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INVARIANT INTEGRATING FACTORS: A new method for integration of ODEs using their symmetries

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Abstract. It is shown that infinitesimal symmetries can be used for constructing *invariant integrating* factors that are different from *Lie's integrating factor*. Knowledge of two linearly independent integrating factors, *Lie's integrating factor* and an *invariant integrating factor,* allows to find the general solution of first-order ordinary equations with known infinitesimal symmetries without additional integration. The method of invariant integrating factors can be applied to higher-order ODEs as well.

Keywords: Ordinary differential equation, Lie's integrating factor, Invariant integrating factor.

MSC: 70S10, 35C99, 35G20
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1 Introduction

It is well known after S. Lie that first-order ordinary differential equations with a known symmetry can be integrated by introducing *canonical variables*, i.e. new independent and dependent variables that reduce the known symmetry group to a *translation group*. Furthermore, S. Lie found a general formula for an integrating factor for first-order ordinary differential equations with known infinitesimal symmetries thus providing an alternative way for integration.

It is shown in the present paper that infinitesimal symmetries can be used for constructing integrating factors different from Lie's integrating factor. Knowledge of two integrating factors, *Lie's integrating factor* and an *invariant integrating factor,* allows to find the general solution of first-order equations with known infinitesimal symmetries without additional integration. The method of invariant integrating factors can be applied to higher-order ODEs as well.

1.1 Method of canonical variables

The following example from [1] clearly illustrates the method of canonical variables. The Riccati equation

$$\frac{dy}{dx} + y^2 - \frac{2}{x^2} = 0 \tag{1.1}$$

is invariant under the dilation group $\bar{x} = x\mathrm{e}^{a},\ \bar{y} = y\mathrm{e}^{-a}$ with the generator

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\,\cdot \tag{1.2}$$

Solution of the equations $X(t) = 1,\ X(t) = 0$ with the above operator X provides the canonical variables

$$t = \ln|x|, \quad u = xy. \tag{1.3}$$

Now Eq. (1.1) should be rewritten in the canonical variables. We have:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{x}\right) = -\frac{u}{x^2} + \frac{1}{x}\frac{du}{dx} = -\frac{u}{x^2} + \frac{1}{x}\frac{du}{dt}\frac{dt}{dx} = -\frac{u}{x^2} + \frac{u'}{x^2}\,\cdot$$

Therefore,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 - \frac{2}{x^2} = \frac{u'}{x^2} - \frac{u}{x^2} + \frac{u^2}{x^2} - \frac{2}{x^2} = \frac{1}{x^2}\left(u' + u^2 - u - 2\right) = 0.$$

Thus, the Riccati equation (1.1) is rewritten in the canonical variables (1.3) in the following separable form:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -(u^2 - u - 2).$$

Let us integrate the above equation. Separating the variables,

$$\frac{du}{u^2 - u - 2} = -dt,$$

decomposing the rational fraction into elementary fractions,

$$\frac{1}{u^2 - u - 2} = \frac{1}{3}\left[\frac{1}{u-2} - \frac{1}{u+1}\right],$$

and integrating we have:

$$\ln\left(\frac{u-2}{u+1}\right) = -3t + \ln C\,\cdot$$

Now we solve this equation with respect to u,

$$u = \frac{C + 2\mathrm{e}^{3t}}{\mathrm{e}^{3t} - C}\,,$$

substitute the expressions (1.3) of $t,\ u$ and obtain the solution of Eq. (1.1):

$$y = \frac{2x^3 + C}{x(x^3 - C)}\,, \quad C = \text{const.} \tag{1.4}$$

Note that the above calculations require that both $xy - 2$ and $xy + 1$ do not vanish. Therefore, we should add to (1.4) the singular solutions of Eq. (1.1):

$$y = \frac{2}{x} \qquad \text{and} \quad y = -\frac{1}{x}\,\cdot \tag{1.5}$$

1.2 Lie's integrating factor

Recall that a function $\mu(x, y)$ is called an *integrating factor* (A. Clairaut, 1739) for a first-order ordinary differential equation

$$M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0 \tag{1.6}$$

if Eq. (1.6) becomes *exact* upon multiplying by $\mu(x, y)$, i.e. if the equation

$$\mu(x, y)(M\mathrm{d}x + N\mathrm{d}y) = \mathrm{d}\Phi \equiv \frac{\partial \Phi}{\partial x}\mathrm{d}x + \frac{\partial \Phi}{\partial y}\mathrm{d}y \tag{1.7}$$

holds with some function[1] $\Phi(x, y)$. It follows from Eq. (1.7) that the integrating factor satisfies the equation

$$\frac{\partial(\mu N)}{\partial x} = \frac{\partial(\mu M)}{\partial y}\,\cdot \tag{1.8}$$

Lie [2] showed that if

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

is a symmetry for Eq. (1.6) and if $\xi M + \eta N \neq 0$, then the function

$$\mu = \frac{1}{\xi M + \eta N} \tag{1.9}$$

is an integrating factor for Eq. (1.6). The integrating factor (1.9) is called *Lie's integrating factor.*

[1] It is presupposed that all functions considered in what follows are sufficiently smooth.

Let us solve the Riccati equation (1.1) by using Lie's integrating factor. We rewrite Eq. (1.1) in the differential form (1.6):

$$dy + \left(y^2 - \frac{2}{x^2}\right) dx = 0, \tag{1.10}$$

and use the symmetry (1.2). Substituting

$$\xi = x, \quad \eta = -y, \quad M = y^2 - \frac{2}{x^2}\,, \quad N = 1$$

in (1.9) we obtain the integrating factor

$$\mu = \frac{x}{x^2y^2 - xy - 2}\,\cdot \tag{1.11}$$

Multiplying Eq. (1.10) by the factor (1.11) we obtain:

$$\frac{xdy}{x^2y^2 - xy - 2} + \frac{1}{x^2y^2 - xy - 2}\,\frac{x^2y^2 - 2}{x}\,dx = 0. \tag{1.12}$$

This equation is exact, i.e. its left-hand side can be written in the form $d\Phi$. To find the function $\Phi(x, y)$, we can use the following simple calculations. Noting that

$$\frac{x^2y^2 - 2}{x} = y + \frac{x^2y^2 - xy - 2}{x}\,,$$

we rewrite the left-hand side of Eq. (1.12) in the form

$$\frac{xdy + ydx}{x^2y^2 - xy - 2} + \frac{dx}{x} = \frac{d(xy)}{x^2y^2 - xy - 2} + \frac{dx}{x}\,\cdot$$

Denoting $z = xy$ and using the decomposition

$$\frac{1}{z^2 - z - 2} = \frac{1}{3}\Big(\frac{1}{z-2} - \frac{1}{z+1}\Big),$$

we obtain

$$\int \frac{dz}{z^2 - z - 2} = \frac{1}{3} \ln \frac{z-2}{z+1}$$

and hence Eq. (1.12) is written

$$\frac{xdy + ydx}{x^2y^2 - xy - 2} + \frac{dx}{x} = d\Big(\ln x + \frac{1}{3} \ln \frac{xy-2}{xy+1}\Big) = 0.$$

The integration yields:

$$\frac{xy-2}{xy+1} = \frac{C}{x^3}\,, \quad C \neq 0.$$

Solving for y, we arrive at the solution (1.4),

$$y = \frac{2x^3 + C}{x(x^3 - C)}\,,$$

of the Riccati equation (1.1). The singular solutions (1.5) are obtained by the reasoning used in the case of the method of canonical variables.

2 Invariance of the equation for integrating factors

2.1 Main statement

The first-order ordinary differential equation

$$\frac{dy}{dx} = f(x, y) \tag{2.1}$$

will be written also in the differential form (1.6):

$$dy - f(x, y)dx = 0. \tag{2.2}$$

The integrating factors $\mu(x, y)$ for Eq. (2.2) are determined by the linear first-order partial differential differential equation (see Eq. (1.8))

$$\mu_x + f(x, y)\, \mu_y + f_y(x, y)\, \mu = 0, \tag{2.3}$$

where the subscripts x and y denote the respective partial derivatives. Namely, μ_x and μ_y are the partial derivatives of the unknown function μ, whereas $f_y(x, y)$ is the partial derivative of the given function $f(x, y)$.

Let Eq. (2.1) admit a one-parameter group with the generator

$$X = \xi(x, y)\, \frac{\partial}{\partial x} + \eta(x, y)\, \frac{\partial}{\partial y} \cdot \tag{2.4}$$

Hence the coefficients ξ and η of the operator (2.4) solve the determining equation

$$\eta_x + f\eta_y - f\xi_x - f^2\xi_y - \xi f_x - \eta f_y = 0. \tag{2.5}$$

Theorem 2.1. Eq. (2.3) admits the operator (2.4) properly extended to the dependent variable μ. Namely, it admits the operator

$$X = \xi(x, y)\, \frac{\partial}{\partial x} + \eta(x, y)\, \frac{\partial}{\partial y} + \sigma(x, y)\mu\, \frac{\partial}{\partial \mu}\,, \tag{2.6}$$

where $\sigma(x, y)$ solves the following non-homogeneous linear first-order partial differential equation

$$\frac{\partial \sigma}{\partial x} + f(x, y)\, \frac{\partial \sigma}{\partial y} = -\xi f_{xy} - \eta f_{yy} - f_y \xi_x - f f_y \xi_y. \tag{2.7}$$

Proof. The prolongation of the operator (2.6) to μ_x and μ_y has the form

$$\begin{aligned} X = \xi\, \frac{\partial}{\partial x} + \eta\, \frac{\partial}{\partial y} + \sigma\mu\, \frac{\partial}{\partial \mu} &+ [(\sigma - \xi_x)\mu_x - \eta_x\mu_y + \mu\sigma_x]\, \frac{\partial}{\partial \mu_x} \\ &+ [(\sigma - \eta_y)\mu_y - \xi_y\mu_x + \mu\sigma_y]\, \frac{\partial}{\partial \mu_y} \cdot \end{aligned} \tag{2.8}$$

Now we write the invariance test for Eq. (2.3) in the form

$$X\left[\mu_x + f(x,y)\,\mu_y + f_y(x,y)\,\mu\right] = \lambda\left[\mu_x + f(x,y)\,\mu_y + f_y(x,y)\,\mu\right], \tag{2.9}$$

where λ is an undetermined coefficient. Substituting (2.8) in (2.9) and equating in the resulting equation the terms with μ_x we obtain

$$\lambda = \sigma - \xi_x - f\xi_y.$$

We substitute this expression for λ in (2.9) and, comparing the terms with μ_y and μ in both sides of Eq. (2.9), obtain the following two equations:

$$\eta_x + f\eta_y - \xi f_x - \eta f_y = (\xi_x - f\xi_y)f, \tag{2.10}$$

and

$$\sigma_x + f\,\sigma_y + \xi f_{xy} + \eta f_{yy} = -f_y\xi_x - f f_y \xi_y. \tag{2.11}$$

Eq. (2.10) is satisfied identically since it coincides with Eq. (2.5). Eq. (2.11) is identical with Eq. (2.7). This completes the proof.

2.2 Examples

Example 2.1. Consider the nonlinear equation

$$y' = \frac{2xy}{3x^2 - y^2}\,\cdot \tag{2.12}$$

It is homogeneous, i.e. admits the dilation group with the generator

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,\cdot \tag{2.13}$$

The calculation shows that Eq. (2.7) has the form

$$\frac{\partial \sigma}{\partial x} + \frac{2xy}{3x^2 - y^2}\,\frac{\partial \sigma}{\partial y} = 0. \tag{2.14}$$

Eq. (2.14) has the trivial solution $\sigma = 0$. Its general solution is an arbitrary function of the first integral of Eq. (2.12), i.e. is given by

$$\sigma = \phi\left(\frac{x^2 - y^2}{y^3}\right).$$

We can take the trivial solution $\sigma = 0$ and conclude that the equation (2.3) defining the integrating factors of Eq. (2.12) admits the operator (2.13).

Example 2.2. The Riccati equation

$$y' + y^2 = \frac{2}{x^2} \tag{2.15}$$

admits the dilation group with the generator

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\,\cdot \tag{2.16}$$

The calculation shows that Eq. (2.7) has the form

$$\frac{\partial \sigma}{\partial x} + \left(\frac{2}{x^2} - y^2\right)\frac{\partial \sigma}{\partial y} = 0.$$

We take its trivial solution $\sigma = 0$ and conclude that the equation (2.3) defining integrating factors of Eq. (2.15) admits the operator (2.16).

Example 2.3. The Riccati equation

$$y' = \frac{y^2}{x^3} + \frac{y}{x} \tag{2.17}$$

admits the conformal transformation group with the generator

$$X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\,\cdot \tag{2.18}$$

The calculation shows that Eq. (2.7) has the form

$$\frac{\partial \sigma}{\partial x} + \left(\frac{y^2}{x^3} + \frac{y}{x}\right)\frac{\partial \sigma}{\partial y} = -1.$$

We take its simple particular solution

$$\sigma = c - x$$

and conclude that the equation (2.3) defining integrating factors of Eq. (2.17) admits the operator

$$X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + (c - x)\mu\frac{\partial}{\partial \mu}\,, \quad c = \text{const}. \tag{2.19}$$

3 Invariant integrating factors

The following examples clearly illustrate the method.

Example 3.1. Let us find the integrating factor for Eq. (2.17),

$$y' = \frac{y^2}{x^3} + \frac{y}{x}\,, \tag{2.17}$$

such that it is invariant under the group generated by the operator (2.19) with $c = 1$:

$$X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + (1-x)\mu\frac{\partial}{\partial \mu}\cdot \tag{3.1}$$

Two functionally independent invariants of the operator (3.1) are

$$\lambda = \frac{y}{x}\,, \quad J = \mu x \mathrm{e}^{1/x}.$$

Letting $J = \phi(\lambda)$ we obtain the following form for the invariant solutions:

$$\mu = \frac{1}{x}\,\mathrm{e}^{-1/x}\,\phi(\lambda), \quad \lambda = \frac{y}{x}\cdot \tag{3.2}$$

Note that Eq. (2.3) defining the integrating factors of Eq. (2.17) has the form

$$\mu_x + \left(\frac{y^2}{x^3} + \frac{y}{x}\right)\mu_y + \left(\frac{2y}{x^3} + \frac{1}{x}\right)\mu = 0. \tag{3.3}$$

Substituting (3.2) in Eq. (3.3) we obtain

$$\lambda^2\frac{d\phi}{d\lambda} + (1+2\lambda)\phi = 0,$$

whence

$$\phi = \lambda^{-2}\,\mathrm{e}^{1/\lambda}.$$

This gives the following integrating factor for Eq. (2.17):

$$\mu = \frac{x}{y^2}\,\mathrm{e}^{(x^2-y)/(xy)}\,. \tag{3.4}$$

Besides, the symmetry (2.18) provides Lie's integrating factor

$$\mu_1 = -\frac{x}{y^2}\cdot \tag{3.5}$$

The formula

$$\frac{\mu}{\mu_1} = C$$

gives

$$\frac{x^2 - y}{xy} = C, \quad C = \text{const.}$$

Solving this equation for y we obtain the following solution of Eq. (2.17):

$$y = \frac{x^2}{1 + Cx} \cdot \tag{3.6}$$

The general solution to Eq. (2.17) is provided by the function (3.6) and by the singular solution $y = 0$.

Example 3.2. Let us apply the method to the Riccati equation (2.15),

$$y' + y^2 = \frac{2}{x^2}\,, \tag{2.15}$$

and to its symmetry (2.16),

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \cdot \tag{2.16}$$

Eq. (2.3) defining the integrating factors of Eq. (2.15) has the form

$$\mu_x + \left(\frac{2}{x^2} - y^2\right)\mu_y - 2y\mu = 0 \tag{3.7}$$

and admits the above operator X. The invariants

$$\lambda = xy, \quad J = \mu$$

give the following form for the invariant solution to Eq. (3.7):

$$\mu = \phi(\lambda), \quad \lambda = xy.$$

Substituting this expression for μ and its derivatives

$$\mu_x = y\phi', \quad \mu_y = x\phi'$$

in Eq. (3.7) we obtain

$$\left(y + \frac{2}{x} - xy^2\right)\phi' - 2y\phi = 0,$$

whence upon multiplying by x and noting that $xy = \lambda$, we see that

$$\left(2 + \lambda - \lambda^2\right)\phi' - 2\lambda\phi = 0.$$

Integration of this separable first-order equation yields

$$\phi = C(\lambda - 2)^{-4/3}(\lambda + 1)^{-2/3}.$$

Ignoring the immaterial constant factor C and substituting $\lambda = xy$ we obtain the following invariant integrating factor:

$$\mu = (xy-2)^{-4/3}(xy+1)^{-2/3}. \tag{3.8}$$

On the other hand, the symmetry (2.16) provides Lie's integrating factor

$$\mu_1 = \frac{x}{x^2y^2 - xy - 2} \cdot \tag{3.9}$$

The formula

$$\frac{\mu_1}{\mu} = C$$

gives

$$\frac{x(xy-2)^{4/3}(xy+1)^{2/3}}{(xy-2)(xy+1)} = x\left(\frac{xy-2}{xy+1}\right)^{1/3} = C.$$

It follows

$$\frac{xy-2}{xy+1} = \frac{C}{x^3}\,,$$

whence upon solving for y, one obtains the solution

$$y = \frac{2x^3 + C}{x(x^3 - C)} \cdot \tag{3.10}$$

The above calculations were made by ignoring the possibilities

$$xy - 2 = 0$$

and

$$xy + 1 = 0.$$

These possibilities give two singular solutions,

$$y = \frac{2}{x}\,, \quad y = -\frac{1}{x} \cdot \tag{3.11}$$

The functions (3.10) and (3.11) provide the general solution to Eq. (2.15).

4 Remark on higher-order equations

The method of invariant integrating factors can be extended to higher-order ordinary differential equations. The main step in this extension requires the proof of a statement similar to Theorem 2.1. For example, in the case of second-order equations

$$a(x,y,y')y'' + b(x,y,y') = 0 \tag{4.1}$$

one has to prove the invariance of the determining equations (see [3], Section 2)

$$
\begin{aligned}
&y'\,(\mu a)_{yy'} + (\mu a)_{xy'} + 2(\mu a)_y - (\mu b)_{y'y'} = 0, \\
&y'^2\,(\mu a)_{yy} + 2y'(\mu a)_{xy} + (\mu a)_{xx} - y'(\mu b)_{yy'} - (\mu b)_{xy'} + (\mu b)_y = 0
\end{aligned}
\tag{4.2}
$$

of integrating factors for Eq. (4.1).

Acknowledgements

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CONSERVATION LAWS OF ANISOTROPIC HEAT EQUATIONS

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Abstract. Nonlinear self-adjointness of the anisotropic nonlinear heat equation is investigated. Mathematical models of heat conduction in anisotropic media with a source are considered and a class of self-adjoint models is identified. Conservation laws corresponding to the symmetries of the equations in question are computed.

Keywords: Anisotropic heat conduction, Nonlinear self-adjointness, Conservation laws.
MSC: 35C99, 70S10, 70G65

1 Three-dimensional equation without source

1.1 Introduction

In this section we deal with the equation

$$u_t = [f(u)u_x]_x + [g(u)u_y]_y + [h(u)u_z], \tag{1.1}$$

where the functions $f(u), g(u), h(u)$ are non-negative according to their physical meaning. When the functions $f(u), g(u)$, and $h(u)$ are not subjected to any restrictions, Eq. (1.1) has five symmetries [1]

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = \frac{\partial}{\partial x}\,, \quad X_3 = \frac{\partial}{\partial y}\,, \\ X_4 = \frac{\partial}{\partial z}\,, \quad X_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\,. \end{gathered} \tag{1.2}$$

1.2 Nonlinear self-adjointness

We will write Eq. (1.1) in the form

$$F \equiv -u_t + f u_{xx} + g u_{yy} + h u_{zz} + f' u_x^2 + g' u_y^2 + h' u_z^2 = 0. \tag{1.1$'$}$$

Using its formal Lagrangian

$$\mathcal{L} = vF \equiv v[-u_t + f u_{xx} + g u_{yy} + h u_{zz} + f' u_x^2 + g' u_y^2 + h' u_z^2] \tag{1.3}$$

we obtain the following adjoint equation to Eq. (1.1):

$$F^* \equiv v_t + f v_{xx} + g v_{yy} + h v_{zz} = 0. \tag{1.4}$$

Eq. (1.1) is said to be *nonlinearly self-adjoint* [2] if the equations (1.4) and (1.1) can be related by the equation

$$F^* = \lambda\, F$$

after the substitution $v = \varphi(t, x, y, z, u)$ with a certain function $\varphi \neq 0$. Here λ is an undetermined variable coefficient; it will be found in the process of calculations.

Thus, the nonlinear self-adjointness condition is written

$$\begin{aligned} & v_t + f(u) v_{xx} + g(u) v_{yy} + h(u) v_{zz} \\ =& \lambda\, \left[-u_t + f(u) u_{xx} + g(u) u_{yy} + h(u) u_{zz} + f'(u) u_x^2 + g'(u) u_y^2 + h'(u) u_z^2\right], \end{aligned} \tag{1.5}$$

where one makes the following replacements of v and its derivatives:

$$\begin{aligned} & v = \varphi(t, x, y, z, u), \\ & v_t = D_t(\varphi), \quad v_x = D_x(\varphi), \quad v_y = D_y(\varphi), \quad v_z = D_z(\varphi), \\ & v_{xx} = D_x^2(\varphi), \quad v_{yy} = D_y^2(\varphi), \quad v_{zz} = D_z^2(\varphi). \end{aligned} \tag{1.6}$$

After this replacement Eq. (1.5) should be satisfied identically in the variables

$$t,\ x,\ y,\ z,\ u,\ u_t,\ u_x,\ u_y,\ u_z,\ u_{xx},\ u_{yy},\ u_{zz}.$$

We write the derivatives of v involved in the adjoint equation (1.4) in the expanded form,

$$\begin{aligned} & v_t = \varphi_u u_t + \varphi_t, \\ & v_{xx} = \varphi_u u_{xx} + \varphi_{uu} u_x^2 + 2\varphi_{xu} u_x + \varphi_{xx}, \\ & v_{yy} = \varphi_u u_{yy} + \varphi_{uu} u_y^2 + 2\varphi_{yu} u_y + \varphi_{yy}, \\ & v_{zz} = \varphi_u u_{zz} + \varphi_{uu} u_z^2 + 2\varphi_{zu} u_z + \varphi_{zz}, \end{aligned} \tag{1.7}$$

and substitute them in the left side of Eq. (1.5). The comparison the coefficients for u_t in both sides of Eq. (1.5) yields

$$\lambda = -\varphi_u.$$

Then the comparison of the coefficients for u_{xx}, u_{yy}, u_{zz} leads to the equations

$$f(u)\varphi_u = -f(u)\varphi_u, \quad g(u)\varphi_u = -g(u)\varphi_u, \quad h(u)\varphi_u = -h(u)\varphi_u.$$

It follows from these equations that $\varphi_u = 0$, since not all functions $f(u), g(u), h(u)$ vanish. Hence,

$$\varphi = \varphi(t, x, y, z)$$

and

$$\lambda = 0, \quad v_t = \varphi_t, \quad v_{xx} = \varphi_{xx}, \quad v_{yy} = \varphi_{yy}, \quad v_{zz} = \varphi_{zz}.$$

Now Eq. (1.5) becomes

$$\varphi_t + f(u)\varphi_{xx} + g(u)\varphi_{yy} + h(u)\varphi_{zz} = 0. \tag{1.8}$$

If the functions $f(u), g(u), h(u)$ are linearly independent and none of them is constant, Eq. (1.8) yields

$$\varphi_t = 0, \quad \varphi_{xx} = 0, \quad \varphi_{yy} = 0, \quad \varphi_{zz} = 0.$$

The general solution of the above system is easily found and provides the following substitution announced in [2]:

$$v = a_1\, xyz + a_2\, xy + a_3\, xz + a_4\, yz + a_5\, x + a_6\, y + a_7\, z + a_8, \tag{1.9}$$

where $a_1, \ldots, a_8$ are arbitrary constants.

If the functions $f(u), g(u), h(u)$ are linearly dependent, i.g.

$$g(u) = rf(u), \quad h(u) = sf(u) \tag{1.10}$$

with positive constant coefficients $r,\ s$, then the substitution (1.9) is replaced by

$$v = \varphi(x, y, z),$$

where $\varphi(x, y, z)$ is an arbitrary solution of the elliptic equation

$$\varphi_{xx} + r\varphi_{yy} + s\varphi_{zz} = 0. \tag{1.11}$$

If one of the functions $f(u), g(u), h(u)$ is a positive constant, e.g. $g = k > 0$, then Eq. (1.8) yields

$$\varphi_t + k\varphi_{yy} = 0, \quad \varphi_{xx} = 0, \quad \varphi_{zz} = 0. \tag{1.12}$$

The second and third equations of the system (1.12) yield

$$\varphi = \alpha(t,y)xz + \beta(t,y)x + \gamma(t,y)z + \sigma(t,y).$$

The first equation (1.12) shows that $\alpha(t,y), \beta(t,y), \gamma(t,y)$ and $\sigma(t,y)$ solve the adjoint equation

$$v_t + kv_{yy} = 0 \tag{1.13}$$

to the linear heat equation $u_t - ku_{yy} = 0$. Hence, Eq. (1.1) of the form

$$u_t = (f(u)u_x)_x + ku_{yy} + (h(u)u_z)_z \tag{1.14}$$

satisfies the nonlinear self-adjointness condition (1.5) with the substitution

$$v = \alpha(t,y)xz + \beta(t,y)x + \gamma(t,y)z + \sigma(t,y), \tag{1.15}$$

where $\alpha(t,y), \beta(t,y), \gamma(t,y)$ and $\sigma(t,y)$ are any solutions of the adjoint equation (1.13) to the linear heat equation.

