A survey
On integration of parabolic equations by reducing them to the heat equation

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Abstract
The present paper is a survey of results [1], [2] on extension of Euler's method for solving hyperbolic equations with one spatial variable to parabolic equations. The new method, based on the invariants of parabolic equations, allows one to identify all linear parabolic equations reducible to the heat equation and find their general solution. The method is illustrated by several examples.

Keywords: Parabolic equations, Semi-invariant, Reducible equations.

1 Introduction

1.1 Two-coefficient form of parabolic equations

The standard form of linear parabolic equations with one spatial variable is

$$u_t + A(t,x)u_{xx} + a(t,x)u_x + c(t,x)u = 0. \tag{1.1}$$

The equivalence transformations of Eqs. (1.1) consists of the linear transformation of the dependent variable

$$v = \sigma(t,x)u, \quad \sigma(t,x) \neq 0, \tag{1.2}$$

and the invertible changes of the independent variables of the form

$$\tau = \phi(t), \quad y = \psi(t,x), \tag{1.3}$$

where $\phi(t), \psi(t,x)$ and $\sigma(t,x)$ are arbitrary functions.

Lemma 1.1. Any parabolic equation (1.1) can be transformed by the change of the independent variables (1.3) to the form

$$u_t - u_{xx} + a(t,x)u_x + c(t,x)u = 0. \tag{1.4}$$

Proof. Under this change of variables (1.3) the derivatives of u undergo the following transformations:

$$u_t = \phi' u_\tau + \psi_t u_y, \quad u_x = \psi_x u_y, \quad u_{xx} = \psi_x^2 u_{yy} + \psi_{xx} u_y,$$

and hence Eq. (1.1) becomes:

$$\phi' u_\tau + A\psi_x^2 u_{yy} + \big(\psi_t + A\psi_{xx} + a\psi_x\big) u_y + cu = 0.$$

By choosing ψ satisfying the condition $|A|\psi_x^2 = 1$, letting $\phi = \pm\, t$ in accordance with the sign of A, and then taking τ and y as the new t and x, respectively, we arrive at Eq. (1.4).

Remark 1.1. By using the linear transformation (1.2) of the dependent variable one can map any parabolic equation to the one-coefficient form ([3], see also [2])

$$u_t - u_{xx} + c(t,x)u = 0. \tag{1.5}$$

But this form is not convenient for our calculations. In what follows, we will use the parabolic equations written in the two-coefficient form (1.4).

1.2 Semi-invariant

It is shown in [4] that the equations (1.1) have the following invariant with respect to the equivalence transformation (1.2):

$$\begin{aligned} K = \frac{1}{2}\, a^2 A_x + \Big(A_t + AA_{xx} - A_x^2\Big) a \\ + (AA_x - Aa)a_x - A^2 a_{xx} - Aa_t + 2A^2 c_x. \end{aligned} \tag{1.6}$$

Since the quantity K is invariant only under the transformation (1.2) of the dependent variable but not under all equivalence transformations (1.2)-(1.3) it is called a *semi-invariant*. Setting in (1.6) $A = -1$ we obtain the following *semi-invariant* for the parabolic equations in the two-coefficient form (1.4):

$$K = aa_x - a_{xx} + a_t + 2c_x. \tag{1.7}$$

2 Reduction by transformation (1.2)

We will use the equivalence transformation (1.2) written in the form

$$u = v\, \mathrm{e}^{-\varrho(t,x)}. \tag{2.1}$$

Theorem 2.1. (See [1]). The parabolic equation (1.4) can be reduced to the heat equation

$$v_t - v_{xx} = 0, \quad t > 0, \tag{2.2}$$

by the linear transformation (2.1) of the dependent variable if and only if the semi-invariant (1.7) vanishes. Namely, if $K = 0$, the function ϱ in the transformation (2.1) mapping Eq. (1.4) to the heat equation (2.2) is obtained by solving the system of equations

$$\frac{\partial \varrho}{\partial x} = -\frac{1}{2}\, a, \quad \frac{\partial \varrho}{\partial t} = \frac{1}{4}\, a^2 - \frac{1}{2}\, a_x + c. \tag{2.3}$$

The equation $K = 0$ guarantees solvability of the over-determined system (2.3).

