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UTILIZATION OF CANONICAL VARIABLES FOR INTEGRATION OF SYSTEMS OF FIRST-ORDER DIFFERENTIAL EQUATIONS

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Abstract. Systems of two nonlinear ordinary differential of the first order admitting nonlinear superpositions are investigated using Lie's enumeration of group on the plane. It is shown that the systems associated with two-dimensional Vessiot-Guldberg-Lie algebras can be integrated by quadrature upon introducing Lie's canonical variables. The knowledge of a symmetry group of a system in question is not needed in this approach. The systems associated with three-dimensional Vessiot-Guldberg-Lie algebras are classified into 13 standard forms 10 of which are integrable by quadratures and three are reduced to Riccati equations. It is also shown that canonical variables furnish a convenient tool for solving systems of two linear partial differential equations of the first order.

Keywords: Two-dimensional Lie algebras L_2, Standard forms of L_2, Canonical variables, Non-linear superposition, First-order linear partial differential equations.

1 Canonical variables for two-dimensional Lie algebras

Consider linearly independent first-order linear partial differential operators

$$X_1 = \xi_1(x, y)\frac{\partial}{\partial x} + \eta_1(x, y)\frac{\partial}{\partial y}\,, \quad X_2 = \xi_2(x, y)\frac{\partial}{\partial x} + \eta_2(x, y)\frac{\partial}{\partial y} \tag{1.1}$$

with two variables x, y. The *commutator* $[X_1, X_2]$ of the operators (1.1) is a linear partial differential operator defined by the formula

$$[X_1, X_2] = X_1X_2 - X_2X_1,$$

or equivalently

$$[X_1, X_2] = \Big(X_1(\xi_2) - X_2(\xi_1)\Big)\frac{\partial}{\partial x} + \Big(X_1(\eta_2) - X_2(\eta_1)\Big)\frac{\partial}{\partial y}\cdot \tag{1.2}$$

The linear space L_2 spanned by the operators (1.1) is a two-dimensional Lie if

$$[X_1, X_2] \in L_2. \tag{1.3}$$

In order to formulate the result on canonical variables in two-dimensional Lie algebras, it is convenient to use, along with the commutator $[X_1, X_2]$ of the operators (1.1), their *pseudo-scalar product*

$$X_1 \vee X_2 = \xi_1 \eta_2 - \eta_1 \xi_2. \tag{1.4}$$

Recall that the operators (1.1) are said to be *linearly connected* if the equation

$$\lambda_1(x, y) X_1 + \lambda_2(x, y) X_2 = 0$$

holds identically in x, y with certain functions $\lambda_1(x, y),\ \lambda_2(x, y)$, not both zero.

A geometrical significance of the pseudo-scalar product is clarified by the following statement: *the operators* (1.1) *are linearly connected if and only if their pseudo-scalar product* (1.4) *vanishes.*

Lie's method of integration of second-order ordinary differential equations by using their symmetries is based on existence of so-called *canonical coordinates* in two-dimensional Lie algebras. These variables provide, for every L_2, the simplest form of its basis and therefore reduce a differential equation admitting L_2, to an integrable form. The basic statement is formulated as follows (for the proof, see[2], Chapter 18, §1; see also [1], Section 12.2.2).

Theorem 1.1. Any two dimensional Lie algebra can be transformed, by a proper choice of its basis and suitable variables t, u, to one and only one of the four non-similar standard forms presented in the following table.

Table 1. Structure and standard forms of L_2

Type	Structure of L_2	Standard form of L_2
I	$[X_1,\ X_2] = 0, \quad X_1 \vee X_2 \neq 0$	$X_1 = \dfrac{\partial}{\partial t},\ X_2 = \dfrac{\partial}{\partial u}$
II	$[X_1,\ X_2] = 0, \quad X_1 \vee X_2 = 0$	$X_1 = \dfrac{\partial}{\partial u},\ X_2 = t\dfrac{\partial}{\partial u}$
III	$[X_1,\ X_2] = X_1,\ X_1 \vee X_2 \neq 0$	$X_1 = \dfrac{\partial}{\partial u},\ X_2 = t\dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial u}$
IV	$[X_1,\ X_2] = X_1,\ X_1 \vee X_2 = 0$	$X_1 = \dfrac{\partial}{\partial u},\ X_2 = u\dfrac{\partial}{\partial u}$

The variables t, u presented in Table 1 are called *canonical variables.* They are found, for each type, by solving the following systems of first-order linear partial differential equations:

$$
\begin{aligned}
\textbf{Type I}: &\quad X_1(t) = 1,\ X_2(t) = 0; \quad X_1(u) = 0,\ X_2(u) = 1.\\
\textbf{Type II}: &\quad X_1(t) = 0,\ X_2(t) = 0; \quad X_1(u) = 1,\ X_2(u) = t.\\
\textbf{Type III}: &\quad X_1(t) = 0,\ X_2(t) = t; \quad X_1(u) = 1,\ X_2(u) = u.\\
\textbf{Type IV}: &\quad X_1(t) = 0,\ X_2(t) = 0; \quad X_1(u) = 1,\ X_2(u) = u.
\end{aligned}
\tag{1.5}
$$

2 Application to systems with nonlinear superposition

2.1 Integration of systems associated with L_2

A method for integrating systems of ordinary differential equations admitting nonlinear superpositions with two-dimensional associated Lie algebras L_2 was suggested in [1], Section 11.2.6. The result is formulated as follows.

