Group analysis of evolutionary integro-differential equations describing nonlinear waves: General model

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Abstract. The paper deals with an evolutionary integro-differential equation describing nonlinear waves. Particular choice of the kernel in the integral leads to well-known equations such as the Khokhlov-Zabolotskaya equation, the Kadomtsev-Petviashvili equation and others. Since solutions of these equations describe many physical phenomena, analysis of the general model studied in the paper equation is important. One of the methods for obtaining solutions differential equations is provided by the Lie group analysis. However, this method is not applicable to integro-differential equations. Therefore we discuss new approaches developed in modern group analysis and apply them to the general model considered in the present paper. Reduced equations and exact solutions are also presented.

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1. Introduction

One of the most general evolution equations used in nonlinear wave physics is the following one [1, 2]:

$$\begin{aligned} (u_x - uu_t - w_{tt})_t &= u_{yy} + u_{zz}, \\ w &= \int_0^\infty K(s) u(t-s) ds \end{aligned} \quad (1)$$

Here the variable t is the time, and x, y, z are the spatial Cartesian coordinates. The coordinate x is distinguished as a "longitudinal" one. It coincides with a preferred orientation of the wave propagation. Other coordinates y, z are identified as "transversal" ones. They are commonly introduced in the cross-section of a wave beam.

Special cases of the equation (1) are well-known. In particular, if the kernel is identically zero, $K(s) \equiv 0$, the general equation (1) is reduced to the Khokhlov-Zabolotskaya (KZ) equation [3, 4], describing wave beams in nonlinear media:

$$(u_x - uu_t)_t = u_{yy} + u_{zz}. \quad (2)$$

If the kernel is the delta-function, $K = 2\delta(s)$, the model (1) leads to the equation

$$(u_x - uu_t - u_{tt})_t = u_{yy} + u_{zz} \quad (3)$$

for nonlinear beams in a dissipative medium [5, 6]. Equation (3) is known as the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation. It is widely used in underwater acoustics for engineering design of parametric radiating and receiving arrays [6].

If the kernel is proportional to the derivative of the delta-function, $K = 2\delta'(s)$, the integro-differential equation (1) becomes the Kadomtsev-Petviashvili (KP) equation

$$(u_x - uu_t - u_{ttt})_t = u_{yy} + u_{zz} \quad (4)$$

for nonlinear beams in a dispersive medium [7, 8]. The similar equation

$$(u_x - uu_t - u_{tttt})_t = u_{yy} + u_{zz} \quad (5)$$

for a scattering medium [9] follows from (1) when $K = 2\delta''(s)$.

There exist other models that specify or generalize the equation (1), e.g. by including (1) in a coupled systems of nonlinear equations [10, 11].

If the wave field $u = u(t, x)$ is a function of a single spatial (longitudinal) coordinate x and does not depend on transverse coordinates y, z , equation (1) is reduced to well-known equations for plane waves [12]. In particular, the Riemann-Hopf equation follows from (2), the Burgers equation follows from (3), and the Korteweg-de Vries equation follows from (4). The one-dimensional equation with fourth-order derivative for scattering medium suggested and solved in [9] follows from equation (5). 1D equations can be obtained by eliminating the y, z derivatives of 3D equations and the subsequent integration over dt , provided that the wave field vanishes at $t \rightarrow \pm\infty$.

A choice of the kernel as a linear combination of the delta-function and its derivatives of different orders gives a possibility to derive from (1) various well-known differential equations of the physics of nonlinear waves. Symmetries of such equations

either have already been studied (many results obtained until 1995 are collected in [13, 14, 15]), or can be studied by the standard Lie group methods [16, 17, 18]. However, to the best of our knowledge, particular versions of the general equation (1) with non-degenerate kernels which maintain the integro-differential feature of the model have not been studied yet.

The exponential kernel $K = \exp(-s)$ is of particular applied interest. Equation (1) with such kernel describes wave beams in relaxing media. In this case the integro-differential equation is also reduced to a differential equation [1]. To derive such an equation, it is sufficient to note that the integral term w in (1) and the variable u for the exponential kernel are related by the following equation:

$$w_t + w = u. \quad (6)$$

Reduction of (1) to a differential equation is also possible for some more complicated kernels. For example, if $K = \exp(-s) \cos(\omega_0 s)$, then the kernel describes internal dynamics of medium with resonant inclusions. In this case, the differential relation between w and u in equation (1) has the form:

$$(w_t + w - u)_t + (w_t + w - u) + \omega_0^2 w = 0. \quad (7)$$

A special class is formed by "model" kernels which are non-zero on the finite segment, for example, within $s \in (0, 1]$. The simplest case is

$$K = \begin{cases} 1 & \text{if } s \leq 1, \\ 0 & \text{if } s > 1. \end{cases}$$

For this kernel the integro-differential equation (1) is reduced to the difference-differential equation:

$$\begin{aligned} (u_x - uu_t - \Delta u_t)_t &= u_{yy} + u_{zz}, \\ \Delta u &\equiv u(t) - u(t-1). \end{aligned} \quad (8)$$

Note that, using the finite shift operator, one rewrite the integral term of equation (1) in the form

$$w(t) = \hat{L}u(t), \quad \hat{L}(\partial_t) = \int_0^\infty K(s) \exp(-s\partial_t) ds, \quad (9)$$

where ∂_t is the partial derivative with respect to time. The second operator $\hat{L}(\partial_t)$ is the Laplace transform of the function $K(s)$ defining the kernel of equation (1).

For example, if a kernel has the form of the Bessel function of zero order, then one has

$$K = J_0(s), \quad \hat{L}(\partial_t) = (1 + \partial_t^2)^{-1/2}.$$

Using the tables of the Laplace transform and physical restrictions of the kernel forms, one can single out all cases when equation (1) can be reduced to a differential equation of a finite order. In the general case, decomposing the exponential function of the integrand (9) into power series, one verify that the resulting differential equation will contain derivatives of an arbitrary order.

Discussion of properties of a kernel is continued below by considering formulations of main physical problems.

2. Physical statement and main physical parameters

For definiteness, a concrete physical object is considered which is most simple and, at the same time, can be adequately described by models like (1). Namely, we will deal with high-intensity acoustic waves. The general equation (1), as well as majority of the particular models (2)-(7) have been written at first for nonlinear acoustic waves.

Note that equation (1) is written in certain dimensionless variables in order to reduce all coefficients of the equation into unity. To discuss a physical meaning of mathematical models we rewrite equation (1) using initial physical notations:

$$\frac{\partial}{\partial \tau} \left[\frac{\partial p}{\partial x} - \frac{\varepsilon}{c^3 \rho} p \frac{\partial p}{\partial \tau} - W \right] = \frac{c}{2} \Delta_{\perp} p, \quad (10)$$

$$\begin{aligned} W &= \frac{m}{2c} \frac{\partial}{\partial \tau} \int_{-\infty}^{\tau} K \left(\frac{\tau - \tau'}{t_0} \right) \frac{\partial p}{\partial \tau'} d\tau' \\ &= \frac{m}{2c} \frac{\partial^2}{\partial \tau^2} \int_0^{\infty} K \left(\frac{\xi}{t_0} \right) p(\vec{r}, \tau - \xi) d\xi. \end{aligned} \quad (11)$$

Here x is the coordinate along the direction of wave propagation; $\Delta_{\perp}$ is the Laplace operator written in the coordinates y, z on the orthogonal plane, $\tau = t - x/c$ is the time in the moving system of coordinates propagating with the sound velocity c , ε is the parameter of nonlinearity, and ρ is the density of a medium. The acoustic pressure p is chosen as the wave field variable. The constant m characterizes the "force" of time delay processes, and t_0 is the typical "memorizing time" of a medium.