Proof. We have from Eq. (2.1) by differentiating:

$$
\begin{aligned}
u_t &= (v_t - v\varrho_t)\,\mathrm{e}^{-\varrho(t,x)},\\
u_x &= (v_x - v\varrho_x)\,\mathrm{e}^{-\varrho(t,x)},\\
u_{xx} &= \big[v_{xx} - 2v_x\varrho_x + (\varrho_x^2 - \varrho_{xx})v\big]\,\mathrm{e}^{-\varrho(t,x)}.
\end{aligned}
$$

Inserting these expressions in the left-hand side of Eq. (1.4), we obtain:

$$
\begin{aligned}
&u_t - u_{xx} + au_x + cu\\
&= \big[v_t - v_{xx} + (a + 2\varrho_x)\,v_x\\
&+ (\varrho_{xx} - \varrho_x^2 - \varrho_t - a\varrho_x + c)\,v\big]\mathrm{e}^{-\varrho(t,x)}.
\end{aligned}
\tag{2.4}
$$

Eq. (2.4) shows that Eq. (1.4) can be reduced to the heat equation

$$
v_t - v_{xx} = 0, \quad t > 0, \tag{2.2}
$$

by a linear transformation (2.1) of the dependent variable if and only if

$$
a + 2\varrho_x = 0, \quad \varrho_{xx} - \varrho_x^2 - \varrho_t - a\varrho_x + c = 0. \tag{2.5}
$$

The first equation (2.5) yields

$$
\varrho_x = -\frac{1}{2}\,a, \quad \varrho_{xx} = -\frac{1}{2}\,a_x,
$$

and then the second equation (2.5) becomes

$$
\frac{1}{4}\,a^2 - \frac{1}{2}\,a_x - \varrho_t + c = 0.
$$

Thus, Eqs. (2.5) can be rewritten as the over-determine system of first-order equations (2.3) for the unknown function $\varrho(t,x)$:

$$
\varrho_x = -\frac{1}{2}\,a, \quad \varrho_t = \frac{1}{4}\,a^2 - \frac{1}{2}\,a_x + c.
$$

The compatibility condition $\varrho_{xt} = \varrho_{tx}$ for the system (2.3) has the form

$$
K \equiv aa_x - a_{xx} + a_t + 2c_x = 0. \tag{2.6}
$$

3 Applications of Theorem 2.1

Theorem 2.1 furnishes us with a practical method for solving by quadrature a wide class of parabolic equations (1.4) by reducing them to the heat equation.

3.1 Examples

Example 3.1. Any Eq. (1.4) with constant coefficients a and c can be reduced to the heat equation. Indeed, the semi-invariant (1.7) vanishes if $a, c = \text{const}$.

Remark 3.1. For hyperbolic equations a similar statement does not valid. For example, the telegraph equation $u_{xy} + u = 0$ cannot be reduced to the wave equation.

Example 3.2. The semi-invariant (1.7) of the equations of the one-coefficient form (1.5) with the coefficient c depending only on t,

$$u_t - u_{xx} + c(t)u = 0,$$

vanishes. Therefore this equation reduces to the heat equation by the equivalence transformation $u = v\,\mathrm{e}^{\int c(t)dt}$.

Example 3.3. The equation

$$u_t - u_{xx} + 2u_x - u = 0 \tag{3.1}$$

has the vanishing semi-invariant (1.7) (see Example 3.1). The system (2.3) yields

$$\varrho(t, x) = -x.$$

Hence, according to Eq. (2.1), the solution to Eq. (3.1) is given by

$$u(t, x) = \mathrm{e}^{x}\, v(t, x),$$

where $v(t, x)$ is any solution of the heat equation (2.2). Taking, e.g. the fundamental solution

$$v(t, x) = \frac{\theta(t)}{2\sqrt{\pi t}}\, \mathrm{e}^{-\frac{x^2}{4t}}$$

of the heat equation, we obtain the fundamental solution for Eq. (3.1):

$$u(t, x) = \frac{\theta(t)}{2\sqrt{\pi t}}\, \mathrm{e}^{x-\frac{x^2}{4t}}\,,$$

where $\theta(t)$ is the Heaviside function.