Theorem 2.1. Consider a system of coupled nonlinear first-order ordinary differential equations of the form

$$
\begin{aligned}
\frac{dx}{dt} &= T_1(t)\,\xi_1^1(x,y) + T_2(t)\,\xi_2^1(x,y),\\
\frac{dy}{dt} &= T_1(t)\,\xi_1^2(x,y) + T_2(t)\,\xi_2^2(x,y)
\end{aligned}
\tag{2.1}
$$

admitting a nonlinear superposition principle. Let the operators

$$
X_1 = \xi_1^1(x,y)\frac{\partial}{\partial x} + \xi_1^2(x,y)\frac{\partial}{\partial y}\,, \quad X_2 = \xi_2^1(x,y)\frac{\partial}{\partial x} + \xi_2^2(x,y)\frac{\partial}{\partial y} \tag{2.2}
$$

associated with the system (2.1) span a two-dimensional Lie algebra L_2, i.e.

$$
[X_1, X_2] = c_1X_1 + c_2X_2, \quad c_1, c_2 = \text{const.}
$$

Then Eqs. (2.1) can be solved by quadratures upon introducing canonical variables.

Proof. After a change of the variables x, y into new variables

$$
\tilde{x} = \tilde{x}(x,y), \quad \tilde{y} = \tilde{y}(x,y) \tag{2.3}
$$

without changing t, the operators (2.2) will be transformed into the operators

$$
X_1 = \tilde{\xi}_1^1(\tilde{x},\tilde{y})\frac{\partial}{\partial \tilde{x}} + \tilde{\xi}_1^2(\tilde{x},\tilde{y})\frac{\partial}{\partial \tilde{y}}\,, \quad X_2 = \tilde{\xi}_2^1(\tilde{x},\tilde{y})\frac{\partial}{\partial \tilde{x}} + \tilde{\xi}_2^2(\tilde{x},\tilde{y})\frac{\partial}{\partial \tilde{y}}\,, \tag{2.4}
$$

where the vectors $(\tilde{\xi}^1_\alpha, \tilde{\xi}^2_\alpha),\ \alpha = 1, 2$ are obtained from the vectors $(\xi^1_\alpha, \xi^2_\alpha)$ by the transformation law for contravariant vectors:

$$\tilde{\xi}^1_\alpha = \frac{\partial \tilde{x}(x,y)}{\partial x}\,\xi^1_\alpha + \frac{\partial \tilde{x}(x,y)}{\partial y}\,\xi^2_\alpha\,,$$

$$\tilde{\xi}^2_\alpha = \frac{\partial \tilde{y}(x,y)}{\partial x}\,\xi^1_\alpha + \frac{\partial \tilde{y}(x,y)}{\partial y}\,\xi^2_\alpha\,.$$

The derivative of (x, y) with respect to t obeys the same transformation law:

$$\frac{d\tilde{x}}{dt} = \frac{\partial \tilde{x}(x,y)}{\partial x}\frac{dx}{dt} + \frac{\partial \tilde{x}(x,y)}{\partial y}\frac{dy}{dt}\,,$$

$$\frac{d\tilde{y}}{dt} = \frac{\partial \tilde{y}(x,y)}{\partial x}\frac{dx}{dt} + \frac{\partial \tilde{y}(x,y)}{\partial y}\frac{dy}{dt}\,.$$

Therefore Eqs. (2.1) will be written in the form

$$\begin{aligned} \frac{d\tilde{x}}{dt} &= T_1(t)\,\tilde{\xi}^1_1(\tilde{x},\tilde{y}) + T_2(t)\,\tilde{\xi}^1_2(\tilde{x},\tilde{y}),\\ \frac{d\tilde{y}}{dt} &= T_1(t)\,\tilde{\xi}^2_1(\tilde{x},\tilde{y}) + T_2(t)\,\tilde{\xi}^2_2(\tilde{x},\tilde{y}) \end{aligned} \tag{2.5}$$

with the same coefficients $T_1(t), T_2(t)$ as those in the system (2.1).

To complete the proof, we chose for $\tilde{x}, \tilde{y}$ canonical variables mapping the operators (2.2) to the standard forms from Table 1, and hence convert Eqs. (2.1) to the simple integrable forms given in the following table, where $\tilde{x}, \tilde{y}$ are denoted again by x, y.