Let us note an important point before passing to further discussion. The natural question arises: why is the coordinate x instead of time t used as a "slow" (evolutionary) variable in equations (1) and (10)? The answer is that the difference between x and t depends only on the way of description which depends on the statement of problem and ease of analysis of results. In the case of non-wave problems (e.g. description of turbulence) the problem is posed as follows. At the initial moment $t = 0$ a distribution of the velocity field in space $u(t = 0, x) = u_0(x)$ is given, and the solution $u(t, x)$ is sought with growing time $t > 0$. In the corresponding experiment, sensors measuring the velocity field are placed in various locations, and the measurement is made by all sensors at the same time t_1 . These results determine the spatial structure of the field $u(t_1, x)$. Then similar measurements performed at t_2 give the field profile $u(t_2, x)$. Repeating the measurements we trace the field evolution with respect to time.

When propagating waves are of interest, the experiment is done in a different way. The only sensor placed at the position x_1 measures the variation of the signal with respect to time: $u(x_1, t)$. Then the sensor is moved to another position $x_2 > x_1$ and the signal $u(x_2, t)$ is measured. By moving the sensor of the vibration velocity (or the acoustic pressure) farther and farther from the source of wave, we trace the evolution of the form of the wave profile as the wave propagates. In real experiments a wave gets distorted at distances of the order of thousand wavelengths, whereas, for a good reconstruction of the wave profile within its each length λ , one has to place no fewer than

ten sensors. In this case the method of the "slow time" is very inconvenient. Moreover, this method is completely inappropriate in those cases when the wave profile contains shock fronts whose extent is very small, e.g. $10^{-4}\lambda$.

But in various acoustic problems, e.g. those dealing with standing waves in a resonator, it is convenient to utilize the "slow time" instead of the "slow coordinate." It is clear that the resonator has a limited length and, by measuring the field at the lowest modes, it is quite realistic to place several sensors along the length of the resonator and perform simultaneous measurement with them at various moments of time.

Let us return to the physical model (10), (11). The integration within the limits $-\infty < \tau' < \tau$ in the first integral (11) means that the wave behavior at a given moment τ is determined by the values of the field variable at the preceding moments from τ to the infinitely distant past. Consequently, the kernel $K(\tau)$ describing the "memory" of a medium, must be nonzero only at positive values of its argument and tend to zero for $\tau \rightarrow +\infty$. Decreasing can be non-monotone and can look like oscillatory damping (see the example leading to formula (7)).

In order to understand how the concrete form of the kernel is related with the measured characteristics of the medium, we shall consider the simplest model of a plane wave moving in a medium without nonlinearity. In other words, let us consider the equation

$$\frac{\partial p}{\partial x} - \frac{m}{2c} \frac{\partial^2}{\partial \tau^2} \int_0^\infty K\left(\frac{\xi}{t_0}\right) p(x, \tau - \xi) d\xi = 0. \quad (12)$$

Let us establish a relation of a kernel with the dispersion law. A solution is sought in the form

$$p = \exp(-i\omega t + ikx), \quad k = k' + ik''. \quad (13)$$

Here k is the wave number, and k' , k'' are its real and imaginary parts. Substituting (13) into (12), we find

$$\begin{aligned} k' &= -\frac{m\omega^2 t_0}{2c} \int_0^\infty K(s) \sin(\omega t_0 s) ds, \\ k'' &= \frac{m\omega^2 t_0}{2c} \int_0^\infty K(s) \cos(\omega t_0 s) ds. \end{aligned} \quad (14)$$

The first formula in (14) gives a frequency-dependent addition to the velocity of the wave propagation: $c(\omega) = c(1 - ck'(\omega)/\omega)$. The second one defines the absorption coefficient or the law of spatial decrease of the wave amplitude: $p_0 \exp(-k''x)$.

Evidently, the integrals (14) must be convergent for physically feasible kernels. The concrete form of a kernel can be reconstructed on the base of corresponding physical model, or on the base of experimental measurements.

A relaxing medium provides an important model known as the Mandelstam-Leontovich model (see [1, 19]). The kernel for this model has the exponential form (see the example leading to formula (6)). In this case

$$k' = -\frac{m\omega}{2c} \frac{(\omega t_0)^2}{1 + (\omega t_0)^2}, \quad k'' = \frac{m\omega}{2c} \frac{\omega t_0}{1 + (\omega t_0)^2}. \quad (15)$$

The frequency dependencies (15) of the dispersion k' and the absorption k'' were confirmed repeatedly in experiments. One could proceed in an opposite way. First establish the dependencies (15) as empirical generalization of measured data, and then reconstruct the kernel by means of a standard procedure. This procedure exploits the causality principle according to which two functions k' and k'' cannot be arbitrary but should be connected by relations of Kramers-Kronig's type [12].

The method of kernel reconstruction has been utilized for deriving mathematical models used in medical applications of ultrasound [20]. It is known that, within the most interesting frequency range, the absorption of the ultrasound in soft tissues behaves like $k'' \sim \omega^{2-\nu}$, $0 < \nu < 1$. It is easy to reconstruct the kernel $K(s) = s^{\nu-1}$ and verify that the corresponding absorption coefficient

$$k'' = \frac{m}{2ct_0} \Gamma(\nu) \cos\left(\frac{\pi}{2}\nu\right) (\omega t_0)^{2-\nu} \quad (16)$$

has the correct frequency dependence. Note that the considered power kernel has a singularity at $s = 0$ and is not integrable in semi-infinite limits. However, the convolution of this kernel with the oscillating function describing a wave provides convergence of the integral for k'' . This example demonstrates wide variety of situations which can be met in applications.

In conclusion of this section we demonstrate how one has to change variables in equation (10), (11) to reduce it to the simplest normalized form (1). One has to set

$$\tau \rightarrow t_0 t, \quad p \rightarrow p_0 u, \quad x \rightarrow x_0 x, \quad y \rightarrow y_0 y, \quad z \rightarrow z_0 z, \quad (17)$$

where the constant t_0 (the "memory" time) is defined by the structure of kernel, and the other constants are:

$$p_0 = \frac{m}{2\varepsilon} c^2 \rho, \quad x_0 = \frac{2}{m} c t_0, \quad y_0 = z_0 = \frac{c t_0}{\sqrt{m}}. \quad (18)$$

3. Admitted Lie group

As for differential equations an admitted Lie group of integro-differential equation (1) is defined by the determining equations. These equations are integro-differential equations for the coordinates of the infinitesimal generator

$$X = \xi^t \partial_t + \xi^x \partial_x + \xi^y \partial_y + \xi^z \partial_z + \zeta^u \partial_u + \zeta^w \partial_w,$$

where the coordinates $\xi^t, \xi^x, \xi^y, \xi^z, \zeta^u$ and ζ^w are functions depending on the variables (t, x, y, z, u, w) . The system (1) comprises a partial differential equation and an integro-differential equation. The determining equation related with the partial differential equation is obtained by the standard procedure:

$$\zeta^{utx} - u \zeta^{utt} - 2u_t \zeta^{ut} - \zeta^{wttt} = \zeta^{uyy} + \zeta^{uzz}, \quad (19)$$

where the coefficients $\zeta^{utx}, \zeta^{utt}, \zeta^{ut}, \zeta^{wttt}, \zeta^{uyy}$ and ζ^{uzz} are the coefficients of the prolonged generator X :

$$\bar{X} = X + \zeta^{utx} \partial_{utx} + \zeta^{utt} \partial_{utt} + \zeta^{ut} \partial_{ut} + \zeta^{wttt} \partial_{wttt} + \zeta^{uyy} \partial_{uyy} + \zeta^{uzz} \partial_{uzz}.$$

The general theory of constructing determining equations for integro-differential equations can be found in [21]. Formerly the determining equation related with integro-differential equation is obtained applying the following strategy. First, one has to construct the canonical Lie-Bäcklund operator equivalent to the generator X :

$$\widetilde{X} = (\zeta^u - \xi^t u_t - \xi^x u_x - \xi^y u_y - \xi^z u_z) \partial_u + (\zeta^w - \xi^t w_t - \xi^x w_x - \xi^y w_y - \xi^z w_z) \partial_w.$$