1.3 Construction of conserved vectors

According to the general theory (see e.g. [2]), any operator

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u}$$

admitted by Eq. (1.1) provides a conserved vector given by

$$C^i = W\left[\frac{\partial \mathcal{L}}{\partial u_i} - D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}}\right)\right] + D_j(W)\frac{\partial \mathcal{L}}{\partial u_{ij}}\,, \tag{1.16}$$

where

$$W = \eta - \xi^1 u_t - \xi^2 u_x - \xi^3 u_y - \xi^4 u_z.$$

The term *conserved vector* refers to the fact that the vector (1.16) satisfies the *conservation equation*

$$D_t(C^1) + D_x(C^2) + D_y(C^3) + D_z(C^4) = 0 \tag{1.17}$$

on the solutions of Eq. (1.1).

In the case of the formal Lagrangian (1.3) Eq. (1.16) yields

$$
\begin{aligned}
C^1 &= W\frac{\partial \mathcal{L}}{\partial u_t} = -Wv,\\
C^2 &= W\left[\frac{\partial \mathcal{L}}{\partial u_x} - D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right)\right] + D_x(W)\frac{\partial \mathcal{L}}{\partial u_{xx}}\\
&= W\left[f' u_x v - f v_x\right] + f v D_x(W),\\
C^3 &= W\left[\frac{\partial \mathcal{L}}{\partial u_y} - D_y\left(\frac{\partial \mathcal{L}}{\partial u_{yy}}\right)\right] + D_y(W)\frac{\partial \mathcal{L}}{\partial u_{yy}}\\
&= W\left[g' u_y v - g v_y\right] + g v D_y(W),\\
C^4 &= W\left[\frac{\partial \mathcal{L}}{\partial u_z} - D_z\left(\frac{\partial \mathcal{L}}{\partial u_{zz}}\right)\right] + D_z(W)\frac{\partial \mathcal{L}}{\partial u_{zz}}\\
&= W\left[h' u_z v - h v_z\right] + h v D_z(W).
\end{aligned} \tag{1.18}
$$

Let us apply the formula (1.18) to the symmetry X_2 from (1.2). The corresponding quantity W equals $-u_x$. We insert it in (1.18) and transfer the terms of the form $D_x(\ldots)$, $D_y(\ldots)$, and $D_z(\ldots)$ from C^1 to C^2, C^3, and C^4, respectively. Then, we iterate the procedure with respect to C^2, C^3, C^4, etc. As a result we obtain the following conserved vector for Eq. (1.1) associated with the symmetry (1.2)

$$
\begin{aligned}
C^1 &= -uv_x,\\
C^2 &= f(u)u_x v_x - g(u)u_y v_y - h(u)u_z v_z,\\
C^3 &= g(u)(u_x v_y + u_y v_x),\\
C^4 &= h(u)(u_x v_z + u_z v_x).
\end{aligned} \tag{1.19}
$$

The conservation equation (1.17) for the vector (1.19) is satisfied in the following form

$$
D_t(C^1) + D_x(C^2) + D_y(C^3) + D_z(C^4) = v_x F, \tag{1.20}
$$

where F is given by Eq. (1.1′).

Using the substitution (1.9) one can write the conserved vector (1.19) as follows:

$$
\begin{aligned}
C^1 &= -(a_1yz + a_2y + a_3z + a_5)u,\\
C^2 &= f(u)(a_1yz + a_2y + a_3z + a_5)u_x - g(u)(a_1xz + a_2x + a_4z + a_6)u_y\\
&\quad - h(u)(a_1xy + a_3x + a_4y + a_7)u_z\\
C^3 &= g(u)(a_1xz + a_2x + a_4z + a_6)u_x + g(u)(a_1yz + a_2y + a_3z + a_5)u_y,\\
C^4 &= h(u)(a_1xy + a_3x + a_4y + a_7)u_x + h(u)(a_1yz + a_2y + a_3z + a_5)u_z.
\end{aligned} \tag{1.21}
$$

Now we apply the formula (1.18) to the symmetry X_3 from (1.2). One can repeat all the above calculations but it is not necessary due to the obvious symmetry of Eq. (1.1) with respect to any permutation of the variables x, y, z and the corresponding permutation f, g, h and C^2, C^3, C^4. Applying the permutation $x \leftrightarrow y$, $f \leftrightarrow g$, $C^2 \leftrightarrow C^3$ to the conserved vector (1.19), we obtain the conserved vector

$$
\begin{aligned}
C^1 &= -uv_y,\\
C^2 &= f(u)(u_y v_x + u_x v_y),\\
C^3 &= g(u)u_y v_y - f(u)u_x v_x - h(u)u_z v_z,\\
C^4 &= h(u)(u_y v_z + u_z v_y)
\end{aligned}
\tag{1.22}
$$

corresponding to the symmetry X_3 from (1.2).

Likewise, using the permutation $x \leftrightarrow z$, $f \leftrightarrow h$, $C^2 \leftrightarrow C^4$, we obtain the following conserved vector associated with the symmetry X_4 from (1.2);

$$
\begin{aligned}
C^1 &= -uv_z,\\
C^2 &= f(u)(u_z v_x + u_x v_z),\\
C^3 &= g(u)(u_z v_y + u_y v_z),\\
C^4 &= h(u)u_z v_z - f(u)u_x v_x - g(u)u_y v_y.
\end{aligned}
\tag{1.23}
$$

Let us consider the conserved vector associated with the time-translational symmetry X_1 from (1.2). In this case $W = -u_t$. Replacing u_t by the right-hand side of Eq. (1.1) and substituting it in the first equation (1.18), we obtain

$$C^1 = v\left[D_x(fu_x) + D_y(gu_y) + D_z(hu_z)\right]. \tag{1.24}$$

We have:

$$vD_x(fu_x) = D_x(vfu_x) - v_x fu_x = D_x(vfu_x) - D_x(v_x F(u)) + v_{xx}F(u),$$

where

$$F(u) = \int f(u)du.$$

Invoking that $v_{xx} = 0$, we arrive at the equation

$$vD_x(fu_x) = D_x(vfu_x - v_x F(u)).$$

Using the similar transformations of the terms $vD_y(gu_y)$ and $vD_z(hu_z)$ we rewrite the expression (1.24) as follows:

$$C^1 = D_x(vfu_x - v_x F(u)) + D_y(vgu_y - v_y G(u)) + D_z(vhu_z - v_z H(u)), \tag{1.25}$$

where

$$G(u) = \int g(u)du, \quad H(u) = \int h(u)du.$$

We have to substitute C^1 given by (1.25) and C^2, C^3, C^4 given by Eqs. (1.18) with $W = -u_t$ in Eq. (1.17),

$$D_t(C^1) + D_x(C^2) + D_y(C^3) + D_z(C^4) = 0.$$

Using the expression (1.25) for C^1 and the commutativity of the total differentiations we have

$$\begin{aligned}D_t(C^1) &= D_t\left[D_x(vfu_x - v_xF(u)) + D_y(vgu_y - v_yG(u)) + D_z(vhu_z - v_zH(u))\right]\\ &= D_xD_t(vfu_x - v_xF(u)) + D_yD_t(vgu_y - v_yG(u)) + D_zD_t(vhu_z - v_zH(u)).\end{aligned}$$

Therefore our conservation equation is equivalent to the following one:

$$\begin{aligned}&D_x(C^2 + D_t(vfu_x - v_xF)) + D_y(C^3 + D_t(vgu_y - v_yG))\\ &+ D_z(C^4 + D_t(vhu_z - v_zH)) = 0.\end{aligned} \tag{1.26}$$

Calculating C^2, C^3, C^4 by Eqs. (1.18), we obtain

$$\begin{aligned}&C^2 + D_t(vfu_x - v_xF)\\ &\quad = -u_t(f'vu_x - fv_x) - fvu_{tx} + vf'u_tu_x + vfu_{tx} - v_xfu_t = 0,\\ &C^3 + D_t(vgu_y - v_yG)\\ &\quad = -u_t(g'vu_y - gv_y) - gvu_{ty} + vg'u_tu_y + vgu_{ty} - v_ygu_t = 0,\\ &C^4 + D_t(vhu_z - v_zH)\\ &\quad = -u_t(h'vu_z - hv_z) - hvu_{tz} + vh'u_tu_z + vhu_{tz} - v_zhu_t = 0.\end{aligned}$$

Hence, the conservation law (1.26) provided by the time-translation symmetry X_1 is trivial.

The dilation symmetry X_5 from (1.2) leads to a conserved vector depending on t, x, y, z explicitly. But we will not dwell on this vector.

2 The equation with a source

2.1 Introduction

The anisotropic heat equation with a source has the form

$$u_t = (f(u)u_x)_x + (g(u)u_y)_y + (h(u)u_z)_z + q(u). \tag{2.1}$$

Its adjoint equation is written

$$v_t + f(u)v_{xx} + g(u)v_{yy} + h(u)v_{zz} + q'(u)v = 0. \tag{2.2}$$

The reckoning shows that Eq. (2.1) with an arbitrary source $q(u)$ is not nonlinearly self-adjointness with the substitution of the form $v = \varphi(t, x, y, z, u)$. However, the self-adjointness occur for particular forms of $q(u)$. We will consider, for the sake of simplicity, the two-dimensional equation (2.1) with a particular source term $q(u)$.

2.2 Two-dimensional model with a particular source term

The two-dimensional equation

$$u_t = (f(u)u_x)_x + (g(u)u_y)_y + q(u) \tag{2.3}$$

with arbitrary functions $f(u), g(u)$ and $q(u) \neq 0$ has only the translational symmetries:

$$X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = \frac{\partial}{\partial x}\,, \quad X_3 = \frac{\partial}{\partial y}\,\cdot \tag{2.4}$$

We will consider here the following particular case of Eq. (2.3):

$$F \equiv -u_t + (f(u)u_x)_x + (g(u)u_y)_y + \omega^2 \mathcal{F}(u), \quad \omega = \text{const.}, \tag{2.5}$$

where

$$\mathcal{F}(u) = \int f(u)du.$$

Then the adjoint equation (2.2) is written

$$F^* \equiv v_t + f(u)v_{xx} + g(u)v_{yy} + \omega^2 f(u)v = 0. \tag{2.6}$$

The investigation of the nonlinear self-adjointness condition

$$F^* \mid_{v=\varphi(t,x,y,u)} = \lambda\, F$$

shows that Eq. (2.5) is nonlinearly self-adjoint with the substitution

$$v = (a_1 y + b_1)\cos(\omega x) + (a_2 y + b_2)\sin(\omega x). \tag{2.7}$$

Therefore one can construct the conserved vector (1.16) associated with the symmetries (2.4).

In particular, the operator X_2 from (2.4),

$$X_2 = \frac{\partial}{\partial x}$$

provides the conserved vector

$$\begin{aligned} C^1 &= -uv_x, \\ C^2 &= f(u)u_x v_x + \omega^2 \mathcal{F}(u)v, \\ C^3 &= g(u)u_y v_x - \mathcal{G}(u)v_{xy}, \end{aligned} \tag{2.8}$$

where v should be replaced by its expression (2.7). Here

$$\mathcal{G}(u) = \int g(u)du.$$

The vector (2.8) satisfies the conservation equation in the following form:

$$D_t\big(C^1\big) + D_x\big(C^2\big) + D_y\big(C^3\big) = v_x\left[(f(u)u_x)_x + (g(u)u_y)_y + \omega^2\mathcal{F}(u) - u_t\right].$$

The expression (2.7) for v contains four arbitrary constants a_1, a_2, b_1, b_2. Accordingly, the vector (2.8) is a linear combination of the following four linearly independent conserved vectors:

$$\begin{aligned} &C^1 = \sin(\omega x)\,u, \quad C^2 = -\sin(\omega x)\,f(u)u_x + \omega\cos(\omega x)\,\mathcal{F}(u), \\ &C^3 = -\sin(\omega x)\,g(u)u_y; \end{aligned} \tag{2.9}$$

$$\begin{aligned} &C^1 = \cos(\omega x)\,u, \quad C^2 = -\cos(\omega x) f(u)u_x - \omega\sin(\omega x)\,\mathcal{F}(u), \\ &C^3 = -\cos(\omega x)\,g(u)u_y; \end{aligned} \tag{2.10}$$

$$\begin{aligned} &C^1 = y\sin(\omega x)\,u, \quad C^2 = -y\sin(\omega x)\,f(u)u_x + \omega y\cos(\omega x)\,\mathcal{F}(u), \\ &C^3 = -y\sin(\omega x)\,g(u)u_y + \sin(\omega x)\mathcal{G}(u); \end{aligned} \tag{2.11}$$

$$\begin{aligned} &C^1 = y\cos(\omega x)\,u, \quad C^2 = -y\cos(\omega x)\,f(u)u_x - \omega y\sin(\omega x)\,\mathcal{F}(u), \\ &C^3 = -y\cos(\omega x)\,g(u)u_y + \cos(\omega x)\mathcal{G}(u). \end{aligned} \tag{2.12}$$

The conserved vectors provided by the generator X_3 of the group of translations of y can be computed likewise. The generator X_1 of the time-translations provides only the trivial conserved vector.

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Solution of a mixed problem by using the invariance principle

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Abstract. Usually a mixed problem is solved for *linear* equations by the method of separation of variables. The aim of this work is to apply the invariance principle to the *nonlinear* heat equation for constructing a mixed problem using the symmetries of the equation in question.

Keywords: Nonlinear heat equation, Mixed problem, Invariance principle.

1 Formulation of the problem

Consider the nonlinear heat equation

$$u_t - (u^\sigma u_x)_x = 0. \tag{1.1}$$

Its symmetries are

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad X_4 = \sigma x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}\cdot \tag{1.2}$$

The aim of this work is to employ the invariance principle [1] using the symmetries (1.2) for constructing the solution of Eq. (1.1) in the domain

$$t > 0, \quad 0 \le x \le \infty,$$

with the side conditions

$$u|_{t=0} = 0, \quad u|_{x=0} = \varphi(t), \quad u|_{x\to\infty} = 0. \tag{1.3}$$

2 Application of the invariance principle

2.1 The symmetry of the mixed problem

The invariance of the equations $t = 0$ and $x = 0$ takes away the symmetries X_1 and X_2, respectively. Therefore the general possible symmetry is

$$X = \alpha X_3 + \beta X_4, \quad \alpha, \beta = \text{const.} \tag{2.1}$$

The invariance condition of the first equation in (1.3) under the operator (2.1) induced on the initial manifold $t = 0$ is written

$$[X_{t=0}(t)]_{t=0} = 0.$$

The invariance condition of the second equation in (1.3) under the operator (2.1) induced on the initial manifold $x = 0$ is written

$$[X_{x=0}(u - \varphi(t))]_{u=\varphi} = 2\left(\beta\phi - \alpha t\frac{d\varphi}{dt}\right) = 0.$$

This equation yields

$$\varphi(t) = Kt^{\gamma}, \quad \gamma = \beta/\alpha, \quad K = \text{const.}.$$

Accordingly, we will deal with the boundary condition

$$u|_{x=0} = Kt^{\gamma} \tag{2.2}$$

with a given positive constant γ.

Since $\beta = \gamma\alpha$ by definition of γ, we see that the mixed problem

$$\begin{aligned} &u_t - u^{\sigma}u_{xx} - \sigma u^{\sigma-1}u_x^2 = 0,\\ &u|_{t=0} = 0, \quad u|_{x=0} = Kt^{\gamma} \end{aligned} \tag{2.3}$$

has the symmetry $X = \alpha\,(X_3 + \gamma X_4)$, or

$$X = 2t\frac{\partial}{\partial t} + (1+\gamma\sigma)x\frac{\partial}{\partial x} + 2\gamma u\frac{\partial}{\partial u}\,\cdot \tag{2.4}$$

The invariance principle tells that the solution of the mixed problem (2.3) should be invariant under the operator (2.4).

2.2 Solution of the problem

The invariants of the operator (2.4) are

$$\lambda = xt^{-(\gamma\sigma+1)/2}, \quad J = t^{-\gamma}u.$$

Letting $J = h(\lambda)$ we obtain the following form of the invariant solution:

$$u = t^{\gamma}h(\lambda). \tag{2.5}$$

Substituting (2.5) in our differential equation, i.e. in the first equation from (2.3), we arrive at the following second-order nonlinear ordinary differential equations:

$$h^{\sigma}h'' + \sigma h^{\sigma-1}h'^{2} + \frac{1}{2}(1+\gamma\sigma)\lambda h' - \gamma h = 0. \tag{2.6}$$

The initial condition, i.e. the second equation (1.3) is satisfied, whereas the boundary condition, i.e. the third equation (2.3) becomes

$$h(0) = K.$$

We also have to satisfy the third condition from (1.3) which is now written

$$h(+\infty) = 0.$$

Thus the problem is to solve Eq. (2.6) under the conditions

$$h(0) = K, \quad h(+\infty) = 0. \tag{2.7}$$

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Group analysis of a model of nonlinear atmospheric zonal flows in a thin rotating spherical shell

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1 Introduction

We consider the third-order nonlinear equation

$$\frac{\partial \Delta_S \psi}{\partial t} + \frac{\psi_\varphi}{R_0} + \frac{F}{\sin\theta}\frac{\partial \Delta_S \psi}{\partial \varphi} - \frac{\psi_\varphi L_1 F}{\sin\theta} + \frac{\varepsilon}{\sin\theta} J\left(\psi_\theta, \Delta_S \psi\right) = 0, \tag{1.1}$$

where

$$\Delta_S = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \tag{1.2}$$

is the Laplace-Beltrami operator on the unit sphere written in the spherical angles, whereas

$$L_1 = \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d}{d\theta}\right) - \frac{1}{\sin^2\theta} \tag{1.3}$$

is the Stourm-Liouville operator for the associated Legendre functions.

Substituting (1.2) and (1.3) in (1.1) we write Equation (1.1) in the expanded form

$$\begin{aligned}
&\psi_{t\theta}\,\cos\theta + \psi_{t\theta\theta}\,\sin\theta + \frac{1}{\sin\theta}\,\psi_{t\varphi\varphi} + \frac{\sin\theta}{R_0}\,\psi_\varphi \\
&+ \left(\frac{\cos\theta}{\sin\theta}\,\psi_{\theta\varphi} + \psi_{\theta\theta\varphi} + \frac{1}{\sin^2\theta}\,\psi_{\varphi\varphi\varphi}\right)[F(\theta) + \varepsilon\psi_\theta] \\
&- \left(F''(\theta) + \frac{\cos\theta}{\sin\theta}\,F'(\theta) - \frac{1}{\sin^2\theta}\,F(\theta)\right)\psi_\varphi \\
&- \varepsilon\left(\frac{\cos\theta}{\sin\theta}\,\psi_{\theta\theta} + \psi_{\theta\theta\theta} + \frac{1}{\sin^2\theta}\,(\psi_{\theta\varphi\varphi} - \psi_\theta) - \frac{2\cos\theta}{\sin^3\theta}\,\psi_{\varphi\varphi}\right)\psi_\varphi = 0.
\end{aligned} \tag{1.4}$$

Here θ, φ and t are three independent variables, ψ is the dependent variable with the partial derivatives $\psi_{t\theta},\ \psi_\varphi$, etc. Equation (1.4) contains an arbitrary function $F(\theta)$ and two parameters ε, R_0 which are supposed in what follows different from zero.

2 Symmetries

Solving the determining equation one can demonstrate that Equation (1.1) with an arbitrary function $F(\theta)$ admits the following Lie algebra.

$$X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = \frac{\partial}{\partial\varphi}\,, \quad X_3 = \lambda(t)\frac{\partial}{\partial\psi}\,, \tag{2.1}$$

$$X_4 = 2R_0\varepsilon t\,\frac{\partial}{\partial t} - \varepsilon t\frac{\partial}{\partial\varphi} + (\cos\theta - 2R_0H(\theta) - 2\varepsilon R_0\psi)\,\frac{\partial}{\partial\psi}\,, \tag{2.2}$$

$$\begin{aligned}
X_5 = {} & \varepsilon\sin\left(\varphi + \frac{t}{2R_0}\right)\frac{\partial}{\partial\theta} + \varepsilon\frac{\cos\theta}{\sin\theta}\cos\left(\varphi + \frac{t}{2R_0}\right)\frac{\partial}{\partial\varphi} \\
& - \left(F(\theta) + \frac{\sin\theta}{2R_0}\right)\sin\left(\varphi + \frac{t}{2R_0}\right)\frac{\partial}{\partial\psi}\,,
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
X_6 = {} & \varepsilon\cos\left(\varphi + \frac{t}{2R_0}\right)\frac{\partial}{\partial\theta} - \varepsilon\frac{\cos\theta}{\sin\theta}\sin\left(\varphi + \frac{t}{2R_0}\right)\frac{\partial}{\partial\varphi} \\
& - \left(F(\theta) + \frac{\sin\theta}{2R_0}\right)\cos\left(\varphi + \frac{t}{2R_0}\right)\frac{\partial}{\partial\psi}\,.
\end{aligned} \tag{2.4}$$

In the operator X_3 given by Equation (2.1), $\lambda(t)$ is an arbitrary function. In the operator X_4 given by Equation (2.2), $H(\theta)$ is the integral of $F(\theta)$, i.e. $H'(\theta) = F(\theta)$.

Computing the commutators of the operators (2.1)–(2.4) one obtains the following commutator table:

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	X_3'	$2\varepsilon R_0 X_1 - \varepsilon X_2$	$\frac{1}{2R_0} X_6$	$-\frac{1}{2R_0} X_5$
X_2	0	0	0	0	X_6	$-X_5$
X_3	$-X_3'$	0	0	X_3''	0	0
X_4	$\varepsilon X_2 - 2\varepsilon R_0 X_1$	0	$-X_3''$	0	0	0
X_5	$-\frac{1}{2R_0} X_6$	$-X_6$	0	0	0	$\varepsilon^2 X_2$
X_6	$\frac{1}{2R_0} X_5$	X_5	0	0	$-\varepsilon^2 X_2$	0

(2.5)

where

$$X_3' = \lambda'(t)\frac{\partial}{\partial\psi}, \quad X_3'' = -2\varepsilon R_0[\lambda(t) + t\lambda'(t)]\frac{\partial}{\partial\psi} \cdot$$

To verify that the operators X_4 and X_5 is admitted by Equation (1.4), we denote the left-hand side of Equation (1.4) by Ω, prolong the operators X_4 and X_5 and obtain after calculations:

$$X_4(\Omega) = -4\varepsilon\, R_0\, \Omega, \quad X_5(\Omega) = \varepsilon\,\frac{\cos\theta}{\sin\theta}\,\sin\left(\varphi + \frac{t}{2R_0}\right)\Omega.$$

This proves that the operators X_4 and X_5 are admitted by Equation (1.4). This fact for the operator X_6 follows from the commutator table (2.5) because the commutator of any two admitted operators is also admitted.

3 Invariant solutions

3.1 Invariant solution based on X_2 and X_4

The commutator table (2.5) shows that X_2 and X_4 span an Abelian two-dimensional subalgebra. Let us find the invariant solution with respect to this subalgebra. The equations

$$X_2 J(t,\varphi,\theta,\psi) = 0, \quad X_4 J(t,\varphi,\theta,\psi) = 0$$

provide two functionally independent invariants

$$\lambda = \theta, \quad \mu = t\left(\psi + \frac{H(\theta)}{\varepsilon} - \frac{\cos\theta}{2\varepsilon R_0}\right).$$

Letting $\mu = \Phi(\lambda)$, we obtain the following general form for the invariant solutions:

$$\psi = \frac{\cos\theta}{2\varepsilon R_0} - \frac{H(\theta)}{\varepsilon} + \frac{1}{t}\Phi(\theta). \tag{3.1}$$

Substituting (3.1) in Equation (1.4) we obtain the second-order ordinary differential equation

$$\Phi'' + \frac{\cos\theta}{\sin\theta}\,\Phi' = 0,$$

whence

$$\Phi = C_1 + C_2 \ln|\tan(\theta/2)|\,.$$

Substituting this in (3.1) we obtain the following invariant solution:

$$\psi = \frac{\cos\theta}{2\varepsilon R_0} - \frac{H(\theta)}{\varepsilon} + \frac{C_1}{t} + \frac{C_2}{t}\,\ln|\tan(\theta/2)|\,, \tag{3.2}$$

where $C_1.C_2$ are arbitrary constants, and $H(\theta) = \int F(\theta)d\theta$.

Remark. The operator X_2 generates the obvious translation $\overline{\varphi} = \varphi + \beta_2$ of the angle φ, where β_2 is the group parameter (the subscript in the group parameter coincides with that of the group generator). The operator X_4 generates the one-parameter group of a more complex form, namely:

$$\begin{aligned} &\bar{t} = t\,\mathrm{e}^{2\varepsilon R_0\beta_4}\,, \quad \bar{\theta} = \theta, \quad \overline{\varphi} = \varphi + \frac{t}{2R_0}\left(1 - \mathrm{e}^{2\varepsilon R_0\beta_4}\right), \\ &\overline{\psi} = \psi\,\mathrm{e}^{-2\varepsilon R_0\beta_4} + \frac{\cos\theta - 2R_0H(\theta)}{2\varepsilon R_0}\left(1 - \mathrm{e}^{-2\varepsilon R_0\beta_4}\right). \end{aligned} \tag{3.3}$$

Thus, the solution (3.2) is invariant under the translation of the angle φ (due to the independence on this angle) and the transformation (3.3).