Table 2. Standard forms of operators (2.2) and systems (2.1)

	Standard forms of operators (2.2)	Standard forms of Eqs. (2.1)
I	$X_1 = \frac{\partial}{\partial x}\,,\ X_2 = \frac{\partial}{\partial y}$	$\frac{\mathrm{d}x}{\mathrm{d}t} = T_1(t),\ \frac{\mathrm{d}y}{\mathrm{d}t} = T_2(t)$
II	$X_1 = \frac{\partial}{\partial y}\,,\ X_2 = x\frac{\partial}{\partial y}$	$\frac{\mathrm{d}x}{\mathrm{d}t} = 0,\ \frac{\mathrm{d}y}{\mathrm{d}t} = T_1(t) + T_2(t)x$
III	$X_1 = \frac{\partial}{\partial y}\,,\ X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$\frac{\mathrm{d}x}{\mathrm{d}t} = T_2(t)x,\ \frac{\mathrm{d}y}{\mathrm{d}t} = T_1(t) + T_2(t)y$
IV	$X_1 = \frac{\partial}{\partial y}\,,\ X_2 = y\frac{\partial}{\partial y}$	$\frac{\mathrm{d}x}{\mathrm{d}t} = 0,\ \frac{\mathrm{d}y}{\mathrm{d}t} = T_1(t) + T_2(t)y$

Example 2.1. Let us apply the method to the following nonlinear system:

$$\frac{dx}{\mathrm{d}t} = xy^2 - \frac{x}{2t}\,, \quad \frac{dy}{\mathrm{d}t} = x^2 y - \frac{y}{2t}\,\cdot \tag{2.6}$$

In this case we have Eqs. (2.1) with

$$\begin{aligned} &T_1(t) = 1, && \xi_1^1(x,y) = xy^2, && \xi_1^2(x,y) = x^2 y,\\ &T_2(t) = -\frac{1}{2t}\,, && \xi_2^1(x,y) = x, && \xi_2^2(x,y) = y. \end{aligned} \tag{2.7}$$

Hence, the operators (2.2) are written:

$$X_1 = xy^2\frac{\partial}{\partial x} + x^2 y\frac{\partial}{\partial y}\,, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,\cdot \tag{2.8}$$

We have:

$$[X_1,\ X_2] = -2X_1, \quad X_1 \vee X_2 \equiv \xi_1\eta_2 - \eta_1\xi_2 = xy(y^2 - x^2) \neq 0.$$

Hence, the operators (2.8) span a two-dimensional Lie algebra of type III. Therefore we can transform the operators (2.8) and Eqs. (2.6) to the form III from Table 2.

Let us find canonical variables $\tilde{x}, \tilde{y}$ for the first operator (2.8) by solving the equations

$$X_1(\tilde{x}) = 0, \quad X_1(\tilde{y}) = 1$$

in accordance with Eqs. (1.5) for Type III. These equations are written

$$xy^2\frac{\partial \tilde{x}}{\partial x} + x^2 y\frac{\partial \tilde{x}}{\partial y} = 0, \quad xy^2\frac{\partial \tilde{y}}{\partial x} + x^2 y\frac{\partial \tilde{y}}{\partial y} = 1.$$

The characteristic equation

$$\frac{dx}{y} - \frac{dy}{x} = 0$$

for the equation $X_1(\tilde{x}) = 0$ has the first integral $x^2 - y^2 = \mathrm{const}$. Hence, $\tilde{x}$ is an arbitrary function of $x^2 - y^2$. One can take it in the simplest form $\tilde{x} = x^2 - y^2$.

Let us solve the equation $X_1(\tilde{y}) = 1$. Consider its characteristic system

$$\frac{dx}{xy^2} = \frac{dy}{x^2 y} = d\tilde{y}.$$

Using the integral given by the first equation of this system in the form $x^2 - y^2 = a^2$, we write the second equation $dx/(xy^2) = d\tilde{y}$ of the characteristic system in the form

$$\frac{dx}{x(x^2 - a^2)} = d\tilde{y}$$

or

$$d\tilde{y} = \frac{1}{a^2}\left[\frac{1}{2(x-a)} + \frac{1}{2(x+a)} - \frac{1}{x}\right]dx.$$

The resulting integral

$$\tilde{y} - \frac{1}{a^2}\left[\ln\sqrt{x^2-a^2} - \ln|x|\right] = C$$

together with the integral $x^2 - y^2 = a^2$ provide the solution to the equation $X_1(\tilde{y}) = 1$:

$$\tilde{y} = \frac{\ln|y| - \ln|x|}{x^2-y^2} + F(x^2-y^2).$$

Letting $F = 0$ and assuming that x, y are positive, we obtain the following variables:

$$\tilde{x} = x^2 - y^2, \quad \tilde{y} = \frac{\ln y - \ln x}{x^2-y^2}\,\cdot \tag{2.9}$$

