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Issue:

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Self-adjointness and conservation laws of a generalized Burgers equation

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June 24, 2011

Abstract

In this paper we consider a $(2 + 1)$ dimensional generalized Burgers equation. After having written this equation as a system in two dependent variables, we show that it is quasi self-adjoint and find a nontrivial additional conservation law for this system.

Keywords: Adjoint system to nonlinear system; Lagrangians; Symmetry and conservation laws; Burgers equation.

1 Introduction

We consider the following $(2 + 1)$ dimensional generalized Burgers equation

$$u_t + u_{xy} + uu_y + u_x \int^x u_y dx' = 0. \quad (1)$$

Eq. (1) reduces to the Burgers equation when $y = x$. These equation had been studied in [1]. Successively in order to find wide classes of solutions in [2] a Lie group analysis has been performed.

After having put $u_y = v_x$, Eq. (1) can be written as the system

$$\begin{aligned} F_1 &\equiv u_t + u_{xy} + uv_x + vu_x = 0, \\ F_2 &\equiv u_y - v_x = 0. \end{aligned} \quad (2)$$

We recall that in [3] it has been observed that the adjoint differential equation to a linear differential equation can be obtained by a new alternative procedure and this latter one can be immediately extended to the nonlinear equations. In [6] it was suggested to extend the same procedure not only to get adjoint system to the linear system but also to get the adjoint system to the nonlinear system.

This paper represents a continuation of the work program started in [3] and successively developed in [4], [5], [6], [7]. It is the first application to a nonlinear system of the procedures previously introduced.

In particular, after having shown that the system (2) is quasi self-adjoint, we provide a conserved vector for it by applying a theorem proved in [4, 5] independently on existence of classical Lagrangians.

The plan of the paper is the following. In the next section we verify that the system (2) is quasi self-adjoint. In section 3 we obtain a nontrivial additional conserved vector for the system (2).

2 Quasi self-adjointness

The *formal Lagrangian* for the system (2) in agreement with [3], [4], [5], [6], [7] reads

$$\mathcal{L} = \alpha(u_t + u_{xy} + uv_x + vu_x) + \beta(u_y - v_x), \quad (3)$$

where α and β are two new dependent variables.

According to [3] (see also [5]) for the system of differential equations (2) the *adjoint equations* (*adjoint system*) are defined by

$$\begin{aligned} F_1^* &\equiv \frac{\delta \mathcal{L}}{\delta u} = 0, \\ F_2^* &\equiv \frac{\delta \mathcal{L}}{\delta v} = 0, \end{aligned} \quad (4)$$

with

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta u} &\equiv \frac{\partial \mathcal{L}}{\partial u} - D_t \frac{\partial \mathcal{L}}{\partial u_t} - D_x \frac{\partial \mathcal{L}}{\partial u_x} - D_y \frac{\partial \mathcal{L}}{\partial u_y} + D_x D_y \frac{\partial \mathcal{L}}{\partial u_{xy}}, \\ \frac{\delta \mathcal{L}}{\delta v} &\equiv \frac{\partial \mathcal{L}}{\partial v} - D_x \frac{\partial \mathcal{L}}{\partial v_x}, \end{aligned} \quad (5)$$

where D_t , D_x and D_y are the total differentiations with respect to t , x and y respectively. Introducing the notation

$$\begin{aligned} x^1 &\equiv t, \quad x^2 \equiv x, \quad x^3 \equiv y, \\ u_i &= \frac{\partial u}{\partial x^i}, \quad u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}, \dots, \quad v_i = \frac{\partial v}{\partial x^i}, \quad v_{ij} = \frac{\partial^2 v}{\partial x^i \partial x^j}, \dots, \\ \alpha_i &= \frac{\partial \alpha}{\partial x^i}, \quad \alpha_{ij} = \frac{\partial^2 \alpha}{\partial x^i \partial x^j}, \dots, \quad \beta_i = \frac{\partial \beta}{\partial x^i}, \quad \beta_{ij} = \frac{\partial^2 \beta}{\partial x^i \partial x^j}, \dots \end{aligned} \quad (6)$$

we can write

$$D_{x^i} = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + v_i \frac{\partial}{\partial v} + \alpha_i \frac{\partial}{\partial \alpha} + \beta_i \frac{\partial}{\partial \beta} + u_{ij} \frac{\partial}{\partial u_j} + v_{ij} \frac{\partial}{\partial v_j} + \alpha_{ij} \frac{\partial}{\partial \alpha_j} + \beta_{ij} \frac{\partial}{\partial \beta_j} + \dots \quad i = 1, 2, 3. \quad (7)$$

