Conservation laws for a coupled variable-coefficient modified Korteweg–de Vries system in a two-layer fluid model

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Abstract

We find the Lie point symmetries of a coupled variable-coefficient modified Korteweg-de Vries system in a two-layer fluid model. Then we establish its quasi self-adjointness and corresponding conservation laws.

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1. Introduction

The purpose of this paper is to study the following coupled system of two equations of Korteweg-de Vries type

$$\begin{cases} u_t - \alpha(t)[u_{xxx} + 6(u^2 - v^2)u_x - 12uvv_x] - 4\beta(t)u_x = 0, \\ v_t - \alpha(t)[v_{xxx} + 6(u^2 - v^2)v_x + 12uvu_x] - 4\beta(t)v_x = 0 \end{cases} \tag{1}$$

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where $u = u(x,t)$, $v = v(x,t)$, from the point of view of the Sophus Lie symmetry theory.

The system (1) was proposed in [1] as an important particular case of the formidable generalized coupled variable-coefficient modified Korteweg-de Vries (CVmKdV) system

$$\begin{cases} u_t + r_1 u_{xxx} + (r_2 u^2 + r_3 uv + r_4 u + r_5 v^2 + r_6 v + r_7) u_x \\ \qquad\qquad + (r_8 u^2 + r_9 uv + r_{10} u + r_{11} v + r_{12}) v_x + r_{13} u = 0, \\ v_t + e_1 v_{xxx} + (e_2 v^2 + e_3 uv + e_4 v + e_5 u^2 + e_6 u + e_7) v_x \\ \qquad\qquad + (e_8 v^2 + e_9 uv + e_{10} v + e_{11} u + e_{12}) u_x + e_{13} v = 0, \end{cases} \tag{2}$$

derived by Gao and Tang in [2] as a two-layer model describing atmospheric and oceanic phenomena like interactions between the atmosphere and ocean, atmospheric blocking, oceanic circulations, hurricanes typhoons, etc.

In (2), r_i e e_i, $i = 1, ..., 13$, are arbitrary functions of t. In the work [2] Gao and Tang found several solutions of (2) under some constraints on its coefficients. In [1] Zhu et al. obtained the system (1) by a reduction different than the constraints used in [2], expecting that it could describe more complex physical properties than those investigated in other works (see [1] and the references therein). In the same paper various wide classes of interesting exact solutions were found.

In the present paper we first get the Lie point symmetries of system (1) and prove that it is quasi self-adjoint. With those two assets available, we obtain conservations laws for that system by using the Noether operator $\mathcal{N}$, see also [3, 4, 5]. Calculating the symmetries of the system (1), obtaining its adjoint system and applying the Noether operator to obtain the conserved vectors are well defined algorithmic procedures. Nevertheless, the calculations involved are usually very difficult and extensive even for the simplest equations. Thus, it may become very tedious and error prone. For that reason the use of computer algebra systems like *Mathematica, Maple, Reduce*, etc and of special symbolic packages that are build based on them are very crucial. One such symbolic package, based on *Mathematica* [6], has been devised and developed by SD as part of his PhD thesis [7]. The package, named SYM [8, 9, 7], was developed from the ground up using the symbolic manipulation power of *Mathematica* and the artificial intelligence capabilities which it offers. It was extensively used for all the results in the present paper, both for obtaining the symmetries of the system and, by employing the symbolic tools provided by it, to get and, simplify, the adjoint system and the conserved vectors that emerge from the use of the Noether operator.

In section 2 we obtain the Lie point symmetries of the CVmKdV system (1). In section 3 we study the self-adjointness of this system. Then in section 4 we establish the conservation laws corresponding to the obtained symmetries exploiting the self-adjointness found [3, 4, 5][1].

2. Lie point symmetries

In this section we calculate the Lie point symmetry group of the system (1) applying the Sophus Lie algorithm [10, 11]. Our basic assumption upon the system (1) is $\alpha \neq 0$ (otherwise we would have a simple system of two uncoupled first order partial differential equations which can be easily solved explicitly). Let the vector field

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial t} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} \tag{3}$$

be an infinitesimal generator of a Lie point symmetry of the CVmKdV system (1), where $\xi^1, \xi^2, \eta^1, \eta^2$ are functions of x, t, u, v.

First we derive the determining equations, a large system of partial differential equations, see Appendix A. To obtain now all the solutions of the determining equations (A.1) we proceed as follows. We first solve a sub-system of the determining equations not containing the arbitrary functions α, β and we substitute its solution back into the system. We repeat the process until the system that remains includes only equations containing the functions α, β.

Particularly, we select from (A.1) the following sub-system:

$$\begin{gathered}
{\xi^2}_{,v} = 0,\ {\xi^2}_{,vv} = 0,\ {\xi^2}_{,vvv} = 0,\ {\xi^2}_{,u} = 0,\ {\xi^2}_{,uv} = 0,\ {\xi^2}_{,uvv} = 0,\ {\xi^2}_{,uu} = 0,\\
{\xi^2}_{,uuv} = 0,\ {\xi^2}_{,uuu} = 0,\ {\xi^2}_{,x} = 0,\ {\xi^2}_{,xv} = 0,\ {\xi^2}_{,xvv} = 0,\ {\xi^2}_{,xu} = 0,\ {\xi^2}_{,xuv} = 0,\\
{\xi^2}_{,xuu} = 0,\ {\xi^2}_{,xx} = 0,\ 2{\xi^2}_{,v} + v{\xi^2}_{,vv} = 0,\ 3{\xi^2}_{,vv} + v{\xi^2}_{,vvv} = 0,\\
2{\xi^2}_{,u} + u{\xi^2}_{,uu} = 0,\ 3{\xi^2}_{,uu} + u{\xi^2}_{,uuu} = 0.
\end{gathered}$$

The solution of the latter can be easily found to be:

$$\xi^2 = \mathcal{F}_1(t), \tag{4}$$

[1] More details when the rest of the paper is complete.

where $\mathcal{F}_1$ is an arbitrary function of t.