One can verify that the variables (2.9) are the canonical variables required for our algebra L_2. Indeed, the operators (2.8) are written in the form of type III of Table 2 (up to nonessential constant factors in X_2):

$$X_1 = \frac{\partial}{\partial\tilde{y}}\,, \quad X_2 = 2\Big(\tilde{x}\,\frac{\partial}{\partial\tilde{x}} - \tilde{y}\frac{\partial}{\partial\tilde{y}}\Big).$$

These operators have the form (2.4) with

$$\tilde{\xi}_1^1 = 0, \quad \tilde{\xi}_1^2 = 1, \quad \tilde{\xi}_2^1 = 2\tilde{x}, \quad \tilde{\xi}_2^2 = -2\tilde{y}.$$

Substituting these expressions in (2.5) (or differentiating (2.9) with respect to t and using Eqs. (2.6)) we see that Eqs. (2.6) are written in the variables (2.9) as the following simple linear equations:

$$\frac{d\tilde{x}}{dt} = -\frac{\tilde{x}}{t}\,, \quad \frac{d\tilde{y}}{dt} = 1 + \frac{\tilde{y}}{t}\,\cdot \tag{2.10}$$

Integration of Eqs. (2.10) yields:

$$\tilde{x} = \frac{C_1}{t}\,, \quad \tilde{y} = C_2\,t + t\ln t. \tag{2.11}$$

Now we solve Eqs. (2.9) with respect to x and y :

$$x = \sqrt{\frac{\tilde{x}}{1 - \mathrm{e}^{2\tilde{x}\,\tilde{y}}}}\,, \quad y = \sqrt{\frac{\tilde{x}}{\mathrm{e}^{-2\tilde{x}\,\tilde{y}} - 1}}\,,$$

substitute here the solutions (2.11) and finally arrive at the following general solution to the system of equations (2.6):

$$x = \sqrt{\frac{k}{t(1-\zeta^2)}}\,, \quad y = \zeta\sqrt{\frac{k}{t(1-\zeta^2)}}\,. \tag{2.12}$$

Here $\zeta = Ct^k$, where C and k are arbitrary constants.

2.2 Lie's classification of L_3

Lie showed that the basis X_1, X_2, X_3 of any three-dimensional algebra of operators in two variables can be mapped, by a complex change of variables, to one of the following 13 *standard forms* (see [2], Chapter 22; see also [1], Section 7.3.8).

Table 3. Standard forms of three-dimensional Lie algebras

A. The first derived algebra has the dimension three :

1) $X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\,, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,, \quad X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\,,$

2) $X_1 = \frac{\partial}{\partial x}\,, \quad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,, \quad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\,,$

3) $X_1 = \frac{\partial}{\partial y}\,, \quad X_2 = y\frac{\partial}{\partial y}\,, \quad X_3 = y^2\frac{\partial}{\partial y}\,\cdot$

B. The first derived algebra has the dimension two :

4) $X_1 = \frac{\partial}{\partial x}\,, \quad X_2 = \frac{\partial}{\partial y}\,, \quad X_3 = x\frac{\partial}{\partial x} + cy\frac{\partial}{\partial y}\ (c \neq 0, \neq 1),$

5) $X_1 = \frac{\partial}{\partial y}\,, \quad X_2 = x\frac{\partial}{\partial y}\,, \quad X_3 = (1-c)x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\ (c \neq 0, \neq 1),$

6) $X_1 = \frac{\partial}{\partial x}\,, \quad X_2 = \frac{\partial}{\partial y}\,, \quad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,,$

7) $X_1 = \frac{\partial}{\partial y}\,, \quad X_2 = x\frac{\partial}{\partial y}\,, \quad X_3 = y\frac{\partial}{\partial y}\,,$

8) $X_1 = \frac{\partial}{\partial x}\,, \quad X_2 = \frac{\partial}{\partial y}\,, \quad X_3 = (x+y)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,,$

9) $X_1 = \frac{\partial}{\partial y}\,, \quad X_2 = x\frac{\partial}{\partial y}\,, \quad X_3 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,\cdot$

C. The first derived algebra has the dimension one :

10) $X_1 = \frac{\partial}{\partial x}\,, \quad X_2 = \frac{\partial}{\partial y}\,, \quad X_3 = x\frac{\partial}{\partial x}\,,$

11) $X_1 = \frac{\partial}{\partial y}\,, \quad X_2 = x\frac{\partial}{\partial y}\,, \quad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,,$

12) $X_1 = \frac{\partial}{\partial x}\,, \quad X_2 = \frac{\partial}{\partial y}\,, \quad X_3 = x\frac{\partial}{\partial y}\,\cdot$

D. The first derived algebra has the dimension zero :

13) $X_1 = \frac{\partial}{\partial y}\,, \quad X_2 = x\frac{\partial}{\partial y}\,, \quad X_3 = p(x)\frac{\partial}{\partial y}\,\cdot$