Taking into account the Eqs. (3), (4), (5) we obtain the following adjoint system for the system (2)

$$\begin{aligned} F_1^* &\equiv \alpha_t + \alpha_x v + \beta_y - \alpha_{xy} = 0 \\ F_2^* &\equiv \alpha_x u - \beta_x = 0. \end{aligned} \quad (8)$$

According to [6] we recall that the nonlinear differential system (2) is said to be quasi self-adjoint if each equations F_i^* of the adjoint system (8) coincides with $\lambda_{i1}F_1 + \lambda_{i2}F_2$ after the following substitution:

$$\alpha = \phi_1(u, v), \quad \beta = \phi_2(u, v). \quad (9)$$

In other words if the adjoint system (8) obeys the condition

$$\begin{aligned} (\alpha_t + \alpha_x v + \beta_y - \alpha_{xy})|_{\alpha=\phi_1(u,v), \beta=\phi_2(u,v)} &= (u_t + u_{xy} + uv_x + vu_x)\lambda_{11} + (u_y - v_x)\lambda_{12}, \\ (\alpha_x u - \beta_x)|_{\alpha=\phi_1(u,v), \beta=\phi_2(u,v)} &= (u_t + u_{xy} + uv_x + vu_x)\lambda_{21} + (u_y - v_x)\lambda_{22} \end{aligned} \quad (10)$$

with regular coefficients λ_{ij} ($i, j = 1, 2$).

Using the differential consequences of (9), the condition (10) for the quasi self-adjointness is written as follows:

$$\begin{aligned} -\phi_u u_t - \phi_v v_t - \phi_u v u_x - \phi_v v v_x - \psi_u u_y - \psi_v v_y + \phi_{uu} u_x u_y + \phi_{uv} u_x v_y + \phi_u u_{xy} + \\ + \phi_{uv} u_y v_x + \phi_{vv} v_x v_y + \phi_v v_{xy} &= \lambda_{11}(u_t + u_{xy} + uv_x + vu_x) + \lambda_{12}(u_y - v_x), \\ -\phi_u u u_x - \phi_v u v_x + \psi_u u_x + \psi_v v_x &= \lambda_{21}(u_t + u_{xy} + uv_x + vu_x) + \lambda_{22}(u_y - v_x). \end{aligned} \quad (11)$$

Eqs. (11) split into the following system

$$\begin{aligned} \phi_u - \lambda_{11} &= 0, \quad \phi_u + \lambda_{11} = 0, \quad \phi_v = 0, \\ \lambda_{21} = \lambda_{22} &= 0, \quad u\lambda_{11} - \lambda_{12} = 0, \quad u\phi_u - \psi_u = 0, \quad \psi_u + \lambda_{12} = 0, \quad \psi_v = 0. \end{aligned} \quad (12)$$

Solving the previous conditions we get

$$\phi = \text{const.}, \quad \psi = \text{const.} \quad (13)$$

So it is verified that the system (2) is quasi self-adjoint and the substitution (9) becomes

$$\alpha = \text{const.}, \quad \beta = \text{const.} \quad (14)$$

3 Conservation laws

In order to get conservation laws we use the following statement proved in [4, 5].

Theorem 1. *Any symmetry (Lie point, Lie-Bäcklund, nonlocal symmetry)*

$$X = \xi^i(x, u, u_{(1)}, \dots) \frac{\partial}{\partial x^i} + \eta^a(x, u, u_{(1)}, \dots) \frac{\partial}{\partial u^a} \quad (15)$$

of a system of m equations

$$F_a(x, u, u_{(1)}, \dots, u_{(s)}) = 0, \quad a = 1, \dots, m, \quad (16)$$

with n independent variables $x = (x^1, \dots, x^n)$ and m dependent variables $u = (u^1, \dots, u^m)$ is inherited by the adjoint system. Specifically the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^a \frac{\partial}{\partial u^a} + \eta_*^a \frac{\partial}{\partial v^a}, \quad (17)$$

with appropriately chosen coefficients η_*^a , is admitted by the system of equations consisting of Eqs. (16) and adjoint equations

$$F_a^*(x, u, v, \dots, u_{(s)}, v_{(s)}) \equiv \frac{\delta(v^b F_b)}{\delta u^a} = 0, \quad a = 1, \dots, m. \quad (18)$$

Furthermore, the combined system (16), (18) has the conservation law $D_i(C^i) = 0$, where

$$\begin{aligned} C^i = & \xi^i \mathcal{L} + W^a \left[\frac{\partial \mathcal{L}}{\partial u_i^a} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^a} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^a} \right) - \dots \right] \\ & + D_j(W^a) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^a} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^a} \right) + \dots \right] + D_j D_k(W^a) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^a} - \dots \right] + \dots, \end{aligned} \quad (19)$$

with

$$W^a = \eta^a - \xi^i u_i^a, \quad a = 1, \dots, m. \quad (20)$$

We apply the previous theorem to look for conservation law of the system (2). Because the system (2) is quasi self-adjoint, we will provide a conserved vector for the system (2) by using the substitution (14) in the vector (19).