By substituting ξ^2 from (4) in the determining equations (A.1) we obtain the reduced system (Appendix A) from which we select another sub-system:

$$\begin{gathered}
{\xi^1}_{,v}=0,\ \eta^1_{,vv}=0,\ \eta^2_{,vv}=3{\xi^1}_{,xv},\ {\xi^1}_{,vv}=0,\ \eta^1_{,vvv}=0,\ \eta^2_{,vvv}=3{\xi^1}_{,xvv},\ {\xi^1}_{,u}=0,\\
{\xi^1}_{,vvv}=0,\ 4uv{\xi^1}_{,u}=\eta^2_{,xuu},\ \eta^1_{,uv}={\xi^1}_{,xv},\ \eta^1_{,uv}=2{\xi^1}_{,xv},\ \eta^2_{,uv}={\xi^1}_{,xu},\ {\xi^1}_{,uv}=0,\\
\eta^2_{,uv}=2{\xi^1}_{,xu},\ \eta^1_{,uvv}={\xi^1}_{,xvv},\ \eta^2_{,uvv}=2{\xi^1}_{,xuv},\ {\xi^1}_{,uvv}=0,\ \eta^1_{,uu}=3{\xi^1}_{,xu},\\
\eta^2_{,uu}=0,\ {\xi^1}_{,uu}=0,\ \eta^1_{,uuv}=2{\xi^1}_{,xuv},\ \eta^2_{,uuv}={\xi^1}_{,xuu},\ {\xi^1}_{,uuv}=0,\ \eta^2_{,uuu}=0,\\
{\xi^1}_{,uuu}=0,\ \eta^1_{,xv}=0,\ \eta^2_{,xv}={\xi^1}_{,xx},\ \eta^1_{,uuu}=3{\xi^1}_{,xuu},\ 4uv{\xi^1}_{,v}+\eta^1_{,xvv}=0,\\
\eta^1_{,xu}={\xi^1}_{,xx},\ \eta^2_{,xu}=0,\ 2\eta^1_{,xuv}={\xi^1}_{,xxv},\ 2\eta^2_{,xuv}={\xi^1}_{,xxu},\\
4uv\eta^1_{,u}+\eta^1_{,xxv}=4\left(v\eta^1+u\left(\eta^2+v\left(\eta^2_{,v}+2{\xi^1}_{,x}\right)\right)\right),\ 4uv{\xi^1}_{,v}+\eta^1_{,xuu}={\xi^1}_{,xxu},\\
4uv{\xi^1}_{,u}+{\xi^1}_{,xxv}=\eta^2_{,xvv},\ 4v\eta^1+4u\eta^2+4uv\eta^1_{,u}+8uv{\xi^1}_{,x}+\eta^2_{,xxu}=4uv\eta^2_{,v}.
\end{gathered}$$

The general solution of the last sub-system is:

$$\xi^1=\mathcal{F}_2(t)-x\mathcal{F}_3(t),\quad \eta^1=u\mathcal{F}_3(t),\quad \eta^2=v\mathcal{F}_3(t), \tag{5}$$

where $\mathcal{F}_2(t)$ and $\mathcal{F}_3(t)$ are arbitrary functions of t. Afterwards, we substitute ξ^1, η^1, η^2 given by (5) in the reduced system (Appendix A). The result is:

$$\begin{gathered}
\mathcal{F}_3'=0,\\
\mathcal{F}_1(t)\alpha'+\alpha(t)\left(3\mathcal{F}_3(t)+\mathcal{F}_1'\right)=0,\\
4\beta(t)\mathcal{F}_1(t)\alpha'=\alpha(t)\left(-8\beta(t)\mathcal{F}_3(t)+4\mathcal{F}_1(t)\beta'+\mathcal{F}_2'-x\mathcal{F}_3'\right).
\end{gathered}$$

Hence, $\mathcal{F}_3=\mathbf{c}_1$, where $\mathbf{c}_1$ is a constant. Putting $\mathcal{F}_3=\mathbf{c}_1$ in the above system we get the following system of two ordinary differential equations

$$\begin{aligned}
&\mathcal{F}_1(t)\alpha'+\alpha(t)\left(3\mathbf{c}_1+\mathcal{F}_1'\right)=0,\\
&4\beta(t)\mathcal{F}_1(t)\alpha'=\alpha(t)\left(-8\mathbf{c}_1\beta(t)+4\mathcal{F}_1(t)\beta'+\mathcal{F}_2'\right)
\end{aligned} \tag{6}$$

The solution of system (6) is

$$\begin{aligned}
\mathcal{F}_1&=\frac{\mathbf{c}_2}{\alpha(t)}-\frac{3\mathbf{c}_1\int\alpha(t)\,\mathrm{d}t}{\alpha(t)},\\
\mathcal{F}_2&=\mathbf{c}_3-4\mathbf{c}_1\int\beta(t)\,\mathrm{d}t+\frac{4\left(3\mathbf{c}_1\int\alpha(t)\,\mathrm{d}t-\mathbf{c}_2\right)\beta(t)}{\alpha(t)}.
\end{aligned}$$

Hence, the Lie point symmetries of system (1) are expressed by the following basis:

$$\begin{aligned}\mathfrak{X}_1 =&\partial_x,\\ \mathfrak{X}_2 =&\frac{1}{\alpha(t)}\partial_t - \frac{4\beta(t)}{\alpha(t)}\partial_x,\\ \mathfrak{X}_3 =& \left(x + 4\int\beta(t)\,\mathrm{d}t + \frac{12(\int\alpha(t)\,dt)\beta(t)}{\alpha(t)}\right)\partial_x + \frac{3\int\alpha(t)\,\mathrm{d}t}{\alpha(t)}\partial_t\\ &- u\partial_u - v\partial_v.\end{aligned}\tag{7}$$

The commutation table of the Lie algebra generated by (7) is:

$$\begin{array}{cccc} [\cdot,\cdot] & \mathfrak{X}_1 & \mathfrak{X}_2 & \mathfrak{X}_3\\ \mathfrak{X}_1 & 0 & 0 & -\mathfrak{X}_1\\ \mathfrak{X}_2 & 0 & 0 & -3\mathfrak{X}_2\\ \mathfrak{X}_3 & \mathfrak{X}_1 & 3\mathfrak{X}_2 & 0\end{array}\tag{8}$$