Remark 2.1. In 13), $p(x)$ is any given function. Lie uses the algebras 1) and 2) also in the following alternative forms:

$$1')\ X_1 = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\,, \quad X_2 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}\,, \quad X_3 = (x^2 - y)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\,,$$
$$2')\ X_1 = x\frac{\partial}{\partial y}\,, \quad X_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\,, \quad X_3 = y\frac{\partial}{\partial x}\,\cdot$$

Recall that the derived algebra L_3' of the Lie algebra L_3 with a basis X_1, X_2, X_3 is the algebra spanned by the commutators $[X_1, X_2], \quad [X_1, X_3], \quad [X_2, X_3]$. The higher derivatives are defined by induction, $L_3'' = (L_3')'$, etc. A Lie algebra is *solvable* if its derivative of a certain order vanishes. It is obvious that L_3 is solvable if $\dim L_3' \le 2$ and not solvable if $\dim L_3' = 3$.

2.3 Integration of systems associated with L_3

Using Lie'sa classification of three-dimensional algebras, I extend Theorem 2.1 from Section 2.1 as follows.

Theorem 2.2. Consider a system of coupled nonlinear first-order ordinary differential equations of the form

$$\begin{aligned} \frac{dx}{dt} &= T_1(t)\,\xi_1^1(x,y) + T_2(t)\,\xi_2^1(x,y) + T_3(t)\,\xi_3^1(x,y),\\ \frac{dy}{dt} &= T_1(t)\,\xi_1^2(x,y) + T_2(t)\,\xi_2^2(x,y) + T_3(t)\,\xi_3^1(x,y) \end{aligned} \tag{2.13}$$

admitting a nonlinear superposition principle. Let the operators

$$\begin{aligned} X_1 &= \xi_1^1(x,y)\frac{\partial}{\partial x} + \xi_1^2(x,y)\frac{\partial}{\partial y}\,,\\ X_2 &= \xi_2^1(x,y)\frac{\partial}{\partial x} + \xi_2^2(x,y)\frac{\partial}{\partial y}\,,\\ X_3 &= \xi_3^1(x,y)\frac{\partial}{\partial x} + \xi_3^2(x,y)\frac{\partial}{\partial y} \end{aligned} \tag{2.14}$$

associated with the system (2.13) span a three-dimensional Lie algebra L_3. Then Eqs. (2.13) can be solved by quadratures if the algebra L_3 is solvable and reduced to integration of Riccati equations if L_3 is not solvable.

Proof. We transform the Lie algebra L_3 associated with the system (2.13) to an appropriate standard form given in Table 3 and map Eqs. (2.13) to the following forms.

Table 4. Standard forms of Eqs. (2.13)

A. $\dim L'_3 = 3:$

1) $\dot{x} = T_1(t) + T_2(t)x + T_3(t)x^2, \qquad \dot{y} = T_1(t) + T_2(t)y + T_3(t)y^2;$

2) $\dot{x} = T_1(t) + 2T_2(t)x + T_3(t)x^2, \qquad \dot{y} = T_2(t)y + T_3(t)xy;$

3) $\dot{x} = 0, \qquad \dot{y} = T_1(t) + T_2(t)y + T_3(t)y^2.$

B. $\dim L'_3 = 2:$

4) $\dot{x} = T_1(t) + 2T_2(t)x + T_3(t)x^2, \qquad \dot{y} = T_2(t)y + T_3(t)xy \ (c \neq 0, \neq 1);$

5) $\dot{x} = (1-c)T_3(t)x \ (c \neq 0, \neq 1), \qquad \dot{y} = T_1/t) + T_2(t)x + T_3(t)y;$

6) $\dot{x} = T_1(t) + T_3(t)x, \qquad \dot{y} = T_2(t) + T_3(t)y;$

7) $\dot{x} = 0, \qquad \dot{y} = T_1(t) + T_2(t)x + T_3(t)y;$

8) $\dot{x} = T_1(t) + T_2(t)(x+y), \qquad \dot{y} = T_2(t) + T_3(t)y;$

9) $\dot{x} = T_3(t), \qquad \dot{y} = T_1(t) + T_2(t)x + T_3(t)y.$

C. $\dim L'_3 = 1:$

10) $\dot{x} = T_1(t) + T_3(t)x, \qquad \dot{y} = T_2(t);$

11) $\dot{x} = T_3(t)x, \qquad \dot{y} = T_1(t) + T_2(t)x + T_3(t)y;$

12) $\dot{x} = T_1(t), \qquad \dot{y} = T_2(t) + T_3(t)x.$

D. $\dim L'_3 = 0:$

13) $\dot{x} = 0, \qquad \dot{y} = T_1(t) + T_2(t)x + T_3(t)p(x).$

In this table, $\dot{x}, \dot{y}$ are the derivatives of x, y with respect to t.

It is manifest that the systems of forms B, C and D can be solved by quadratures. It is also obvious that the systems in A require integration of Riccati equations and, in general, cannot be solved by quadratures. This completes the proof of the theorem.