Then the Lie-Bäcklund operator has to be prolonged up to the maximum order of derivatives of the equation. Finally, the determining equation is obtained by applying the prolonged Lie-Bäcklund operator to the equation, where the actions of the derivatives are considered in terms of the Frechet derivatives:

$$\psi^w(t, x, y, z) = \int_0^\infty K(s) \psi^u(t - s, x, y, z) ds. \quad (20)$$

Here

$$\begin{aligned} \psi^u(h_1) &= \zeta^u(h_2) - \xi^t(h_2) u_t(h_1) - \xi^x(h_2) u_x(h_1) - \xi^y(h_2) u_y(h_1) - \xi^z(h_2) u_z(h_1), \\ \psi^w(h_1) &= \zeta^w(h_2) - \xi^t(h_2) w_t(h_1) - \xi^x(h_2) w_x(h_1) - \xi^y(h_2) w_y(h_1) - \xi^z(h_2) w_z(h_1), \end{aligned}$$

where for the sake of simplicity of the presentation we denoted

$$h_1 = (t, x, y, z), \quad h_2 = (t, x, y, z, u(t, x, y, z), w(t, x, y, z)).$$

The determining equations (19) and (20) have to be satisfied for any solution of equations (1). Notice that the determining equation (20) is still an integral equation.

Since it is difficult to find the general solution of the determining equations (19) and (20), the following simplification is considered. One can assume that the determining equation (19) is valid for any functions $u(t, x, y, z)$ and $w(t, x, y, z)$ only satisfying the first equation of (1). This allows to use standard procedure for solving determining equations developed for partial differential equations. After solving the determining equation (19), one can use the found solution for solving the integral determining equation (20). It has to be noticed that this way of solving the determining equations (19) and (20) can give a particular solution. In the present paper this method is used.

The described method of solving the determining equations (19) and (20) will be illustrated on the one-dimensional case of equations (1). For the other cases final results will be presented in next sections.

3.1. One-dimensional case

In the one-dimensional case equations (1) are

$$\frac{\partial}{\partial t} (u_x - uu_t - w_{tt}) = 0, \quad w(t, x) = \int_0^\infty K(s) u(t - s, x) ds. \quad (21)$$

The admitted generator is sought in the form

$$X = \xi^t(t, x, u, w) \partial_t + \xi^x(t, x, u, w) \partial_x + \zeta^u(t, x, u, w) \partial_u + \zeta^w(t, x, u, w) \partial_w.$$

Applying the group analysis method to the first equation, one finds that

$$\begin{aligned}\xi^t &= t(\xi'/2 + k_1)t + g, \quad \zeta^u = u(k_1 - \xi'/2) - t\xi''/2 - g', \\ \zeta^w &= -(6\xi'w + t^3\xi''')/12 + 3wk_1 + t^2\mu + t\eta + \zeta.\end{aligned}\quad (22)$$

where k_1 is constant, $\xi = \xi(x)$, $\zeta = \zeta(x)$, $\eta = \eta(x)$, $\mu = \mu(x)$ and $g = g(x)$ are arbitrary functions.

Remark. There is another representation of the first equation of (21). This equation is obtained by integrating with respect to t and setting the arbitrary function of the integration to zero:

$$u_x - uu_t - w_{tt} = 0, \quad w(t, x) = \int_0^\infty K(s)u(t-s, x) ds. \quad (23)$$

In this case the first step in finding admitted Lie group leads to (22) with the particular case of the function $\mu = -\xi''/2$.

The determining equation for the second equation of (21)

$$w(t, x) = \int_0^\infty K(s)u(t-s, x) ds.$$

is the equation

$$(\tilde{\zeta}^w - \tilde{\xi}^t w_t - \tilde{\xi}^x w_x)(t, x) = \int_0^\infty K(s)(\tilde{\zeta}^u - \tilde{\xi}^t u_t - \tilde{\xi}^x u_x)(t-s, x) ds, \quad (24)$$

where

$$\begin{aligned}\tilde{\xi}^t(t, x) &= \xi^t(t, x, u(t, x), w(t, x)), \quad \tilde{\xi}^x(t, x) = \xi^x(t, x, u(t, x), w(t, x)), \\ \tilde{\zeta}^w(t, x) &= \zeta^w(t, x, u(t, x), w(t, x)), \quad \tilde{\zeta}^u(t, x) = \zeta^u(t, x, u(t, x), w(t, x)).\end{aligned}$$

Substituting the coefficients (22) into (24), let us satisfy equation (24). Notice that

$$(\tilde{\xi}^x w_x)(t, x) = \int_0^\infty K(s)(\xi^x u_x)(t-s, x) ds,$$

and

$$\begin{aligned}\tilde{\xi}^t(t-s) &= \xi^t(t) - s(k_1 + \tfrac{1}{2}\xi'), \\ \tilde{\zeta}^u(t-s) &= (k_1 - \tfrac{1}{2}\xi')u(t-s) - \tfrac{1}{2}(t-s)\xi'' - g'.\end{aligned}$$

Here and further, the argument x is omitted, further tilde is also omitted. The determining equation (24) becomes

$$\begin{aligned}& -\zeta^w(t) + \int_0^\infty K(s) \left(\zeta^u(t-s) - s(k_1 + \tfrac{1}{2}\xi') u_t(t-s) \right) ds \\ &= (6\xi'w + t^3\xi''')/12 - 3wk_1 - t^2\mu - t\eta - \zeta + (k_1 - \tfrac{1}{2}\xi')w - g' \int_0^\infty K(s) ds \\ & - \tfrac{1}{2}\xi''(t \int_0^\infty K(s) ds - \int_0^\infty sK(s) ds) + (\tfrac{1}{2}\xi' + k_1) \int_0^\infty sK(s)u_t(t-s) ds \\ & = t^3\xi'''/12 - 2wk_1 - t^2\mu - t\eta - \zeta - g' \int_0^\infty K(s) ds \\ & - \tfrac{1}{2}\xi''(t \int_0^\infty K(s) ds - \int_0^\infty sK(s) ds) + (\tfrac{1}{2}\xi' + k_1) \int_0^\infty sK(s)u_t(t-s) ds.\end{aligned}$$

Let us calculate

$$\begin{aligned}\int_0^\infty sK(s)u_t(t-s) ds &= -\int_0^\infty sK(s) du(t-s) \\ &= -sK(s)u(t-s)|_0^\infty + \int_0^\infty (K(s) + sK'(s))u(t-s) ds \\ &= w + \int_0^\infty sK'(s)u(t-s) ds.\end{aligned}$$

Here it is assumed that

$$sK(s)u(t-s)|_0^\infty = 0.$$

The determining equation becomes

$$\begin{aligned} & t^3\xi'''/12 - 2wk_1 - t^2\mu - t\eta - \zeta - g' \int_0^\infty K(s) ds \\ & - \frac{1}{2}\xi'' (t \int_0^\infty K(s) ds - \int_0^\infty sK(s) ds) + \left(\frac{1}{2}\xi' + k_1\right) \int_0^\infty sK(s)u_t(t-s) ds \\ & = t^3\xi'''/12 - 2wk_1 - t^2\mu - t\eta - \zeta - g' \int_0^\infty K(s) ds - \frac{1}{2}\xi'' (t \int_0^\infty K(s) ds - \int_0^\infty sK(s) ds) \\ & \quad + \left(\frac{1}{2}\xi' + k_1\right) (w + \int_0^\infty sK'(s)u(t-s) ds) \\ & = t^3\xi'''/12 + w(\frac{1}{2}\xi' - k_1) - t^2\mu - t\eta - \zeta - g' \int_0^\infty K(s) ds - \frac{1}{2}\xi'' (t \int_0^\infty K(s) ds - \int_0^\infty sK(s) ds) \\ & \quad + \left(\frac{1}{2}\xi' + k_1\right) \int_0^\infty sK'(s)u(t-s) ds \\ & = t^3\xi'''/12 - t^2\mu - t\eta - \zeta - g' \int_0^\infty K(s) ds - \frac{1}{3}\xi'' (t \int_0^\infty K(s) ds - \int_0^\infty sK(s) ds) \\ & \quad + \int_0^\infty \left(\left(\frac{1}{2}\xi' + k_1\right) sK'(s) + \left(\frac{1}{2}\xi' - k_1\right) K(s)\right) u(t-s) ds = 0. \end{aligned}$$