3. Self-adjointness

In accordance to [3, 4, 5] we introduce the formal Lagrangian

$$\begin{aligned}\mathcal{L} =&z\left(u_{,t} - 4\beta(t)u_{,x} - \alpha(t)\left(6\left(u^2-v^2\right)u_{,x} - 12uvv_{,x} + u_{,xxx}\right)\right)\\ &+ w\left(v_{,t} - 4\beta(t)v_{,x} - \alpha(t)\left(12uvu_{,x} + 6\left(u^2-v^2\right)v_{,x} + v_{,xxx}\right)\right)\end{aligned}\tag{9}$$

where $z = z(x,t)$ and $w = w(x,t)$ are the nonlocal variables. Then the adjoint system to the system (1) reads

$$\begin{aligned}-z_{,t} + \alpha(t)z_{,xxx} + 6\alpha(t)(u^2-v^2)z_{,x} + 12\alpha(t)uvw_{,x} + 4\beta(t)z_{,x} = 0,\\ -w_{,t} + \alpha(t)w_{,xxx} + 6\alpha(t)(u^2-v^2)w_{,x} - 12\alpha(t)uvz_{,x} + 4\beta(t)w_{,x} = 0.\end{aligned}\tag{10}$$

It is evident that the system (1) is not strictly self-adjoint.

Now we look for a substitution

$$z = \Phi(u,v), \quad w = \Psi(u,v)\tag{11}$$

such that the system (1) becomes quasi self-adjoint. For this purpose we substitute (11) into (10) and expressing u_t and v_t from (1) to obtain two identities:

$$\begin{aligned}&-\Phi_{,vvv}{v_{,x}}^3 + 12uv(-v_{,x}(\Psi_{,v}+\Phi_{,u}) + u_{,x}(\Phi_{,v}-\Psi_{,u})) - 3v_{,x}(\Phi_{,uv}u_{,xx} + \Phi_{,vv}v_{,xx})\\ &- 3(u_{,x}({v_{,x}}^2\Phi_{,uvv} + \Phi_{,uv}v_{,xx} + u_{,xx}\Phi_{,uu}) + {u_{,x}}^2v_{,x}\Phi_{,uuv}) - {u_{,x}}^3\Phi_{,uuu} = 0,\end{aligned}$$

$$-\Psi_{,vvv}{v_{,x}}^3+12uv(u_{,x}(\Psi_{,v}+\Phi_{,u})+v_{,x}(\Phi_{,v}-\Psi_{,u}))-3v_{,x}(\Psi_{,uv}u_{,xx}+\Psi_{,vv}v_{,xx})$$
$$-3(u_{,x}({v_{,x}}^2\Psi_{,uvv}+\Psi_{,uv}v_{,xx}+u_{,xx}\Psi_{,uu})+{u_{,x}}^2v_{,x}\Psi_{,uuv})-{u_{,x}}^3\Psi_{,uuu}=0.$$

Equalizing to zero the coefficients of the derivatives $u_{,x}, v_{,x}, u_{,xx}, v_{,xx}$ in the above identities and simplifying we find that

$$\Phi_{,vv}=0,\ \Psi_{,vv}=0,\ \Phi_{,vvv}=0,\ \Psi_{,vvv}=0,\ \Psi_{,v}+\Phi_{,u}=0,\ \Phi_{,v}-\Psi_{,u}=0,$$
$$\Phi_{,uv}=0,\ \Psi_{,uv}=0,\ \Phi_{,uvv}=0,\ \Psi_{,uvv}=0,\ \Phi_{,uu}=0,\ \Psi_{,uu}=0,\ \Phi_{,uuv}=0,$$
$$\Psi_{,uuv}=0,\ \Phi_{,uuu}=0,\ \Psi_{,uuu}=0.$$

The solution of the above system can be easily found to be

$$\begin{aligned}\Phi&=\mathbf{c}_1-\mathbf{c}_4u+\mathbf{c}_3v,\\ \Psi&=\mathbf{c}_2+\mathbf{c}_3u+\mathbf{c}_4v.\end{aligned}\tag{12}$$

Hence the CVmKdV system (1) is quasi self-adjoint. This property will enable us in the next section to construct conservation laws for system (1).

4. Conservation laws

The fact that the system (1) is quasi self-adjoint allow us, with the help of its point symmetries, to use the Noether operator $\mathcal{N}$ to obtain conserved vectors, (C^1, C^2), for it [3, 4, 5]. Such vectors will satisfy the conservation equation $D_xC^1+D_tC^2|_{(1)}=0$.

For this purpose the nonlocal variables appearing in that formula must be substituted according to (12), namely

$$\begin{aligned}z&=\mathbf{c}_1-\mathbf{c}_4u+\mathbf{c}_3v,\\ w&=\mathbf{c}_2+\mathbf{c}_3u+\mathbf{c}_4v.\end{aligned}$$

We now use each one of the three symmetries (7) to obtain the following conserved vectors[2].

- The symmetry $\mathfrak{X}_1$ determines the conserved vector:

[2]We know beforehand, because as on can easily see from the commutation table (8) the abelian Lie subalgebra of $\mathfrak{X}_1,\ \mathfrak{X}_2$ is an ideal of the Lie algebra, that the conservation laws from the first two symmetries will be trivial, see also [10, Section 22.4].

$$
\begin{aligned}
C^1 &= (\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)u_{,t} + (\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)v_{,t},\\
C^2 &= -(\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)u_{,x} - (\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)v_{,x}
\end{aligned}
$$

one can easily see that $D_xC^1 + D_tC^2 = 0$ and therefore it is a trivial conservation law.

- The symmetry $\mathfrak{X}_2$ determines the conserved vector:

$$
\begin{aligned}
C^1 =& -6(\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)v^2v_{,t} + (\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)(-6v^2u_{,t} + u_{,xxt})\\
&+ 6u^2((\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)u_{,t} + (\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)v_{,t})\\
&+ 12uv((\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)u_{,t} - (\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)v_{,t})\\
&+ (\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)v_{,xxt},\\
C^2 =& 12uv(-(\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)u_{,x} + (\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)v_{,x})\\
&- 6u^2((\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)u_{,x} + (\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)v_{,x})\\
&+ (\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)(6v^2u_{,x} - u_{,xxx}) + (\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)(6v^2v_{,x} - v_{,xxx}).
\end{aligned}
$$

Again, after some calculations one can see that $D_xC^1 + D_tC^2 = 0$ and hence it is a trivial conservation law.