Remark 2.2. The alternative forms in Remark 2.1 provide the following alternative standard forms of Eqs. (2.13):

1′) $\dot{x} = T_1(t) + T_2(t)x + T_3(t)(x^2 - 1), \qquad \dot{y} = T_1(t)x + 2T_2(t)y + T_3(t)xy;$

2′) $\dot{x} = T_2(t)x + T_3(t)y, \qquad \dot{y} = T_1(t)x - T_2(t)y.$

In certain particular cases the systems in A can be integrated either by quadratures or in terms of special functions. The simplest case is $T_3(t) = 0$. Then the equations 1) - 3) become easily solvable linear systems. Furthermore, if $T_1(t) = 0$ the Riccati equations in systems 1) - 3) can be linearized by a change of the dependent variables (see [3], Chapter 1). Moreover, it is demonstrated in [3] that the Riccati equations in systems 1), 3) can be linearized by a change of the dependent variables if

$$T_3(t) = k[T_2/t) - kT_1(t)],\ k = \text{const}.$$

In the case of the system 2) this condition is replaced by $T_3(t) = k[2T_2/t) - kT_1(t)]$.

It is well known that if $T_3(t) \neq 0$, one can transform the Riccati equations in question to the equivalent form with $T_3(t) = -1,\ T_2(t) = 0$. Assuming that this transformation has been done, let us consider, e.g., the system 2),

$$\dot{x} + x^2 = T_1(t), \quad \dot{y} + xy = 0. \tag{2.15}$$

We set $x = (\ln|u|)'$ and rewrite the first equation of this system in the form of a linear second-order equation

$$u'' = T_1(t),$$

where u' is the derivative of u with respect to t. The above equation can be solved in terms of special function if $T_1(t)$ is a linear function. Indeed, let $T_1(t) = \alpha t + \beta,\ \alpha \neq 0$. Then our equation

$$u'' = \alpha t + \beta$$

becomes the *Airy equation*

$$\frac{d^2u}{d\tau^2} - \tau\, u = 0$$

upon introducing the new independent variable

$$\tau = \alpha^{-2/3}[\alpha t + \beta].$$

The general solution to the Airy equation is given by the linear combination

$$u = C_1 \text{Ai}(\tau) + C_2 \text{Bi}(\tau)$$

of the *Airy functions*

$$\text{Ai}(\tau) = \frac{1}{\pi}\int_0^\infty \cos\Big(s\tau + \frac{1}{3}\,s^3\Big)ds,$$

$$\text{Bi}(\tau) = \frac{1}{\pi}\int_0^\infty \Big[\exp\Big(s\tau - \frac{1}{3}\,s^3\Big) + \sin\Big(s\tau + \frac{1}{3}\,s^3\Big)\Big]ds.$$

Assuming that $C_1 \neq 0$ and introducing the new constant $K_1 = C_2/C_1$ we obtain

$$x(t) = \frac{d}{dt}\ln\left|\mathrm{Ai}\big(\alpha^{-2/3}[\alpha t+\beta]\big) + K_1\mathrm{Bi}\big(\alpha^{-2/3}[\alpha t+\beta]\big)\right|. \tag{2.16}$$

Now we substitute (2.16) in the second equation of the system (2.15) and obtain upon integration:

$$y(t) = K_2\left\{\mathrm{Ai}\big(\alpha^{-2/3}[\alpha t+\beta]\big) + K_1\mathrm{Bi}\big(\alpha^{-2/3}[\alpha t+\beta]\big)\right\}^{-1}. \tag{2.17}$$

Thus, the solution of the system (2.15) with $T_1(t) = \alpha t + \beta$ is given by the special functions (2.16), (2.17).

Example 2.2. Consider the nonlinear system

$$\begin{aligned} \frac{dx}{dt} &= -T_1(t)\, y\, \mathrm{e}^{\arctan(y/x)} + T_2(t)x + T_3(t)y,\\ \frac{dy}{dt} &= \ T_(t)\, x\, \mathrm{e}^{\arctan(y/x)} + T_2(t)y - T_3(t)x \end{aligned} \tag{2.18}$$

with arbitrary coefficients $T_1(t), T_2(t), T_3(t)$. The operators (2.14) associated with Eqs. (2.18) have the form

$$\begin{aligned} X_1 &= \mathrm{e}^{\arctan(y/x)}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right),\\ X_2 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\\ X_3 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \end{aligned} \tag{2.19}$$

and span a three-dimensional Lie algebra L_3 with the following commutator relations:

$$[X_1,\ X_2] = 0, \quad [X_1,\ X_3] = X_1, \quad [X_2,\ X_3] = 0. \tag{2.20}$$

It follows that the derived algebra L_3' has the dimension one, and hence our algebra L_3 belongs to the category C of Table 3. Specifically, comparison of the commutator relations (2.20) with the commutators of the standard operators 10), 11) or 12) from Table 3 shows that the operators (2.19) can be mapped by a change of variables (2.3) either to 10) or to 11). However, it is easy to show that they cannot be mapped to the form 11). Indeed, the change of variables (2.3) converts (2.19) to the form 11),