3.2 Utilization of Poisson's formula

Consider the solutions $v(t, x)$ of the heat equation (2.2) by assuming that they do not grow extremely rapidly as $x \to \infty$. Specifically, we assume that $v(t, x)$ is defined and continuous on a strip

$$0 \le t \le T < +\infty, \quad -\infty < x < +\infty,$$

and satisfies the following condition:

$$\max_{0\le t\le T} \big|v(t,x)\big|\,\mathrm{e}^{-\beta x^2} \to 0 \quad \text{as} \quad |x| \to \infty, \tag{3.2}$$

where $\beta = \text{const}$. If in addition we impose the initial condition

$$v(0,x) = f(x), \tag{3.3}$$

where $f(x)$ is any continuous and bounded function, then the solution of the heat equation (2.2) is unique and is given by Poisson's formula

$$v(t,x) = \frac{1}{2\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz, \quad t > 0. \tag{3.4}$$

Taking arbitrary continuous and bounded functions $f(x)$ one obtains an integral representation (3.4) of all solutions of the heat equation in the class of functions satisfying the condition (3.2).

Eqs. (2.1), (3.2) and (3.4) lead to the following statement.

Theorem 3.1. Let the semi-invariant (1.7) of Eq. (1.4) vanish. Then the solutions to Eq. (1.4) that belong to the class of functions satisfying the condition

$$\max_{0\le t\le T} \big|u(t,x)\,\mathrm{e}^{\varrho(t,x)}\big|\,\mathrm{e}^{-\beta x^2} \to 0 \quad \text{as} \quad |x| \to \infty, \tag{3.5}$$

where $\varrho(t,x)$ is determined by Eqs. (2.3), admit the integral representation

$$u(t,x) = \frac{1}{2\sqrt{\pi t}}\,\mathrm{e}^{-\varrho(t,x)} \int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz, \quad t > 0. \tag{3.6}$$

Thus, Eq. (3.6) furnishes the solution to Eq. (1.4) with the vanishing semi-invariant K, provided that the condition (3.5) is satisfied.

3.3 Utilization of Tikhonov's formula

A.N. Tikhonov showed in 1935 (see [5] or [6]) that if we do not impose the restriction (3.2) on the growth of solutions, the solution of the initial value problem (2.2), (3.3) is not unique. He gave an example of non-uniqueness by using the infinite series representation

$$\begin{aligned} v(t,x) = &F(t) + xF_1(t) + \frac{x^2}{2!}\,F'(t) + \frac{x^3}{3!}\,F_1'(t) + \frac{x^4}{4!}\,F''(t) + \cdots \\ &+ \frac{x^{2n}}{(2n)!}\,F^{(n)}(t) + \frac{x^{2n+1}}{(2n+1)!}\,F_1^{(n)}(t) + \cdots \end{aligned} \tag{3.7}$$

of the solutions $v(t,x)$ to the heat equation. In Eq. (3.7) $F(t)$ and $F_1(t)$ are any C^∞ functions such that the series (3.7) is uniformly convergent. Tikhonov's example is mentioned in [7], Chapter IV, and [8], Chapter 12.

Let us verify that the function $v(t,x)$ given by the series (3.7) solves the heat equation (2.2). Since the series (3.7) is uniformly convergent, we can differentiate it termwise and obtain:

$$\begin{aligned}
v_t =& F'(t) + xF_1'(t) + \frac{x^2}{2!}\,F''(t) + \frac{x^3}{3!}\,F_1''(t) + \cdots \\
&+ \frac{x^{2n}}{(2n)!}\,F^{(n+1)}(t) + \frac{x^{2n+1}}{(2n+1)!}\,F_1^{(n+1)}(t) + \cdots\,, \\
v_x =& F_1(t) + xF'(t) + \frac{x^2}{2!}\,F_1'(t) + \frac{x^3}{3!}\,F''(t) + \frac{x^4}{4!}\,F_1''(t) + \cdots \\
&+ \frac{x^{2n+1}}{(2n+1)!}\,F^{(n+1)}(t) + \frac{x^{2n+2}}{(2n+2)!}\,F_1^{(n+1)}(t) + \cdots\,, \\
v_{xx} =& F'(t) + x\,F_1'(t) + \frac{x^2}{2!}\,F''(t) + \frac{x^3}{3!}\,F_1''(t) + \cdots \\
&+ \frac{x^{2n}}{(2n)!}\,F^{(n+1)}(t) + \frac{x^{2n+1}}{(2n+1)!}\,F_1^{(n+1)}(t) + \cdots\,.
\end{aligned}$$

Subtracting term by term we obtain $v_t - v_{xx} = 0$.