- The symmetry $\mathfrak{X}_3$ determines the conserved vector:

$$
\begin{aligned}
C^1 =& -6(\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)u^3\alpha - (\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)\left(x + 4\int\beta(t)\,\mathrm{d}t\right)u_{,t}\\
&+ 18(\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)v^2u_{,t}\int\alpha(t)\,\mathrm{d}t - 18u^2(\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)v\alpha\\
&- 18u^2\int\alpha(t)\,\mathrm{d}t((\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)u_{,t} + (\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)v_{,t})\\
&- 36u\int\alpha(t)\,\mathrm{d}t(\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)vu_{,t} - 3(\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)\alpha u_{,xx}\\
&+ 2u(\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)(-2\beta + 9v\left(v\alpha + 2\int\alpha(t)\,\mathrm{d}tv_{,t})\right)\\
&- 3\int\alpha(t)\,\mathrm{d}t(\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)u_{,xxt} + 6v^3\alpha(\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)\\
&- 4v\beta(\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4) - (\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)\left(x + 4\int\beta(t)\,\mathrm{d}t\right)v_{,t}\\
&+ (\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)\left(18\int\alpha(t)\,\mathrm{d}tv^2v_{,t} - 3(\alpha v_{,xx} + \int\alpha(t)\,\mathrm{d}tv_{,xxt})\right),
\end{aligned}
$$

$$
\begin{aligned}
C^2 =& (\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)v + 36u \int \alpha(t)\,\mathrm{d}t(\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)vu_{,x} \\
& + 18 \int \alpha(t)\,\mathrm{d}tu^2((\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)u_{,x} + (\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)v_{,x}) \\
& + u(\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)\left(1 - 36 \int \alpha(t)\,\mathrm{d}tvv_{,x}\right) \\
& + (\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)\left(x + 4\int \beta(t)\,\mathrm{d}t - 18 \int \alpha(t)\,\mathrm{d}tv^2\right) u_{,x} \\
& + (\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)\left(x + 4\int \beta(t)\,\mathrm{d}t - 18 \int \alpha(t)\,\mathrm{d}tv^2\right) v_{,x} \\
& + 3 \int \alpha(t)\,\mathrm{d}t((\mathbf{c}_1 + v\mathbf{c}_3 - u\mathbf{c}_4)u_{,xxx} + (\mathbf{c}_2 + u\mathbf{c}_3 + v\mathbf{c}_4)v_{,xxx})
\end{aligned}
$$

After simplifying, the latter conserved vector assumes the following form:

$$
\begin{aligned}
C^1 =& \mathbf{c}_4 \left(-9u^2v^2\alpha + (u_{,x}u_{,xt} - v_{,x}v_{,xt} - 6u^3u_{,t} - 6v^3v_{,t}) \int \alpha(t)\,\mathrm{d}t - v\alpha v_{,xx} \right. \\
& \left. +4vv_{,t} \int \beta(t)\,\mathrm{d}t + u(-4\int \beta(t)\,\mathrm{d}tu_{,t} + \alpha u_{,xx}) \right) \\
& + \mathbf{c}_3(-6u^3v\alpha + \alpha(u_{,x}v_{,x} - vu_{,xx}) + u(6v^3\alpha - 4v\beta - \alpha v_{,xx})), \\
C^2 =& \mathbf{c}_4 \left(-\frac{1}{2}u^2 + \frac{1}{2}v^2 + 4(uu_{,x} - vv_{,x}) \int \beta(t)\,\mathrm{d}t + 6(u^3u_{,x} + v^3v_{,x}) \int \alpha(t)\,\mathrm{d}t \right. \\
& \left. +(v_{,x}v_{,xx} - u_{,x}u_{,xx}) \int \alpha(t)\,\mathrm{d}t \right) + \mathbf{c}_3uv
\end{aligned}
$$

It is worth stating explicitly two particular cases:

First, let $\mathbf{c}_1 = \mathbf{c}_2 = \mathbf{c}_4 = 0$ and $\mathbf{c}_3 = 1$, e.g. we use the substitution $z = v$ and $w = u$. The conserved vector is:

$$
\begin{aligned}
C^1 &= -6u^3v\alpha + \alpha(u_{,x}v_{,x} - vu_{,xx}) + u(6v^3\alpha - 4v\beta - \alpha v_{,xx}), \\
C^2 &= uv
\end{aligned}
$$

Second, let $\mathbf{c}_1 = \mathbf{c}_2 = \mathbf{c}_3 = 0$ and $\mathbf{c}_4 = 1$. In this case the substitution is $z = -u$ and $w = v$. Then the symmetry $\mathfrak{X}_3$ determines the conserved vector given in a simplified form:

$$
\begin{aligned}
C^1 =& -9u^2v^2\alpha + (u_{,x}u_{,xt} - v_{,x}v_{,xt} - 6u^3u_{,t} - 6v^3v_{,t})\int\alpha(t)\,\mathrm{d}t - v\alpha v_{,xx} \\
&+ 4vv_{,t}\int\beta(t)\,\mathrm{d}t + u(-4\int\beta(t)\,\mathrm{d}t u_{,t} + \alpha u_{,xx}), \\
C^2 =& -\frac{1}{2}u^2 + \frac{1}{2}v^2 + 4(uu_{,x} - vv_{,x})\int\beta(t)\,\mathrm{d}t + 6(u^3u_{,x} + v^3v_{,x})\int\alpha(t)\,\mathrm{d}t \\
&+ (v_{,x}v_{,xx} - u_{,x}u_{,xx})\int\alpha(t)\,\mathrm{d}t
\end{aligned}
$$

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Appendix A.