$$X_1 = \frac{\partial}{\partial \tilde{y}}, \quad X_2 = \tilde{x}\frac{\partial}{\partial \tilde{y}}, \quad X_3 = \tilde{x}\frac{\partial}{\partial \tilde{x}} + \tilde{y}\frac{\partial}{\partial \tilde{y}},$$

if $\tilde{x}$ and $\tilde{y}$ solve the following over-determined systems:

$$\begin{aligned} X_1(\tilde{x}) &= 0, \qquad & X_1(\tilde{y}) &= 1,\\ X_2(\tilde{x}) &= 0, \qquad & X_2(\tilde{y}) &= \tilde{x},\\ X_3(\tilde{x}) &= \tilde{x}, \qquad & X_3(\tilde{y}) &= \tilde{y}, \end{aligned}$$

where X_1, X_2, X_3 are the operators (2.19). These equations are not compatible. For example, the equations $X_1(\tilde{x}) = 0$ and $X_3(\tilde{x}) = \tilde{x}$ contradict each other because X_1 differs from X_3 by a non-vanishing factor only. For another reasoning, see the general construction of similarity transformations given in [1], Section 7.3.7.

Let us find the change of variables (2.3) mapping (2.19) to the form 10),

$$X_1 = \frac{\partial}{\partial \tilde{x}}\,, \quad X_2 = \frac{\partial}{\partial \tilde{y}}\,, \quad X_3 = \tilde{x}\frac{\partial}{\partial \tilde{x}}\cdot \tag{2.21}$$

Now $\tilde{x}$ and $\tilde{y}$ should solve the following over-determined systems:

$$\begin{aligned} X_1(\tilde{x}) &= 1, \qquad & X_1(\tilde{y}) &= 0,\\ X_2(\tilde{x}) &= 0, \qquad & X_2(\tilde{y}) &= 1,\\ X_3(\tilde{x}) &= \tilde{x}, \qquad & X_3(\tilde{y}) &= 0. \end{aligned}$$

Substituting the expressions (2.19) for X_1, X_2, X_3 we write these equations in the form

$$\begin{aligned} x\frac{\partial \tilde{x}}{\partial y} - y\frac{\partial \tilde{x}}{\partial x} &= \mathrm{e}^{-\arctan(y/x)}, \qquad & x\frac{\partial \tilde{y}}{\partial y} - y\frac{\partial \tilde{y}}{\partial x} &= 0,\\ x\frac{\partial \tilde{x}}{\partial x} + y\frac{\partial \tilde{x}}{\partial y} &= 0, \qquad & x\frac{\partial \tilde{y}}{\partial x} + y\frac{\partial \tilde{y}}{\partial y} &= 1,\\ x\frac{\partial \tilde{x}}{\partial y} - y\frac{\partial \tilde{x}}{\partial x} &= \tilde{x}, \qquad & x\frac{\partial \tilde{y}}{\partial y} - y\frac{\partial \tilde{y}}{\partial x} &= 0. \end{aligned} \tag{2.22}$$

Comparison of the first and third equations for $\tilde{x}$ yields $\tilde{x} = \mathrm{e}^{-\arctan(y/x)}$. One can readily verify that this function solves all three equations (2.22) for $\tilde{x}$. The equations for $\tilde{y}$ are easy to solve and yield $\tilde{y} = \ln\sqrt{x^2+y^2}$. Thus, the canonical variables mapping the operators (2.19) to the standard form (2.21) are given by

$$\tilde{x} = \mathrm{e}^{-\arctan(y/x)}, \quad \tilde{y} = \ln\sqrt{x^2+y^2}. \tag{2.23}$$

In these variables Eqs. (2.18) are written in the integrable form 10) from Table 4:

$$\frac{d\tilde{x}}{dt} = T_1(t) + T_3(t)\,\tilde{x}, \quad \frac{d\tilde{y}}{dt} = T_1(t). \tag{2.24}$$

3 Calculation of invariants using canonical variables

I will explain the essence of the approach by discussing the problem on invariants of a linear representation of the two-parameter group composed by the homogeneous dilations and rotations on the (x, y) plain. Namely, let us find the invariants $J(x, y, u, v, w)$ of the two-parameter group with the generators

$$\begin{aligned} X_1 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + kw\frac{\partial}{\partial w}\,, \\ X_2 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + (v + \alpha x + \beta y)\frac{\partial}{\partial u} - (u + \gamma x + \delta y)\frac{\partial}{\partial v}\,, \end{aligned} \tag{3.1}$$

where $k, \alpha, \beta, \gamma, \delta$ are any constants.