The solution (3.7) satisfies the conditions

$$v(t,0) = F(t), \quad v_x(t,0) = F_1(t). \tag{3.8}$$

Furthermore, the Cauchy-Kowalewsky theorem guarantees that the solution of the heat equation satisfying the conditions (3.8) is unique. Consequently, the representation (3.7) is obtained by expanding the solution $v(t,x)$ to the heat equation into Taylor series with respect to x,

$$v(t,x) = v(t,0) + x\,v_x(t,0) + \frac{x^2}{2!}\,v_{xx}(t,0) + \frac{x^3}{3!}\,v_{xxx}(t,0) + \frac{x^4}{4!}\,v_{xxxx}(t,0) + \cdots\,, \tag{3.9}$$

and using the conditions (3.8). Indeed, Eqs. (2.2), (3.8) yield:

$$\begin{aligned}
v_{xx}(t,0) &= v_t(t,0) = \frac{dv(t,0)}{dt} = F'(t), \\
v_{xxx}(t,0) &= (v_{xx})_x(t,0) = (v_t)_x(t,0) = \frac{dv_x(t,0)}{dt} = F_1'(t), \\
v_{xxxx}(t,0) &= (v_{xx})_{xx}(t,0) = (v_t)_{xx}(t,0) = \frac{dv_{xx}(t,0)}{dt} = F''(t), \ldots\,.
\end{aligned} \tag{3.10}$$

Substituting (3.8) and (3.10) in the expansion (3.9) we obtain Tikhonov's representation (3.7).

The infinite series representation (3.7) is particularly useful for obtaining approximate solutions to the heat equation (2.2) and to the equivalent equations, e.g. by truncating the infinite series. Tikhonov's series representation is also convenient for obtaining solutions in closed forms, in particular, in terms of elementary functions. One of such cases is obtained by taking for $F(t)$ and $F_1(t)$ any polynomials.

Example 3.4. Letting $F(t) = 0$ and

$$F_1(t) = a + bt + ct^2 + kt^3,$$

we obtain the following polynomial solution:

$$v(t,x) = (a + bt + ct^2 + kt^3)x + \frac{1}{3!}\left(b + 2ct + 3kt^2\right)x^3 + \frac{1}{5!}\left(2c + 6kt\right)x^5 + \frac{6k}{7!}\,x^7.$$

Substituting in Eq. (2.1) Tikhonov's representation (3.7) we arrive at the following statement.

Theorem 3.2. Let the semi-invariant (1.7) of Eq. (1.4) vanish. Then the series

$$\begin{aligned} u(t,x) =& \mathrm{e}^{-\varrho(t,x)}\Big[F(t) + xF_1(t) + \frac{x^2}{2!}\,F'(t) + \frac{x^3}{3!}\,F_1'(t) + \cdots \\ &+ \frac{x^{2n}}{(2n)!}\,F^{(n)}(t) + \frac{x^{2n+1}}{(2n+1)!}\,F_1^{(n)}(t) + \cdots\Big], \end{aligned} \tag{3.11}$$

where $\varrho(t,x)$ is determined by Eqs. (2.3), solves the equation (1.4).