3.2 Invariant solution based on X_4 and X_5

These operators also span an Abelian subalgebra. Let us find a basis of invariants of this subalgebra. Three functionally independent invariants for X_4 are

$$\theta, \quad \nu = t + 2R_0\varphi, \quad \mu = t\left(\psi + \frac{H(\theta)}{\varepsilon} - \frac{\cos\theta}{2\varepsilon R_0}\right).$$

Acting on the invariants by the operator X_5 we obtain:

$$X_5(\theta) = \varepsilon\sin\left(\varphi + \frac{t}{2R_0}\right) = \varepsilon\sin\frac{\nu}{2R_0}\,,$$

$$X_5(\nu) = 2\varepsilon R_0 \frac{\cos\theta}{\sin\theta} \cos\left(\varphi + \frac{t}{2R_0}\right) = 2\varepsilon R_0 \frac{\cos\theta}{\sin\theta} \cos\frac{\nu}{2R_0}\,\cdot$$

The reckoning also shows that $X_5(\mu) = 0$. Hence, X_5 is written in terms of the invariants as follows:

$$X_5 = \varepsilon \sin\frac{\nu}{2R_0}\frac{\partial}{\partial\theta} + 2\varepsilon R_0 \frac{\cos\theta}{\sin\theta} \cos\frac{\nu}{2R_0}\frac{\partial}{\partial\nu}\,\cdot \tag{3.4}$$

Solving the equation $X_5 J(\theta, \nu, \mu) = 0$ with the operator X_5 given by (3.4) one readily obtains the invariants μ and

$$\lambda = \sin\theta\cos\frac{\nu}{2R_0}.$$

Thus, the subalgebra spanned by X_4, X_5 has the following two functionally independent invariants:

$$\lambda = \sin\theta\cos\left(\varphi + \frac{t}{2R_0}\right), \quad \mu = t\left(\psi + \frac{H(\theta)}{\varepsilon} - \frac{\cos\theta}{2\varepsilon R_0}\right). \tag{3.5}$$

Letting $\mu = \Phi(\lambda)$, we obtain the following general form for the invariant solutions:

$$\psi = \frac{\cos\theta}{2\varepsilon R_0} - \frac{H(\theta)}{\varepsilon} + \frac{1}{t}\Phi(\lambda). \tag{3.6}$$

Here λ is given in (3.5) and therefore

$$\begin{aligned} \lambda_t &= -\frac{1}{2R_0}\sin\theta\sin\left(\varphi + \frac{t}{2R_0}\right), \\ \lambda_\theta &= \cos\theta\cos\left(\varphi + \frac{t}{2R_0}\right), \\ \lambda_\varphi &= -\sin\theta\sin\left(\varphi + \frac{t}{2R_0}\right). \end{aligned} \tag{3.7}$$

For the sake of brevity, we will use the notation

$$a = \cos\theta, \quad b = \sin\theta, \quad A = \cos\left(\varphi + \frac{t}{2R_0}\right), \quad B = \sin\left(\varphi + \frac{t}{2R_0}\right). \tag{3.8}$$

In this notation Equations (3.7) are written:

$$\lambda = bA, \quad \lambda_t = -\frac{bB}{2R_0}, \quad \lambda_\theta = aA, \quad \lambda_\varphi = -bB. \tag{3.9}$$

Using (3.6), (3.9) and the notation (3.8) we obtain:

$$\psi_\theta = -\frac{b}{2\varepsilon R_0} - \frac{F}{\varepsilon} + \frac{aA}{t}\,\Phi', \quad \psi_\varphi = -\frac{bB}{t}\,\Phi', \tag{3.10}$$

$$\psi_{\theta\theta} = -\frac{a}{2\varepsilon R_0} - \frac{F'}{\varepsilon} - \frac{bA}{t}\,\Phi' + \frac{a^2A^2}{t}\,\Phi'', \quad \psi_{\varphi\varphi} = -\frac{bA}{t}\,\Phi' + \frac{b^2B^2}{t}\,\Phi'',$$
$$\psi_{\theta\varphi} = -\frac{aB}{t}\,\Phi' - \frac{abAB}{t}\,\Phi'', \quad \psi_{t\theta} = -\frac{aA}{t^2}\,\Phi' - \frac{1}{2tR_0}\,(aB\Phi' + abAB\Phi''), \tag{3.11}$$

$$\psi_{t\varphi\varphi} = \frac{bB}{2tR_0}\,\Phi' + \frac{bA}{t^2}\,\Phi' - \frac{b^2B^2}{t^2}\,\Phi'' + \frac{3b^2AB}{2tR_0}\,\Phi'' - \frac{b^3B^3}{2tR_0}\,\Phi''', \tag{3.12}$$

$$\psi_{t\theta\theta} = \frac{bB}{2tR_0}\,\Phi' + \frac{bA}{t^2}\,\Phi' - \frac{a^2A^2}{t^2}\,\Phi'' + \frac{b^2AB}{2tR_0}\,\Phi'' - \frac{a^2AB}{tR_0}\,\Phi'' - \frac{a^2bA^2B}{2tR_0}\,\Phi''', \tag{3.13}$$

$$\psi_{\theta\theta\theta} = \frac{b}{2\varepsilon R_0} - \frac{F''}{\varepsilon} - \frac{aA}{t}\,\Phi' - \frac{3abA^2}{t}\,\Phi'' + \frac{a^3A^3}{t}\,\Phi''', \tag{3.14}$$

$$\psi_{\theta\theta\varphi} = \frac{bB}{t}\,\Phi' + \frac{b^2AB}{t}\,\Phi'' - \frac{2a^2AB}{t}\,\Phi'' - \frac{a^2bA^2B}{t}\,\Phi''', \tag{3.15}$$

$$\psi_{\theta\varphi\varphi} = -\frac{aA}{t}\,\Phi' - \frac{abA^2}{t}\,\Phi'' + \frac{2abB^2}{t}\,\Phi'' + \frac{ab^2AB^2}{t}\,\Phi''', \tag{3.16}$$

$$\psi_{\varphi\varphi\varphi} = \frac{bB}{t}\,\Phi' + \frac{3b^2AB}{t}\,\Phi'' - \frac{b^3B^3}{t}\,\Phi'''. \tag{3.17}$$

Substituting (3.10)-(3.17) in the left side of (1.4) and collecting the terms with $\Phi''',\ \Phi'', \Phi'^2$ and Φ' one can easily verify that it is written:

$$-\frac{b}{t^2}\left[B^2 + a^2A^2\right]\Phi'' + \frac{2b}{t^2}\,bA\Phi'.$$

Hence, Equation (1.4) becomes

$$\left[B^2 + a^2A^2\right]\Phi'' - bA\Phi' = 0. \tag{3.18}$$

According to the notation (3.8), we have

$$B^2 = 1 - A^2, \quad a^2 = 1 - b^2.$$

Substituting these expressions in (3.18) and invoking that $bA = \lambda$ (the first equation (3.9)) we reduce Equation (1.4) to the following equation for $\Phi(\lambda)$:

$$(1 - \lambda^2)\Phi'' - 2\lambda\Phi' = 0. \tag{3.19}$$

One can readily integrate this second-order linear ordinary differential equation. Since $|\lambda| \leq 1$ by definition of λ, the solution is given by

$$\Phi = C_1 + C_2 \ln\frac{1-\lambda}{1+\lambda} \tag{3.20}$$

when $|\lambda| \neq 1$. Here C_1, C_2 are arbitrary constants.

4 Self-adjointness

For investigating the self-adjointness, it is convenient to write Equation (1.4) in the form

$$\Omega_1 + \Omega_2 + \varepsilon\Omega_3 = 0, \tag{4.1}$$

where

$$\Omega_1 = \psi_{t\theta}\,\cos\theta + \psi_{t\theta\theta}\,\sin\theta + \frac{1}{\sin\theta}\,\psi_{t\varphi\varphi} + \frac{\sin\theta}{R_0}\,\psi_\varphi, \tag{4.2}$$

$$\begin{aligned}\Omega_2 = &\left(\frac{\cos\theta}{\sin\theta}\,\psi_{\theta\varphi} + \psi_{\theta\theta\varphi} + \frac{1}{\sin^2\theta}\,\psi_{\varphi\varphi\varphi}\right)F(\theta)\\ &-\left(F''(\theta) + \frac{\cos\theta}{\sin\theta}\,F'(\theta) - \frac{1}{\sin^2\theta}\,F(\theta)\right)\psi_\varphi,\end{aligned} \tag{4.3}$$

$$\begin{aligned}\Omega_3 = &\left(\frac{\cos\theta}{\sin\theta}\,\psi_{\theta\varphi} + \psi_{\theta\theta\varphi} + \frac{1}{\sin^2\theta}\,\psi_{\varphi\varphi\varphi}\right)\psi_\theta\\ &-\left(\frac{\cos\theta}{\sin\theta}\,\psi_{\theta\theta} + \psi_{\theta\theta\theta} + \frac{1}{\sin^2\theta}\,(\psi_{\theta\varphi\varphi} - \psi_\theta) - \frac{2\cos\theta}{\sin^3\theta}\,\psi_{\varphi\varphi}\right)\psi_\varphi.\end{aligned} \tag{4.4}$$

Then the adjoint equation to Equation (4.1) will be written

$$\Omega_1^* + \Omega_2^* + \varepsilon\Omega_3^* = 0 \tag{4.5}$$

with

$$\Omega_1^* = \frac{\delta(v\Omega_1)}{\delta\psi}\,, \quad \Omega_2^* = \frac{\delta(v\Omega_2)}{\delta\psi}\,, \quad \Omega_3^* = \frac{\delta(v\Omega_3)}{\delta\psi}\,, \tag{4.6}$$

where v is a new dependent variable. By definition of the variational derivative $\delta/\delta\psi$, we have, e.g.

$$\Omega_1^* = D_t D_\theta(v\cos\theta) - D_t D_\theta^2(v\sin\theta) - D_t D_\varphi^2\left(\frac{v}{\sin\theta}\right) - D_\varphi\left(\frac{\sin\theta}{R_0}\,v\right).$$

Working out the differentiations, we obtain

$$\Omega_1^* = -\left[v_{t\theta}\,\cos\theta + v_{t\theta\theta}\,\sin\theta + \frac{1}{\sin\theta}\,v_{t\varphi\varphi} + \frac{\sin\theta}{R_0}\,v_\varphi\right].$$

It is manifest that letting here $v = \psi$ we obtain

$$\Omega_1^* = -\Omega_1. \tag{4.7}$$

The similar calculation for Ω_2^* show that the equation

$$\Omega_2^* = -\Omega_2 \tag{4.8}$$

holds when $v = \psi$ if and only if the function $F(\theta)$ solves the equation

$$\frac{dF}{d\theta} = \frac{\cos\theta}{\sin\theta} F. \tag{4.9}$$

The general solution to Equation (4.9) is

$$F(\theta) = k \sin\theta, \quad k = \text{const}. \tag{4.10}$$

Finally, the reckoning shows that the equation

$$\Omega_3^* = -\Omega_3 \tag{4.11}$$

holds when $v = \psi$.

Equations (4.7), (4.8) and (4.11) yield to the following result.

Theorem 4.1. The adjoint equation (4.5) coincides with Equation (4.1) upon the substitution

$$v = \psi \tag{4.12}$$

provided that $F(\theta)$ has the form (4.10). In other words, Equation (4.1) is self-adjoint if the function $F(\theta)$ is given by Equation (4.10) with an arbitrary constant k.

5 Conservation laws

The conserved vectors associated with every symmetry

$$X = \xi^i(x, \psi)\frac{\partial}{\partial x^i} + \eta(x, \psi)\frac{\partial}{\partial \psi} \tag{5.1}$$

of Equation (1.4) will be computed by the following general formula:

$$\begin{aligned} C^i = W &\left[\frac{\partial \mathcal{L}}{\partial \psi_i} - D_j\left(\frac{\partial \mathcal{L}}{\partial \psi_{ij}}\right) + D_j D_k\left(\frac{\partial \mathcal{L}}{\partial \psi_{ijk}}\right)\right] \\ &+ D_j(W)\left[\frac{\partial \mathcal{L}}{\partial \psi_{ij}} - D_k\left(\frac{\partial \mathcal{L}}{\partial \psi_{ijk}}\right)\right] + D_j D_k(W)\frac{\partial \mathcal{L}}{\partial \psi_{ijk}}, \end{aligned} \tag{5.2}$$

where $x^1 = t,\ x^2 = \theta,\ x^3 = \varphi$, and

$$W = \eta - \xi^j \psi_j. \tag{5.3}$$

In order to apply Equation (5.2) one has to write the formal Lagrangian $\mathcal{L}$ for Equation (1.4) in the symmetric form

$$\begin{aligned}\mathcal{L} =& v \bigg[\frac{1}{2}\,\psi_{t\theta}\,\cos\theta + \frac{1}{2}\,\psi_{\theta t}\,\cos\theta + \frac{1}{3}\,\psi_{t\theta\theta}\,\sin\theta + \frac{1}{3}\,\psi_{\theta t\theta}\,\sin\theta + \frac{1}{3}\,\psi_{\theta\theta t}\,\sin\theta \\ &+ \frac{1}{3\sin\theta}\,\psi_{t\varphi\varphi} + \frac{1}{3\sin\theta}\,\psi_{\varphi t\varphi} + \frac{1}{3\sin\theta}\,\psi_{\varphi\varphi t} + \frac{\sin\theta}{R_0}\,\psi_\varphi \\ &+ \left(\frac{\cos\theta}{2\sin\theta}\,\psi_{\theta\varphi} + \frac{\cos\theta}{2\sin\theta}\,\psi_{\varphi\theta} + \frac{1}{3}\,\psi_{\theta\theta\varphi} + \frac{1}{3}\,\psi_{\theta\varphi\theta} + \frac{1}{3}\,\psi_{\varphi\theta\theta}\right. \\ &\left.+\frac{1}{\sin^2\theta}\,\psi_{\varphi\varphi\varphi}\right)(F(\theta)+\varepsilon\psi_\theta) - \left(F''(\theta) + \frac{\cos\theta}{\sin\theta}\,F'(\theta) - \frac{1}{\sin^2\theta}\,F(\theta)\right)\psi_\varphi \\ &- \varepsilon\left(\frac{\cos\theta}{\sin\theta}\,\psi_{\theta\theta} + \psi_{\theta\theta\theta} - \frac{1}{\sin^2\theta}\,\psi_\theta - \frac{2\cos\theta}{\sin^3\theta}\,\psi_{\varphi\varphi}\right. \\ &\left.+\frac{1}{3\sin^2\theta}\,\psi_{\theta\varphi\varphi} + \frac{1}{3\sin^2\theta}\,\psi_{\varphi\theta\varphi} + \frac{1}{3\sin^2\theta}\,\psi_{\varphi\varphi\theta}\right)\psi_\varphi\bigg].\end{aligned} \tag{5.4}$$

We consider the nonlinearly self-adjoint case, i.e. we assume that $F(\theta)$ has the form (4.10) so that we have to substitute in $\mathcal{L}$ the expressions

$$F(\theta) = k\sin\theta, \quad F'(\theta) = k\cos\theta, \quad F''(\theta) = -k\sin\theta \tag{5.5}$$

and let $v = \psi$ according to Equation (4.12).

In our notation the conservation equation for the vector (5.2) is written in the form

$$D_t(C^1) + D_\theta(C^2) + D_\varphi(C^3) = 0.$$

Accordingly, the component C^1 of the conserved vector is termed the *conserved density*. In what follows, we will write down only the conserved density C^1. It is obtained by substituting (5.4) in (5.2). The result is written as follows:

$$\begin{aligned}C^1 =& \left[-\frac{1}{2}\,D_\theta(\psi\cos\theta) + \frac{1}{3}\,D_\theta^2(\psi\sin\theta) + \frac{1}{3\sin\theta}\,\psi_{\varphi\varphi}\right]W \\ &+ \left[\frac{1}{2}\,\psi\cos\theta - \frac{1}{3}\,D_\theta(\psi\sin\theta)\right]D_\theta(W) - \frac{\psi_\varphi}{3\sin\theta}\,D_\varphi(W) \\ &+ \frac{\psi}{3}\,\sin\theta D_\theta^2(W) + \frac{\psi}{3\sin\theta}\,D_\varphi^2(W).\end{aligned} \tag{5.6}$$

The expression (5.6) can be simplified significantly by transferring the terms of the forms $D_\theta(A)$ and $D_\varphi(B)$ from C^1 to C^2 and C^3, respectively. Namely, the reckoning shows that (5.6) can be written in the form

$$C^1 = \left[D_\theta(\psi_\theta \sin\theta) + \frac{1}{\sin\theta}\,\psi_{\varphi\varphi}\right] W$$

$$+ D_\theta\left[\frac{1}{2}\,W\psi\cos\theta + \frac{1}{3}\,D_\theta(W)\psi\sin\theta - \frac{2}{3}\,W D_\theta(\psi\sin\theta)\right] \tag{5.7}$$

$$+ D_\varphi\left[\frac{1}{3\sin\theta}\,\{\psi D_\varphi(W) - 2W\psi_\varphi\}\right].$$

Thus, the symmetry (5.1) leads to the conserved density

$$C^1 = \left[D_\theta(\psi_\theta \sin\theta) + \frac{1}{\sin\theta}\,\psi_{\varphi\varphi}\right] W. \tag{5.8}$$

We will see that in each particular case the expression (5.8) can be simplified further. If the simplification leads to $C^1 = 0$, the conserved vector is said to be *trivial*.

Let us find the conserved density (5.6) associated with the first symmetry from (2.1), i.e. with the time translation generator X_1. Here $W = -\psi_t$ and Equation (5.8) yields

$$C^1 = -\psi_t\left(D_\theta(\psi_\theta\sin\theta) + \frac{1}{\sin\theta}\,\psi_{\varphi\varphi}\right)$$

$$= \psi_\theta\psi_{t\theta}\sin\theta + \frac{1}{\sin\theta}\,\psi_\varphi\psi_{t\varphi} - D_\theta(\psi_t\psi_\theta\sin\theta) - D_\varphi\left(\frac{1}{\sin\theta}\,\psi_t\psi_\varphi\right).$$

Hence the symmetry X_1 leads to the conserved density

$$C^1 = \psi_\theta\psi_{t\theta}\sin\theta + \frac{1}{\sin\theta}\,\psi_\varphi\psi_{t\varphi}. \tag{5.9}$$

In terms of the velocity vector with the spherical components

$$v_\varphi = \psi_\theta, \quad v_\theta = -\frac{1}{\sin\theta}\,\psi_\varphi \tag{5.10}$$

it is written

$$C^1 = \frac{1}{2}\left(v_\varphi^2 + v_\theta^2\right)_t \sin\theta. \tag{5.11}$$

We can show, however, that this conserved vector is trivial. Indeed, let us write the expression (5.9) for C^1 in the form

$$C^1 = D_\theta\left(\psi\psi_{t\theta}\sin\theta\right) + D_\varphi\left(\frac{1}{\sin\theta}\,\psi\psi_{t\varphi}\right) - \left[\psi_{t\theta}\cos\theta + \psi_{t\theta\theta}\sin\theta + \frac{1}{\sin\theta}\,\psi_{t\varphi\varphi}\right]\psi.$$

Then we note that, due to (5.5), Equation (1.4) is written

$$\Lambda \equiv \left[\psi_{t\theta}\cos\theta + \psi_{t\theta\theta}\sin\theta + \frac{1}{\sin\theta}\,\psi_{t\varphi\varphi}\right] \tag{5.12}$$
$$+\left(\frac{1}{R_0} + 2k\right)\psi_\varphi \sin\theta + k\left(\psi_{\theta\varphi}\cos\theta + \psi_{\theta\theta\varphi}\sin\theta + \frac{1}{\sin\theta}\,\psi_{\varphi\varphi\varphi}\right)$$
$$+\,\varepsilon\left(\frac{\cos\theta}{\sin\theta}\,\psi_{\theta\varphi} + \psi_{\theta\theta\varphi} + \frac{1}{\sin^2\theta}\,\psi_{\varphi\varphi\varphi}\right)\psi_\theta$$
$$-\,\varepsilon\left(\frac{\cos\theta}{\sin\theta}\,\psi_{\theta\theta} + \psi_{\theta\theta\theta} + \frac{1}{\sin^2\theta}\,(\psi_{\theta\varphi\varphi} - \psi_\theta) - \frac{2\cos\theta}{\sin^3\theta}\,\psi_{\varphi\varphi}\right)\psi_\varphi = 0.$$

We eliminate the expression in the square brackets in the above formula for C^1 by using Equation (5.12) and after some calculations arrive at the equation

$$C^1 = D_\theta\,(\Theta) + D_\varphi\,(\Phi) - \psi\,\Lambda, \tag{5.13}$$

where Λ is defined in Equation (5.12) and

$$\Theta = \psi\psi_{t\theta}\sin\theta + k\psi\psi_{\theta\varphi}\sin\theta - \varepsilon\left(\psi\psi_\varphi\psi_{\theta\theta} + \frac{\cos\theta}{\sin\theta}\,\psi\psi_\theta\psi_\varphi - \frac{1}{2\sin^2\theta}\,\psi_\varphi^3\right),$$

$$\Phi = \frac{1}{\sin\theta}\,\psi\psi_{t\varphi} + \left(\frac{1}{2R_0} + k\right)\psi^2\sin\theta + \frac{k}{2\sin\theta}\left(2\psi\psi_{\varphi\varphi} - \psi_\varphi^2\right) - \frac{k}{2}\,\psi_\theta^2\sin\theta$$
$$+\,\varepsilon\left[\psi\psi_\theta\psi_{\theta\theta} + \frac{1}{2\sin^2\theta}\left(2\psi\psi_\theta\psi_{\varphi\varphi} - 2\psi\psi_\varphi\psi_{\theta\varphi} - \psi_\theta\psi_\varphi^2\right) + \frac{\cos\theta}{\sin\theta}\,\psi\psi_\theta^2 + \frac{\cos\theta}{\sin^3\theta}\,\psi\psi_\varphi^2\right].$$

Therefore C^1 vanishes on Equation (5.12) after transferring the terms $D_\theta\,(\Theta)$ and $D_\varphi\,(\Phi)$ into C^2 and C^3, respectively. This proves our statement.

For the symmetry X_2 from (2.1) we have $W = -\psi_\varphi$, and Equation (5.8) yields

$$C^1 = -\psi_\varphi\left[D_\theta(\psi_\theta\sin\theta) + \frac{1}{\sin\theta}\,\psi_{\varphi\varphi}\right] = \frac{1}{2}D_\varphi\left[\psi_\theta^2\sin\theta - \frac{1}{\sin\theta}\psi_\varphi^2\right] - D_\theta(\psi_\varphi\psi_\theta\sin\theta).$$

Hence the symmetry X_2 provides a trivial conserved vector.

We conclude likewise that the symmetry X_3 from (2.1) also provides a trivial conserved vector. Indeed, in this case we have $W = \lambda(t)$, and Equation (5.8) yields

$$C^1 = D_\theta\left[\lambda(t)\psi_\theta\sin\theta\right] + D_\varphi\left[(\lambda(t)/\sin\theta)\,\psi_\varphi\right].$$

The similar calculations for X_4 from (2.2) provide a nontrivial conserved vector. Its density is written in terms of the velocity vector (5.10) as follows:

$$C^1 = \left[(1 + 2kR_0)v_\varphi\sin\theta + 2\varepsilon R_0\left(v_\theta^2 + v_\varphi^2\right) + \varepsilon t R_0\left(v_\theta^2 + v_\varphi^2\right)_t\right]\sin\theta.$$

It can be reduced to the form

$$C^1 = (1+2kR_0)v_\varphi \sin^2\theta + 2\varepsilon R_0\left(v_\varphi^2 + v_\theta^2\right)\sin\theta$$
$$\equiv (1+2kR_0)\psi_\theta \sin^2\theta + 2\varepsilon R_0\left(\psi_\theta^2\sin\theta + \frac{1}{\sin\theta}\,\psi_\varphi^2\right) \tag{5.14}$$

because

$$\varepsilon t R_0\left(v_\theta^2 + v_\varphi^2\right)_t \sin\theta = 2\varepsilon R_0\left[D_\theta\,(t\Theta) + D_\varphi\,(t\Phi) - t\psi\,\Lambda\right],$$

where Θ and Φ are the same as in Equation (5.13).

Consider now X_5 from (2.2). Equation (5.3) gives

$$W = -\left(k + \frac{1}{2R_0}\right)\sin\theta\sin\left(\varphi + \frac{t}{2R_0}\right)$$
$$-\varepsilon\sin\left(\varphi + \frac{t}{2R_0}\right)\psi_\theta - \varepsilon\frac{\cos\theta}{\sin\theta}\cos\left(\varphi + \frac{t}{2R_0}\right)\psi_\varphi.$$

Substituting this expression into (5.8) and applying the simplification procedure employed above we obtain the conserved density

$$C^1 = \left(k + \frac{1}{2R_0}\right)\cos\theta\sin\theta\sin\left(\varphi + \frac{t}{2R_0}\right)\psi_\theta + \left(k + \frac{1}{2R_0}\right)\cos\left(\varphi + \frac{t}{2R_0}\right)\psi_\varphi$$
$$-\frac{\varepsilon}{2}\cos\theta\sin\left(\varphi + \frac{t}{2R_0}\right)\left[\psi_\theta^2 + \frac{1}{\sin^2\theta}\,\psi_\varphi^2\right] - \varepsilon\frac{\cos^2\theta}{\sin\theta}\cos\left(\varphi + \frac{t}{2R_0}\right)\psi_\varphi\psi_\theta$$
$$-\varepsilon\cos\theta\cos\left(\varphi + \frac{t}{2R_0}\right)\psi_\varphi\psi_{\theta\theta} - \frac{\varepsilon}{\sin\theta}\sin\left(\varphi + \frac{t}{2R_0}\right)\psi_\theta\psi_{\varphi\varphi}. \tag{5.15}$$

It can be easily rewritten in terms of the the velocity vector (5.10) as well.

Finally, using the symmetry X_6 one obtains a similar conserved density.