The determining equations for the CVmKdV system (1) are:

$$
\begin{aligned}
{\xi^2}_{,u} &= 0, \qquad\qquad (A.1)\\
{\xi^2}_{,x} &= 0,\\
{\xi^2}_{,v} &= 0,\\
{\xi^2}_{,xx} &= 0,\\
{\xi^2}_{,xu} &= 0,\\
{\xi^2}_{,uu} &= 0,\\
{\xi^2}_{,xv} &= 0,\\
{\xi^2}_{,uv} &= 0,\\
{\xi^2}_{,vv} &= 0,\\
{\xi^2}_{,uuu} &= 0,\\
{\xi^2}_{,xuu} &= 0,
\end{aligned}
$$

$$\xi^2{}_{,xuv} = 0,$$
$$\xi^2{}_{,uuv} = 0,$$
$$\xi^2{}_{,xvv} = 0,$$
$$\xi^2{}_{,uvv} = 0,$$
$$\xi^2{}_{,vvv} = 0,$$
$$2\xi^2{}_{,v} + v\xi^2{}_{,vv} = 0,$$
$$2\xi^2{}_{,u} + u\xi^2{}_{,uu} = 0,$$
$$3\xi^2{}_{,vv} + v\xi^2{}_{,vvv} = 0,$$
$$3\xi^2{}_{,uu} + u\xi^2{}_{,uuu} = 0,$$
$$\eta^1_{,xv} + 12uv\alpha\xi^2{}_{,xx} = 0,$$
$$12uv\alpha\xi^2{}_{,xx} - \eta^2_{,xu} = 0,$$
$$\eta^1_{,vv} + 36u\alpha\left(\xi^2{}_{,x} + v\xi^2{}_{,xv}\right) = 0,$$
$$36v\alpha\left(\xi^2{}_{,x} + u\xi^2{}_{,xu}\right) - \eta^2_{,uu} = 0,$$
$$\eta^1_{,vvv} + 36u\alpha\left(2\xi^2{}_{,xv} + v\xi^2{}_{,xvv}\right) = 0,$$
$$36v\alpha\left(2\xi^2{}_{,xu} + u\xi^2{}_{,xuu}\right) - \eta^2_{,uuu} = 0,$$
$$\xi^1{}_{,u} + 2\left(3\left(u^2 - v^2\right)\alpha + 2\beta\right)\xi^2{}_{,u} = 0,$$
$$\xi^1{}_{,v} + 2\left(3\left(u^2 - v^2\right)\alpha + 2\beta\right)\xi^2{}_{,v} = 0,$$
$$\xi^1{}_{,xx} + 2\left(3\left(u^2 - v^2\right)\alpha + 2\beta\right)\xi^2{}_{,xx} - \eta^2_{,xv} = 0,$$
$$\xi^1{}_{,xx} + 2\left(3\left(u^2 - v^2\right)\alpha + 2\beta\right)\xi^2{}_{,xx} - \eta^1_{,xu} = 0,$$
$$\xi^1{}_{,v} + 4\beta\xi^2{}_{,v} + 6\alpha\left(\left(u^2 - v^2\right)\xi^2{}_{,v} - 2uv\xi^2{}_{,u}\right) = 0,$$
$$\xi^1{}_{,u} + 4\beta\xi^2{}_{,u} + 6\alpha\left(2uv\xi^2{}_{,v} + \left(u^2 - v^2\right)\xi^2{}_{,u}\right) = 0,$$
$$\xi^1{}_{,uu} + 4\beta\xi^2{}_{,uu} + 6\alpha\left(4u\xi^2{}_{,u} + \left(u^2 - v^2\right)\xi^2{}_{,uu}\right) = 0,$$
$$\xi^1{}_{,vv} + 4\beta\xi^2{}_{,vv} - 6\alpha\left(4v\xi^2{}_{,v} + \left(-u^2 + v^2\right)\xi^2{}_{,vv}\right) = 0,$$
$$\eta^1_{,t} - 4\beta\eta^1_{,x} - \alpha\left(6\left(u^2 - v^2\right)\eta^1_{,x} - 12uv\eta^2_{,x} + \eta^1_{,xxx}\right) = 0,$$
$$\eta^2_{,t} - 4\beta\eta^2_{,x} - \alpha\left(12uv\eta^1_{,x} + 6\left(u^2 - v^2\right)\eta^2_{,x} + \eta^2_{,xxx}\right) = 0,$$
$$\xi^1{}_{,v} + 4\beta\xi^2{}_{,v} + \alpha\left(6\left(u^2 - v^2\right)\xi^2{}_{,v} + 4uv\xi^2{}_{,u} + \xi^2{}_{,xxv}\right) = 0,$$
$$\xi^1{}_{,u} + 4\beta\xi^2{}_{,u} + \alpha\left(-4uv\xi^2{}_{,v} + 6\left(u^2 - v^2\right)\xi^2{}_{,u} + \xi^2{}_{,xxu}\right) = 0,$$
$$\eta^1_{,uu} - 3\left(\xi^1{}_{,xu} + 4\beta\xi^2{}_{,xu} + 6\alpha\left(2u\xi^2{}_{,x} + \left(u^2 - v^2\right)\xi^2{}_{,xu}\right)\right) = 0,$$
$$\xi^1{}_{,uuu} + 4\beta\xi^2{}_{,uuu} + 6\alpha\left(6\xi^2{}_{,u} + 6u\xi^2{}_{,uu} + \left(u^2 - v^2\right)\xi^2{}_{,uuu}\right) = 0,$$
$$\eta^2_{,vv} - 3\left(\xi^1{}_{,xv} + 4\beta\xi^2{}_{,xv} - 6\alpha\left(2v\xi^2{}_{,x} + \left(-u^2 + v^2\right)\xi^2{}_{,xv}\right)\right) = 0,$$