Since $u = 0$, $w = 0$ is a solution of equations (21), the determining equation has to be satisfied on this solution. Thus, one obtains

$$\begin{aligned} & -t^3\xi'''/12 + t^2\mu + t\eta + \zeta + g' \int_0^\infty K(s) ds \\ & + \frac{1}{3}\xi'' (t \int_0^\infty K(s) ds - \int_0^\infty sK(s) ds) = 0, \end{aligned} \tag{25}$$

and, hence,

$$\int_0^\infty \left(\left(\frac{1}{2}\xi' + k_1\right) sK'(s) + \left(\frac{1}{2}\xi' - k_1\right) K(s) \right) u(t-s) ds = 0.$$

Since $u(t, x)$ is an arbitrary, the last equation gives

$$\left(\frac{1}{2}\xi' + k_1\right) sK'(s) + \left(\frac{1}{2}\xi' - k_1\right) K(s) = 0,$$

which means that ξ' is constant, for example, $\xi = 2kx + k_0$. Thus,

$$(k + k_1) sK'(s) + (k - k_1)K(s) = 0,$$

or $K(s) = K_0 s^\alpha$, and

$$(k + k_1) \alpha + (k - k_1) = 0.$$

Equation (25) becomes

$$t^2\mu + t\eta + \zeta + g' \int_0^\infty K(s) ds = 0.$$

This equation implies

$$\mu = 0, \quad \eta = 0, \quad \zeta = -g' \int_0^\infty K(s) ds.$$

To prevent problems with convergency of the integral $\int_0^\infty K(s) ds$, one can assume that

$$g' = 0.$$

Remark. One can call the transformations corresponding to the generators

$$X = g\partial_t - g'\partial_u$$

formally admitted. These generators can be used for constructing invariant solutions.

Thus, one obtains that the Lie group corresponding to the generators

$$X_1 = \partial_x, \quad X_2 = \partial_t,$$

$$X_3 = k^t t \partial_t + k^x x \partial_x + k^u u \partial_u + k^w w \partial_w,$$

is admitted by equations (21). Here

$$\begin{aligned} k^t &= k + k_1, \quad k^x = 2k, \quad k^u = k_1 - k, \quad k^w = 3k_1 - k, \\ (k + k_1)\alpha + (k - k_1) &= 0. \end{aligned} \tag{26}$$

Notice also that

$$k^u = \alpha k^t, \quad k^t \neq 0.$$

Since the integro-differential equations are nonlocal, not any admitted Lie group has the property to transform a solution of integro-differential equations into a solution. However, for the transformations corresponding to the generators X_1 and X_2 it is trivial to check that these transformations possess this property. Let us also check that the scaling group corresponding to the generator X_3 maps any solution of equations (21) into a solution of the same equations.

The transformation corresponding to the generator X_3 is

$$t' = te^{ak^t}, \quad x' = xe^{ak^x}, \quad u' = ue^{ak^u}.$$

which maps a function $u(t, x)$ into the function

$$u'(t', x') = e^{ak^u} u(t'e^{-ak^t}, x'e^{-ak^x}).$$

Let us consider the transformation of the integral

$$\begin{aligned} & \int_0^\infty K(s') u'(t' - s', x') ds' \\ &= e^{ak^u} \int_0^\infty K(s') u((t' - s')e^{-ak^t}, x'e^{-ak^x}) ds' \\ &= e^{ak^u} \int_0^\infty K(s') u(t'e^{-ak^t} - s'e^{-ak^t}) ds' \\ &= e^{a(k^u + k^t)} \int_0^\infty K(s'e^{-ak^t} e^{ak^t}) u(t - s'e^{-ak^t}) d(s'e^{-ak^t}) \\ &= e^{a(k^u + k^t)} \int_0^\infty K(se^{ak^t}) u(t - s) ds \\ &= e^{a(k^u + k^t)} \int_0^\infty K_0(se^{ak^t})^\alpha u(t - s) ds \\ &= e^{a(k^u + (\alpha+1)k^t)} \int_0^\infty K_0 s^\alpha u(t - s) ds \\ &= e^{a(k^u + (\alpha+1)k^t)} w(t, x) \end{aligned}$$

Thus, for checking one only needs to check that

$$-k^w + k^u + (\alpha + 1)k^t = 0.$$

Indeed

$$\begin{aligned}
& -k^w + k^u + (\alpha + 1)k^t = \\
& = -3k_1 + k + k_1 - k + (\alpha + 1)(k + k_1) \\
& = k_1(-3 + 1 + \alpha + 1) + k(1 - 1 + \alpha + 1) \\
& = k_1(\alpha - 1) + k(\alpha + 1) = 0.
\end{aligned}$$

Here the condition (26) was used. One also needs to check the other conditions:

$$k^u - k^t - k^x = 2k^u - 2k^t = k^w - 3k^t$$

Indeed,

$$\begin{aligned}
k^u - k^t - k^x &= k_1 - k - (k + k_1) - 2k = -4k, \\
2k^u - 2k^t &= 2(k_1 - k) - 2(k + k_1) = -4k, \\
k^w - 3k^t &= 3k_1 - k - 3(k + k_1) = -4k.
\end{aligned}$$

3.2. Classification of subalgebras

The commutator table of the Lie algebra $L_3 = \{X_1, X_2, X_3\}$ is

	X_1	X_2	X_3
X_1	0	0	$k^x X_1$
X_2	0	0	$k^t X_2$
X_3	$-k^x X_1$	$-k^t X_2$	0

The set of automorphisms is defined by the commutators table:

$$\begin{aligned}
A_1 : \quad & x'_1 = x_1 + a_1 k^x x_3, \\
A_2 : \quad & x'_2 = x_2 + a_2 k^t x_3, \\
A_3 : \quad & x'_1 = x_1 e^{a_3 k^x}, \quad x'_2 = x_2 e^{a_3 k^t}.
\end{aligned}$$

where only changeable coordinates of the automorphisms are presented.

If $\alpha = 1$, then $k^x = 0$, and the operator X_1 composes a center of the Lie algebra. Thus the one-dimensional optimal system of subalgebras consists of the subalgebras

$$\{X_3 + \lambda X_1, X_2 + \gamma X_1, X_1\},$$

where $\gamma = \pm 1$.

If $\alpha \neq 1$, then $k^x \neq 0$, and the one-dimensional optimal system of subalgebras consists of the subalgebras

$$\{X_3, X_2 + \lambda X_1, X_1\}.$$

For $\alpha \neq 0$, using the automorphism A_1 the subalgebra $X_2 + \lambda X_1$ can be also reduced to $X_2 \pm X_1$.

3.3. Invariant solutions

The optimal system of subalgebras of the Lie algebra L_3 defines the complete set of representations of solutions invariant with respect to L_3 .

