infty} \exp(-n^2\epsilon\sigma) n(-1)^n I_n(\frac{1}{2\epsilon}) \sin n\theta}{I_0(\frac{1}{2\epsilon}) + 2\sum_{n=1}^{\infty} \exp(-n^2\epsilon\sigma)(-1)^n I_n(\frac{1}{2\epsilon}) \cos n\theta} \tag{5}$$

The quotient between two Fourier series may be replaced by an analytic single Fourier series [5].

Any solution for a periodic boundary condition may have the form

$$V(\sigma,\theta) = i2\epsilon \frac{\sum_{k=-\infty}^{\infty} k c_k e^{-k^2\epsilon\sigma} e^{ik\theta}}{\sum_{k=-\infty}^{\infty} c_k e^{-k^2\epsilon\sigma} e^{ik\theta}} \tag{6}$$

Let (6) be the solution to a boundary condition consisting of a number of L frequencies, the α_ls are integers:

$$V^*(\sigma^*=0,\theta^*) = \sum_{l=1}^{L} a_l \sin(\alpha_l\theta^* + \gamma_l) = \sum_{l=1}^{L} \frac{a_l}{2}[\exp(i(\alpha_l\theta^* + \gamma_l)) - \exp(-i(\alpha_l\theta^* + \gamma_l,))] \tag{7}$$

Each one of the L frequencies has on their own a known solution V_l and the results will be consistent in the same dimensionless parameters through making the following replacements in

$$V = \frac{V^*}{a} \qquad \theta = \alpha\theta^* + \gamma \tag{8}$$

$$\sigma = a\alpha\sigma^* \qquad \epsilon = \frac{\alpha}{a}\epsilon^* \tag{9}$$

The solutions to the individual frequencies will be

$$\begin{aligned} V^* &= aV(\sigma^*,\theta^*) \\ &= -4\alpha\epsilon^* \frac{\sum_{n=1}^{\infty} \exp(-n^2\alpha^2\epsilon^*\sigma^*) n(-1)^n I_n(\frac{a}{2\alpha\epsilon^*}) \sin n(\alpha\theta^* + \gamma)}{I_0(\frac{a}{2\alpha\epsilon^*}) + 2\sum_{n=1}^{\infty} \exp(-n^2\alpha^2\epsilon^*\sigma^*)(-1)^n I_n(\frac{1}{2\alpha\epsilon^*}) \cos n(\alpha\theta^* + \gamma)} \end{aligned} \tag{10}$$

Through superposition at zero propagated distance the coefficients become

$$c_k = \sum_{k=\sum n^{(l)}\alpha_l} I_n(\frac{a_1}{2\alpha_1\epsilon^*})(-1)^n \exp(in\gamma_1) \ldots I_n(\frac{a_L}{2\alpha_L\epsilon^*})(-1)^n \exp(in\gamma_L) \tag{11}$$

, which inserted into (6) expressed in the common variables is the solution to Burgers' equation (1):

$$V(\sigma^*,\theta^*) = i2\epsilon^* \frac{\sum_{k=-\infty}^{\infty} k c_k e^{-k^2\epsilon^*\sigma^*} e^{ik\theta^*}}{\sum_{k=-\infty}^{\infty} c_k e^{-k^2\epsilon^*\sigma^*} e^{ik\theta^*}} \tag{12}$$

This expresson is valid for any number of frequencies with arbitrary amplitudes and phases according to the boundary condition (7)

The individual frequencies in a straightforward Fourier series may be obtained analytically in the same way as for a single frequency [5].

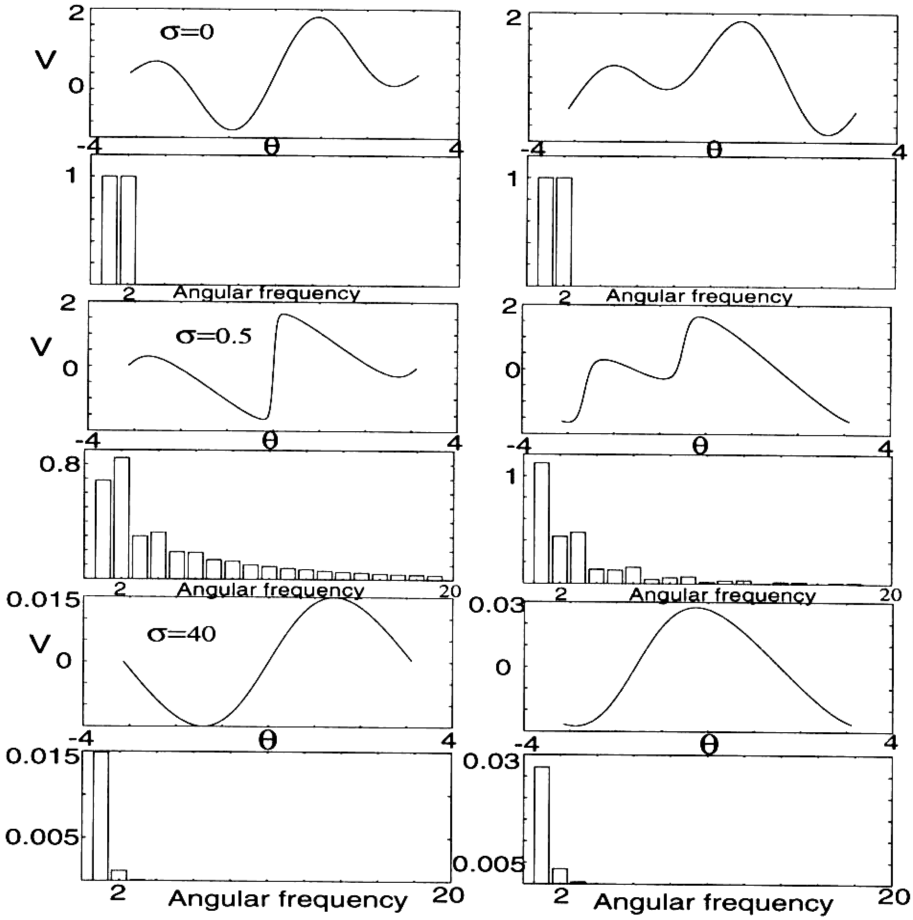

Figure 1: One of the original frequencies is the difference frequency and equal to the lowest frequency. Different phases affect the wave spectra.
$V_0 = \sin(\theta + \gamma) + \sin 2\theta$, $\epsilon^* = 0.05$. Left: $\gamma = 0$. Right: $\gamma = \pi/2$