$$\xi^1{}_{,vvv} + 4\beta\xi^2{}_{,vvv} - 6\alpha\left(6\xi^2{}_{,v} + 6v\xi^2{}_{,vv} + \left(-u^2+v^2\right)\xi^2{}_{,vvv}\right) = 0,$$
$$\xi^1{}_{,xu} - \eta^2_{,uv} + 4\beta\xi^2{}_{,xu} + 6\alpha\left(6u\xi^2{}_{,x} + 4uv\xi^2{}_{,xv} + \left(u^2-v^2\right)\xi^2{}_{,xu}\right) = 0,$$
$$\eta^1_{,uv} - \xi^1{}_{,xv} - 4\beta\xi^2{}_{,xv} + 6\alpha\left(6v\xi^2{}_{,x} + \left(-u^2+v^2\right)\xi^2{}_{,xv} + 4uv\xi^2{}_{,xu}\right) = 0,$$
$$\eta^2_{,uv} - 2\left(\xi^1{}_{,xu} + 4\beta\xi^2{}_{,xu} + 6\alpha\left(3u\xi^2{}_{,x} + uv\xi^2{}_{,xv} + \left(u^2-v^2\right)\xi^2{}_{,xu}\right)\right) = 0,$$
$$\xi^1{}_{,vv} + 4\beta\xi^2{}_{,vv} - 6\alpha\left(6v\xi^2{}_{,v} + \left(-u^2+v^2\right)\xi^2{}_{,vv} + 2u\left(\xi^2{}_{,u} + v\xi^2{}_{,uv}\right)\right) = 0,$$
$$\xi^1{}_{,uv} + 4\beta\xi^2{}_{,uv} + 6\alpha\left(6u\xi^2{}_{,v} + 2uv\xi^2{}_{,vv} - 2v\xi^2{}_{,u} + u^2\xi^2{}_{,uv} - v^2\xi^2{}_{,uv}\right) = 0,$$
$$\xi^1{}_{,uv} + 4\beta\xi^2{}_{,uv} + 6\alpha\left(2u\xi^2{}_{,v} - 6v\xi^2{}_{,u} + u^2\xi^2{}_{,uv} - v^2\xi^2{}_{,uv} - 2uv\xi^2{}_{,uu}\right) = 0,$$
$$\eta^1_{,uv} - 2\left(\xi^1{}_{,xv} + 4\beta\xi^2{}_{,xv} - 6\alpha\left(3v\xi^2{}_{,x} + \left(-u^2+v^2\right)\xi^2{}_{,xv} + uv\xi^2{}_{,xu}\right)\right) = 0,$$
$$\xi^1{}_{,vv} + 4\beta\xi^2{}_{,vv} - 6\alpha\left(10v\xi^2{}_{,v} + \left(-u^2+v^2\right)\xi^2{}_{,vv} + 6u\left(\xi^2{}_{,u} + v\xi^2{}_{,uv}\right)\right) = 0,$$
$$\eta^1_{,uuu} - 3\left(\xi^1{}_{,xuu} + 4\beta\xi^2{}_{,xuu} + 6\alpha\left(2\xi^2{}_{,x} + 4u\xi^2{}_{,xu} + \left(u^2-v^2\right)\xi^2{}_{,xuu}\right)\right) = 0,$$
$$\xi^2\alpha' - \alpha\left(-\xi^2{}_{,t} + 3\xi^1{}_{,x} + 24u^2\alpha\xi^2{}_{,x} - 24v^2\alpha\xi^2{}_{,x} + 16\beta\xi^2{}_{,x} + \alpha\xi^2{}_{,xxx}\right) = 0,$$
$$\eta^2_{,vvv} - 3\left(\xi^1{}_{,xvv} + 4\beta\xi^2{}_{,xvv} - 6\alpha\left(2\xi^2{}_{,x} + 4v\xi^2{}_{,xv} + \left(-u^2+v^2\right)\xi^2{}_{,xvv}\right)\right) = 0,$$
$$\xi^1{}_{,xxu} - 2\eta^2_{,xuv} + 4\beta\xi^2{}_{,xxu} + 6\alpha\left(4u\xi^2{}_{,xx} + 2uv\xi^2{}_{,xxv} + \left(u^2-v^2\right)\xi^2{}_{,xxu}\right) = 0,$$
$$\xi^1{}_{,uu} + 4\beta\xi^2{}_{,uu} - 6\alpha\left(-2v\xi^2{}_{,v} - 6u\xi^2{}_{,u} - 2uv\xi^2{}_{,uv} - u^2\xi^2{}_{,uu} + v^2\xi^2{}_{,uu}\right) = 0,$$
$$\xi^1{}_{,uu} + 4\beta\xi^2{}_{,uu} - 6\alpha\left(-6v\xi^2{}_{,v} - 10u\xi^2{}_{,u} - 6uv\xi^2{}_{,uv} - u^2\xi^2{}_{,uu} + v^2\xi^2{}_{,uu}\right) = 0,$$
$$\xi^1{}_{,uv} + 4\beta\xi^2{}_{,uv} + 2\alpha\left(10u\xi^2{}_{,v} + 2uv\xi^2{}_{,vv} - 6v\xi^2{}_{,u} + 3u^2\xi^2{}_{,uv} - 3v^2\xi^2{}_{,uv}\right) = 0,$$
$$\xi^1{}_{,uv} + 4\beta\xi^2{}_{,uv} + 2\alpha\left(6u\xi^2{}_{,v} - 10v\xi^2{}_{,u} + 3u^2\xi^2{}_{,uv} - 3v^2\xi^2{}_{,uv} - 2uv\xi^2{}_{,uu}\right) = 0,$$
$$2\eta^1_{,xuv} - \xi^1{}_{,xxv} - 4\beta\xi^2{}_{,xxv} + 6\alpha\left(4v\xi^2{}_{,xx} + \left(-u^2+v^2\right)\xi^2{}_{,xxv} + 2uv\xi^2{}_{,xxu}\right) = 0,$$
$$\begin{aligned}16uv\beta\xi^2{}_{,v} + \eta^1_{,xvv} + 12u\alpha\left(2v\left(u^2-v^2\right)\xi^2{}_{,v} + 4uv^2\xi^2{}_{,u} + \xi^2{}_{,xx} + v\xi^2{}_{,xxv}\right)&\\ + 4uv\xi^1{}_{,v} = 0,&\end{aligned}$$
$$\begin{aligned}16uv\beta\xi^2{}_{,u} - \eta^2_{,xuu} + 12v\alpha\left(-4u^2v\xi^2{}_{,v} + 2\left(u^3-uv^2\right)\xi^2{}_{,u} + \xi^2{}_{,xx} + u\xi^2{}_{,xxu}\right)&\\ + 4uv\xi^1{}_{,u} = 0,&\end{aligned}$$
$$\begin{aligned}\xi^1{}_{,uuu} + 6\alpha\left(18\xi^2{}_{,u} + 12v\xi^2{}_{,uv} + 12u\xi^2{}_{,uu} + 6uv\xi^2{}_{,uuv} + u^2\xi^2{}_{,uuu} - v^2\xi^2{}_{,uuu}\right)&\\ + 4\beta\xi^2{}_{,uuu} = 0,&\end{aligned}$$
$$\begin{aligned}\xi^1{}_{,uvv} + 2\alpha\left(12u\xi^2{}_{,vv} + 2uv\xi^2{}_{,vvv} - 6\xi^2{}_{,u} - 12v\xi^2{}_{,uv} + 3u^2\xi^2{}_{,uvv} - 3v^2\xi^2{}_{,uvv}\right)&\\ + 4\beta\xi^2{}_{,uvv} = 0,&\end{aligned}$$
$$\begin{aligned}\xi^1{}_{,uuv} + 2\alpha\left(6\xi^2{}_{,v} + 12u\xi^2{}_{,uv} - 12v\xi^2{}_{,uu} + 3u^2\xi^2{}_{,uuv} - 3v^2\xi^2{}_{,uuv} - 2uv\xi^2{}_{,uuu}\right)&\\ + 4\beta\xi^2{}_{,uuv} = 0,&\end{aligned}$$
$$\begin{aligned}\xi^1{}_{,vvv} - 6\alpha\left(18\xi^2{}_{,v} + 12v\xi^2{}_{,vv} - u^2\xi^2{}_{,vvv} + v^2\xi^2{}_{,vvv} + 12u\xi^2{}_{,uv} + 6uv\xi^2{}_{,uvv}\right)&\\ + 4\beta\xi^2{}_{,vvv} = 0,&\end{aligned}$$