3.3.1. Case $\alpha = 1$

The subalgebra $\{X_3 + \lambda X_1\}$. The generator $\{X_3 + \lambda X_1\}$ is

$$t\partial_t + u\partial_u + \lambda\partial_x.$$

It is convenient to separate on two cases (a) $\lambda = 0$, (b) $\lambda \neq 0$. The invariants are

$$\begin{aligned} \lambda = 0 : & \quad u/t, x; \\ \lambda \neq 0 : & \quad u/t, te^{x\lambda_o}, \quad (\lambda_o = -1/\lambda). \end{aligned}$$

Invariant solutions have the representations

$$\begin{aligned} \lambda = 0 : & \quad u = t\varphi(x); \\ \lambda \neq 0 : & \quad u = t\varphi(y), \quad (y = te^{x\lambda_o}). \end{aligned}$$

Notice that for $\lambda = 0$ the integral

$$w = \int_0^\infty K_0 s(t-s)\varphi(x) ds = K_0\varphi(x) \int_0^\infty s(t-s) ds,$$

is divergent. Hence, one has only to consider the case $\lambda \neq 0$. In this case

$$\begin{aligned} w(t, x) &= K_0 \int_0^\infty s(t-s)\varphi((t-s)e^{x\lambda_o}) ds = K_0 e^{-3x\lambda_o} \int_y^{-\infty} (y-z)z\varphi(z) d(y-z) \\ &= K_0 e^{-3x\lambda_o} \int_0^\infty s(y-s)\varphi(y-s) ds = e^{-3x\lambda_o} W(y). \\ w_t &= e^{-2x\lambda_o} W', \quad w_{tt} = e^{-x\lambda_o} W'', \end{aligned}$$

where we used the relations

$$s = t - ze^{-x\lambda_o} = e^{-x\lambda_o}(te^{x\lambda_o} - z) = e^{-x\lambda_o}(y - z).$$

Equation (23) becomes

$$y \left((\lambda\varphi + 1)\varphi' - \varphi^2 \right) = K_0 \frac{d^2}{dy^2} \left(\int_0^\infty s(y-s)\varphi(y-s) ds \right).$$

The subalgebra $\{X_2 + \gamma X_1\}$. The generator $\{X_2 + \gamma X_1\}$ is $\partial_t + \gamma\partial_x$. The invariants are

$$u, y = x - \gamma t.$$

An invariant solution has the representation

$$u = \varphi(y).$$

Hence, one has

$$\begin{aligned} w(t, x) &= K_0 \int_0^\infty s\varphi(x - \gamma(t-s)) ds = K_0 \int_y^{-\infty} (y-z)\varphi(z) d(y-z) \\ &= K_0 \int_0^\infty s\varphi(y-s) ds = W(y). \\ w_t &= -\gamma W', \quad w_{tt} = W'', \end{aligned}$$

where we used the relations

$$s = t + \gamma(z - x) = -\gamma(x - \gamma t) + \gamma z = -\gamma(y - z).$$

Equation (23) becomes

$$(1 + \gamma\varphi)\varphi' = K_0 \frac{d^2}{dy^2} \left(\int_0^\infty s(y-s)\varphi(y-s) ds \right).$$

The subalgebra $\{X_1\}$. The generator $\{X_1\}$ is ∂_x . The invariants are

$$u, t.$$

An invariant solution has the representation

$$u = \varphi(t).$$

Hence, one has

$$w(t, x) = K_0 \int_0^\infty s \varphi(t - s) ds.$$

Equation (23) becomes

$$-\varphi \varphi' = K_0 \frac{d^2}{dt^2} \left(\int_0^\infty s \varphi(t - s) ds \right).$$

3.4. Case $\alpha \neq 1$.

The subalgebra $\{X_3\}$. The generator $\{X_3\}$ is

$$X_3 = k^t t \partial_t + k^x x \partial_x + k^u u \partial_u, \quad (k^x \neq 0),$$

where the constants k^t , k^x and k^u are defined by (26). Substituting the representation of the invariant solution

$$u = x^{k^u/(2k)} \varphi(y), \quad t = y x^{k^t/(2k)}$$

into the second equation of (23), one has

$$\begin{aligned} w &= \int_0^\infty K_0 s^\alpha u(t - s) ds = K_0 x^{k^u/(2k)} \int_0^\infty s^\alpha \varphi((t - s) x^{-k^t/(2k)}) ds \\ &= -K_0 x^{((1+\alpha)k^t + k^u)/(2k)} \int_{tx^{-k^t/(2k)}}^{-\infty} (y - z)^\alpha \varphi(z) dz \\ &= K_0 x^{((1+\alpha)k^t + k^u)/(2k)} \int_{-\infty}^y (y - z)^\alpha \varphi(z) dz \\ &= K_0 x^{((1+\alpha)k^t + k^u)/(2k)} W(y) = K_0 x^{(k^t + 2k^u)/(2k)} W(y), \end{aligned}$$

where we used the relations

$$s = t - z x^{k^t/(2k)} = x^{k^t/(2k)} (t x^{-k^t/(2k)} - z) = x^{k^t/(2k)} (y - z), \quad (1 + \alpha)k^t + k^u = k^t + 2k^u,$$

and

$$W(y) = \int_{-\infty}^y (y - z)^\alpha \varphi(z) dz.$$

The derivatives are changed as follows

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= \frac{y}{t} \varphi', \quad \frac{\partial \varphi}{\partial x} = -\frac{k^t}{2k} \frac{y}{x} \varphi', \\ \frac{\partial}{\partial t} \left(\frac{\partial W}{\partial t} \right) &= \frac{\partial}{\partial t} \left(\frac{y}{t} W' \right) = \frac{1}{t^2} (-y W' + y (y W')') = \frac{y^2}{t^2} W''. \end{aligned}$$

Thus, the first equation of (23) becomes

$$\begin{aligned}
& u_x - uu_t - v_{tt} = u_x - uu_t - \frac{\partial^2}{\partial t^2} \int_0^\infty K_0 s^\alpha u(t-s) ds \\
&= \frac{k^u}{2k} \frac{x^{k^u/(2k)}}{x} \varphi - \frac{k^t}{2k} \frac{x^{k^u/(2k)} y}{x} \varphi' - x^{k^u/(2k)} \varphi x^{k^u/(2k)} \frac{y}{t} \varphi' - K_0 \frac{x^{(k^t+2k^u)/(2k)} y^2}{t^2} W'' \\
&= \frac{k^u}{2k} \frac{x^{k^u/(2k)}}{x} \varphi - \frac{k^t}{2k} \frac{x^{k^u/(2k)} y}{x} \varphi' - x^{k^u/(2k)} \varphi x^{k^u/(2k)} \frac{y}{t} \varphi' - K_0 y x^{(2k^u-k^t)/(2k)} W'' \\
&= \frac{x^{k^u/(2k)}}{x} \left(\frac{k^u}{2k} \varphi - \frac{k^t}{2k} y \varphi' - \varphi x^{(k^u+k^u-k^t)/(2k)} \varphi' - K_0 x^{(k^u-k^t+k^x)/(2k)} W'' \right) \\
&= \frac{x^{k^u/(2k)}}{x} \left(\frac{k^u}{2k} \varphi - \frac{k^t}{2k} y \varphi' - \varphi \varphi' - K_0 W'' \right) = 0.
\end{aligned}$$

Here, the following relation was used

$$k^u + k^x - k^t = k_1 - k + 2k - (k + k_1) = 0.$$

Thus, the reduced equation is

$$\alpha \varphi + ((\alpha - 1)\varphi - y)\varphi' = (1 - \alpha)K_0 \int_0^\infty s^\alpha \varphi''(y-s) ds,$$

where

$$W''(y) = \frac{d^2}{dy^2} \left(\int_{-\infty}^y (y-z)^\alpha \varphi(z) dz \right) = \frac{d^2}{dy^2} \left(\int_0^\infty s^\alpha \varphi(y-s) ds \right) = \int_0^\infty s^\alpha \varphi''(y-s) ds.$$