Figure 1 shows the specific situation when the difference frequency is the lowest frequency and also the same as one of the original frequencies: $V_0 = \sin(\theta + \gamma) + \sin 2\theta$ with $\epsilon^* = 0.05$. The explicit expression for the coefficients c_k is

$$c_k = \sum_{n=-\infty}^{\infty} (-1)^{k-n} e^{i(k-2n)\gamma} I_{k-2n}(\frac{1}{2\epsilon^*}) I_n(\frac{1}{4\epsilon^*}) \tag{13}$$

In Figure 1 is seen the evolution for the phase $\gamma = 0$ in the left column, and the phase $\gamma = \pi/2$ in the right column. To begin with, at $\sigma = 0$, the two cases of course have the same frequency content. At the distance $\sigma = 0.5$ is seen how the frequency 1 is larger when the phase $\gamma = \pi/2$ and has increased its amplitude to above its original one. The frequency 2 on the other hand is larger when the phase $\gamma = 0$, although it is not above its original amplitude. At a larger distance $\sigma = 40$, the frequency 1 is twice the size when the phase $\gamma = \pi/2$. But surprisingly enough, also the frequency 2 is larger for the same phase. It is approximately three times as large as for when the phase $\gamma = 0$.

We shall now compare the influence of a high frequency added to a low frequency with the evolution of the pure low frequency. The signals are shown for different distances

$\sigma^* = 0, 0.2, 0.5, 2, 20, 200$ in Figure 2.

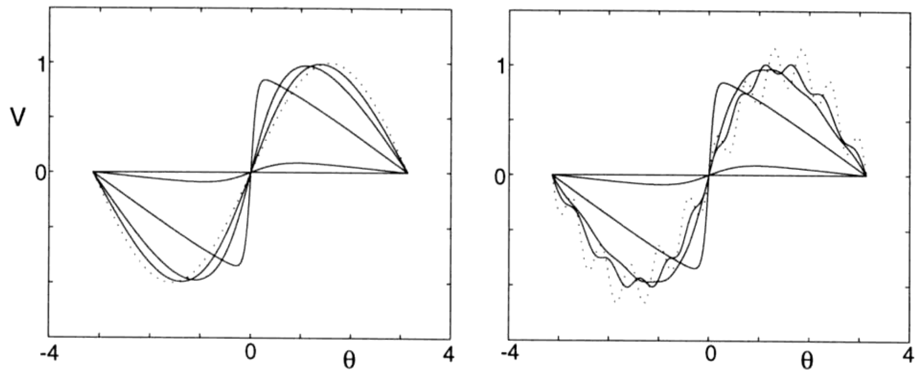

Figure 2: The evolutions of $V_0 = \sin\theta$ - left, and $V_0 = \sin\,\theta + 0.2\sin 11\theta$ - right.

The difference between these signals, $V(V_0 = \sin\,\theta + 0.2\sin 11\,\theta\,) - V(V_0 = \sin\theta)$, are pictured in Figure 3 for the propagation distances $\sigma^* = 0.2, 2, 20, 200$ with $\epsilon^* = 0.05$.

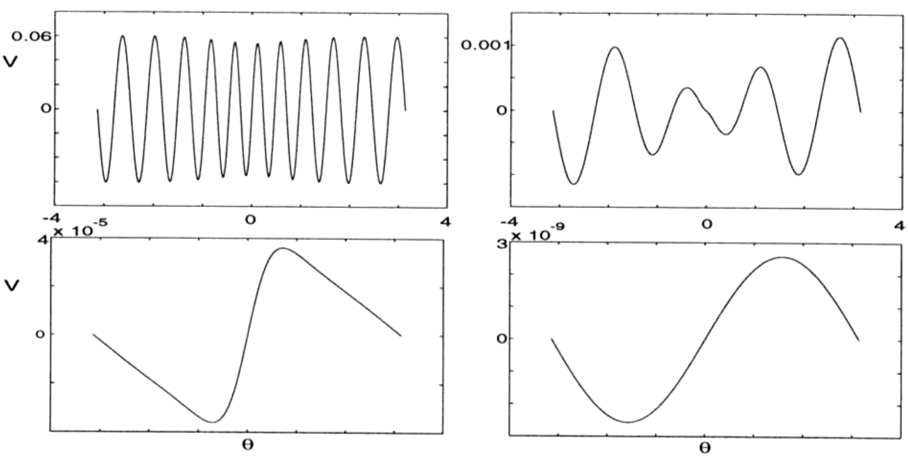

Figure 3: The difference $V(V_0 = \sin\theta + 0.2\sin 11\,\theta) - V(V_0 = \sin\theta)$ for the distances $\sigma^* = 0.2, 2, 20, 200$

The difference is at the final stage in a pure *low* frequency form. The peak values of the signals at the distance 200 is approximately 10^{-5} which makes the relative difference as small as $10^{-9}/10^{-4} = 10^{-4}$. It is anyway a positive number.

References

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