$$\begin{aligned}
&\eta^2_{,uvv} - 2(\xi^1{}_{,xuv} - 8\beta\xi^2{}_{,xuv} \\
&\qquad - 12\alpha\left(4u\xi^2{}_{,xv} + uv\xi^2{}_{,xvv} - 2v\xi^2{}_{,xu} + u^2\xi^2{}_{,xuv} - v^2\xi^2{}_{,xuv}\right) = 0,\\
&\eta^1_{,uuv} - 2\xi^1{}_{,xuv} - 8\beta\xi^2{}_{,xuv} \\
&\qquad - 12\alpha\left(2u\xi^2{}_{,xv} - 4v\xi^2{}_{,xu} + u^2\xi^2{}_{,xuv} - v^2\xi^2{}_{,xuv} - uv\xi^2{}_{,xuu}\right) = 0,\\
&\eta^1_{,xxv} - 4v\eta^1 - 4u\eta^2 - 4uv\eta^2_{,v} + 4uv\eta^1_{,u} - 8uv\xi^1{}_{,x} - 48u^3v\alpha\xi^2{}_{,x} \\
&\qquad\qquad + 48uv^3\alpha\xi^2{}_{,x} - 32uv\beta\xi^2{}_{,x} = 0,\\
&4v\eta^1 + 4u\eta^2 - 4uv\eta^2_{,v} + 4uv\eta^1_{,u} + 8uv\xi^1{}_{,x} + 48u^3v\alpha\xi^2{}_{,x} \\
&\qquad\qquad - 48uv^3\alpha\xi^2{}_{,x} + 32uv\beta\xi^2{}_{,x} + \eta^2_{,xxu} = 0,
\end{aligned}$$

$$\begin{aligned}
&\eta^1_{,uvv} - \xi^1{}_{,xvv} - 4\beta\xi^2{}_{,xvv} \\
&\qquad + 6\alpha\left(6\xi^2{}_{,x} + 8v\xi^2{}_{,xv} - u^2\xi^2{}_{,xvv} + v^2\xi^2{}_{,xvv} + 4u\xi^2{}_{,xu} + 4uv\xi^2{}_{,xuv}\right) = 0,\\
&\eta^2_{,uuv} - \xi^1{}_{,xuu} - 4\beta\xi^2{}_{,xuu} \\
&\qquad - 6\alpha\left(6\xi^2{}_{,x} + 4v\xi^2{}_{,xv} + 8u\xi^2{}_{,xu} + 4uv\xi^2{}_{,xuv} + u^2\xi^2{}_{,xuu} - v^2\xi^2{}_{,xuu}\right) = 0,\\
&\xi^1{}_{,uvv} + 4\beta\xi^2{}_{,uvv} + 6\alpha u^2\xi^2{}_{,uvv} \\
&\qquad + 6\alpha\left(2u\xi^2{}_{,vv} - 6\xi^2{}_{,u} - 8v\xi^2{}_{,uv} - v^2\xi^2{}_{,uvv} - 2u\xi^2{}_{,uu} - 2uv\xi^2{}_{,uuv}\right) = 0,\\
&\xi^1{}_{,uuv} + 4\beta\xi^2{}_{,uuv} + 12\alpha uv\xi^2{}_{,uvv} \\
&\qquad + 6\alpha\left(6\xi^2{}_{,v} + 2v\xi^2{}_{,vv} + 8u\xi^2{}_{,uv} - 2v\xi^2{}_{,uu} + u^2\xi^2{}_{,uuv} - v^2\xi^2{}_{,uuv}\right) = 0,
\end{aligned}$$

$$\begin{aligned}
&4uv\xi^1{}_{,u} + 16uv\beta\xi^2{}_{,u} - \eta^2_{,xvv} + \xi^1{}_{,xxv} + 4\beta\xi^2{}_{,xxv} - 48\alpha u^2v^2\xi^2{}_{,v} \\
&\qquad + 6\alpha\left(4uv\left(u^2 - v^2\right)\xi^2{}_{,u} - 2v\xi^2{}_{,xx} + u^2\xi^2{}_{,xxv} - v^2\xi^2{}_{,xxv}\right) = 0,\\
&4uv\xi^1{}_{,v} + 16uv\beta\xi^2{}_{,v} + \eta^1_{,xuu} - \xi^1{}_{,xxu} - 4\beta\xi^2{}_{,xxu} + 48\alpha u^2v^2\xi^2{}_{,u} \\
&\qquad + 6\alpha\left(4uv\left(u^2 - v^2\right)\xi^2{}_{,v} - 2u\xi^2{}_{,xx} - u^2\xi^2{}_{,xxu} + v^2\xi^2{}_{,xxu}\right) = 0,\\
&36\left(3u^4 - 2u^2v^2 + 3v^4\right)\alpha^3\xi^2{}_{,x} + \alpha\left(4\xi^2\beta' + \xi^1{}_{,t} + 8\beta\left(\xi^1{}_{,x} + 6\beta\xi^2{}_{,x}\right)\right) \\
&\quad - 4\beta\xi^2\alpha' + \alpha^2(12u\eta^1 - 12v\eta^2 + 12uv\eta^1_{,v} + 12uv\eta^2_{,u} + 12u^2\xi^1{}_{,x} \\
&\qquad - 12v^2\xi^1{}_{,x} + 144u^2\beta\xi^2{}_{,x} - 144v^2\beta\xi^2{}_{,x} + 3\eta^2_{,xxv} - \xi^1{}_{,xxx}) = 0,\\
&36\left(3u^4 - 2u^2v^2 + 3v^4\right)\alpha^3\xi^2{}_{,x} + \alpha\left(4\xi^2\beta' + \xi^1{}_{,t} + 8\beta\left(\xi^1{}_{,x} + 6\beta\xi^2{}_{,x}\right)\right) \\
&\quad - 4\beta\xi^2\alpha' + \alpha^2(12u\eta^1 - 12v\eta^2 - 12uv\eta^1_{,v} - 12uv\eta^2_{,u} + 12u^2\xi^1{}_{,x} \\
&\qquad - 12v^2\xi^1{}_{,x} + 144u^2\beta\xi^2{}_{,x} - 144v^2\beta\xi^2{}_{,x} + 3\eta^1_{,xxu} - \xi^1{}_{,xxx}) = 0.
\end{aligned}$$