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Self-adjointness, conservation laws and invariant solutions of Kelvin waves

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1 Model and its symmetries

We employ the Kelvin's assumption which, in the present context, says that the radial component of the velocity vector is zero everywhere. Thus the nonlinear model under the Kelvin's assumption is written as

$$\frac{u_\theta^2}{r} + f u_\theta = \frac{\partial p}{\partial r}, \tag{1.1}$$

$$\frac{\partial u_\theta}{\partial t} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + w\frac{\partial u_\theta}{\partial z} = -\frac{1}{r}\frac{\partial p}{\partial \theta}, \tag{1.2}$$

$$\frac{\partial w}{\partial t} + \frac{u_\theta}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} - \rho g, \tag{1.3}$$

$$\frac{\partial \rho}{\partial t} + \frac{u_\theta}{r}\frac{\partial \rho}{\partial \theta} = \frac{N^2}{g}\, w, \tag{1.4}$$

$$\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \tag{1.5}$$

where f, g and N^2 are constants.

The symmetries of Eqs. contain two arbitrary functions, $\varphi(z)$, $\psi(t)$, and are spanned by the following operators:

$$
\begin{aligned}
& X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = \frac{\partial}{\partial \theta}\,, \quad X_3 = \frac{\partial}{\partial z}\,, \\
& X_4 = r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z} + u_\theta \frac{\partial}{\partial u_\theta} + w\frac{\partial}{\partial w} + 2p\frac{\partial}{\partial p} + \rho\frac{\partial}{\partial \rho}\,, \\
& X_5 = 2(ft + 2\theta)\frac{\partial}{\partial \theta} - 4r\frac{\partial}{\partial r} + 2fr\frac{\partial}{\partial u_\theta} + f^2 r^2 \frac{\partial}{\partial p}\,, \\
& X_\varphi = g\varphi(z)\frac{\partial}{\partial p} - \varphi'(z)\frac{\partial}{\partial \rho}\,, \quad X_\psi = \psi(t)\frac{\partial}{\partial p}\,\cdot
\end{aligned}
\tag{1.6}
$$

2 Self-adjointness

Our aim is to construct conservation laws for the system (1.1)-(1.5) using the method developed in [1]. According to this method, one can associate conservation laws with symmetries on any system of differential equations provided that the system under consideration is *nonlinearly self-adjoint* in the terminology of [1]. The system in question can be *determined* (the number of the equations in the system is equal to the number of dependent variables), *over-determined* (the number of the equations in the system is more than the number of dependent variables) or *sub-definite* (the number of the equations in the system is less than the number of dependent variables).

We will show in this section that the *over-determined* system of equations (1.1)-(1.5) is nonlinearly self-adjoint.

2.1 Adjoint system

Let us rewrite Eqs. (1.1)-(1.5), denoting $u_\theta = u$, in the following form:

$$
\begin{aligned}
& F_1 \equiv \frac{\partial u}{\partial t} + \frac{u}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial p}{\partial \theta} = 0, \\
& F_2 \equiv \frac{\partial w}{\partial t} + \frac{u}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} + \rho g = 0, \\
& F_3 \equiv \frac{\partial p}{\partial r} - \frac{u^2}{r} - f\,u = 0, \\
& F_4 \equiv \frac{\partial \rho}{\partial t} + \frac{u}{r}\frac{\partial \rho}{\partial \theta} - \frac{N^2}{g}\,w = 0, \\
& F_5 \equiv \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial z} = 0.
\end{aligned}
\tag{2.1}
$$

The formal Lagrangian for the system (2.1) is

$$\mathcal{L} = U\left(\frac{\partial u}{\partial t} + \frac{u}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial p}{\partial \theta}\right) + V\left(\frac{\partial w}{\partial t} + \frac{u}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} + \rho g\right)$$
$$+ P\left(\frac{\partial p}{\partial r} - \frac{u^2}{r} - f\,u\right) + R\left(\frac{\partial \rho}{\partial t} + \frac{u}{r}\frac{\partial \rho}{\partial \theta} - \frac{N^2}{g}\,w\right) + Q\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial z}\right), \quad (2.2)$$

where U, V, P, R and Q are new dependent variables.

The adjoint system to Eqs. (2.1) is written

$$F_1^* \equiv \frac{\delta \mathcal{L}}{\delta u} = 0, \quad F_2^* \equiv \frac{\delta \mathcal{L}}{\delta w} = 0, \quad F_3^* \equiv \frac{\delta \mathcal{L}}{\delta p} = 0, \quad F_4^* \equiv \frac{\delta \mathcal{L}}{\delta \rho} = 0. \quad (2.3)$$

Using the formal Lagrangian (2.2) we obtain:

$$F_1^* = -U_t - \frac{u}{r}\,U_\theta - wU_z - U\frac{\partial w}{\partial z} + \frac{V}{r}\,\frac{\partial w}{\partial \theta} + \frac{R}{r}\,\frac{\partial \rho}{\partial \theta} - \frac{2u}{r}\,P - \frac{1}{r}\,Q_\theta - fP,$$
$$F_2^* = -V_t - \frac{u}{r}\,V_\theta - \frac{V}{r}\,\frac{\partial u}{\partial \theta} - wV_z - Q_z + U\frac{\partial u}{\partial z} - \frac{N^2}{g}\,R, \quad (2.4)$$
$$F_3^* = -P_r - \frac{1}{r}\,U_\theta - V_z, \qquad F_4^* = -R_t - \frac{u}{r}\,R_\theta - \frac{R}{r}\,\frac{\partial u}{\partial \theta} + gV,$$

where the subscripts in $U_t, U_\theta, U_z, \ldots$ denote the differentiations.

2.2 Proof of nonlinear self-adjointness

According to [1] the system (2.1) is nonlinearly self-adjoint if there exists a substitution

$$U = \varphi^1, \quad V = \varphi^2, \quad P = \varphi^3, \quad R = \varphi^4, \quad Q = \varphi^5, \quad (2.5)$$

where $\varphi^1, \ldots, \varphi^5$ are functions of $t, \theta, r, z, u, w, p, \rho$ not vanishing simultaneously, such that the following equations are satisfied (Eqs. (3.5) from [1]):

$$F_\alpha^* \big|_{(2.5)} = \lambda_\alpha^{\bar\beta} F_{\bar\beta}, \quad \alpha = 1, \ldots, 4. \quad (2.6)$$

Here $F_{\bar\beta}$ ($\bar\beta = 1, \ldots, 5$) and F_α^* ($\alpha = 1, \ldots, 4$) are given by the equations (2.1) and (2.4), respectively, $\lambda_\alpha^{\bar\beta}$ are undetermined coefficients, the symbol $|_{(2.5)}$ indicates that the non-physical variables $U, \ldots, Q$ and their derivatives are eliminated by means of the substitution (2.5). The derivatives are computed in a usual way, e.g.

$$
\begin{aligned}
U_t = D_t(\varphi^1) &\equiv \varphi^1_t + \varphi^1_u \frac{\partial u}{\partial t} + \varphi^1_w \frac{\partial w}{\partial t} + \varphi^1_p \frac{\partial p}{\partial t} + \varphi^1_\rho \frac{\partial \rho}{\partial t}\,,\\
U_\theta = D_\theta(\varphi^1) &\equiv \varphi^1_\theta + \varphi^1_u \frac{\partial u}{\partial \theta} + \varphi^1_w \frac{\partial w}{\partial \theta} + \varphi^1_p \frac{\partial p}{\partial \theta} + \varphi^1_\rho \frac{\partial \rho}{\partial \theta}\,,\\
U_z = D_z(\varphi^1) &\equiv \varphi^1_z + \varphi^1_u \frac{\partial u}{\partial z} + \varphi^1_w \frac{\partial w}{\partial z} + \varphi^1_p \frac{\partial p}{\partial z} + \varphi^1_\rho \frac{\partial \rho}{\partial z}\,,\\
Q_\theta = D_\theta(\varphi^5) &\equiv \varphi^5_\theta + \varphi^5_u \frac{\partial u}{\partial \theta} + \varphi^5_w \frac{\partial w}{\partial \theta} + \varphi^5_p \frac{\partial p}{\partial \theta} + \varphi^5_\rho \frac{\partial \rho}{\partial \theta}\,,
\end{aligned}
\tag{2.7}
$$

where

$$\varphi^1_t,\ \ldots \varphi^1_\rho \quad \text{and} \quad \varphi^5_\theta,\ \ldots \varphi^5_\rho$$

are the respective partial derivatives of the function

$$\varphi^1(t, \theta, r, z, u, w, p, \rho) \quad \text{and} \quad \varphi^5(t, \theta, r, z, u, w, p, \rho).$$

The coefficients $\lambda^{\bar{\beta}}_\alpha$ and the functions $\varphi^1, \ldots, \varphi^5$ are found by solving four equations (2.6) corresponding to $\alpha = 1, \ldots, 4$. The first equation corresponding to $\alpha = 1$ is written

$$F^*_1 = \lambda^1_1 F_1 + \lambda^2_1 F_2 + \lambda^3_1 F_3 + \lambda^4_1 F_4 + \lambda^5_1 F_5. \tag{2.8}$$

We substitute in F^*_1 given by the first equation (2.4) the expressions (2.7) for $U_t, \ldots, Q_\theta$, equate the coefficients for

$$\frac{\partial u}{\partial t}\,, \quad \frac{\partial w}{\partial t}\,, \quad \frac{\partial p}{\partial t}\,, \quad \frac{\partial \rho}{\partial t}$$

in both sides of Eq. (2.8) and obtain:

$$\lambda^1_1 = -\varphi^1_u, \quad \lambda^2_1 = -\varphi^1_w, \quad \varphi^1_p = 0, \quad \lambda^4_1 = -\varphi^1_\rho. \tag{2.9}$$

Likewise, considering the c coefficients for

$$\frac{\partial p}{\partial r}\,, \quad \frac{\partial p}{\partial z}\,, \quad \frac{\partial \rho}{\partial z}$$

we obtain:

$$\lambda^3_1 = 0, \quad \lambda^2_1 = 0, \quad \varphi^1_\rho = 0. \tag{2.10}$$

Summarizing the equations (2.9) and (2.10) we see that

$$\varphi^1 = \varphi^1(t, \theta, r, z, u) \tag{2.11}$$

and that

$$\lambda^1_1 = -\varphi^1_u, \quad \lambda^2_1 = \lambda^3_1 = \lambda^4_1 = 0. \tag{2.12}$$

In view of (2.12) Eq. (2.8) becomes

$$F_1^* = -\varphi_u^1 F_1 + \lambda_1^5 F_5. \tag{2.13}$$

We continue the similar calculations with Eq. (2.13) and with the remaining three equations (2.6) corresponding to $\alpha = 2, 3, 4$. After lengthy but regular calculations we conclude that the substitution (2.5) satisfying Eqs. (2.6) has the following form:

$$\begin{aligned} &U = a(t,\theta)r + b(t,r), \quad V = 0,\\ &P = -\frac{\partial a(t,\theta)}{\partial \theta}\, r, \qquad R = 0,\\ &Q = [a(t,\theta)r + b(t,r)]\, u + c(t,\theta,r), \end{aligned} \tag{2.14}$$

where $a(t,\theta),\ b(t,r)$ are arbitrary functions, and $c(t,\theta,r)$ is determined by the differential equation

$$\frac{1}{r}\frac{\partial c}{\partial \theta} = fr\,\frac{\partial a(t,\theta)}{\partial \theta} - \left[r\,\frac{\partial a(t,\theta)}{\partial t} + \frac{\partial b(t,r)}{\partial t}\right]. \tag{2.15}$$

Substituting (2.14) in (2.4) one can verify that the nonlinear self-adjointness conditions (2.6) of Eqs. (2.1) are satisfied in the following form:

$$F_1^* = -[a(t,\theta)r + b(t,r)]\, F_5, \quad F_2^* = 0, \quad F_3^* = 0, \quad F_4^* = 0.$$

Thus the system (2.1) is nonlinearly self-adjoint.

3 Conservation laws

Conserved vectors associated with symmetries

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}$$

of any nonlinearly self-adjoint system of differential equations

$$F_{\bar\alpha}\left(x, u, u_{(1)}, \ldots, u_{(s)}\right) = 0, \quad \bar\alpha = 1, \ldots, \overline{m},$$

is given by the following formula (see [1]), Eq. (8.23)):

$$\begin{aligned} C^i = W^\alpha &\left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\right) + D_j D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right) - \ldots\right]\\ &+ D_j\left(W^\alpha\right)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right) + \ldots\right] + D_j D_k\left(W^\alpha\right)\left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \ldots\right]. \end{aligned} \tag{3.1}$$

Here $\mathcal{L} = v^{\bar\beta} F_{\bar\beta}$ is the formal Lagrangian for the system in question and

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha.$$

The "non-physical variables" $v^{\bar\alpha}$ should be eliminated from the vector (3.1) by using the substitution

$$v^{\bar\alpha} = \varphi^{\bar\alpha}(x,u), \quad \bar\alpha = 1,\ldots,\overline{m},$$

connecting the adjoint system with the system under consideration ([1]), Section 8.2).

We will apply the formula (3.1) to our system (2.1) by using the notation

$$\begin{aligned} &x^1 = t, \quad x^2 = \theta, \quad x^3 = r, \quad x^4 = z,\\ &u^1 = u, \quad u^2 = w, \quad u^3 = p, \quad u^4 = \rho. \end{aligned} \tag{3.2}$$

According to the notation (3.2), the conservation laws will be written in the form

$$\big[D_t(C^1) + D_\theta(C^2) + D_r(C^3) + D_z(C^4)\big]_{(2.1)} = 0, \tag{3.3}$$

where $D_t, \ldots, D_z$ denote the total differentiations in $t, \ldots, z$.

In the case of the first-order formal Lagrangian (2.2) the formula (3.1) is written

$$C^i = W^\alpha \, \frac{\partial \mathcal{L}}{\partial u_i^\alpha} \cdot \tag{3.4}$$

Substituting in (3.4) the expression (2.2) of the formal Lagrangian by taking into account the notation (3.2) and the equation $V = R = 0$ due to Eqs. (2.14) we obtain:

$$\begin{aligned} C^1 &\equiv W^\alpha \, \frac{\partial \mathcal{L}}{\partial u_t^\alpha} = UW^1,\\ C^2 &\equiv W^\alpha \, \frac{\partial \mathcal{L}}{\partial u_\theta^\alpha} = \frac{1}{r}\,(uU + Q)W^1 + \frac{1}{r}\,UW^3,\\ C^3 &\equiv W^\alpha \, \frac{\partial \mathcal{L}}{\partial u_r^\alpha} = PW^3,\\ C^4 &\equiv W^\alpha \, \frac{\partial \mathcal{L}}{\partial u_z^\alpha} = wUW^1 + QW^2. \end{aligned}$$

We replace here U, P and Q with their values given in (2.14) and arrive at the following final formula for calculating the conserved vectors:

$$\begin{aligned} C^1 &= (ar + b)W^1,\\ C^2 &= \frac{1}{r}\,[2(ar+b)u + c]W^1 + \frac{1}{r}\,(ar+b)W^3,\\ C^3 &= -ra_\theta\, W^3,\\ C^4 &= (ar+b)wW^1 + [(ar+b)u + c]W^2, \end{aligned} \tag{3.5}$$

where $a_\theta = \partial a(t,\theta)/\partial\theta$.

3.1 Time translation: Energy

Let us compute the conserved vector provided by the time translation symmetry X_1 from (1.6). In this case we have

$$W^1 = -\frac{\partial u}{\partial t}\,, \quad W^2 = -\frac{\partial w}{\partial t}\,, \quad W^3 = -\frac{\partial p}{\partial t}\,\cdot$$

Inserting these expressions in (3.5), eliminating in C^1 the derivative $\partial u/\partial t$ via the first equation (2.1) and using Eqs. (2.14) we write the conserved vector in the following form:

$$\begin{aligned}
C^1 &= -a_\theta\,(u^2+p) + D_\theta\left[\left(a+\frac{b}{r}\right)(u^2+p)\right] + D_z[(ar+b)uw],\\
C^2 &= -\frac{1}{r}\,(ar+b)\left(\frac{\partial p}{\partial t} + 2u\frac{\partial u}{\partial t}\right) - \frac{c}{r}\,\frac{\partial u}{\partial t}\,,\\
C^3 &= r\,a_\theta\,\frac{\partial p}{\partial t}\,,\\
C^4 &= -(ar+b)w\left(\frac{\partial u}{\partial t} + u\,\frac{\partial w}{\partial t}\right) - c\,\frac{\partial w}{\partial t}\,\cdot
\end{aligned}$$

Finally, removing the terms $D_\theta(\cdots)$ and $D_z(\cdots)$ from C^1 to C^2 and C^4 we obtain:

$$\begin{aligned}
C^1 &= -(u^2+p)\,a_\theta,\\
C^2 &= \frac{1}{r}\,\left[(r\,a_t+b_t)\,(u^2+p) - c\,\frac{\partial u}{\partial t}\right],\\
C^3 &= r\,a_\theta\,\frac{\partial p}{\partial t}\,,\\
C^4 &= (r\,a_t+b_t)\ wu^2 - c\,\frac{\partial w}{\partial t}\,,
\end{aligned} \tag{3.6}$$

where a_θ, a_t, b_t denote the partial derivatives.

The calculation shows that the conservation law (3.3) is satisfied for the vector (3.6) in the following form:

$$\begin{aligned}
&D_t(C^1) + D_\theta(C^2) + D_r(C^3) + D_z(C^4)\\
&= (r\,a_t+b_t)\left(\frac{\partial u}{\partial t} + \frac{u}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial p}{\partial \theta}\right)\\
&+ r\,a_\theta\left(\frac{\partial p}{\partial r} - \frac{u^2}{r} - f\,u\right) + \left(u - c\frac{\partial}{\partial t}\right)\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial z}\right).
\end{aligned} \tag{3.7}$$

Invoking that $u = u_\theta$, we obtain the conserved vector for Eqs. (1.1)-(1.5):

$$
\begin{aligned}
C^1 &= -\frac{\partial a(t,\theta)}{\partial \theta}\,(u_\theta^2 + p),\\
C^2 &= \frac{1}{r}\,\left[\left(r\,\frac{\partial a(t,\theta)}{\partial t} + \frac{\partial b(t,r)}{\partial t}\right)(u_\theta^2 + p) - c(t,\theta,r)\,\frac{\partial u_\theta}{\partial t}\right],\\
C^3 &= r\,\frac{\partial a(t,\theta)}{\partial \theta}\,\frac{\partial p}{\partial t}\,,\\
C^4 &= \left(r\,\frac{\partial a(t,\theta)}{\partial t} + \frac{\partial b(t,r)}{\partial t}\right)\,w u_\theta^2 - c(t,\theta,r)\,\frac{\partial w}{\partial t}\,\cdot
\end{aligned}
\tag{3.8}
$$

The integral form of the conservation law with the vector (3.8) gives the following *energy conservation* for the system (1.1)-(1.5):

$$
\frac{d}{dt}\int_{R^3} \frac{\partial a(t,\theta)}{\partial \theta}\,(u_\theta^2 + p)\,d\theta\,dr\,dz = 0. \tag{3.9}
$$

3.2 Rotation: Angular momentum

For the horizontal rotation, i.e. to the θ-translation symmetry X_2 from (1.6) we have

$$
W^1 = -\frac{\partial u}{\partial \theta}\,, \quad W^2 = -\frac{\partial w}{\partial \theta}\,, \quad W^3 = -\frac{\partial p}{\partial \theta}\,\cdot
$$

Substituting these expressions in the formula (3.5) and making simplifications as above we arrive at the following conserved vector:

$$
\begin{aligned}
C^1 &= r a_\theta\, u,\\
C^2 &= -\frac{1}{r}\,[(ar+b)u + c]\,\frac{\partial u}{\partial \theta} + (ar+b)w\frac{\partial u}{\partial z} - (a_t r + b_t)u,\\
C^3 &= r\,a_\theta\,\frac{\partial p}{\partial \theta}\,,\\
C^4 &= -(ar+b)w\,\frac{\partial u}{\partial \theta} - [(ar+b)u + c]\,\frac{\partial w}{\partial \theta}\,\cdot
\end{aligned}
\tag{3.10}
$$

The vector (3.10) satisfies the conservation law (3.3) in the following form:

$$
\begin{aligned}
&D_t(C^1) + D_\theta(C^2) + D_r(C^3) + D_z(C^4)\\
&= r\,a_\theta\left(\frac{\partial u}{\partial t} + \frac{u}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial p}{\partial \theta}\right) + r\,a_\theta\,\frac{\partial}{\partial \theta}\left(\frac{\partial p}{\partial r} - \frac{u^2}{r} - f\,u\right)\\
&- (ar+b)\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial z}\right)\frac{\partial u}{\partial \theta} - [(ar+b)u + c]\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial z}\right).
\end{aligned}
\tag{3.11}
$$

Remark 3.1. One can simplify the vector (3.10) by removing the term of the form $D_z(\cdots)$ from C^2 to C^4. Then the vector (3.10) becomes

$$\begin{aligned} C^1 &= ra_\theta\, u, \\ C^2 &= -\frac{c}{r}\frac{\partial u}{\partial \theta} - (a_t r + b_t)u, \\ C^3 &= r\, a_\theta \frac{\partial p}{\partial \theta}, \\ C^4 &= ra_\theta\, uw - c\frac{\partial w}{\partial \theta} \end{aligned} \tag{3.10'}$$

and the conservation equation (3.11) is written

$$\begin{aligned} & D_t(C^1) + D_\theta(C^2) + D_r(C^3) + D_z(C^4) \\ &= r\, a_\theta \left(\frac{\partial u}{\partial t} + \frac{u}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial p}{\partial \theta}\right) \\ &+ r\, a_\theta \frac{\partial}{\partial \theta}\left(\frac{\partial p}{\partial r} - \frac{u^2}{r} - f\, u\right) + \left(r\, a_\theta\, u - c\frac{\partial}{\partial \theta}\right)\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial z}\right). \end{aligned} \tag{3.11'}$$

Writing the integral form of the conservation law with the vector (3.11) and returning to the physical notation $u = u_\theta$ we obtain the following law of *conservation of the angular momentum* for the system (1.1)-(1.5):

$$\frac{d}{dt}\int_{R^3} \frac{\partial a(t,\theta)}{\partial \theta}\, ru_\theta\, d\theta dr dz = 0. \tag{3.12}$$

3.3 Symmetries X_3, X_φ and X_ψ provide trivial conserved vectors

The conserved vector (3.5) associated with the z-translation symmetry X_3 vanishes on the solution manifold of the system (2.1). Hence, it is a trivial conserved vector. Moreover, the symmetries X_φ and X_ψ also provide trivial conserved vectors.

3.4 Conserved vector associated with scaling transformation

For the scaling transformation generator X_4 from (1.6) we have

$$\begin{aligned} W^1 &= u - r\frac{\partial u}{\partial r} - z\frac{\partial u}{\partial r}, \\ W^2 &= w - r\frac{\partial w}{\partial r} - z\frac{\partial w}{\partial r}, \\ W^3 &= 2p - r\frac{\partial p}{\partial r} - z\frac{\partial p}{\partial r} \cdot \end{aligned}$$

Substituting these expressions in the formula (3.5) and making simplifications as above we arrive at the following conserved vector:

$$\begin{aligned}
C^1 &= (4a\,r + 3b + rb_r)\,u,\\
C^2 &= \frac{1}{r}\,(4a\,r + 3b + rb_r)(u^2 + p) + \left(\frac{2c}{r} + c_r\right)u,\\
C^3 &= -4ra_\theta p - c\,\frac{\partial u}{\partial \theta}\,,\\
C^4 &= (4a\,r + 3b + rb_r)\,uw + c\left(w - r\,\frac{\partial w}{\partial r}\right).
\end{aligned} \tag{3.13}$$

The vector (3.13) satisfies the conservation law (3.3) in the following form:

$$\begin{aligned}
D_t(C^1) + D_\theta(C^2) + D_r(C^3) + D_z(C^4) = c\left(\frac{2}{r} - \frac{\partial}{\partial r}\right)\left(\frac{\partial u}{\partial \theta} + r\frac{\partial w}{\partial z}\right)\\
+ (4a\,r + 3b + rb_r)\left[\frac{\partial u}{\partial t} + \frac{u}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial p}{\partial \theta} + u\left(\frac{1}{r}\,\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial z}\right)\right].
\end{aligned} \tag{3.14}$$

The corresponding integral conservation law for the system (1.1)-(1.5) is written:

$$\frac{d}{dt}\int_{R^3}\left[4a(t,\theta)r + 3b(t,r) + r\,\frac{\partial b(t,r)}{\partial r}\right]u_\theta\,d\theta dr dz = 0. \tag{3.15}$$

3.5 Conserved vectors given by other symmetries from (1.6)

The symmetry X_5 from (1.6) gives a conserved vector with the vanishing density C^1.

3.6 Conclusion

Thus, the symmetries (1.6) provide three infinite sets of nontrivial integral conservation laws, (3.9), (3.12) and (3.15), depending on three arbitrary functions, namely $a(t,\theta), b(t,r)$ and an arbitrary function involved in $c(t,\theta,r)$.

4 Invariants solutions

The symmetries (1.6) can be used for obtaining exact solutions of the system (2.1) by computing the *invariant and partially invariant solutions* (see, e.g. [2]). In order to obtain all possible invariant and partially invariant solutions one has to construct *optimal systems of subalgebras* of the Lie algebra with the basis (1.6). Note that invariant solutions based on three-dimensional subalgebras are of a particular interest because they are described by systems of ordinary differential equations. We will consider two solutions of this type.