The subalgebra $\{X_2 + \lambda X_1\}$. The generator $\{X_2 + \lambda X_1\}$ is $\partial_t + \lambda \partial_x$. The invariants are

$$u, y = x - \lambda t.$$

An invariant solution has the representation of a traveling wave type

$$u = \varphi(y).$$

As in the previous case one only needs to study the case $\lambda \neq 0$. If $\lambda > 0$, then one has

$$\begin{aligned}
w(t, x) &= K_0 \int_0^\infty s^\alpha \varphi(x - \lambda(t-s)) ds = \lambda^{-\alpha-1} K_0 \int_y^\infty (z-y)^\alpha \varphi(z) d(z-y) \\
&= \lambda^{-\alpha-1} K_0 \int_0^\infty s^\alpha \varphi(y+s) ds = W(y). \\
w_t &= -\lambda W', \quad w_{tt} = \lambda^2 W'',
\end{aligned}$$

where

$$s = t + \frac{z-x}{\lambda} = -\frac{x-\lambda t}{\lambda} + \frac{z}{\lambda} = \frac{z-y}{\lambda}.$$

The reduced equation is

$$(1 + \lambda \varphi) \varphi' = \lambda^{1-\alpha} K_0 \int_0^\infty s^\alpha \varphi''(y+s) ds. \quad (27)$$

If $\lambda < 0$, then

$$\begin{aligned}
w(t, x) &= K_0 \int_0^\infty s^\alpha \varphi(x - \lambda(t-s)) ds = K_0 (-\lambda)^{-\alpha-1} \int_y^{-\infty} (y-z)^\alpha \varphi(z) d(y-z) \\
&= (-\lambda)^{-\alpha-1} K_0 \int_0^\infty s^\alpha \varphi(y+s) ds = W(y).
\end{aligned}$$

Equation (23) becomes

$$(1 + \lambda \varphi) \varphi' = (-\lambda)^{1-\alpha} K_0 \int_0^\infty s^\alpha \varphi''(y+s) ds. \quad (28)$$

Combining equations (27) and (28), one has

$$(1 + \lambda \varphi) \varphi' = |\lambda|^{1-\alpha} K_0 \int_0^\infty s^\alpha \varphi''(y+s) ds. \quad (29)$$

The subalgebra $\{X_1\}$. The generator $\{X_1\}$ is ∂_x . The invariants are u, t . An invariant solution has the representation

$$u = \varphi(t).$$

Hence, one has

$$w(t, x) = K_0 \int_0^\infty s^\alpha \varphi(t - s) ds.$$

Equation (23) becomes

$$-\varphi\varphi' = K_0 \left(\int_0^\infty s^\alpha \varphi''(t - s) ds \right).$$

4. Two-dimensional equation

The studied equations are

$$\frac{\partial}{\partial t} (u_x - upu_t - w_{tt}) = u_{yy}, \quad w(t, x, y) = \int_0^\infty K(s)u(t - s, x, y) ds.$$

The first step gives the generator

$$X = \xi^t(t, x, y, p, w)\partial_t + \xi^x(t, x, y, p, w)\partial_x + \xi^y(t, x, y, p, w)\partial_y \\ + \zeta^u(t, x, y, p, w)\partial_p + \zeta^v(t, x, y, p, w)\partial_w,$$

where the coefficients are

$$\begin{aligned} \xi^t &= (2t\xi' + \xi''y^2 + 3h'y)/6 + tk_1 + g, \quad \xi^y = \xi, \quad \xi^y = (4y\xi' + 3yk_1)/6 + h, \\ \zeta^u &= ((6k_1 - 4\xi') - \xi'''y^2 - 2t\xi'' - 6g' - 3yh'')/6, \\ \zeta^v &= (3k_1 - \xi')w + t^2\mu + t\eta + \zeta, \end{aligned} \quad (30)$$

Here k_1 is constant, $\xi = \xi(x)$, $h = h(x)$, $g = g(x)$, $\zeta = \zeta(t, x, y)$, $\eta = \eta(t, x, y)$ and $\mu = \mu(t, x, y)$ are arbitrary functions.

The determining equation for the second equation

$$w(t, x, y) = \int_0^\infty K(s)u(t - s, x, y) ds.$$

is the equation

$$(\zeta^v - \xi^t w_t - \xi^x w_x - \xi^y w_y)(t, x, y) = \int_0^\infty K(s)(\zeta^u - \xi^t u_t - \xi^x u_x - \xi^y u_y)(t - s, x, y) ds.$$

Substituting the coefficients (30) into the last equation, one obtains

$$\mu = 0, \quad \eta = 0, \quad \zeta = 0, \quad g' = 0, \quad h'' = 0,$$

$$\left(\frac{1}{3}\xi' + k_1\right) sK'(s) + \left(\frac{2}{3}\xi' - k_1\right) K(s) = 0,$$

which means that ξ' is constant, for example, $\xi = 3kx + k_0$. Thus,

$$(k + k_1) sK'(s) + (2k - k_1)K(s) = 0,$$

or

$$K(s) = K_0 s^\alpha,$$

and

$$(k + k_1)\alpha + (2k - k_1) = 0.$$

Therefore, the admitted Lie algebra is defined by the generators

$$\begin{aligned} X_1 &= \partial_x, \quad X_2 = \partial_t, \quad X_3 = y\partial_t + 2x\partial_y, \quad X_4 = \partial_y, \\ X_5 &= 2t(k + k_1)\partial_t + 6kx\partial_x + (4k + k_1)y\partial_y + 2(k_1 - 2k)u\partial_u + 6v(k_1 - k)\partial_w. \end{aligned}$$

5. 3-dimensional case

The studied equations are

$$\frac{\partial}{\partial t}(u_x - uu_t - w_{tt}) = u_{yy} + u_{zz}, \quad w(t, x, y, z) = \int_0^\infty K(s)u(t - s, x, y, z) ds.$$

The first step gives the generator

$$\begin{aligned} X &= \xi^t(t, x, y, u, z, w)\partial_t + \xi^x(t, x, y, u, z, w)\partial_x + \xi^y(t, x, y, u, z, w)\partial_y \\ &\quad + \zeta^u(t, x, y, u, z, w)\partial_u + \zeta^w(t, x, y, z, u, w)\partial_w \end{aligned}$$

with the coefficients

$$\begin{aligned} \xi^t &= t(\xi' + 2k_1) + 3\xi''(z^2 + y^2)/4 + (h'y + f'z)/2 + g, \\ \xi^x &= 5\xi, \quad \xi^y = 3\xi'y + zk_2 + yk_1 + h, \quad \xi^z = 3\xi'z + zk_1 - yk_2 + f, \\ \zeta^u &= 2(k_1 - 2\xi')u - 3\xi'''(z^2 + y^2)/4 - \xi''t - g' - (h''y + f''z)/2, \\ \zeta^w &= w(6k_1 - 7\xi') + \xi'''t^3/3 + t^2\mu + t\eta + \zeta. \end{aligned}$$

where k_1, k_2 are constant, $\xi = \xi(x)$, $h = h(x)$, $g = g(x)$, $f = f(x)$, $\zeta = \zeta(x, y, z)$, $\eta = \eta(x, y, z)$ and $\mu = \mu(x, y, z)$ are arbitrary functions. It is obvious that the generator corresponding to the rotation in the plain y and z is admitted. This generator is defined by k_2 :

$$X = z\partial_y - y\partial_z.$$

Excluding this generator one obtains

$$\begin{aligned} \xi^t &= t(\xi' + 2k_1) + 3\xi''(z^2 + y^2)/4 + (h'y + f'z)/2 + g, \\ \xi^x &= 5\xi, \quad \xi^y = y(3\xi' + k_1) + h, \quad \xi^z = z(3\xi' + k_1) + f, \\ \zeta^u &= 2(k_1 - 2\xi')u - 3\xi'''(z^2 + y^2)/4 - \xi''t - g' - (h''y + f''z)/2, \\ \zeta^w &= w(6k_1 - 7\xi') + \xi'''t^3/3 + t^2\mu + t\eta + \zeta. \end{aligned}$$