The determining equations for the CVmKdV system (1) after substituting (4) are:

$$\xi^1{}_{,u} = 0,$$
$$\xi^1{}_{,v} = 0,$$

$$
\begin{aligned}
\eta^2_{,uu} &= 0,\\
\eta^1_{,xv} &= 0,\\
\eta^1_{,vv} &= 0,\\
\eta^2_{,uuu} &= 0,\\
\eta^2_{,xu} &= 0,\\
\xi^1{}_{,uuu} &= 0,\\
\xi^1{}_{,uu} &= 0,\\
\xi^1{}_{,uv} &= 0,\\
\xi^1{}_{,vv} &= 0,\\
\xi^1{}_{,uvv} &= 0,\\
\eta^1_{,vvv} &= 0,\\
\xi^1{}_{,vvv} &= 0,\\
\xi^1{}_{,uuv} &= 0,\\
\eta^1_{,xu} &= \xi^1{}_{,xx},\\
\eta^2_{,xv} &= \xi^1{}_{,xx},\\
\eta^1_{,uv} &= \xi^1{}_{,xv},\\
\eta^1_{,uv} &= 2\xi^1{}_{,xv},\\
\eta^2_{,uv} &= \xi^1{}_{,xu},\\
\eta^2_{,uv} &= 2\xi^1{}_{,xu},\\
2\eta^1_{,xuv} &= \xi^1{}_{,xxv},\\
2\eta^2_{,xuv} &= \xi^1{}_{,xxu},\\
\eta^1_{,uvv} &= \xi^1{}_{,xvv},\\
\eta^2_{,uvv} &= 2\xi^1{}_{,xuv},\\
\eta^1_{,uu} &= 3\xi^1{}_{,xu},\\
\eta^1_{,uuv} &= 2\xi^1{}_{,xuv},\\
\eta^2_{,uuv} &= \xi^1{}_{,xuu},\\
\eta^1_{,uuu} &= 3\xi^1{}_{,xuu},\\
\eta^2_{,vv} &= 3\xi^1{}_{,xv},\\
\eta^2_{,vvv} &= 3\xi^1{}_{,xvv},\\
4uv\xi^1{}_{,u} &= \eta^2_{,xuu},
\end{aligned}
\tag{A.2}
$$

$$
\begin{gathered}
4uv\xi^1{}_{,v} + \eta^1_{,xvv} = 0, \\
4uv\xi^1{}_{,v} + \eta^1_{,xuu} = \xi^1{}_{,xxu}, \\
\mathcal{F}_1(t)\alpha' + \alpha(t)\mathcal{F}_1' = 3\alpha(t)\xi^1{}_{,x}, \\
4uv\eta^1_{,u} + \eta^1_{,xxv} = 4\left(v\eta^1 + u\left(\eta^2 + v\left(\eta^2_{,v} + 2\xi^1{}_{,x}\right)\right)\right), \\
\eta^1_{,t} = 4\beta(t)\eta^1_{,x} + \alpha(t)\left(6\left(u^2 - v^2\right)\eta^1_{,x} - 12uv\eta^2_{,x} + \eta^1_{,xxx}\right), \\
\eta^2_{,t} = 4\beta(t)\eta^2_{,x} + \alpha(t)\left(12uv\eta^1_{,x} + 6\left(u^2 - v^2\right)\eta^2_{,x} + \eta^2_{,xxx}\right), \\
4uv\xi^1{}_{,u} + \xi^1{}_{,xxv} = \eta^2_{,xvv}, 4v\eta^1 + 4u\eta^2 + 4uv\eta^1_{,u} + 8uv\xi^1{}_{,x} + \eta^2_{,xxu} = 4uv\eta^2_{,v}, \\
\alpha(t)\left(4\mathcal{F}_1(t)\beta' + \xi^1{}_{,t} + 8\beta(t)\xi^1{}_{,x}\right) - 4\beta(t)\mathcal{F}_1(t)\alpha' + 12\alpha^2(t)\left(u\eta^1 - v\eta^2\right) \\
+ \alpha^2(t)\left(12uv\eta^1_{,v} + 12uv\eta^2_{,u} + 12u^2\xi^1{}_{,x} - 12v^2\xi^1{}_{,x} + 3\eta^2_{,xxv} - \xi^1{}_{,xxx}\right) = 0, \\
4\beta(t)\mathcal{F}_1(t)\alpha' - \alpha(t)\left(4\mathcal{F}_1(t)\beta' + \xi^1{}_{,t} + 8\beta(t)\xi^1{}_{,x}\right) + 12\alpha(t)^2\left(v\eta^2 + uv\eta^1_{,v}\right) \\
+ \alpha(t)^2\left(12uv\eta^2_{,u} - 12u\eta^1 - 12u^2\xi^1{}_{,x} + 12v^2\xi^1{}_{,x} - 3\eta^1_{,xxu} + \xi^1{}_{,xxx}\right) = 0
\end{gathered}
$$

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