4.1 Non-stationary solution

Let us construct the invariant solutions based on the three-dimensional subalgebra spanned by the operators X_3, X_4, X_5 from (1.6). According to the theorem on representation of non-singular invariant manifolds ([2], Section 14.3), the invariant solutions can be represented via invariants $J(t, \theta, r, z, u, w, p, \rho)$ of the operators X_3, X_4, X_5. The invariance under X_3 requires that J does not depend on z. Thus, we have to solve the system

$$\begin{aligned} X_4(J) &\equiv r\frac{\partial J}{\partial r} + u\frac{\partial J}{\partial u} + w\frac{\partial J}{\partial w} + 2p\frac{\partial J}{\partial p} + \rho\frac{\partial J}{\partial \rho} = 0, \\ X_5(J) &\equiv 2(ft + 2\theta)\frac{\partial J}{\partial \theta} - 4r\frac{\partial J}{\partial r} + 2fr\frac{\partial J}{\partial u} + f^2r^2\frac{\partial J}{\partial p} = 0 \end{aligned} \tag{4.1}$$

for the function $J = J(t, \theta, r, u, w, p, \rho)$. Integration of the system (4.1) gives the following basis of invariants:

$$\begin{aligned} &J_1 = t, \quad J_2 = \frac{2u + fr}{(ft + 2\theta)r}, \quad J_3 = \frac{w}{(ft + 2\theta)r}, \\ &J_4 = \frac{8p + f^2r^2}{(ft + 2\theta)^2r^2}, \qquad J_5 = \frac{\rho}{(ft + 2\theta)r}. \end{aligned} \tag{4.2}$$

The representation of the invariant solutions via the basic invariants (4.2) is obtained by assuming that J_2, J_3, J_4, J_5 are unknown functions of J_1. This yields the following candidates for the invariant solutions:

$$\begin{aligned} &u = -\frac{fr}{2} + (ft + 2\theta)r\,F(t), \qquad w = (ft + 2\theta)r\,G(t), \\ &p = -\frac{f^2r^2}{8} + (ft + 2\theta)^2)r^2\,H(t), \quad \rho = (ft + 2\theta)r\,K(t). \end{aligned} \tag{4.3}$$

Substituting (4.3) in Eqs. (2.1) and solving the resulting equations for the unknown function $F(t), G(t), H(t), K(t)$ we obtain

$$\begin{aligned} &F(t) = 0, \quad G(t) = A\cos(Nt) + B\sin(Nt), \\ &H(t) = 0, \quad K(t) = \frac{N}{g}\left[A\sin(Nt) - B\cos(Nt)\right], \end{aligned} \tag{4.4}$$

where A and B are arbitrary constants. Substituting (4.4) in (2.3) and returning to the notation $u = u_\theta$ we obtain the following transient-state solution of the system (1.1)-(1.5):

$$\begin{aligned} &u_\theta = -\frac{fr}{2}, \quad w = (ft + 2\theta)\,r\,[A\cos(Nt) + B\sin(Nt)], \\ &p = -\frac{f^2r^2}{8}, \quad \rho = \frac{N}{g}\,(ft + 2\theta)\,r\left[A\sin(Nt) - B\cos(Nt)\right]. \end{aligned} \tag{4.5}$$

4.2 Stationary solution

Let us construct the invariant solutions based on the three-dimensional subalgebra spanned by the operators X_1, X_3, X_4 from (1.6). Proceeding as in Section 4.1 we obtain the following candidates for the invariant solutions:

$$u = r\,F(\theta), \quad w = r\,G(\theta), \quad p = r^2\,H(\theta), \quad \rho = r\,K(\theta). \tag{4.6}$$

Substituting (4.6) in Eqs. (2.1) we obtain

$$\begin{aligned} &F(\theta) = F_0, \qquad G(t) = C_1\cos(k\theta) + C_2\sin(k\theta),\\ &H(t) = \frac{1}{2}\,(F_0^2 + fF_0), \quad K(t) = \frac{N}{g}\,\left[C_1\sin(k\theta) - C_2\cos(k\theta)\right], \end{aligned} \tag{4.7}$$

where C_1, C_2 are arbitrary constants, F_0 is an arbitrary constant different from zero, and

$$k = \frac{N}{F_0}\,\cdot$$

If $F_0 = 0$ the solution collapses to the trivial solution $u = w = p = \rho = 0.$

Substituting (4.7) in (2.15) and returning to the notation $u = u_\theta$ we obtain the following steady-state solution of the system (1.1)-(1.5):

$$\begin{aligned} &u_\theta = F_0\,r\,, \qquad\qquad w = r\,[C_1\cos(k\theta) + C_2\sin(k\theta)],\\ &p = \frac{1}{2}\,(F_0^2 + fF_0)\,r^2, \quad \rho = \frac{N}{g}\,r\,[C_1\sin(k\theta) - C_2\cos(k\theta)]\,. \end{aligned} \tag{4.8}$$

4.3 Invariant solution based on X_2, X_4, X_5

Constructing as above the invariant solutions based on the three-dimensional subalgebra spanned by the operators X_2, X_4, X_5 from (1.6) we obtain the following candidates for the invariant solutions:

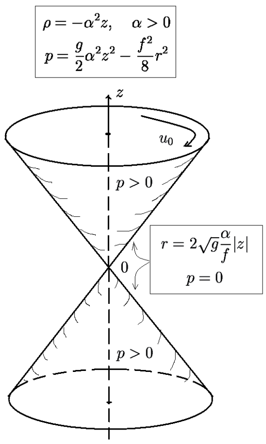
$$u = -\frac{f\,r}{2} + z\,F(t), \quad w = z\,G(t), \quad p = -\frac{f^2\,r^2}{8} + z^2\,H(t), \quad \rho = z\,K(t).$$

Substituting these expressions in Eqs. (2.1) we obtain

$$F(t) = 0, \quad G(t) = 0, \quad K(t) = k = \text{const.}, \quad H(t) = -\frac{k}{2}\,g.$$

Thus, we have the following solution of the system (1.1)-(1.5):

$$u_\theta = -\frac{f\,r}{2}\,, \quad w = 0, \quad p = -\frac{f^2\,r^2}{8} - \frac{k}{2}\,gz^2, \quad \rho = kz, \tag{4.9}$$



where k is an arbitrary constant. If we take the coordinate system so that z is directed upward then k should be positive. In this coordinate system $g < 0$, and hence the pressure deviation p vanishes on the surface of the funnel (whirlpool) with the apex on the bottom of the basin:

$$r = \frac{2}{f}\sqrt{-gk}\, z \tag{4.10}$$

and is positive inside of the funnel, i.e. under the condition

$$r < \frac{2}{f}\sqrt{-gk}\, z. \tag{4.11}$$

4.4 Invariant solution based on the subalgebra $L_2 =< X_2, X_3 >$

Invariance with respect to the translations in θ and z generated by X_2 and X_3, respectively requires that the dependent variables are functions of t and r only. Accordingly, the system (1.1)-(1.5) reduces to the form

$$\frac{\partial u_\theta}{\partial t} = 0, \tag{4.12}$$

$$\frac{\partial p}{\partial r} = \frac{u_\theta^2}{r} + f u_\theta\,, \tag{4.13}$$

$$\frac{\partial w}{\partial t} = -\rho g, \tag{4.14}$$

$$\frac{\partial \rho}{\partial t} = \frac{N^2}{g}\, w. \tag{4.15}$$

Integration of Eqs. (4.12)-(4.15) gives the invariant solution

$$u_\theta = U(r), \tag{4.16}$$

$$p = \int \left[\frac{1}{r} U^2(r) + f U(r)\right] dr + V(t)\,, \tag{4.17}$$

$$w = W_1(r)\cos(Nt) + W_2(r)\sin(Nt), \tag{4.18}$$

$$\rho = \frac{N^2}{g}\,[W_1(r)\sin(Nt) - W_2(r)\cos(Nt)] \tag{4.19}$$

with arbitrary functions $U(r),\ W_1(r),\ W_2(r)$ and $V(t)$.

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CONSERVATION LAWS OF NONLINEAR HEAT AND FILTRATION EQUATIONS

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Abstract. Using the method developed by N.H. Ibragimov, conservation laws for the nonlinear heat and filtration equations have been constructed.

Keywords: Conservation laws, Nonlinear heat equation, Nonlinear filtration equation.

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1 Introduction

Conservation laws are very significant during construction and research of mathematical models described by the ordinary and partial differential equations. Therefore, their study provides a more extended analysis of various differential equations.

Recently, N.H. Ibragimov [1] developed a new method which generalizes Noether's theorem. In particular, it allows one to associate conservation laws with symmetries of partial differential equations having no classical Lagrangian.

In the present paper, we apply this method to nonlinear heat and filtration equations. In section 2 we consider the nonlinear heat equation

$$u_t = (k(u)u_x)_x, \tag{1.1}$$

where heat conductivity $k(u)$ depends on temperature u only. This equation is well studied. In particular, it is known that for arbitrary $k(u)$ it has three-dimensional symmetry group (see, for example, [2]). This group is extended to four- and five-dimensions for $k(u)$ of special types.

In section 3 we consider the nonlinear filtration equation

$$u_t = k(u_x)u_{xx}. \tag{1.2}$$

In particular, this equation describes a pressure distribution in a porous medium. Similarly to the equation (1.1), for arbitrary $k(u_x)$ this equation admits a four-dimensional symmetry group (see [2]). For special $k(u_x)$ this group is extended up to five-dimensional symmetry group.

We have found all conservation laws of these equations, using the admitted point symmetry.

2 Conservation laws of the nonlinear heat equation

The nonlinear heat equation (1.1) with arbitrary $k(u)$ admits the three-dimensional Lie algebra L_3 with the basis

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\cdot$$

If $k(u) = e^u$, the algebra L_3 extends by the operator

$$X_4 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}.$$

If $k(u) = u^\sigma, \ (\sigma \neq 0, -\frac{4}{3})$, the algebra L_3 extends by the operator

$$X_4 = \frac{\sigma}{2}x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

If $k(u) = u^{-4/3}$, the algebra L_3 extends by the operators

$$X_4 = -\frac{2}{3}x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}\,, X_5 = -x^2\frac{\partial}{\partial x} + 3xu\frac{\partial}{\partial u}.$$

The adjoint equation has the form

$$F^* = v_t + k(u)v_{xx}.$$

Let us find a function $\varphi(t, x, u)$ such that after the substitution $v = \varphi(t, x, u)$ we obtain

$$F^* \mid_{v=\varphi(t,x,u)} = \lambda(u_t - k(u)u_{xx} - k'(u)u_x^2). \tag{2.1}$$

Equation (2.1) yields that $\varphi(t, x, u) = Ax + B$ with arbitrary constants A and B.

Using the algorithm given in [1] we construct the conservation laws

$$D_t\left(C^1\right) + D_x\left(C^2\right) = 0$$

for all symmetries. The result is as follows:

for X_1

$$C^1 = Ak(u)u_x, \quad C^2 = Ak(u)\left(k(u)u_x\right)_x;$$

for X_2

$$C^1 = Au, \quad C^2 = -Ak(u)u_x;$$

for X_3

$$C^1 = (2Ax + B)u + 2Atk(u)u_x,$$

$$C^2 = -(Ax + B)k(u)u_x - Ak(u)\left(2t\left(k(u)u_x\right)_x + xu_x\right);$$

Remark. We have $2Atk(u)u_x = D_x[2At\mathcal{K}(u)]$, where $\mathcal{K}'(u) = k(u)$. Therefore the above conserved vector is equivalent to

$$C^1 = (2Ax + B)u, \quad C^2 = 2A\mathcal{K}(u) - (Ax + B)k(u)u_x.$$

In the cases of extended symmetries we have the following conserved vectors.
If $k(u) = e^u$, the operator X_4 yields

$$C^1 = (2Ax + B)(u + 1) + B, \quad C^2 = -(Ax + B)e^u u_x + Ae^u(2 - xu_x).$$

If $k(u) = u^\sigma$, $(\sigma \neq 0, -\frac{4}{3})$, the operator X_4 yields

$$C^1 = 2A(\sigma + 1)xu + B(\sigma + 2)u,$$
$$C^2 = 2Au^\sigma[u - (\sigma + 1)xu_x] - B(\sigma + 2)u^\sigma u_x.$$

The conservation equation for this vector is written

$$D_t(C^1) + D_x(C^2) = [2A(\sigma + 1)x + B(\sigma + 2)][u_t - (u^\sigma u_x)_x];$$

Let $k(u) = u^{-4/3}$. In this case the operator X_4 leads to the conserved vector

$$C^1 = \frac{u}{3}\left(B - Ax\right),$$

$$C^2 = -\frac{1}{3}u^{-4/3}u_x(Ax + B) + Au^{-4/3}\left(u + \frac{2}{3}xu_x\right)$$

whereas the operator X_5 yields

$$C^1 = Bxu, \quad C^2 = -(Ax + B)\left(xu^{-4/3}u_x + 3u^{-1/3}\right) + A\left(3xu^{-1/3} + x^2u^{-4/3}u_x\right).$$

3 Nonlinear filtration equation

3.1 Nonlinear self-adjointness

Let us consider the nonlinear filtration equation (1.2),

$$F \equiv -u_t + k(u_x)u_{xx} = 0. \tag{3.1}$$

Its adjoint equation has the form

$$F^* \equiv v_t + k(u_x)v_{xx} + k'(u_x)v_x u_{xx} = 0. \tag{3.2}$$

Let us find a function $\varphi(t, x, u)$ satisfying the nonlinear self-adjointness condition

$$F^* \mid_{v=\varphi(t,x,u)} = \lambda \left[u_t - k(u_x)u_{xx}\right]. \tag{3.3}$$

The expanded form of Eq. (3.3) is

$$\begin{gathered}\varphi_u u_t + \varphi_t + k(u_x)\left[\varphi_u u_{xx} + \varphi_{uu}u_x^2 + 2\varphi_{xu}u_x + \varphi_{xx}\right] \\ + k'(u_x)\left[\varphi_u u_x + \varphi_x\right]u_{xx} = \lambda\left[u_t - k(u_x)u_{xx}\right].\end{gathered} \tag{3.4}$$

Equating the terms with u_t in both sides of Eq. (3.4) we obtain

$$\lambda = \varphi_u.$$

Taking this into account and equating the terms with u_{xx} in both sides of Eq. (3.4) we arrive at the equation

$$\varphi_u\left[2k(u_x) + u_x k'(u_x)\right] + \varphi_x k'(u_x) = 0. \tag{3.5}$$

Then Eq. (3.4) reduces to the following:

$$\varphi_t + k(u_x)\left[\varphi_{uu}u_x^2 + 2\varphi_{xu}u_x + \varphi_{xx}\right] = 0. \tag{3.6}$$

In the case of an arbitrary function $k(u_x)$ the *determining equations* (3.5)-(3.6) for $\varphi(t, x, u)$ are satisfied only if $\varphi =$ const. We can let

$$\varphi = 1. \tag{3.7}$$

3.2 A special case

We will find now the particular form of $k(u_x)$ when Eqs. (3.5)-(3.6) are satisfied for a non-constant function $\varphi(t, x, u)$. Separating the variables in Eq. (3.5) we have:

$$\frac{2k(u_x)}{k'(u_x)} + u_x = -\frac{\varphi_x}{\varphi_u}\cdot$$

It follows that

$$\frac{2k(u_x)}{k'(u_x)} + u_x = -a, \quad -\frac{\varphi_x}{\varphi_u} = -a, \quad a = \text{const.} \tag{3.8}$$

The first equation (3.8) written in the form

$$\frac{dk}{du_x} = -\frac{2k}{u_x + a}$$

gives

$$k(u_x) = \frac{m}{(u_x + a)^2}\,, \quad m = \text{const.} \tag{3.9}$$

The solution of the second equation (3.8), i.e. of the partial differential equation

$$a\frac{\partial\varphi}{\partial u} - \frac{\partial\varphi}{\partial x} = 0,$$

has the form

$$\varphi = \phi(t, z), \quad z = u + ax. \tag{3.10}$$

The substitution of (3.9) and (3.10) in Eq. (3.6) yields:

$$\phi_t + m\phi_{zz} = 0. \tag{3.11}$$

We further simplify Eqs. (3.9)-(3.11) by using the equivalence transformation

$$\bar{u} = u + ax \tag{3.12}$$

of Eq. (3.1). Applying this transformation and denoting $\bar{u}$ again by u we conclude that the nonlinear filtration equation

$$u_t = \frac{m}{u_x^2}\, u_{xx} \tag{3.13}$$

satisfies the nonlinear self-adjointness condition (3.3) with the function

$$\varphi = \phi(t, u), \tag{3.14}$$

where $\phi(t, u)$ is an arbitrary solution of the equation

$$\phi_t + m\,\phi_{uu} = 0. \tag{3.15}$$

3.3 Conservation laws

The nonlinear filtration equation (1.2) admits the four-dimensional Lie algebra L_4 with the basis

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial u}, \quad X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} \cdot \tag{3.16}$$

The algebra L_4 extends by one additional operator X_5 in the following cases [2]:
if $k(u_x) = e^{u_x}$

$$X_5 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial u},$$

if $k(u_x) = u_x^n \ (n \geq -1, n \neq 0)$

$$X_5 = nt\frac{\partial}{\partial t} - u\frac{\partial}{\partial u},$$

if $k(u_x) = \dfrac{e^{(n \arctan u_x)}}{u_x^2 + 1} \ (n \geq 0)$

$$X_5 = nt\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}.$$

Let us construct the conservation laws

$$D_t\left(C^1\right) + D_x\left(C^2\right) = 0$$

for all symmetries using the algorithm given in [1]. Namely, writing the formal Lagrangian in the form

$$\mathcal{L} = v\left[u_t - k(u_x)u_{xx}\right] \tag{3.17}$$

we have the following expressions for the components of the conserved vectors:

$$\begin{aligned} C^1 &= W\frac{\partial \mathcal{L}}{\partial u_t} = Wv, \\ C^2 &= W\left[\frac{\partial \mathcal{L}}{\partial u_x} - D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right)\right] + D_x(W)\frac{\partial \mathcal{L}}{\partial u_{xx}} \\ &= Wk(u_x)v_x - D_x(W)k(u_x)v, \end{aligned} \tag{3.18}$$

where we should make the substitution $v = \varphi(t, x, u)$.

In the general case we have $\varphi = 1$ (see Eq. (3.7)). One can verify (see also Remark in Section 2) that X_1, X_2 and X_3 provide only trivial conserved vectors whereas X_4 yields the following conserved vector :

$$C^1 = u, \quad C^2 = -\mathcal{K}(u_x),$$

where

$$\mathcal{K}'(u_x) = k(u_x).$$

In the case

$$k(u_x) = e^{u_x}$$

the operator X_5 provides the conserved vector

$$C^1 = -x - te^{u_x} u_{xx}, \qquad C^2 = e^{u_x} + te^{2u_x} \left(u_{xx}^2 + u_{xxx} \right).$$

In the case

$$k(u_x) = u_x^n$$

the operator X_5 yields

$$\begin{array}{lll} \text{at } n > -1, n \neq 0 & C^1 = -u, & C^2 = \dfrac{u_x^{n+1}}{n+1}, \\ \text{at } n = -1 & C^1 = -u, & C^2 = \ln u_x. \end{array}$$

In the case

$$k(u_x) = \frac{e^{n \arctan u_x}}{u_x^2 + 1}, \quad n \geq 0,$$

the operator X_5 yields the trivial conserved vector

$$C^1 = -x, \qquad C^2 = 0.$$

3.4 Conserved vectors in the special case

Let us turn to Eq. (3.13). In this case $\varphi(t, x, u)$ is given by Eq. (3.14). The symmetries of Eq. (3.13) are given by (3.16).

Let us begin with Consider the symmetry X_3. We have $W = 1$, and Eqs. (3.18) give the infinite set of conserved vectors

$$C^1 = \phi, \quad C^2 = \frac{m}{u_x} \phi_u \tag{3.19}$$

involving an arbitrary solution $\phi = \phi(t, u)$ of Eq. (3.15). We have:

$$D_t(C^1) + D_x(C^2) = \phi_t + m\phi_{uu} + \left[u_t - \frac{m}{u_x^2} u_{xx} \right] \phi_u.$$

Hence, invoking Eq. (3.15), we obtain the conservation equation

$$D_t(C^1) + D_x(C^2) = \left(u_t - \frac{m}{u_x^2} u_{xx} \right) \phi_u.$$

Consider now the symmetry X_1. Eqs. (3.18) give

$$C^1 = -\phi\, u_t, \quad C^2 = -\frac{m}{u_x}\phi_u\, u_t + \frac{m}{u_x^2}\phi\, u_{tx}.$$

Since

$$-\phi\, u_t = -\phi\,\frac{m}{u_x^2}\, u_{xx} = mD_x\left(\frac{\phi}{u_x}\right) - m\phi_u$$

we can write the above conserved vector in the form

$$C^1 = \phi_u, \quad C^2 = -\frac{1}{u_x}\,\phi_t. \tag{3.20}$$

This vector satisfies the conservation equation due Eq. (3.15) because

$$D_t(C^1) + D_x(C^2) = (\phi_t + m\phi_{uu})\frac{u_{xx}}{u_x^2} + \left(u_t - \frac{m}{u_x^2}\, u_{xx}\right)\phi_{uu}.$$

For X_2 we obtain

$$C^1 = -\phi\, u_x, \quad C^2 = -m\phi_u + \frac{m}{u_x^2}\,\phi\, u_{xx}.$$

We have:

$$\phi\, u_x = D_x[\Phi(t,u)],$$

where $\Phi(t,u)$ is defined by the equation

$$\Phi_u = \phi(t,u).$$

Therefore the above conserved vector is equivalent to

$$C^1 = 0, \quad C^2 = -m\phi_u(t,u) - \Phi_t(t,u). \tag{3.21}$$

The conservation equation for this vector is satisfied due to Eq. (3.15). Namely, we have:

$$D_t(C^1) + D_x(C^2) = -(\phi_t + m\phi_{uu})u_x.$$

For X_4 we obtain

$$C^1 = (u - 2tu_t - xu_x)\phi,$$
$$C^2 = \frac{m}{u_x}\,(u - 2tu_t - xu_x)\,\phi_u + \frac{m}{u_x^2}\,(2tu_{tx} + xu_{xx})\,\phi.$$

We have:

$$-2t\phi\, u_t = -2t\phi\,\frac{m}{u_x^2}\, u_{xx} = D_x\left(2mt\frac{\phi}{u_x}\right) - 2mt\phi_u,$$

and

$$-x\phi\, u_x = -D_x(x\Phi) + \Phi,$$

where $\Phi = \Phi(t, u)$ has been defined in the previous case. Therefore the above conserved vector is equivalent to

$$C^1 = u\phi - 2mt\phi_u + \Phi, \tag{3.22}$$
$$C^2 = -x\Phi_t + \frac{2m}{u_x}(\phi + t\phi_t) + \frac{m\phi_u}{u_x}(u - xu_x).$$

The conservation equation for this vector is satisfied in the following form:

$$D_t(C^1) + D_x(C^2) = \left(u - xu_x - \frac{2mt}{u_x^2}\, u_{xx}\right)(\phi_t + m\phi_{uu})$$
$$+ (2\phi + u\phi_u - 2mt\phi_{uu})\left(u_t - \frac{m}{u_x^2}\, u_{xx}\right).$$

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Conserved vectors for a model of atmospheric flows

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1 Introduction

The densities of the conserved vectors associated with the symmetries of the third-order nonlinear equation

$$\begin{aligned}
&\psi_{t\theta}\,\cos\theta+\psi_{t\theta\theta}\,\sin\theta+\frac{1}{\sin\theta}\,\psi_{t\varphi\varphi}+\frac{\sin\theta}{R_0}\,\psi_\varphi\\
&+\left(\frac{\cos\theta}{\sin\theta}\,\psi_{\theta\varphi}+\psi_{\theta\theta\varphi}+\frac{1}{\sin^2\theta}\,\psi_{\varphi\varphi\varphi}\right)[F(\theta)+\varepsilon\psi_\theta]\\
&-\left(F''(\theta)+\frac{\cos\theta}{\sin\theta}\,F'(\theta)-\frac{1}{\sin^2\theta}\,F(\theta)\right)\psi_\varphi\\
&-\varepsilon\left(\frac{\cos\theta}{\sin\theta}\,\psi_{\theta\theta}+\psi_{\theta\theta\theta}+\frac{1}{\sin^2\theta}\,(\psi_{\theta\varphi\varphi}-\psi_\theta)-\frac{2\cos\theta}{\sin^3\theta}\,\psi_{\varphi\varphi}\right)\psi_\varphi=0
\end{aligned}\tag{1.1}$$

are constructed in paper [1] in this volume provided that the function $F(\theta)$ satisfies the condition of nonlinear self-adjointness of Equation (1.1), namely

$$F(\theta)=k\sin\theta,\quad k=\text{const}.\tag{1.2}$$

Substituting the expressions

$$F(\theta)=k\sin\theta,\quad F'(\theta)=k\cos\theta,\quad F''(\theta)=-k\sin\theta\tag{1.3}$$

in Equation (1.1) we write it in the form

$$\begin{aligned} \Lambda \equiv & \left[\psi_{t\theta}\cos\theta + \psi_{t\theta\theta}\sin\theta + \frac{1}{\sin\theta}\,\psi_{t\varphi\varphi}\right] \\ & + \left(\frac{1}{R_0} + 2k\right)\psi_\varphi \sin\theta + k\left(\psi_{\theta\varphi}\cos\theta + \psi_{\theta\theta\varphi}\sin\theta + \frac{1}{\sin\theta}\,\psi_{\varphi\varphi\varphi}\right) \\ & + \varepsilon\left(\frac{\cos\theta}{\sin\theta}\,\psi_{\theta\varphi} + \psi_{\theta\theta\varphi} + \frac{1}{\sin^2\theta}\,\psi_{\varphi\varphi\varphi}\right)\psi_\theta \\ & - \varepsilon\left(\frac{\cos\theta}{\sin\theta}\,\psi_{\theta\theta} + \psi_{\theta\theta\theta} + \frac{1}{\sin^2\theta}\left(\psi_{\theta\varphi\varphi} - \psi_\theta\right) - \frac{2\cos\theta}{\sin^3\theta}\,\psi_{\varphi\varphi}\right)\psi_\varphi = 0. \end{aligned} \tag{1.4}$$

We present here all components $C^1,\ C^2,\ C^3$ of the conservation equation

$$D_t(C^1) + D_\theta(C^2) + D_\varphi(C^3) = 0. \tag{1.5}$$

The conserved vectors have been computed using the package MAPLE.