The determining equation for the second equation

$$w(t, x, y, z) = \int_0^\infty K(s)u(t - s, x, y, z) ds$$

is the equation

$$\begin{aligned} & (\zeta^w - \xi^t w_t - \xi^x w_x - \xi^y w_y - \xi^z w_z)(t, x, y, z) \\ &= \int_0^\infty K(s)(\zeta^u - \xi^t u_t - \xi^x u_x - \xi^y u_y - \xi^z u_z)(t-s, x, y, z) ds. \end{aligned}$$

Substituting the coefficients into the last equation, one obtains

$$\begin{aligned} g &= k_3, \quad h = 2k_4x + k_5, \quad f = 2k_6x + k_7 \\ \xi^t &= t(k + 2k_1) + k_4y + k_6z + k_3, \quad \xi^x = 5(kx + k_0), \quad \xi^y = y(3k + k_1) + 2k_4x + k_5, \\ \xi^z &= z(3k + k_1) + 2k_6x + k_7, \quad \zeta^u = 2(k_1 - 2k)u, \quad \zeta^w = w(6k_1 - 7k). \\ (k + 2k_1) s K'(s) &+ 2(2k - k_1)K(s) = 0, \end{aligned}$$

which means that

$$K(s) = K_0 s^\alpha,$$

and

$$k(\alpha + 4) + 2k_1(\alpha - 1) = 0.$$

The admitted generators are

$$\begin{aligned} X_1 &= \partial_x, \quad X_2 = \partial_t, \quad X_3 = z\partial_y - y\partial_z, \quad X_4 = y\partial_t + 2x\partial_y, \\ X_5 &= z\partial_t + 2x\partial_z, \quad X_6 = \partial_y, \quad X_7 = \partial_z, \\ X_8 &= t(k + 2k_1)\partial_t + 5kx\partial_x + (3k + k_1)y\partial_y + (3k + k_1)z\partial_z \\ &\quad + 2(k_1 - 2k)u\partial_u + w(6k_1 - 7k)\partial_w. \end{aligned}$$

6. Exponential kernel

There exists one known exact solution to 1D equation for an exponential kernel [1, 19]:

$$\begin{aligned} u_x - uu_t - w_{tt} &= 0, \\ w &= \int_0^\infty \exp(-s) u(t-s) ds. \end{aligned}$$

One can seek for it in the form of a traveling wave:

$$u = u(t + \alpha x).$$

The solution has the form:

$$t + \gamma = \frac{1}{\Delta} \ln \frac{|u - \alpha + \Delta|^{1-\Delta}}{|u - \alpha - \Delta|^{1+\Delta}}.$$

Here α , Δ and γ are the constants. This solution has evident physical meaning for the parameters $\Delta > |\alpha|$, $0 < \Delta < 1$. This is shown in Fig. 1 and the solution describes the shape of a single shock front in a relaxing medium.

It is interesting to derive this solution using computed symmetries.

Figure 1. The solution describing the shape of front in a relaxing medium. Curves 1, 2 and 3 are constructed for $\alpha = 0$ and $\Delta=1, 0.75$ and 0.5 , correspondingly. The dashed curve 2' is constructed for values $\Delta=0.75$ and $\alpha = 0.15$

Since for the exponential kernel $K(s) = e^{-s}$ one has the relation (6):

$$w_t = u - w,$$

equations

$$u_{xt} - uu_{tt} - u_t^2 = w_{ttt} \quad (31)$$

and

$$u_x - uu_t = w_{tt} \quad (32)$$

can be reduced to the partial differential equations, respectively,

$$u_{xtt} - (1+u)u_{ttt} - 3u_t u_{tt} + u_{xt} - uu_{tt} - u_t^2 = 0, \quad (33)$$

and

$$u_{xt} - (1+u)u_{tt} - u_t^2 + u_x - uu_t = 0. \quad (34)$$

Admitted Lie groups of these equations are as follows. The admitted Lie group of equation (33) is defined by the generators

$$X_1 = \partial_x, \quad X_2 = g(x)\partial_t - g'(x)\partial_u,$$

where the function $g = g(x)$ is an arbitrary function. The admitted Lie group of equation (34) is defined by the generators

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = x\partial_t - \partial_u.$$

The commutators table of the algebra $\{X_1, X_2, X_3\}$ is

	X_1	X_2	X_3
X_1	0	0	X_2
X_2	0	0	0
X_3	$-X_2$	0	0

The generator X_2 is a center of the Lie algebra. The set of automorphisms is defined by the commutators table:

$$\begin{aligned} A_1 : \quad x'_2 &= x_2 + a_1 x_3, \\ A_3 : \quad x'_2 &= x_2 - a_3 x_1. \end{aligned}$$

The one-dimensional optimal system of subalgebras consists of the subalgebras

$$\{X_3 + \lambda X_1, X_2, X_1\}.$$

Representations of invariant solutions are

$$\begin{aligned} X_3 + \lambda X_1 : \quad u &= -\frac{x}{\lambda} + \varphi(x^2 - 2\lambda t), \quad (\lambda \neq 0), \\ X_3 : \quad u &= -\frac{t}{x} + \varphi(x), \\ X_2 : \quad u &= \varphi(x), \\ X_1 : \quad u &= \varphi(t). \end{aligned}$$

and the reduced equations are

$$\begin{aligned} X_3 + \lambda X_1 : \quad (4\lambda^3(\varphi' + \varphi\varphi') - \lambda^2\varphi^2)' + 1 &= 0, \\ X_3 : \quad \varphi' + \frac{1}{x}\varphi &= 0, \\ X_2 : \quad \varphi' &= 0, \\ X_1 : \quad (\varphi' + \varphi\varphi' + \varphi^2/2)' &= 0. \end{aligned}$$

The solution

$$u = u(t + \alpha x)$$

is invariant with respect to the operator $X_1 - \alpha X_2 = \partial_x - \alpha \partial_t$. The subalgebra corresponding to this operator is equivalent to the subalgebra with the generator: X_1 . The reduced equation of an invariant solution corresponding to the generator X_1 is

$$\varphi' = \frac{k - \varphi^2}{2(1 + \varphi)}.$$

where k is an arbitrary constant of the integration. The general solution of this equation depends on the constant k :

$$\begin{aligned} k = \alpha^2 > 0 : \quad t + c_0 &= \frac{1}{\alpha} \ln \left(\frac{|\varphi + \alpha|^{1-\alpha}}{|\varphi - \alpha|^{1+\alpha}} \right), \\ k = -\alpha^2 < 0 : \quad t + c_0 &= -\ln(\varphi^2 + \alpha^2) - \frac{2}{\alpha} \arctan\left(\frac{\varphi}{\alpha}\right), \\ k = 0 : \quad t + c_0 &= -2\ln(\varphi) + \frac{2}{\varphi}. \end{aligned}$$

Figure 2. Solution describing the shape of wave front in the medium with the constant "memory" inside the time interval $[0, 1]$. Curves 1, 2 are constructed for $\alpha = 0$ and values of $\beta=1$ and 0.5 , correspondingly. Dashed curve 2' is constructed for values $\beta=0.5$ and $\alpha = 0.25$

7. Delay equation

It is desirable to derive an exact solution for any kernel which does not permit the reduction of integral equation to the differential one. Such an example exists. That is the model kernel which is nonzero on the finite segment, say, $s \in [0, 1]$. The simplest case is $K = 1, s \leq 1; K = 0, s > 1$. For this kernel the integral equation is reduced to the difference-differential equation

$$u_x - uu_t - \Delta u_t, \quad \Delta u \equiv u(t) - u(t-1).$$

Its solution can be sought in the form of a traveling wave: $u = u(t + \alpha x)$. The reduced equation can be integrated one time. The solution is $u(t-1) = \frac{1}{2} [u^2(t) + 2(1-\alpha)u(t) - \beta^2]$. Here α and β are constant. This formula defines a mapping $u(t) \rightarrow u(t-1)$ which offers easy possibility to construct curves representing profiles of the wave. These profiles are shown in Fig. 2. They display the image of a shock front in medium with constant "memory" within the segment $[0, 1]$.