By considering the symmetry operator of system (2) in the form

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial v} \quad (21)$$

and the formal Lagrangian (3) for the system (2), the Eqs. (19) become

$$\begin{aligned} C^1 &= W^1 \left(\frac{\partial \mathcal{L}}{\partial u_t} \right), \\ C^2 &= W^1 \left(\frac{\partial \mathcal{L}}{\partial u_x} \right) + W^2 \left(\frac{\partial \mathcal{L}}{\partial v_x} \right) + \left(\frac{\partial \mathcal{L}}{\partial u_{xy}} \right) D_y(W^1), \\ C^3 &= W^1 \left(\frac{\partial \mathcal{L}}{\partial u_y} \right) + \left(\frac{\partial \mathcal{L}}{\partial u_{yx}} \right) D_x(W^1), \end{aligned} \quad (22)$$

where

$$\begin{aligned} W^1 &\equiv \eta_1 - \xi_1 u_t - \xi_2 u_x - \xi_3 u_y, \\ W^2 &\equiv \eta_2 - \xi_1 v_t - \xi_2 v_x - \xi_3 v_y. \end{aligned} \quad (23)$$

Considering the following symmetry operator obtained in [2]

$$X = -t^2 \frac{\partial}{\partial t} - tx \frac{\partial}{\partial x} - ty \frac{\partial}{\partial y} + (tu - y) \frac{\partial}{\partial u} + (tv - x) \frac{\partial}{\partial v}, \quad (24)$$

we obtain

$$\begin{aligned} W^1 &= tu - y + t^2 u_t + tx u_x + ty u_y, \\ W^2 &= tv - x + t^2 v_t + tx v_x + ty v_y. \end{aligned} \quad (25)$$

Taking into account that α and β are constants, from (22) and (3) we obtain the conservation vector

$$C \equiv (C^1, C^2, C^3) = (\alpha tu, (tu + y)\alpha + \beta t^2 v_t, (tu_x - yu)\alpha - \beta t^2 u_t). \quad (26)$$

The vector (26) obeys the equation

$$D_t(C^1) + D_x(C^2) + D_y(C^3) = 0 \quad (27)$$

for all solutions of the system (2), that is

$$D_t(C^1) + D_x(C^2) + D_y(C^3) \big|_{u_t + u_{xy} + uv_x + vu_x = 0, u_y - v_x = 0} = 0, \quad (28)$$

because

$$D_t(C^1) + D_x(C^2) + D_y(C^3) = \alpha(u_t + u_x + u_{xy} - u_y) + \beta t^2(v_{tx} - u_{ty}). \quad (29)$$

4 Conclusion

We have considered the $(2 + 1)$ dimensional generalized Burgers equation (1) wrote as a system (2) in two dependent variables.

Using the recent concepts of adjoint system to nonlinear system and quasi self-adjoint nonlinear system [6], we have shown that the system (2) is quasi self-adjoint.

Finally as an application, taking into account [4, 5] we get additional conservation law for the system (2) by using the knowledge of a Lie symmetry independently on existence of classical Lagrangians.

Acknowledgments

This work was done during the visit of Prof. Ibragimov at Dipartimento di Matematica e Informatica of the University of Catania. The work has been supported by the University of Catania through P.R.A. (ex 60%).

References

- [1] Yan Zhenya, Singularity Structure Analysis and Abundant New Dromion-like Structures for the (2+1)-Dimensional Generalized Burgers Equation, *Chinese Journal of Physics*, **3**, 203-213, 2002.
- [2] Senthilvelan M., Torrisi M., Symmetry analysis and linearization of the (2+1) dimensional Burgers equation. “WASCOM 2005” - 13th Conference on Waves and Stability in Continuous Media, World Sci. Publ., Hackensack, NJ, 493-504, 2006.
- [3] Ibragimov N.H., Integrating factors, adjoint equations and Lagrangians, *J. Math. Anal. Appl.* 318, 742-757, 2006.
- [4] Ibragimov N.H., The answer to the question put to me by L.V. Ovsiannikov 33 years ago, *Archives of ALGA*, Vol. 3, 53-80, 2006.
- [5] Ibragimov N.H., A new conservation theorem, *J. Math. Anal. Appl.* 333, 311-328, 2007.
- [6] Ibragimov N.H., Quasi-self-adjoint differential equations, *Archives of ALGA*, Vol. 4, 55-60, 2007.
- [7] Ibragimov N.H., Torrisi M., Tracinà R., Quasi-self-adjoint nonlinear wave equations, *J. Phys. A: Math. Theor.*, **43**, 2010.