Following [2], the conserved vectors associated with every symmetry

$$X = \xi^i(x,\psi)\frac{\partial}{\partial x^i} + \eta(x,\psi)\frac{\partial}{\partial \psi} \tag{1.6}$$

of Equation (1.4) will be computed by the following formula:

$$\begin{aligned} C^i = & W\ \left[\frac{\partial \mathcal{L}}{\partial \psi_i} - D_j\left(\frac{\partial \mathcal{L}}{\partial \psi_{ij}}\right) + D_j D_k\left(\frac{\partial \mathcal{L}}{\partial \psi_{ijk}}\right)\right] \\ & + D_j(W)\ \left[\frac{\partial \mathcal{L}}{\partial \psi_{ij}} - D_k\left(\frac{\partial \mathcal{L}}{\partial \psi_{ijk}}\right)\right] + D_j D_k(W)\frac{\partial \mathcal{L}}{\partial \psi_{ijk}}\,, \end{aligned} \tag{1.7}$$

where $x^1 = t,\ x^2 = \theta,\ x^3 = \varphi$, and

$$W = \eta - \xi^j \psi_j.$$

Recall that $\mathcal{L}$ is the formal Lagrangian for Equation (1.4) and that it should be written in the symmetric form

$$
\begin{aligned}
\mathcal{L} =& v \left[\frac{1}{2}\,\psi_{t\theta}\,\cos\theta + \frac{1}{2}\,\psi_{\theta t}\,\cos\theta + \frac{1}{3}\,\psi_{t\theta\theta}\,\sin\theta + \frac{1}{3}\,\psi_{\theta t\theta}\,\sin\theta + \frac{1}{3}\,\psi_{\theta\theta t}\,\sin\theta \right.\\
&+ \frac{1}{3\sin\theta}\,\psi_{t\varphi\varphi} + \frac{1}{3\sin\theta}\,\psi_{\varphi t\varphi} + \frac{1}{3\sin\theta}\,\psi_{\varphi\varphi t} + \frac{\sin\theta}{R_0}\,\psi_{\varphi}\\
&+ \left(\frac{\cos\theta}{2\sin\theta}\,\psi_{\theta\varphi} + \frac{\cos\theta}{2\sin\theta}\,\psi_{\varphi\theta} + \frac{1}{3}\,\psi_{\theta\theta\varphi} + \frac{1}{3}\,\psi_{\theta\varphi\theta} + \frac{1}{3}\,\psi_{\varphi\theta\theta}\right.\\
&\left.+\frac{1}{\sin^2\theta}\,\psi_{\varphi\varphi\varphi}\right)(F(\theta) + \varepsilon\psi_\theta) - \left(F''(\theta) + \frac{\cos\theta}{\sin\theta}\,F'(\theta) - \frac{1}{\sin^2\theta}\,F(\theta)\right)\psi_\varphi\\
&- \varepsilon\left(\frac{\cos\theta}{\sin\theta}\,\psi_{\theta\theta} + \psi_{\theta\theta\theta} - \frac{1}{\sin^2\theta}\,\psi_\theta - \frac{2\cos\theta}{\sin^3\theta}\,\psi_{\varphi\varphi}\right.\\
&\left.\left.+\frac{1}{3\sin^2\theta}\,\psi_{\theta\varphi\varphi} + \frac{1}{3\sin^2\theta}\,\psi_{\varphi\theta\varphi} + \frac{1}{3\sin^2\theta}\,\psi_{\varphi\varphi\theta}\right)\psi_\varphi\right].
\end{aligned}
$$

Since we consider the nonlinearly self-adjoint case, we assume that $F(\theta)$ has the form (1.2). Therefore we substitute in $\mathcal{L}$ the expressions (1.3), let $v = \psi$ due to the self-adjointness of Equation (1.4) and obtain:

$$
\begin{aligned}
\mathcal{L} = \psi &\left[\frac{\cos\theta}{2}\,(\psi_{t\theta} + \psi_{\theta t}) + \frac{\sin\theta}{3}\,(\psi_{t\theta\theta} + \psi_{\theta t\theta} + \psi_{\theta\theta t})\right.\\
&+ \left(\frac{1}{R_0} + 2k\right)\psi_\varphi\sin\theta + \frac{k\cos\theta}{2}\,(\psi_{\theta\varphi} + \psi_{\varphi\theta}) + \frac{\psi_{t\varphi\varphi} + \psi_{\varphi t\varphi} + \psi_{\varphi\varphi t}}{3\sin\theta}\\
&+ \frac{k\sin\theta}{3}\,(\psi_{\theta\theta\varphi} + \psi_{\theta\varphi\theta} + \psi_{\varphi\theta\theta}) + \frac{k}{\sin\theta}\,\psi_{\varphi\varphi\varphi} \qquad (1.8)\\
&+ \varepsilon\psi_\theta\left(\frac{\cos\theta}{2\sin\theta}(\psi_{\theta\varphi} + \psi_{\varphi\theta}) + \frac{1}{3}\,(\psi_{\theta\theta\varphi} + \psi_{\theta\varphi\theta} + \psi_{\varphi\theta\theta}) + \frac{1}{\sin^2\theta}\,\psi_{\varphi\varphi\varphi}\right)\\
&\left.-\varepsilon\psi_\varphi\left(\frac{\cos\theta}{\sin\theta}\,\psi_{\theta\theta} + \psi_{\theta\theta\theta} - \frac{1}{\sin^2\theta}\,\psi_\theta - \frac{2\cos\theta}{\sin^3\theta}\,\psi_{\varphi\varphi} + \frac{\psi_{\theta\varphi\varphi} + \psi_{\varphi\theta\varphi} + \psi_{\varphi\varphi\theta}}{3\sin^2\theta}\right)\right].
\end{aligned}
$$

2 Calculation of conserved vectors

We calculate in this section all conserved vectors associated with the Lie point symmetries of Equation (1.1).

2.1 Symmetries

Equation (1.4) has the following symmetries (see [1]):

$$
\begin{aligned}
& X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial \varphi}, \quad X_3 = \lambda(t) \frac{\partial}{\partial \psi}, \\
& X_4 = 2R_0 \varepsilon t \frac{\partial}{\partial t} - \varepsilon t \frac{\partial}{\partial \varphi} + [(1 + 2kR_0)\cos\theta - 2\varepsilon R_0 \psi] \frac{\partial}{\partial \psi}, \\
& X_5 = \varepsilon \sin\left(\varphi + \frac{t}{2R_0}\right) \frac{\partial}{\partial \theta} + \varepsilon \frac{\cos\theta}{\sin\theta} \cos\left(\varphi + \frac{t}{2R_0}\right) \frac{\partial}{\partial \varphi} \\
& - \left(F(\theta) + \frac{\sin\theta}{2R_0}\right) \sin\left(\varphi + \frac{t}{2R_0}\right) \frac{\partial}{\partial \psi}, \\
& X_6 = \varepsilon \cos\left(\varphi + \frac{t}{2R_0}\right) \frac{\partial}{\partial \theta} - \varepsilon \frac{\cos\theta}{\sin\theta} \sin\left(\varphi + \frac{t}{2R_0}\right) \frac{\partial}{\partial \varphi} \\
& - \left(F(\theta) + \frac{\sin\theta}{2R_0}\right) \cos\left(\varphi + \frac{t}{2R_0}\right) \frac{\partial}{\partial \psi}.
\end{aligned} \tag{2.1}
$$

2.2 Conserved vector associated with X_1

Let us find the conserved vector (1.7) associated with the symmetry X_1. Here $W = -\psi_t$ and the computation gives the conserved vector with the components

$$
C^1 = \psi_\theta \psi_{t\theta} \sin\theta + \frac{1}{\sin\theta} \psi_\varphi \psi_{t\varphi}, \tag{2.2}
$$

$$
\begin{aligned}
C^2 = & - (\psi_t \psi_{t\theta} + \psi \psi_{tt\theta} + k\psi_t \psi_{\theta\varphi} + k\psi \psi_{t\theta\varphi}) \sin\theta \\
& + \varepsilon \left[\psi \psi_\varphi \psi_{t\theta\theta} - \psi \psi_t \psi_{\theta\theta\varphi} + \frac{\cos\theta}{\sin\theta} (\psi \psi_\varphi \psi_{t\theta} - \psi \psi_t \psi_{\theta\varphi}), \right. \\
& \left. + \frac{1}{\sin^2\theta} (\psi \psi_\varphi \psi_{t\varphi\varphi} - \psi \psi_t \psi_{\varphi\varphi\varphi}) \right],
\end{aligned} \tag{2.3}
$$

$$C^3 = -\frac{\sin\theta}{R_0}\,\psi\psi_t - \frac{1}{\sin\theta}\left(\psi_t\psi_{t\varphi} + \psi\psi_{tt\varphi}\right)$$
$$- k\left[(2\psi\psi_t - \psi_\theta\psi_{t\theta})\sin\theta + \frac{1}{\sin\theta}\left(\psi_t\psi_{\varphi\varphi} - \psi_\varphi\psi_{t\varphi} + \psi\psi_{t\varphi\varphi}\right)\right]$$
$$+ \varepsilon\left[\psi\psi_t\psi_{\theta\theta\theta} - \psi\psi_\theta\psi_{t\theta\theta} + \frac{1}{\sin\theta}\left(\psi\psi_t\psi_{\theta\theta} - \psi\psi_\theta\psi_{t\theta}\right)\right.$$
$$\left. + \frac{1}{\sin^2\theta}\left(\psi\psi_t\psi_{\theta\varphi\varphi} - \psi\psi_\theta\psi_{t\varphi\varphi} - \psi\psi_t\psi_\theta\right) - \frac{2\cos\theta}{\sin^3\theta}\,\psi\psi_t\psi_{\varphi\varphi}\right]. \tag{2.4}$$

It is shown in [1] that the conserved density (2.2) is trivial, namely it can be written in the divergent form

$$C^1 = -\psi\,\Lambda + D_\theta\left(\Theta\right) + D_\varphi\left(\Phi\right), \tag{2.5}$$

where Λ is defined in Equation (1.4) and

$$\Theta = \psi\psi_{t\theta}\sin\theta + k\psi\psi_{\theta\varphi}\sin\theta - \varepsilon\left(\psi\psi_\varphi\psi_{\theta\theta} + \frac{\cos\theta}{\sin\theta}\,\psi\psi_\theta\psi_\varphi - \frac{1}{2\sin^2\theta}\,\psi_\varphi^3\right),$$

$$\Phi = \frac{1}{\sin\theta}\,\psi\psi_{t\varphi} + \left(\frac{1}{2R_0} + k\right)\psi^2\sin\theta + \frac{k}{2\sin\theta}\left(2\psi\psi_{\varphi\varphi} - \psi_\varphi^2\right) - \frac{k}{2}\,\psi_\theta^2\sin\theta$$
$$+ \varepsilon\left[\psi\psi_\theta\psi_{\theta\theta} + \frac{1}{2\sin^2\theta}\left(2\psi\psi_\theta\psi_{\varphi\varphi} - 2\psi\psi_\varphi\psi_{\theta\varphi} - \psi_\theta\psi_\varphi^2\right) + \frac{\cos\theta}{\sin\theta}\,\psi\psi_\theta^2 + \frac{\cos\theta}{\sin^3\theta}\,\psi\psi_\varphi^2\right].$$

Therefore one can nullify C^1 by transferring the terms $D_\theta\left(\Theta\right)$ and $D_\varphi\left(\Phi\right)$ into C^2 and C^3, respectively. As a result, we first obtain

$$\widetilde{C}^2 = C^2 + D_t\left(\Theta\right) = D_\varphi\left(\varepsilon\,\widetilde{\Phi}\right), \tag{2.6}$$

$$\widetilde{C}^3 = C^3 + D_t\left(\Phi\right) = D_\theta\left(-\varepsilon\,\widetilde{\Phi}\right), \tag{2.7}$$

where

$$\widetilde{\Phi} = -\psi\psi_t\psi_{\theta\theta} - \frac{\cos\theta}{\sin\theta}\,\psi\psi_t\psi_\theta + \frac{1}{2\sin^2\theta}\left(2\psi\psi_\varphi\psi_{t\varphi} - 2\psi\psi_t\psi_{\varphi\varphi} + \psi_t\psi_\varphi^2\right).$$

It follows that the vector (2.2) - (2.4) is equivalent to the trivial conserved vector

$$\widetilde{C} = (-\psi\,\Lambda, 0, 0).$$

2.3 Conserved vector associated with X_2

For the symmetry X_2 from (2.1) we obtain

$$C^1 = -\psi_\varphi \left(D_\theta(\psi_\theta \sin\theta) + \frac{1}{\sin\theta}\,\psi_{\varphi\varphi} \right)$$

$$= D_\varphi \left[\frac{1}{2}\psi_\theta^2 \sin\theta - \frac{1}{2\sin\theta}\psi_\varphi^2 \right] - D_\theta(\psi_\varphi \psi_\theta \sin\theta).$$

Calculating further we see that the conserved vector has the form

$$C^1 = 0,$$
$$C^2 = D_\varphi(\Phi),$$
$$C^3 = -D_\theta(\Phi) - \psi\Lambda,$$

where

$$\Phi = \varepsilon\bigg(\frac{1}{3\sin^2\theta}({\psi_\varphi}^3 - \psi_{\varphi\varphi}\psi_\varphi\psi) - \frac{\cos\theta}{2\sin\theta}\psi\psi_\theta\psi_\varphi + \frac{1}{3}\psi\psi_\varphi\psi_{\theta\theta} + \frac{1}{3}{\psi_\theta}^2\psi_\varphi - \frac{2}{3}\psi\psi_\theta\psi_{\theta\varphi}\bigg) - \frac{k}{6}\psi\psi_\varphi\cos\theta + \frac{1}{3}(k\psi_\theta\psi_\varphi - 2k\psi\psi_{\theta\varphi} - 3\psi\psi_{t\theta})\sin\theta.$$

Hence the symmetry X_2 provides the trivial conserved vector

$$(0, 0, -\psi\Lambda).$$

2.4 Conserved vector associated with X_3

For the symmetry X_3 we obtain the following nontrivial conserved vector:

$$C^1 = 0,$$

$$C^2 = \lambda(t)\left[\varepsilon\left(-\frac{1}{\sin^2\theta}\psi_\varphi\psi_{\varphi\varphi} - \psi_\varphi\psi_{\theta\theta} - \frac{\cos\theta}{\sin\theta}\psi_\theta\psi_\varphi \right) + \psi_{t\theta}\sin\theta + k\psi_{\theta\varphi}\sin\theta \right],$$

$$C^3 = \lambda(t)\bigg[\varepsilon\left(\frac{1}{\sin^2\theta}\psi_\theta\psi_{\varphi\varphi} + \psi_\theta\psi_{\theta\theta} + \frac{\cos\theta}{\sin\theta}\psi_\theta^2 \right) + \frac{1}{\sin\theta}(\psi_{t\varphi} + k\psi_{\varphi\varphi}) + \left(\frac{1}{R_0} + 2k \right)\psi\sin\theta \bigg]. \tag{2.8}$$

For this vector the conservation equation is satisfied in the form

$$D_t(C^1) + D_\theta(C^2) + D_\varphi(C^3) = \lambda(t)\Lambda.$$

2.5 Conserved vector associated with X_4

The symmetry X_4 provides a nontrivial conserved vector. Namely,

$$C^1 = \left[(\sin\theta + 2R_0F)v_\varphi + 2\varepsilon R_0\left(v_\theta^2 + v_\varphi^2\right) + \varepsilon t R_0\left(v_\theta^2 + v_\varphi^2\right)_t\right]\sin\theta, \tag{2.9}$$

The conserved density (2.9) can be reduced to the form

$$\begin{aligned} C^1 &= (1 + 2kR_0)v_\varphi \sin^2\theta + 2\varepsilon R_0\left(v_\varphi^2 + v_\theta^2\right)\sin\theta \\ &\equiv (1 + 2kR_0)\psi_\theta \sin^2\theta + 2\varepsilon R_0\left(\psi_\theta^2 \sin\theta + \frac{1}{\sin\theta}\,\psi_\varphi^2\right) \end{aligned} \tag{2.10}$$

because

$$\varepsilon t R_0\left(v_\theta^2 + v_\varphi^2\right)_t \sin\theta = 2\varepsilon R_0\left[D_\theta\left(t\Theta\right) + D_\varphi\left(t\Phi\right) - t\psi\,\Lambda\right],$$

where Θ and Φ are the same as in Equation (2.5). For C^2, C^3 we obtain

$$\begin{aligned} C^2 = 4\varepsilon^2 R_0\psi\left(\psi_\varphi\psi_{\theta\theta} + \frac{\cos\theta}{\sin\theta}\psi_\theta\psi_\varphi + \frac{1}{\sin^2\theta}\psi_\varphi\psi_{\varphi\varphi}\right) - \varepsilon\Bigg[4R_0\psi(k\psi_{\theta\varphi} + \psi_{t\theta})\sin\theta \\ + (2R_0k + 1)\left(\frac{1}{\sin\theta}\psi_\theta\psi_\varphi + \psi_\varphi\psi_{\theta\theta}\cos\theta\right)\Bigg] + (2R_0k + 1)\psi_{t\theta}\sin\theta\cos\theta, \end{aligned}$$

$$\begin{aligned} C^3 = -4\varepsilon^2 R_0\psi\left(\psi_\theta\psi_{\theta\theta} + \frac{\cos\theta}{\sin\theta}\psi_\theta^2 + \frac{1}{\sin^2\theta}\psi_\theta\psi_{\varphi\varphi}\right) + \varepsilon\Bigg[-\frac{4R_0}{\sin\theta}(k\psi_{\varphi\varphi} + \psi_{t\varphi})\psi \\ + (2R_0k + 1)\Bigg(\frac{1}{\sin^3\theta}\psi_\varphi^2 + \psi_\theta\psi_{\theta\theta}\cos\theta + \frac{\cos\theta}{\sin^2\theta}(\psi_\theta\psi_{\varphi\varphi} - \psi_\varphi\psi_{\theta\varphi}) + \frac{1}{\sin\theta}\psi_\theta^2 \\ - 2\psi^2\sin\theta\Bigg) - \frac{1}{\sin\theta}\psi_\varphi^2 + \frac{1}{2}(2R_0k - 1)\psi_\theta^2\sin\theta\Bigg] + (2R_0k + 1)\Bigg[k\psi_\theta(1 - \sin^2\theta) \\ + \frac{\cos\theta}{\sin\theta}(k\psi_{\varphi\varphi} + \psi_{t\varphi}) + \left(\left(2k + \frac{1}{R_0}\right)\psi + k\psi_{\theta\theta}\right)\sin\theta\cos\theta\Bigg]. \end{aligned}$$

In this case the conservation equation (1.5) has the form

$$D_t(C^1) + D_\theta(C^2) + D_\varphi(C^3) = ((2kR_0 + 1)\cos\theta - 4\varepsilon R_0\psi)\Lambda.$$

2.6 Conserved vector associated with X_5

Considering X_5 from (2.1) we obtain the conserved vector

$$
\begin{aligned}
C^1 &= \left(k + \frac{1}{2R_0}\right)(\psi_\theta \sin\theta\cos\theta - \psi_{\varphi\varphi})\sin h, \qquad (2.11)\\
C^2 &= \varepsilon^2(\psi_\varphi \cos h + \psi \sin h)\left[-\frac{2\cos\theta}{\sin^3\theta}\psi_\varphi\psi_{\varphi\varphi} + \frac{1}{\sin^2\theta}(\psi_\varphi\psi_{\theta\varphi\varphi} - \psi_\theta\psi_\varphi - \psi_\theta\psi_{\varphi\varphi\varphi})\right.\\
&\left. + \frac{\cos\theta}{\sin\theta}(\psi_\varphi\psi_{\theta\theta} - \psi_\theta\psi_{\theta\varphi}) + \psi_\varphi\psi_{\theta\theta\theta} - \psi_\theta\psi_{\theta\theta\varphi}\right] + \varepsilon\left(k + \frac{1}{2R_0}\right)\left(\frac{2}{\sin\theta}\psi_\varphi\psi_{\varphi\varphi}\right.\\
&\left. + (\psi_\varphi\psi_{\theta\theta} - \psi_\theta\psi_{\theta\varphi})\sin\theta\right)\sin h - \varepsilon(\psi_\varphi\cos h + \psi\sin h)\left[\frac{1}{\sin\theta}(k\psi_{\varphi\varphi\varphi} + \psi_{t\varphi\varphi})\right.\\
&\left. + \left(\left(k + \frac{1}{2R_0}\right)2\psi_\varphi + k\psi_{\theta\theta\varphi} + \psi_{t\theta\theta}\right)\sin\theta + (k\psi_{\theta\varphi} + \psi_{t\theta})\cos\theta\right]\\
&+ \sin\theta\left(k + \frac{1}{2R_0}\right)\left[k\psi_\varphi\cos\theta\sin h - (k\psi_{\theta\varphi} + \psi_{t\theta})\sin\theta\sin h - \frac{1}{2R_0}\psi\cos\theta\cos h\right],\\
C^3 &= \frac{\varepsilon}{\sin\theta}\left(k + \frac{1}{2R_0}\right)[\psi_\theta\psi_\varphi\cos h - (\psi_\varphi\psi_{\theta\varphi} + \psi_\theta\psi_{\varphi\varphi})\sin h]\\
&+ \left(k + \frac{1}{2R_0}\right)^2\psi_\varphi\cos h + \left(k + \frac{1}{2R_0}\right)\left(\frac{1}{2R_0}\psi(1 - 2\sin^2\theta) - k\psi_{\varphi\varphi}\right)\sin h,
\end{aligned}
$$

where

$$
h = \varphi + \frac{t}{2R_0}.
$$

The conservation equation (1.5) has the form

$$
\begin{aligned}
D_t(C^1) + D_\theta(C^2) + D_\varphi(C^3) &= -\varepsilon(\psi\sin h + \psi_\varphi\cos h)D_\theta(\Lambda)\\
&- \left(\varepsilon(\psi_\theta\sin h + \psi_{\theta\varphi}\cos h) + \left(k + \frac{1}{2R_0}\right)\sin\theta\sin h\right)\Lambda.
\end{aligned}
$$

2.7 Conserved vector associated with X_6

Finally, using the symmetry X_6 we obtain the similar conserved vector:

$$C^1 = \left(k + \frac{1}{2R_0}\right)(\psi_\theta \sin\theta\cos\theta - \psi_{\varphi\varphi})\cos h, \tag{2.12}$$

$$C^2 = \varepsilon^2(\psi_\varphi \sin h - \psi\cos h)\left[\frac{2\cos\theta}{\sin^3\theta}\psi_\varphi\psi_{\varphi\varphi} + \frac{1}{\sin^2\theta}(\psi_\theta\psi_\varphi - \psi_\varphi\psi_{\theta\varphi\varphi} + \psi_\theta\psi_{\varphi\varphi\varphi})\right.$$

$$\left. + \frac{\cos\theta}{\sin\theta}(\psi_\theta\psi_{\theta\varphi} - \psi_\varphi\psi_{\theta\theta}) + \psi_\theta\psi_{\theta\theta\varphi} - \psi_\varphi\psi_{\theta\theta\theta}\right] + \varepsilon\left(k + \frac{1}{2R_0}\right)\left[\frac{2}{\sin\theta}\psi_\varphi\psi_{\varphi\varphi}\right.$$

$$\left. + (\psi_\varphi\psi_{\theta\theta} - \psi_\theta\psi_{\theta\varphi})\sin\theta\right]\cos h + \varepsilon(\psi_\varphi\sin h - \psi\cos h)\left[\frac{1}{\sin\theta}(k\psi_{\varphi\varphi\varphi} + \psi_{t\varphi\varphi})\right.$$

$$\left. + \left(\left(k + \frac{1}{2R_0}\right)2\psi_\varphi + k\psi_{\theta\theta\varphi} + \psi_{t\theta\theta}\right)\sin\theta + (k\psi_{\theta\varphi} + \psi_{t\theta})\cos\theta\right]$$

$$+ \sin\theta\left(k + \frac{1}{2R_0}\right)\left[\frac{1}{2R_0}\psi\cos\theta\sin h + k\psi_\varphi\cos\theta\cos h - (k\psi_{\theta\varphi} + \psi_{t\theta})\sin\theta\cos h\right],$$

$$C^3 = -\frac{\varepsilon}{\sin\theta}\left(k + \frac{1}{2R_0}\right)[\psi_\theta\psi_\varphi\sin h + (\psi_\varphi\psi_{\theta\varphi} + \psi_\theta\psi_{\varphi\varphi})\cos h]$$

$$- \left(k + \frac{1}{2R_0}\right)^2\psi_\varphi\sin h + \left(k + \frac{1}{2R_0}\right)\left(\frac{1}{2R_0}\psi(1 - 2\sin^2\theta) - k\psi_{\varphi\varphi}\right)\cos h,$$

where

$$h = \varphi + \frac{t}{2R_0}.$$

The conservation equation (1.5) has the form

$$D_t(C^1) + D_\theta(C^2) + D_\varphi(C^3) = \varepsilon(\psi_\varphi\sin h - \psi\cos h)D_\theta(\Lambda)$$

$$+ \left(\varepsilon(\psi_{\theta\varphi}\sin h - \psi_\theta\cos h) - \left(k + \frac{1}{2R_0}\right)\sin\theta\cos h\right)\Lambda.$$

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GROUP ANALYSIS OF ACOUSTIC WAVES IN A FLUID WITH BUBBLES

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Abstract
It is shown that two systems of differential equations describing acoustic waves in a fluid with bubbles are nonlinearly self-adjoint [1]. Using this important property local conservation laws are derived.

Keywords: Acoustic waves, fluid with bubbles, formal Lagrangian, nonlinear self-adjointness, conservation law.