Example 3.5. Consider again Eq. (3.1). We know that here $\varrho(t,x) = -x$. Therefore Eq. (3.11) yields:

$$u(t,x) = \mathrm{e}^{x}\left[F(t) + xF_1(t) + \frac{x^2}{2!}\,F'(t) + \frac{x^3}{3!}\,F_1'(t) + \cdots\right].$$

4 Reduction by general equivalence transformation

The general equivalence group for Eq. (1.4) comprises the following change of the independent variables:

$$t = H(s), \quad x = \varphi_1(s)\,y + \varphi_0(s), \tag{4.1}$$

where $\varphi_1(s) \neq 0$ and $H(s)$ is defined by the equation

$$H'(s) = \varphi_1^2(s), \tag{4.2}$$

and the linear transformation (1.2) of the dependent variable which we will write now in the form

$$u = V(s,y)\,v, \quad V(s,y) \neq 0. \tag{4.3}$$

The generators of the transformations (4.1) and (4.3) are

$$Y_\alpha = \alpha \frac{\partial}{\partial x} + \alpha' \frac{\partial}{\partial a}, \quad Y_\gamma = 2\gamma \frac{\partial}{\partial t} + \gamma' x \frac{\partial}{\partial x} + (x\gamma'' - a\gamma') \frac{\partial}{\partial a} - c\gamma' \frac{\partial}{\partial c} \tag{4.4}$$

and

$$Y_\sigma = \sigma u \frac{\partial}{\partial u} + 2\sigma_x \frac{\partial}{\partial a} + (\sigma_{xx} - \sigma_t - a\sigma_x) \frac{\partial}{\partial c}\,, \tag{4.5}$$

respectively.

Theorem 4.1. (See [2]). Eq. (1.4) can be mapped into the heat equation

$$v_s - v_{yy} = 0 \tag{4.6}$$

by the general group of equivalence transformations (4.1), (4.3) if and only if the semi-invariant K satisfies the equation

$$K_{xx} = 0. \tag{4.7}$$

If the condition (4.7) is satisfied, the transformations (4.1), (4.3) mapping Eq. (1.4) into the heat equation (4.6) are obtained by solving the following differential equations:

$$\begin{aligned} &2\,\varphi_1\,V_y + (\varphi_1'\,y + \varphi_0' - a\,\varphi_1^2)V = 0,\\ &(V_{yy} - V_s)\varphi_1 - a\,\varphi_1^2\,V_y - c\,\varphi_1^3\,V + (\varphi_1'\,y + \varphi_0')V_y = 0. \end{aligned} \tag{4.8}$$

The solvability of Eqs. (4.8) is guaranteed by the condition (4.7).

Example 4.1. Consider the following simple equation of the form (1.4):

$$u_t - u_{xx} - xu_x = 0. \tag{4.9}$$

In the example we have $a = -x,\ c = 0$ and Eq. (1.7) yields $K = x$. It is manifest that the reducibility condition (4.7) is satisfied.

Let us investigate the reducibility of Eq. (4.9) to the heat equation by a change of the independent variables, without transforming the dependent variable, i.e. by letting $V = 1$. Then Eqs. (4.8) reduce to one equation, namely

$$\varphi_1'\,y + \varphi_0' - a\varphi_1^2 = 0.$$

Hence, invoking that $a = -x$ and using the second equation (4.1), we have:

$$\varphi_1'(s)\,y + \varphi_0'(s) + [\varphi_1(s)\,y + \varphi_0(s)]\varphi_1^2(s) = 0.$$

Upon separating the variables,

$$[\varphi_1'(s) + \varphi_1^3(s)]\,y + \varphi_0'(s) + \varphi_0(s)\varphi_1^2(s) = 0,$$

this equation provides the following system of two ordinary differential equations:

$$\varphi_1' + \varphi_1^3 = 0, \quad \varphi_0' + \varphi_0 \varphi_1^2 = 0. \tag{4.10}$$

It suffices to find any particular solution of this system with $\varphi_1 \neq 0$. We can let, e.g. $\varphi_0 = 0$, integrate the first equation (4.10), ignore the constant of integration and obtain:

$$\varphi_1 = \frac{1}{\sqrt{2s}} \cdot \tag{4.11}$$

According to Eq. (4.2), we have

$$H' = \frac{1}{2s},$$

and hence

$$H(s) = \ln \sqrt{s}\,. \tag{4.12}$$

Thus, we have arrived at the following change of the independent variables (4.1):

$$t = \ln \sqrt{s}\,, \quad x = \frac{y}{\sqrt{2s}} \cdot \tag{4.13}$$

It maps Eq. (4.9) into the heat equation

$$u_s - u_{yy} = 0. \tag{4.14}$$

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