The equation

$$u_x(t, x) = (u(t, x) + 1)u_t(t, x) - u_t(t-1, x)$$

is a delay differential equation. Algorithm for applying the group analysis method to delay differential equations is given in [21, 22, 23]. Calculations show that the admitted Lie group is defined by the generators

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = x\partial_t - \partial_u.$$

Representations of invariant solutions are given in the previous case. The reduced equations are

$$\begin{aligned} X_3 + \lambda X_1 : \quad & 2\lambda^2(\varphi'(z)(\varphi(z) + 1) + \varphi'(z + 2\lambda)) = 1, \\ X_3 : \quad & \varphi' + \frac{1}{x}\varphi = 0, \\ X_2 : \quad & \varphi' = 0, \\ X_1 : \quad & \varphi'(t)(\varphi(t) + 1) + \varphi'(t - 1) = 0. \end{aligned}$$

The reduced equations can be integrated

$$\begin{aligned} X_3 + \lambda X_1 : \quad & \lambda^2(\varphi^2(z) + 2\varphi(z) + \varphi^2(z + 2\lambda)) = z + c_0, \\ X_3 : \quad & \varphi = c_0/x, \\ X_2 : \quad & \varphi = c_0, \\ X_1 : \quad & \varphi^2(t) + 2\varphi(t) + 2\varphi(t - 1) = c_0. \end{aligned}$$

8. Conclusion

The nonlinear *integro-differential* evolution equation (1) considered in the present paper is not an exotic model. It encapsulates numerous mathematical models formulated by *differential* evolution equations and differs from them significantly not only in its form, but mostly due to its physical content meaning. Namely, any dispersion (frequency-dependent phase velocity) must be strongly connected with frequency-dependent absorption. Such connection follows from the causality principle. For example, waves having infinite velocities of propagation which are allowed by differential equations of Burgers and Korteweg-de Vries type must disappear on their way, since otherwise a cause appears at a certain point later than its effect. The causality principle is given in physical models by integral Kramers-Kronig relations. Consequently, a consistent model must contain integral terms, in other words, be represented in an integro-differential form. Though this conclusion is well known, mathematical models described by purely differential evolution equations have been widely accepted in the nonlinear wave physics due to their simplicity compare to the integro-differential models. It seems that the consistent integro-differential nonlinear models will meet more applications in future.

The present paper provides a first step in application of the Lie group analysis to Equation (1). The approach used in this paper is described in [21]. The analysis of the determining equation for the integro-differential equation allows, in particular, to single out a class of kernels used for deriving mathematical models in medical applications of ultrasound [20].

Note that for particular kernels the integro-differential equation (1) becomes a partial differential equation or a delay partial differential equation. In these cases the complete group classification of equation (1) can be obtained. In the case of partial differential equations the classical group analysis is used. For delay partial differential equations the analysis developed in [22, 23] and described in [21] is applied. A complete study of particular cases is given in the paper. This provides a new result

in the application of the group analysis method to partial and delay partial differential equations.

Along with admitted Lie groups, representations of exact solutions and reduced equations are constructed in the paper. A complete solution and a physical interpretation of some of them is presented.

We hope that more results will be obtained in future by applying the above approach for solving concrete models of physical significance as well as for new mathematical developments. In particular, it is interesting to make the preliminary group classification of *exceptional kernels* by applying the method of an *a priori* use of symmetries [24] to the integro-differential equations of the form (1).

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References

- [1] O. V. Rudenko and S. I. Soluyan. *Theoretical Foundations of Nonlinear Acoustics*. Plenum, Consultants Bureau, New York, 1977.
- [2] O. V. Rudenko, S. I. Soluyan, and R. V. Khokhlov. Problems of the theory of nonlinear acoustics. *Sov. Phys. Acoustics*, 20(3):356–359, 1974.
- [3] E. A. Zabolotskaya and R. V. Khokhlov. Quasi-plane waves in the nonlinear acoustics of confined beams. *Sov. Phys. Acoustics*, 15(1):35–40, 1969.
- [4] O. V. Rudenko. The 40th anniversary of the Khokhlov-Zabolotskaya equation. *Acoustical Physics*, 56(4):457–466, 2010.
- [5] V. P. Kuznetsov. Equations of nonlinear acoustics. *Sov. Phys. Acoustics*, 16(4):548–553, 1970.
- [6] B. K. Novikov, O. V. Rudenko, and V. I. Timoshenko. *Nonlinear Underwater Acoustics*. American Inst. of Physics, New York, 1987.
- [7] B. B. Kadomtsev and V. I. Petviashvili. On the stability of solitary waves in weakly dispersive media. *Sov. Phys. Doklady*, 15:539–541, 1970.
- [8] U. A. Danilov. Kadomtsev-Petviashvili equation. In *Encyclopedia of Physics*, page 229. Soviet Enceklopedia, Moscow, 1990.
- [9] O. V. Rudenko and V. A. Robsman. Equation of nonlinear waves in a scattering medium. *Doklady Physics*, 47(6):443–446, 2002.
- [10] O. V. Rudenko. Nonlinear sawtooth-shaped waves. *Physics-Uspekhi (Advances in Physical Sciences)*, 38(9):965–989, 1995.
- [11] O. V. Rudenko and O. A. Sapozhnikov. Self-action effects for wave beams containing shock fronts. *Physics-Uspekhi (Advances in Physical Sciences)*, 47(9):907–922, 2004.
- [12] M. B. Vinogradova, O. V. Rudenko, and A. P. Sukhorukov. *Theory of Waves*. Nauka, Moscow, 1990. 2nd ed. (in Russian).
- [13] N. H. Ibragimov, editor. *CRC Handbook of Lie Group Analysis of Differential Equations*, volume 1. CRC Press, Boca Raton, 1994.
- [14] N. H. Ibragimov, editor. *CRC Handbook of Lie Group Analysis of Differential Equations*, volume 2. CRC Press, Boca Raton, 1995.
- [15] N. H. Ibragimov, editor. *CRC Handbook of Lie Group Analysis of Differential Equations*, volume 3. CRC Press, Boca Raton, 1996.

- [16] L. V. Ovsiannikov. *Group analysis of differential equations*. Nauka, Moscow, 1978. English translation, Ames, W.F., Ed., published by Academic Press, New York, 1982.
- [17] N. H. Ibragimov. *Transformation Groups Applied to Mathematical Physics*. Nauka, Moscow, 1983. English translation, Reidel, D., Ed., Dordrecht, 1985.
- [18] P. J. Olver. *Applications of Lie groups to differential equations*. Springer-Verlag, New York, 1986.
- [19] O. V. Rudenko, S. N. Gurbatov, and C. M. Hedberg. *Nonlinear Acoustics through Problems and Examples*. Trafford, 2010.
- [20] A. P. Sarvazyan, O. V. Rudenko, S. D. Swanson, J. B. Fowlkes, and S. Y. Emelianov. Shear wave elasticity imaging: a new ultrasonic technology of medical diagnostics. *Ultrasound Med. Biol.*, 24(9):1419–1435, 1998.
- [21] Yu. N. Grigoriev, N. H. Ibragimov, V. F. Kovalev, and S. V. Meleshko. *Symmetries of integro-differential equations and their applications in mechanics and plasma physics*. Lecture Notes in Physics, Vol. 806. Springer, Berlin / Heidelberg, 2010.
- [22] J. Tanthanuch and S. V. Meleshko. On definition of an admitted lie group for functional differential equations. *Communications in nonlinear science and numerical simulations*, 9(1):117–125, 2004.
- [23] S. V. Meleshko. *Methods for Constructing Exact Solutions of Partial Differential Equations*. Mathematical and Analytical Techniques with Applications to Engineering. Springer, New York, 2005.
- [24] N. H. Ibragimov and O. V. Rudenko. Principle of an a priori use of symmetries in the theory of nonlinear waves. *Acoustical Physics*, 50(4):406–419, 2004.