1 Introduction

The equation (see [2]), [3], [4])

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -\rho\nu\frac{\partial^2 w}{\partial t^2}\,.$$

describes a nonlinear propagation of sound beams in media with gas bubbles. Here p is pressure, $w = V'/V_0$ is the relative disturbance of the equilibrium volume $V_0 = \frac{4}{3}\pi R_0^3$ of a bubble, where R_0 is the radius of the bubble in equilibrium, and V' is a variation of the volume. The constants c and ρ denote the sound velocity and the density of the water, ν is the volume concentration of air equal to the product of volume of one bubble, V_0, times the number of bubbles per unit volume of the medium, and x is the coordinate measured in the direction of wave propagation.

Vibrations of a single bubble are governed by the equation (see [4])

$$\frac{\partial^2 w}{\partial t^2} + \omega_0^2 w(1 + \epsilon_a w) = -\frac{\omega_0^2}{c_a^2 \rho_a} p, \quad \epsilon_a = \text{const.},$$

where c_a and ρ_a are the sound velocity and the density of air, respectively. The resonant frequency of the bubble is $\omega_0^2 = \frac{3c_a^2}{R_0^2}\frac{\rho_a}{\rho}$. The equations given above should be considered together. We will consider the low frequency approximation given in [4].

For low frequency cases variation of volume of the bubble can be neglected in the second equation, therefore the corresponding term can be omitted. Setting

$$\alpha = \rho\nu, \quad \beta = \omega_0^2, \quad \delta = \frac{\omega_0^2}{c_a^2 \rho_a}$$

and denoting the pressure and the coefficient of nonlinearity of the air by u and γ, respectively, one obtains the following system

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^2 w}{\partial t^2} = 0,$$

$$\beta w(1 - \gamma w) + \delta u = 0. \tag{1.1}$$

The first equation does not contain the term responsible for nonlinear effects in the water. If the nonlinearity is taken into account, one has the following modification of the system (1.1):

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^2 w}{\partial t^2} + \epsilon \frac{\partial^2 u^2}{\partial t^2} = 0,$$

$$\frac{\partial^2 w}{\partial t^2} + \beta w(1 - \gamma w) + \delta u = 0. \tag{1.2}$$

2 Investigation for nonlinear self-adjointness

Following the notation of the paper [1] and letting

$$x^1 = t, \quad x^2 = x, \quad u^1 = u, \quad u^2 = w$$

we write the system (1.1) in the form

$$F_1 \equiv u_{xx} - c^{-2} u_{tt} + \alpha w_{tt} = 0,$$

$$F_2 \equiv \beta w(1 - \gamma w) + \delta u = 0. \tag{2.1}$$

The formal Lagrangian of the system (2.1) is

$$\mathcal{L} = v^1[u_{xx} - c^{-2}u_{tt} + \alpha w_{tt}] + v^2[\beta w(1-\gamma w) + \delta u],$$

where v^1 and v^2 are new dependent variables.

The adjoint system to Equations (2.1) has the form

$$\begin{aligned} F_1^* &= \delta v^2 - c^{-2}v^1_{tt} + v^1_{xx} = 0,\\ F_2^* &= \beta v^2 - 2\beta\gamma w v^2 + \alpha v^1_{tt} = 0. \end{aligned}$$

The condition of the nonlinear self-adjointness of the the system (2.1) is written

$$\begin{aligned} F_1^* &= AF_1 + BF_2,\\ F_2^* &= CF_1 + DF_2, \end{aligned} \tag{2.2}$$

where one should substitute

$$v^1 = \varphi^1(t,x,u,w), \quad v^2 = \varphi^2(t,x,u,w)$$

together with their derivatives

$$v^1_{tt} = \varphi^1_{tt} + u_{tt}\varphi^1_u + 2u_t\varphi^1_{ut} + u_t^2\varphi^1_{uu} + 2u_tw_t\varphi^1_{uw} + w_{tt}\varphi^1_w + 2w_t\varphi^1_{wt} + w_t^2\varphi^1_{ww},$$

$$v^1_{xx} = \varphi^1_{xx} + u_{xx}\varphi^1_u + 2u_x\varphi^1_{ux} + u_x^2\varphi^1_{uu} + 2u_xw_x\varphi^1_{uw} + w_{xx}\varphi^1_w + 2w_x\varphi^1_{wx} + w_x^2\varphi^1_{ww}.$$

Solving Equations (2.2) we conclude that the system (1.1) is nonlinearly self-adjoint with the substitution

$$\begin{aligned} v^1 &= A_1tx + A_2t + A_3x + A_4,\\ v^2 &= 0. \end{aligned} \tag{2.3}$$

For the system (1.2),

$$\begin{aligned} F_1 &\equiv u_{xx} - c^{-2}u_{tt} + \alpha w_{tt} + 2\epsilon u_t^2 + 2\epsilon u u_{tt} = 0,\\ F_2 &\equiv w_{tt} + \beta w(1-\gamma w) + \delta u = 0, \end{aligned} \tag{2.4}$$

we have the following adjoint system

$$\begin{aligned} F_1^* &= \delta v^2 - c^{-2}v^1_{tt} + 2\epsilon u v^1_{tt} + v^1_{xx} = 0,\\ F_2^* &= \beta v^2 - 2\beta\gamma w v^2 + \alpha v^1_{tt} + v^2_{tt} = 0. \end{aligned} \tag{2.5}$$

The calculation shows that the system (2.4) is nonlinearly self-adjoint with v^1 and v^2 given by Equations (2.3).

3 Conservation laws

System (1.1) admits the three-dimensional Lie algebra with the basis

$$X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = \frac{\partial}{\partial x}\,, \quad X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\,\cdot \tag{3.1}$$

The system (1.2) admits only the operators X_1 and X_2. The conservation laws corresponding to these operators are calculated using nonlinear self-adjointness (detailed calculations can be found in [5]). The result is as follows.

The system (1.1) has four conserved vectors given below.

Time translation $X_1 = \frac{\partial}{\partial t}$ **gives**

$$C^1 = x[-c^{-2}u_t + \alpha w_t],$$
$$C^2 = xu_x - u;$$

$$C^1 = -c^{-2}u_t + \alpha w_t,$$
$$C^2 = u_x.$$

Spatial translation $X_2 = \frac{\partial}{\partial x}$ **gives**

$$C^1 = t[-c^{-2}u_t + \alpha w_t] - [-c^{-2}u + \alpha w],$$
$$C^2 = tu_x.$$

Dilation $X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$ **gives**

$$C^1 = tx[-c^{-2}u_t + \alpha w_t] - x[-c^{-2}u + \alpha w],$$
$$C^2 = txu_x - tu.$$

The system (1.2) has three conserved vectors:

Time translation $X_1 = \frac{\partial}{\partial t}$ **gives**

$$C^1 = x[2\epsilon u u_t - c^{-2}u_t + \alpha w_t],$$
$$C^2 = xu_x - u;$$

$$C^1 = 2\epsilon u u_t - c^{-2}u_t + \alpha w_t,$$
$$C^2 = u_x.$$

Spatial translation $X_2 = \frac{\partial}{\partial x}$ **gives**

$$
\begin{aligned}
C^1 &= t[2\epsilon u u_t - c^{-2} u_t + \alpha w_t] - [\epsilon u^2 - c^{-2} u + \alpha w], \\
C^2 &= t u_x.
\end{aligned}
\tag{3.2}
$$

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HIERARCHY OF SUBMODELS OF DIFFERENTIAL EQUATIONS

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Abstract
A system of differential equations admitting a group of transformations is considered. One can construct a hierarchy of submodels with the help of the Lie algebra L of this group. The hierarchy may be chosen so that the solutions of any submodel are the solutions of another submodel of the same hierarchy. For that one must construct an optimal system of subalgebras and produce the graph–tree of the embedded subalgebras, then calculate the differential invariants and the invariant differentiations for each subalgebra. The invariants of the algebra L will be functions of the invariants of the subalgebra. Moreover, the invariant differentiations of the algebra L will be linear combinations of the invariant differentiations of the subalgebra over the field of invariants of the same subalgebra. The comparison of representations of group solutions gives a connection between solutions of the submodels for the algebra L and the subalgebra. Examples of embedded submodels for the gasdynamic equations are given.

Key words: Differential invariant submodels, hierarchy of submodels, gas dynamics

ИЕРАРХИЯ ПОДМОДЕЛЕЙ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

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Аннотация: Рассматривается система дифференциальных уравнений допускающая группу преобразований. По алгебре Ли этой группы можно построить иерархию подмоделей. Эту иерархию можно выбрать так, что решения любой подмодели будут решениями некоторой другой подмодели этой же иерархии. Для этого надо вычислить оптимальную систему подалгебр и построить граф – дерево вложенных подалгебр, а затем вычислить дифференциальные инварианты и операторы инвариантного дифференцирования для каждой подалгебры. Инварианты надалгебры будут функциями инвариантов подалгебры. Операторы инвариантного дифференцирования надалгебры линейно выражаются через операторы инвариантного дифференцирования подалгебры над полем инвариантов подалгебры. Сравнение представлений групповых решений дает связь между решениями подмоделей надалгебры и подалгебры. Приведены примеры вложенных подмоделей для уравнений газовой динамики.

Ключевые слова: Дифференциально инвариантные подмодели, иерархия подмоделей, газовая динамика

1 Введение

Система дифференциальных уравнений, связанная с приложениями (модель), всегда допускает группу преобразований. Использование этой группы для нахождения точных решений, для классификации классов точных решений (редукции системы) занимается групповой анализ дифференциальных уравнений [1]. В абстрактном виде классификацию редукций задает оптимальная система не подобных подалгебр алгебры Ли, соответствующей допускаемой группе преобразований [2]. По каждой подалгебре можно построить подмодель, которая связывает лишь инварианты подалгебры и поэтому имеет меньшее число переменных. Если инварианты точечные, то получаются инвариантные и частично инвариантные подмодели [3]. Для подалгебры большой размерности точечных инвариантов не хватает для конструктивного построения подмодели, поэтому привлекаются базис дифференциальных инвариантов и операторы инвариантного дифференцирования для построения дифференциально инвариантных подмоделей [4].

Подмодель построенная по подалгебре допускает фактор нормализатора этой подалгебры, что позволяет строить подмодели следующего уровня. Получается иерархия подмоделей [5, 6]. Но иерархия подмоделей строится не только по нормализаторам. Если взять две подалгебры, одна есть подалгебра другой, вторая — надалгебра первой, то их инварианты можно выбрать так, что решение любой подмодели надалгебры будет решением некоторой подмодели подалгебры. Это относится к любым подмоделям: инвариантным, частично инвариантным или дифференциально инвариантным.

Такое вложение показано на примере графа–дерева вложенных подалгебр одной самонормализованной пятимерной подалгебры, допускаемой уравнениями газовой динамики.

Подмодели задаются системами уравнений, которые проще исходной потому, что содержат меньшее число переменных, но они же более сложные, так как появляются дополнительные нелинейности. Предлагаемая иерархия подмоделей позволяет находить точные решения подмоделей регулярным образом.

Кроме этого предлагаемая идеология дает возможность находить новые решения у моделей, вкладывая их в объемлющие модели (например, с большим числом независимых переменных) с расширенной группой симметрий.

Из сказанного следует, что оптимальная система нужна для построения графа вложенных подалгебр. Представить граф в целом сложно. Он представляется фрагментами, используя преобразования внутренних автоморфизмов алгебры для выделения нужного представителя из класса подобных.

2 Дифференциальные инварианты и операторы инвариантного дифференцирования

Пусть система дифференциальных уравнений E (модель) с m искомыми функциями $u \in R^m$ от n независимых переменных $x \in R^n$ допускает в смысле Ли локальную группу Ли преобразований G. Группе G соответствует алгебра Ли L операторов дифференцирования первого порядка [1]. Каждой подалгебре $H \subset L$ может соответствовать множество решений системы E, которое называется подмоделью. В зависимости от того, какие инварианты имеет подалгебра H, подмодели бывают разных типов. Инварианты могут быть точечными $J^0(x, u)$ и дифференциальными $J^i(x, u, u_1, u_2, ..., u_i)$, где u_i — производные порядка i. Для их вычисления надо продолжить базисные операторы X_α подалгебры H на производные и найти полный набор функционально независимых решений переопределенной системы линейных дифференциальных уравнений первого порядка $X_\alpha I = 0$. Существует конечный базис дифференциальных инвариантов, из которого все дифференциальные инварианты получаются действием n операторами инвариантного дифференцирования Y_j и взятия функций от них [1].

Лемма 2.1. О последовательном вычислении инвариантов подалгебры. Дифференциальные инварианты подалгебры можно вычислить последовательно, на каждом шаге вычисляя инварианты одного оператора, не приводя систему операторов к полной абелевой системе.

Доказательство. Пусть подалгебра H задается базисом $X_1, ...X_k$ (порядок операторов не важен). Уравнение $X_1 I = 0$ имеет полный набор функционально независимых интегралов (инвариантов) $I_1, ..., I_{n+m-1}$. Вводятся новые переменные $x^1, I_1, ...I_{n+m-1}$, где $X_1 x^1 \neq 0$. Базисные операторы принимают вид $X_1 = (X_1 x^1)\partial_{x^1}$, $X_2 = (X_2 x^1)\partial_{x^1} + F_{2j}(x^1, I)\partial_{I_j}$,..., где $F_{2j} = X_2 I_j$,... Уравнения для нахождения инвариантов принимают вид $I_{x^1} = 0$, $F_{2j} I_{I_j} = 0$,... Здесь x^1 — свободный параметр, от которого I не зависит. Значит, можно расщепить уравнения по переменной x^1, т.е. приравнять нулю коэффициенты при линейно независимых функциях от x^1. Получится не более $k-1$ линейно не связного уравнения. Их не может быть больше, иначе подалгебра H имела бы большую размерность. Индуктивный переход доказан. Далее надо взять один из оставшихся операторов, записанный в инвариантах первого и у него вычислить инварианты, записать другие операторы через инварианты выбранного, расщепить по свободной переменной и т. д. Если исходные операторы были линейно связные, то число уравнений, которые надо интегрировать, будет меньше k.

При непосредственном вычислении инвариантов порядок выбора операторов играет существенную роль из-за того, что интегралы-инварианты приходится находить, решая системы обыкновенных дифференциальных уравнений, но это есть некоторое искусство, и не всегда интеграл можно записать явно через элементарные функции.

Лемма 2.2. О вычислении операторов инвариантного дифференцирования. Операторы инвариантного дифференцирования (ОИД) подалгебры H могут быть найдены с помощью дифференциальных инвариантов достаточно высокого (но конечного) порядка.

Доказательство. Пусть $I_1(x)$,...,$I_{k_1}(x)$; $J^0_1(x,u)$,..., $J^0_l(x,u)$; $J^1_1(x,u,u_1)$,..., $J^1_{l_1}(x,u,u_1)$,... — полный набор дифференциальных инвариантов подалгебры H. Существует базис дифференциальных инвариантов $J^0_b(x,u)$, $J^1_b(x,u,u_1)$,..., $J^k_b(x,u,u_1,...,u_k)$, из которого все остальные инварианты получаются с помощью ОИД и функциональных операций [1,§3]. Для вычисления ОИД надо взять дифференциальные инварианты, не входящие в базис возможно меньшего порядка и представить его как действие ОИД на инвариант из базиса. Тогда для нахождения ОИД Y_j получаются тождества $Y_j J_b^0(x,u) = J^1(x,u,u_1)$, $Y_j J_b^1(x,u,u_1) = J^2(x,u,u_1,u_2),...$. Из теоремы о существовании базиса следует, что Y_j обязательно найдутся с точностью до линейной комбинации над полем инвариантов, порожденным базисом.

Лемма 2.3. Об инвариантах надалгебры. Инварианты надалгебры есть функции инвариантов подалгебры.

Доказательство. Базис операторов подалгебры дополняется до базиса операторов надалгебры. Сначала вычисляются инварианты подалгебры. В новых переменных, связанных с этими инвариантами, записываются операторы надалгебры (лемма 2.1). После расщепления по свободным переменным получим уравнения для инвариантов надалгебры, записанные только через переменные-инварианты подалгебры. Отсюда следует, что инварианты надалгебры есть функции инвариантов подалгебры.

Лемма 2.4. Об операторах инвариантого дифференцирования надалгебры. ОИД надалгебры есть линейная комбинация ОИД подалгебры над полем инвариантов подалгебры.

Доказательство. Пусть базис инвариантов подалгебры имеет порядок инвариантов не более чем k, а базис инвариантов надалгебры имеет порядок инвариантов не более чем $\bar{k} \geq k$. Тогда справедливы тождества с операторами инвариантного дифференцирования $Y_j J^{\bar{k}} = J^{\bar{k}+1}(x,u,u_1,...,u^{\bar{k}+1})$ — линейные функции производных $u^{\bar{k}+1}$, $\bar{Y}_j \bar{J}^{\bar{k}} = \bar{J}^{\bar{k}+1}(x,u,u_1,...,u^{\bar{k}+1})$ — линейные функции производных $u^{\bar{k}+1}$. По лемме 3 $\bar{J}^{\bar{k}} = \Phi(J^0,...,J^{\bar{k}})$, $\bar{J}^{\bar{k}+1} = \Psi(J^0,...,J^{\bar{k}+1})$. В силу предыдущих равенств получаем тождество $\Phi_{J^{\bar{k}}} \bar{Y}_j J^{\bar{k}} + ... = \Psi(J^0,...,J^{\bar{k}}, Y_j J^{\bar{k}})$. Отсюда следует, что функция Ψ линейна по $J^{\bar{k}+1}$, а коэффициенты этой зависимости будут функциями инвариантов подалгебры по лемме 3. Значит, операторы $\bar{Y}_j$ будут линейными комбинациями операторов Y_j над полем инвариантов подалгебры.

3 Дифференциально инвариантные подмодели

Базис дифференциальных инвариантов подалгебры H представляется в виде: $I(x) = (I_1,...,I_k)$ — функционально независимые инварианты, зависящие от x; $J^0 = (J_1^0,...,J_{l_0}^0)$ — функционально независимые инварианты, зависящие от x, u; $J^i = (J_1^i,...,J_{l_i}^i)$, $i = 1,...,p$ — функционально независимые инварианты i - го порядка, зависящие от $x,u,u_1,...,u_i$ (сюда не входят инварианты, полученные из инвариантов меньшего порядка действием операторами инвариантного дифференцирования). Инварианты I, J^0 — инварианты нулевого порядка. Для любой подалгебры число p конечно [1]. ОИД Y_j, $j = 1, ..., n$ задают все возможные инварианты порядка p: $Y_{j_1}...Y_{j_p}J^0$, $Y_{j_1}...Y_{j_{p-1}}J^1, ..., J^p$ (здесь p — любое неотрицательное целое число).

По теореме о представлении инвариантного многообразия [1] система E записывается через дифференциальные инварианты, тем самым определяются независимые дифференциальные инварианты базиса.

Определение 3.1. [4]. Дифференциально инвариантной подмоделью (ДИП) ранга $r + r_1$ называется представление системы E как $(r + r_1)$ - мерного многообразия в пространстве всех дифференциальных инвариантов, проекция которого в пространство инвариантов нулевого порядка есть r - мерное многообразие.

Если $r + r_1 = k$, то происходит редукция к инвариантным подмоделям.

Если $r + r_1 > k$, то возникает переопределенная система уравнений, которую можно разбить на две подсистемы. Первая подсистема получается после действия ОИД Y_j на $r + r_1 - k$ независимых дифференциальных инварианта базиса и приравнивания результата новым функциям ψ от $r + r_1$ переменных-инвариантов. Так как операторы Y_j образуют алгебру Ли дифференцирований со структурными постоянными, зависящими от базисных инвариантов, то условия совместности задают инволютивную переопределенную систему на функции ψ. Общее решение этой системы задает представление решения исходной системы E.

Вторая подсистема определяется тем, что некоторые дифференциальные инварианты связаны исходной системой E. После подстановки представления решения и изучения совместности, получается дифференциально инвариантная подмодель.

Если $r = k$ (инварианты J^0 зависят только от I), а r_1 равно числу независимых дифференциальных инвариантов, то получается регулярная частично инвариантная подмодель (РЧИП). Нерегулярная частично инвариантная подмодель (НЧИП) получается, когда $J^0 = (J^0)' \cup (J^0)''$ и инварианты $(J^0)''$ зависят от $I, (J^0)'$, а r_1 равно числу независимых дифференциальных инвариантов порядка большего нуля. Число $r - k$ есть дефект инвариантности [3]. Если у представления $J^0(I)$ определяются все m функций, то получается инвариантная подмодель (ИП).

Таким образом, ДИП может быть подмоделью РЧИП или НЧИП, когда дифференциальные инварианты порядка большего нуля есть функции общего вида, т.е. зависят от n переменных.

4 Основная теорема

Теорема 4.1. О вложении подмодели надалгебры в подмодель подалгебры. Пусть подалгебра вложена в надалгебру большей размерности. Любая ДИП надалгебры задает семейство точных решений некоторой ДИП подалгебры. Для определения точных решений ДИП подалгебры надо получить представление решения ДИП подалгебры из представления решения ДИП надалгебры.

Доказательство. Пусть $H \subset \bar{H}$, H — подалгебра, $\bar{H}$ — надалгебра. Дифференциальные инварианты подалгебры H: $I(x)$, $J^0(x,u)$, $J^i(x,u,u_1,...,u_i)$, $i=1,...,p$; операторы инвариантного дифференцирования: Y_j, $j=1,...,n$. В силу лемм 1–4 дифференциальные инварианты надалгебры $\bar{H}$ представляются в виде $\bar{I}(I)$, $\bar{J}^0(I,J^0)$, $\bar{J}^i(I,J^0,...,J^i)$; а операторы инвариантного дифференцирования - в виде $\bar{Y}_j=\sum_{i=1}^n g_{ji}(I,J^0,J^i)Y_i$. Решение ДИП ранга $\bar{r}+\bar{r}_1$ надалгебры имеет параметрическое представление вида $\bar{\alpha}=(\bar{\alpha}',\bar{\alpha}'')\in R^{\bar{r}+\bar{r}_1}$, $\alpha'=(\bar{I},(\bar{J}^0)')$, $\bar{J}^0=(\bar{J}^0)'\cup(\bar{J}^0)''$, $\dim(\bar{I}\cup(\bar{J}^0)')=\bar{r}$, $(\bar{J}^0)''=\bar{\Psi}(\bar{\alpha}')$, $\bar{J}^i=\bar{\xi}(\bar{\alpha})$. Это представление можно записать через инварианты подалгебры H: $(\bar{J}^0)''(I,J^0)=\bar{\Psi}(\bar{I}(I),(\bar{J}^0)'(I,J^0))$, $\bar{J}^i(I,J^0,J^i)=\bar{\xi}(\bar{I}(I),(\bar{J}^0)'(I,J^0),$ $\bar{\alpha}''(I,J^0,J^i))$. Отсюда определяется представление решения ДИП подалгебры H: $(J^0)''=\Psi(I,(J^0)')=\Psi(\alpha')$, $\dim(I,(J^0)')=r\geq\bar{r}$, $J^i=\xi(I,(J^0)',\alpha'')$, $\dim(\alpha'')=r_1\geq\bar{r}_1$. Так как ОИД надалгебры линейно выражаются через ОИД подалгебры, то новые инвариантные функции $\bar{\Psi}$ для надалгебры и инвариантные функции для подалгебры Ψ удовлетворяют системам одного типа (условие алгебры для ОИД), но с разным числом независимых переменных. Для надалгебры число переменных должно быть меньше. Итак, по представлению решения ДИП надалгебры определяется представление решения ДИП подалгебры и следовательно решения первой подмодели могут быть решениями второй подмодели.

Построение вложенной подмодели по надалгебре отличается от построения вложенной подмодели по фактору нормализатора. Переменные–инварианты подмоделей должны быть согласованы, а именно, переменные–инварианты надалгебры должны быть функциями переменных–инвариантов подалгебры; групповой природы здесь нет. Для нормализатора такое соответствие получается за счет индуцированного действия фактора нормализатора на подмоделе подалгебры (групповая основа).

5 Примеры построения вложенных подмоделей

Для построения иерархии подмоделей надо сначала построить оптимальную систему неподобных подалгебр алгебры Ли [2], далее представить граф–

дерево вложенных друг в друга подалгебр с учетом подобия в классе подобных подалгебр. Для каждой ветки дерева можно построить вложенные подмодели. Возможность такого вложения для инвариантных подмоделей и только по нормализатору подалгебры была доказана в работе [5], а для частично инвариантных подмоделей в работе [6]. Теорема 4.1 утверждает большее: например, существование вложенных подмоделей для подмоделей самонормализованной подалгебры.

Групповой анализ наиболее продвинут для уравнений газовой динамики (УГД) [7]. На примере этой модели с общим уравнением состояния строится иерархия подмоделей на графе вложенных подалгебр самонормализованной пятимерной подалгебры из оптимальной системы работы [8].

Модель УГД записывается в координатах, связанных с декартовыми по формулам $y = r\cos\theta$, $z = r\sin\theta$, $v = q\cos\sigma$, $w = q\sin\sigma$:

$$u_t + uu_x + qY_1u + \rho^{-1}p_x = 0,$$

$$q_t + uq_x + qY_1q + \rho^{-1}Y_1p = 0,$$
$$q(\sigma_t + u\sigma_x + qY_1\sigma) + \rho^{-1}Y_2p = 0,$$
$$\rho_t + u\rho_x + qY_1\rho + \rho(u_x + Y_1q + qY_2\sigma) = 0,$$
$$S_t + uS_x + qY_1S = 0,$$

где x, y, z — декартовы координаты частицы; u, v, w — декартовы координаты скорости частицы, давление определяется из уравнения состояния $p = f(\rho, S)$, ρ —плотность, S — энтропия, $Y_1 = \cos\sigma D_y + \sin\sigma D_z = \cos(\sigma-\theta)D_r + r^{-1}\sin(\sigma-\theta)D_\theta$, $Y_2 = -\sin\sigma D_y + \cos\sigma D_z = -\sin(\sigma-\theta)D_r + r^{-1}\cos(\sigma-\theta)D_\theta$. Модель УГД допускает 11–мерную алгебру Ли [8]. Рассматривается пятимерная самонормализованная подалгебра с базисом (обозначения взяты из [8]):

$$X_2 = \partial_y = \cos\theta\partial_r - r^{-1}\sin\theta\partial_\theta, \quad X_3 = \partial_z = \sin\theta\partial_r + r^{-1}\cos\theta\partial_\theta,$$

$$X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v = \partial_\theta + \partial_\sigma, \quad X_{10} = \partial_t,$$
$$X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z = t\partial_t + x\partial_x + r\partial_r.$$

Оптимальная система [8] позволяет построить граф–дерево вложенных подалгебр (операторы изображены своими индексами, параметр $a \neq 0$) (см. Рис. 5.1).

Дифференциальные инварианты (ДИ) и операторы инвариантного дифференцирования (ОИД) для подалгебр из графа–дерева вычисляются согласно леммам 2.1–2.4. Результаты вычислений сведены в таблицу, в которой сначала записаны базисные операторы подалгебры с помощью своих номеров,

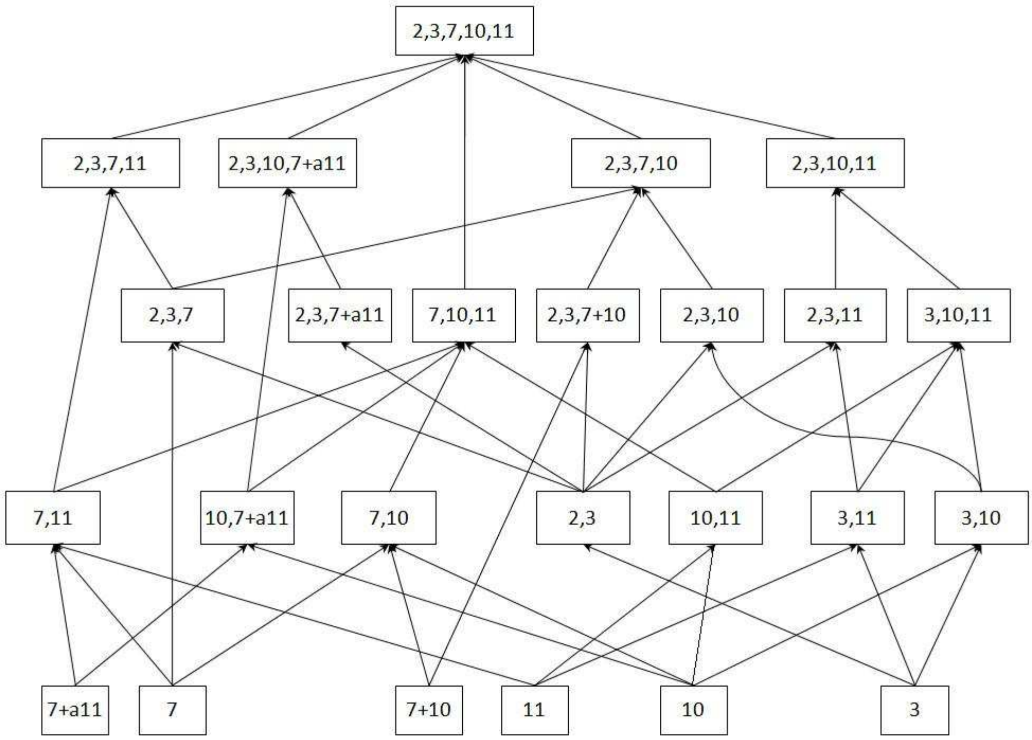

Рис. 5.1: Граф-дерево вложенных подалгебр

затем ДИ (общие для всех подалгебр инварианты ρ, S не записаны), наконец выписана ОИД.

$$7; \quad t, x, r, u, q, \sigma - \theta; \quad D_t, D_x, D_r, D_\theta.$$

$$10; \quad x, y, z, u, v, w; \quad D_t, D_x, D_y, D_z.$$

$$7 + 10; \quad x, r, \theta - t, u, q, \sigma - \theta; \quad D_t, D_x, Y_1, Y_2.$$

$$7 + a11; \quad xt^{-1}, rt^{-1}, \theta - a^{-1}\ln|t|, u, q, \sigma - a^{-1}\ln|t|; \quad tD_t, tD_x, tD_r, D_\theta.$$

$$11; \quad xt^{-1}, yt^{-1}, zt^{-1}, u, v, w; \quad tD_t, tD_x, tD_y, tD_z.$$

$$3; \quad t, x, y, u, v, w; \quad D_t, D_x, D_y, D_z.$$

$$3, 10; \quad x, y, u, v, w; \quad D_t, D_x, D_y, D_z.$$

$$3, 11; \quad xt^{-1}, yt^{-1}, u, v, w; \quad tD_t, tD_x, tD_y, tD_z.$$

$$10, 11; \quad xr^{-1}, \theta, u, q, \sigma - \theta; \quad rD_t, rD_x, rD_r, D_\theta.$$

$$7, 10; \quad x, r, , u, q, \sigma - \theta; \quad D_t, D_x, D_r, D_\theta.$$

$$7, 11; \quad xt^{-1}, rt^{-1}, u, q, \sigma - \theta; \quad tD_t, tD_x, tD_r, D_\theta.$$

$$10, 7+a11; \quad xe^{-a\theta}, re^{-a\theta}, u, q, \sigma-\theta; \quad rD_t, rD_x, rD_r, D_\theta.$$

$$2, 3, 10; \quad x, u, v, w; \quad D_t, D_x, D_y, D_z.$$

$$2, 3, 11; \quad xt^{-1}, u, v, w; \quad tD_t, tD_x, tD_y, tD_z.$$

$$3, 10, 11; \quad xy^{-1}, u, v, w; \quad yD_t, yD_x, yD_y, yD_z.$$

$$2, 3, 7; \quad t, x, u, q, \sigma_t, \sigma_x, Y_1\sigma, Y_2\sigma; \quad D_t, D_x, Y_1, Y_2.$$

$$2, 3, 7+a11; \quad xt^{-1}, u, q, \sigma - a^{-1}\ln|t|; \quad tD_t, tD_x, tD_r, D_\theta.$$

$$7, 10, 11; \quad xr^{-1}, u, q, \sigma-\theta; \quad rD_t, rD_x, rD_r, D_\theta.$$

$$2, 3, 7+10; \quad x, u, q, \sigma - t; \quad D_t, D_x, Y_1, Y_2.$$

$$2, 3, 7, 10; \quad x, u, q, \sigma_t, \sigma_x, Y_1\sigma, Y_2\sigma; \quad D_t, D_x, Y_1, Y_2.$$

$$2, 3, 7, 11; \quad xt^{-1}, u, q, \sigma_t, \sigma_x, Y_1\sigma, Y_2\sigma; \quad D_t, D_x, Y_1, Y_2.$$

$$2, 3, 10, 11; \quad u, v, w; \quad D_t, D_x, D_y, D_z.$$

$$2, 3, 10, 7+a11; \quad u, q, \sigma - a^{-1}\ln|x|; \quad xD_t, xD_x, xD_r, D_\theta.$$

$$2, 3, 7, 10, 11; \quad u, q, \sigma_t, \sigma_x, Y_1\sigma, Y_2\sigma; \quad D_t, D_x, Y_1, Y_2.$$

Рассматриваются подмодели (ИП, ЧИП, ДИП) подалгебр из графа-дерева.

Для подалгебры 7 представление инвариантного решения таково: величины $\vartheta = \sigma - \theta$, u, q, ρ, S зависят от t, x, r. УГД дают подмодель ИП(7) вращательных симметричных движений

$$u_t + uu_x + q\cos\vartheta u_r + \rho^{-1}p_x = 0,$$

$$q_t + uq_x + q\cos\vartheta q_r + \rho^{-1}\cos\vartheta p_r = 0,$$

$$\rho q[\vartheta_t + u\vartheta_x + q(\cos\vartheta\vartheta_r - r^{-1}\sin\vartheta)] - \sin\vartheta p_r = 0,$$

$$\rho_t + u\rho_x + q\cos\vartheta\rho_r + \rho[u_x + \cos\vartheta q_r + q(-\sin\vartheta\vartheta_r + r^{-1}\cos\vartheta)] = 0,$$

$$S_t + uS_x + q\cos\vartheta S_r = 0.$$

При $\vartheta = 0$ получается подмодель осесимметричных движений.

Для подалгебры 10 получается подмодель ИП(10) стационарных течений, в которой все функции не завися от времени t.

Подалгебра 7+10 задает ИП(7+10) вращательных движений с представлением решения: функции u, q, $\vartheta = \sigma - \theta$, ρ, S зависят от x, r, $\phi = \theta - t$;

$$-u_\phi + uu_x + qY_1u + \rho^{-1}p_x = 0,$$

$$-q_\phi + uq_x + qY_1q + \rho^{-1}Y_1p = 0,$$

$$\rho q[-\vartheta_\phi + u\vartheta_x + q(Y_1\vartheta + r^{-1}\sin\vartheta)] + \rho^{-1}Y_2p = 0,$$

$$-\rho_\phi + u\rho_x + qY_1\rho + \rho[u_x + Y_1 q + q(Y_2\vartheta + r^{-1}\cos\vartheta)] = 0,$$
$$-S_\phi + uS_x + qY_1 S = 0,$$

где $Y_1 = \cos\vartheta\partial_r + r^{-1}\sin\vartheta\partial_\phi$, $Y_2 = -\sin\vartheta\partial_r + r^{-1}\cos\vartheta\partial_\phi$.

Подалгебра 7+a11, $a \neq 0$, задает ИП(7+a11) с коническими спиральными линиями уровня с представлением решения: функции u, q, $\vartheta = \sigma - \theta$, ρ, S зависят от $x_1 = xt^{-1}$, $r_1 = rt^{-1}$, $\theta_1 = \theta - a^{-1}\ln|t|$;

$$(u - x_1)u_{x_1} - a^{-1}u_{\theta_1} + qY_1 u + \rho^{-1}p_{x_1} = 0,$$
$$(u - x_1)q_{x_1} - a^{-1}q_{\theta_1} + qY_1 q + \rho^{-1}Y_1 p = 0,$$
$$\rho q[(u - x_1)\vartheta_{x_1} - a^{-1}\vartheta_{\theta_1} + q(Y_1\vartheta + r_1^{-1}\sin\vartheta)] + Y_2 p = 0,$$
$$(u - x_1)\rho_{x_1} - a^{-1}\rho_{\theta_1} + qY_1\rho + \rho[u_{x_1} + Y_1 q + q(Y_2\vartheta + r_1^{-1}\cos\vartheta)] = 0,$$
$$(u - x_1)S_{x_1} - a^{-1}S_{\theta_1} + qY_1 S = 0,$$

где $Y_1 = \cos\vartheta\partial_{r_1} + r_1^{-1}\sin\vartheta\partial_{\theta_1}$, $Y_2 = -\sin\vartheta\partial_{r_1} + r_1^{-1}\cos\vartheta\partial_{\theta_1}$.

Подалгебра 11 задает ИП(11) конических движений с представлением решения: функции u, q, σ, ρ, S зависят от $x_1 = xt^{-1}$, $r_1 = rt^{-1}$, θ;

$$(u - x_1)u_{x_1} - r_1 u_{r_1} + qY_1 u + \rho^{-1}p_{x_1} = 0,$$
$$(u - x_1)q_{x_1} - r_1 q_{r_1} + qY_1 q + \rho^{-1}Y_1 p = 0,$$
$$\rho q[(u - x_1)\sigma_{x_1} - r_1\sigma_{r_1} + qY_1\sigma] + Y_2 p = 0,$$
$$(u - x_1)\rho_{x_1} - r_1\rho_{r_1} + qY_1\rho + \rho(u_{x_1} + Y_1 q + qY_2\sigma) = 0,$$
$$(u - x_1)S_{x_1} - r_1 S_{r_1} + qY_1 S = 0,$$

где $Y_1 = \cos(\vartheta - \theta)\partial_{r_1} + r_1^{-1}\sin(\vartheta - \theta)\partial_\theta$, $Y_2 = -\sin(\vartheta - \theta)\partial_{r_1} + r_1^{-1}\cos(\vartheta - \theta)\partial_\theta$.

Подалгебра 3 задает ИП(3) плоских движений с представлением решения: функции u, q, σ, ρ, S зависят от t, x, y;

$$u_t + uu_x + q\cos\sigma u_y + \rho^{-1}p_x = 0,$$
$$q_t + uq_x + q\cos\sigma q_y + \rho^{-1}\cos\sigma p_y = 0,$$
$$\rho q(\sigma_t + u\sigma_x + q\cos\sigma\sigma_y) - \rho^{-1}\sin\sigma p_y = 0,$$
$$\rho_t + u\rho_x + q\cos\sigma\rho_y + \rho(u_x + \cos\sigma q_y - q\sin\sigma\sigma_y) = 0,$$
$$S_t + uS_x + q\cos\sigma S_y = 0.$$

Полученные инвариантные подмодели имеют множество точных решений, удовлетворяющих более простым системам уравнений. Для их получения рассматривается надалгебра той подалгебры, по который строилась подмодель, и сравниваются представления решений подалгебры и надалгебры.

Подалгебра 3,10 есть надалгеброй подалгебр 3 и 10. Представление инвариантного решения надалгебры: функции u, q, σ, ρ, S зависят от x, y. ИП(3,10) плоских установившихся движений очевидно задает точные решения ИП(10) стационарных течений и ИП(3) плоских движений.

Подалгебра 3,11 есть надалгебра подалгебр 3 и 11. Представление инвариантного решения надалгебры: функции u, q, σ, ρ, S зависят от $x_1 = xt^{-1}$, $y_1 = yt^{-1}$. Так как подалгебра 3 - идеал надалгебры 3,11, то ИП(3) допускает оператор X_{11} [1] и поэтому ИП(3,11) есть подмодель ИП(3) [5]:

$$(u - x_1)u_{x_1} - y_1 u_{y_1} + q\cos\sigma u_{y_1} + \rho^{-1} p_{x_1} = 0,$$

$$(u - x_1)q_{x_1} - y_1 q_{y_1} + q\cos\sigma q_{y_1} + \rho^{-1}\cos\sigma p_{y_1} = 0,$$
$$\rho q[(u - x_1)\sigma_{x_1} - y_1\sigma_{y_1} + q\cos\sigma\sigma_{y_1}) - \rho^{-1}\sin\sigma p_{y_1} = 0,$$
$$(u - x_1)\rho_{x_1} - y_1\rho_{y_1} + q\cos\sigma\rho_{y_1} + \rho(u_{x_1} + \cos\sigma q_{y_1} - q\sin\sigma\sigma_{y_1}) = 0,$$
$$(u - x_1)S_{x_1} - y_1 S_{y_1} + q\cos\sigma S_{y_1} = 0.$$

ИП(3,11) задает так же точные решения ИП(11), если записать ее в переменных $y_1 = yt^{-1}$, $z_1 = zt^{-1}$.

Подалгебра 10,11 есть надалгебра подалгебры 11 и идеала 10. Представление инвариантного решения надалгебры: функции u, q, σ, ρ, S зависят от $x_2 = xr^{-1}$, θ. Стационарная ИП(10) допускает оператор X_{11}, и поэтому ИП(10,11) есть подмодель ИП(10):

$$uu_{x_2} + qY_1 u + \rho^{-1} p_{x_2} = 0,$$

$$uq_{x_2} + qY_1 q + \rho^{-1} Y_1 p = 0,$$
$$\rho q(u\sigma_{x_2} + qY_1\sigma) + Y_2 p = 0,$$
$$u\rho_{x_2} + qY_1\rho + \rho(u_{x_2} + Y_1 q + qY_2\sigma) = 0,$$
$$uS_{x_2} + qY_1 S = 0,$$

где $Y_1 = -x_2\cos(\sigma - \theta)\partial_{x_2} + \sin(\sigma - \theta)\partial_\theta$, $Y_2 = x_2\sin(\sigma - \theta)\partial_{x_2} + \cos(\sigma - \theta)\partial_\theta$.

ИП(10,11) задает так же точные решения ИП(11), если сделать замену переменных $x_2 = x_1 r_1^{-1}$, которая следует из сравнения инвариантов.

Абелева подалгебра 7,10 есть надалгебра идеалов 7, 7+10, 10 и поэтому ИП(7,10) с представлением решения: u, q, $\vartheta = \sigma - \theta$, ρ, S — функции переменных x, r,

$$uu_x + q\cos\vartheta u_r + \rho^{-1} p_x = 0,$$
$$uq_x + q\cos\vartheta q_r + \rho^{-1}\cos\vartheta p_r = 0,$$

$$\rho q(u\vartheta_x + q\cos\vartheta\vartheta_r) - \sin\vartheta p_r = 0,$$
$$u\rho_x + q\cos\vartheta\rho_r + \rho(u_x + \cos\vartheta q_r - q\sin\vartheta\vartheta_r) = 0,$$
$$uS_x + q\cos\vartheta S_r = 0$$

является подмоделью для ИП(7), ИП(7+10), ИП(10).

Абелева подалгебра 7,11 есть надалгебра идеалов 7, $7 + a11$, 11 задает представление: u, q, $\vartheta = \sigma - \theta$, ρ, S — функции переменных $x_1 = xt^{-1}$, $r_1 = rt^{-1}$. ИП(7,11) есть подмодель для ИП(7), ИП(7+10), ИП(11):

$$(u - x_1)u_{x_1} - r_1 u_{r_1} + q\cos\vartheta u_{r_1} + \rho^{-1}p_{x_1} = 0,$$
$$(u - x_1)q_{x_1} - r_1 q_{r_1} + q\cos\vartheta q_{r_1} + \rho^{-1}\cos\vartheta p_{r_1} = 0,$$
$$\rho q[(u - x_1)\vartheta_{x_1} - r_1\vartheta_{r_1} + q(\cos\vartheta\vartheta_{r_1} + r_1^{-1}\sin\vartheta)] - \sin\vartheta p_{r_1} = 0,$$
$$(u - x_1)\rho_x - r_1\rho_{r_1} + q\cos\vartheta\rho_{r_1} + \rho(u_{x_1} + \cos\vartheta q_{r_1} - q\sin\vartheta\vartheta_{r_1} + r_1^{-1}q\cos\vartheta) = 0,$$
$$(u - x_1)S_{x_1} - r_1 S_{r_1} + q\cos\vartheta S_{r_1} = 0.$$

Подалгебра $10, 7 + a11$ есть надалгебра своего идеала 10 и подалгебры $7 + a11$. Представление инвариантного решения: u, q, $\vartheta = \sigma - \theta$, ρ, S — функции переменных $x_2 = xe^{-a\theta}$, $r_2 = re^{-a\theta}$ задает ИП$(10, 7 + a11)$

$$uu_{x_2} + q\bar{Y}_1 u + \rho^{-1}p_{x_2} = 0,$$
$$uq_{x_2} + q\bar{Y}_1 q + \rho^{-1}\bar{Y}_1 p = 0,$$
$$\rho q[u\vartheta_{x_2} + q(\bar{Y}_1\vartheta + r_2^{-1}\sin\vartheta)] + \bar{Y}_2 p = 0,$$
$$u\rho_{x_2} + q\bar{Y}_1\rho + \rho[u_{x_2} + \bar{Y}_1 q + q(\bar{Y}_2\vartheta + r_2^{-1}\cos\vartheta)] = 0,$$
$$uS_{x_2} + q\bar{Y}_1 S = 0,$$

где $\bar{Y}_1 = \cos\vartheta\partial_{r_2} - ar_2^{-1}\sin\vartheta(x_2\partial_{x_2} + r_2\partial_{r_2})$, $\bar{Y}_2 = -\sin\vartheta\partial_{r_2} - ar_2^{-1}\cos\vartheta(x_2\partial_{x_2} + r_2\partial_{r_2})$, которая есть подмодель для ИП(10) и задает точные решения подмодели ИП$(7 + a11)$ после замены $x_2 = x_1 e^{-a\theta_1}$, $r_2 = r_1 e^{-a\theta_1}$. Замена получена из сравнения инвариантов надалгебры и подалгебры.

Подалгебра 2,3 есть надалгебра идеала 3, поэтому ИП(2,3) одномерных движений есть подмодель ИП(3) плоских движений.

Для трехмерных подалгебр рассматриваются только собственные подалгебры, не являющиеся идеалами.

Подалгебра 2,3,7 есть надалгебра собственных подалгебр 7; 3. Инвариантных подмоделей для надалгебры нет. Можно рассмотреть регулярную частично инвариантную подмодель ранга 2 дефекта 1 с представлением решения: u, q, ρ, S — функции переменных t, x; σ — функция общего вида.

РИЧП(2,3,7) задается переопределенной системой уравнений, которая приведена в инволюцию:

$$u_t + uu_x + \rho^{-1}p_x = 0, \quad q_t + uq_x = 0, \quad S_t + uS_x = 0,$$

$$\rho_t + u\rho_x + \rho(u_x + kq) = 0, \quad k_t + uk_x + qk^2 = 0,$$

$$-\sin\sigma\sigma_y + \cos\sigma\sigma_z = k(t,x), \quad \sigma_t + u\sigma_x + q(\cos\sigma\sigma_y + \sin\sigma\sigma_z) = 0.$$

Общее решение этой системы определяются равенствами

$$u = x_t, \quad S = S(\xi), \quad q = q(\xi), \quad k^{-1} = tq, \quad \rho tqx_\xi = n(\xi),$$

$$p = f(\rho, S), \quad nx_{tt} + tqp_\xi = 0,$$

$$F(\xi, y - tq\cos\sigma, z - tq\sin\sigma) = 0,$$

где S, q, n, F — произвольные функции, $x = x(t,\xi)$ — обратная функция к лагранжевой замене $\xi = \xi(t,x) : \xi_t + u\xi_x = 0$.

Если рассмотреть РЧИП ранга 3 дефекта 1 для подалгебры 7 или 3 с функцией σ общего вида, то, очевидно, решения РЧИП(2,3,7) будут точными решениями этих подмоделей. При переходе к подалгебре ранг увеличивается дефект остается прежним. Среди решений РЧИП(2,3,7) найдутся решения ИП(3) или ИП(7), для этого функцию σ надо представить с помощью соответствующих инвариантов подалгебры 3 или 7. Таким образом, при переходе к подалгебре ранг увеличивается, а дефект для некоторых решений уменьшается [1].

Аналогичные рассуждения могут быть проведены для НЧИП или ДИП, которые вкладываются в ЧИП.

Подалгебра 3,10,11 есть надалгебра собственных подалгебр 3,10; 10,11; 3,11. ИП надалгебры с представлением: $\vec{u}$, ρ, S — функции переменной $s = xy^{-1}$, задается системой обыкновенных дифференциальных уравнений

$$(u - sv)u_s + \rho^{-1}p_s = 0, \quad (u - sv)(v_s + su_s) = 0, \quad (u - sv)w_s = 0,$$

$$(u - sv)S_s = 0, \quad (u - sv)\rho^{-1}\rho_s + u_s - sv_s = 0.$$

Система имеет два типа решений. Если $u = sv$, то получается изобарическое сдвиговое по оси z, центрированное на оси z течение: $u = v = 0$, $p = p_0 = f(\rho(s), S(s))$; где $w(s)$, $\rho(s)$ — произвольные функции.

Если $u \neq sv$, то получается либо покой, либо простая изэнтропическая волна: $S = S_0$, $w = w_0$, $u = m\cos\psi$, $v = m\sin\psi$, $\psi = C - \int m'\sqrt{a^{-2} - m^{-2}}d\rho$, $m(\cos\psi - s\sin\psi) = a\sqrt{1 + s^2}$, где $m^2 = B - \int a^2\rho^{-1}d\rho$, $a^2 = f_\rho$ — квадрат скорости звука; S_0, w_0, B, C — постоянные.

Полученные решения являются точными решениями для ИП(3,10), для ИП(3,11) с заменой $s = x_1 y_1^{-1}$, для ИП(10,11) с заменой $x_2 = s\cos\theta$.

НЧИП(3,10,11) ранга 2 дефекта 1 имеет представление: u, v, w, ρ, S — функции переменной $s = xy^{-1}$ и параметра $\alpha = \alpha(t, x, y, z)$, который может быть любой комбинацией инвариантных функций;

$$S_\alpha D\alpha + y^{-1} S_s (u - sv) = 0,$$

$$\vec{u}_\alpha D\alpha + \rho^{-1} p_\alpha \nabla\alpha + y^{-1}[\vec{u}_s(u - sv) + \rho^{-1} p_s(1, -s, 0)] = 0,$$
$$\rho_\alpha D\alpha + \rho \vec{u}_\alpha \cdot \nabla\alpha + y^{-1}[\rho_s(u - sv) + \rho(u_s - sv_s)] = 0,$$

где $D_\alpha = \alpha_t + \vec{u} \cdot \nabla\alpha$. Эта система переопределена. Если из нее определяются величины $D\alpha$ и $\nabla\alpha$, то подмодель редуцируется к ИП некоторой подалгебры [1].

Если рассмотреть НЧИП ранга 3 дефекта 1 для подалгебр, то решения НЧИП(3,10,11) ранга 2 дефекта 1 являются их решениями.

Надалгебра 2,3,10,11 порождает точные решения НЧИП(3,10,11) ранга 2 дефекта 1, если рассмотреть НЧИП(2,3,10,11) ранга 1 дефекта 1. Это будут решения НЧИП(3,10,11) ранга 2 дефекта 1, независящие от s, которые задают изэнтропическую простую волну.

НЧИП(2,3,7,10,11) ранга 0 дефекта 1 дает решения НЧИП(2,3,10,11) ранга 1 дефекта 2, решения которой дают решения НЧИП(3,10,11) ранга 2 дефекта 2 и т.д.

Итак, сравнение представления решений надалгебры с представлением решений подалгебры дает связь между решениями соответствующих подмоделей.

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