3.1 Computation in original variables

We have to solve the following system of linear homogeneous first-order partial differential equations:

$$X_1(J) \equiv x\frac{\partial J}{\partial x} + y\frac{\partial J}{\partial y} + u\frac{\partial J}{\partial u} + v\frac{\partial J}{\partial v} + kw\frac{\partial J}{\partial w} = 0, \tag{3.2}$$

$$X_2(J) = y\frac{\partial J}{\partial x} - x\frac{\partial J}{\partial y} + (v + \alpha x + \beta y)\frac{\partial J}{\partial u} - (u + \gamma x + \delta y)\frac{\partial J}{\partial v} = 0. \tag{3.3}$$

The integration of the characteristic system for Eq. (3.2),

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u} = \frac{dv}{v} = \frac{dw}{kw}\,,$$

provides the following invariants for X_1 :

$$\lambda = \frac{y}{x}\,, \quad U = \frac{u}{x}\,, \quad V = \frac{v}{x}\,, \quad W = \frac{w}{x^k}\,\cdot \tag{3.4}$$

Hence, the general solution to Eq. (3.2) has the form

$$J = J(\lambda, U, V, W).$$

Therefore we write the action of the operator X_2 on functions depending on λ, U, V, W. This action is obtained by the formula

$$X_2 = X_2(\lambda)\frac{\partial}{\partial \lambda} + X_2(U)\frac{\partial}{\partial U} + X_2(V)\frac{\partial}{\partial V} + X_2(W)\frac{\partial}{\partial W}\,\cdot$$

The reckoning yields:

$$X_2(\lambda) = -(1+\lambda^2), \quad X_2(U) = V - \lambda U + \alpha + \beta\lambda,$$
$$X_2(V) = -(U + \lambda V + \gamma + \delta\lambda), \quad X_2(W) = -k\lambda W,$$

and hence

$$X_2 = -(1+\lambda^2)\frac{\partial}{\partial\lambda} + (V - \lambda U + \alpha + \beta\lambda)\frac{\partial}{\partial U} - (U + \lambda V + \gamma + \delta\lambda)\frac{\partial}{\partial V} - k\lambda W\frac{\partial}{\partial W}\,\cdot$$

Now the characteristic system for Eq. (3.3) is written

$$-\frac{d\lambda}{1+\lambda^2} = \frac{dU}{V - \lambda U + \alpha + \beta\lambda} = -\frac{dV}{U + \lambda V + \gamma + \delta\lambda} = -\frac{dW}{k\lambda W}\,\cdot$$

Thus, we have arrived at the problem of integration of the following system of first-order linear ordinary differential equations with variable coefficients:

$$\begin{aligned} \frac{dU}{d\lambda} &= \frac{\lambda}{1+\lambda^2}\,U - \frac{1}{1+\lambda^2}\,(V + \alpha + \beta\lambda),\\ \frac{dV}{d\lambda} &= \frac{\lambda}{1+\lambda^2}\,V + \frac{1}{1+\lambda^2}\,(U + \gamma + \delta\lambda),\\ \frac{dW}{d\lambda} &= \frac{k\lambda}{1+\lambda^2}\,W. \end{aligned} \tag{3.5}$$

3.2 Computation in canonical variables

First we compute the commutator (1.2) and the pseudoscalar product (1.4) of the operators (3.1) and obtain:

$$[X_1, X_2] = 0, \quad X_1 \vee X_2 = -(x^2 + y^2) \neq 0.$$

Therefore, according to Table 1, the operators (3.1) span a Lie algebra L_2 of type I. In order to find canonical variables for this algebra, we have to determine a change of variables of the form (2.3),

$$\tilde{x} = \tilde{x}(x, y), \quad \tilde{y} = \tilde{y}(x, y),$$

by solving Eqs. (1.5) for type I,

$$X_1(\tilde{x}) \equiv x\frac{\partial \tilde{x}}{\partial x} + y\frac{\partial \tilde{x}}{\partial y} = 1, \quad X_2(\tilde{x}) \equiv y\frac{\partial \tilde{x}}{\partial x} - x\frac{\partial \tilde{x}}{\partial y} = 0; \tag{3.6}$$

$$X_1(\tilde{y}) \equiv x\frac{\partial \tilde{y}}{\partial x} + y\frac{\partial \tilde{y}}{\partial y} = 0, \quad X_2(\tilde{y}) \equiv y\frac{\partial \tilde{y}}{\partial x} - x\frac{\partial \tilde{y}x}{\partial y} = 1. \tag{3.7}$$

Writing the characteristic equation for the equation $X_2(\tilde{x}) = 0$ from (3.6) in the form

$$xdx + ydy = 0$$

we obtain the first integral

$$x^2 + y^2 = \text{const.}$$

Hence, the general solution to the equation $X_2(\tilde{x}) = 0$ has the form

$$\tilde{x} = \tilde{x}(r), \quad \text{where} \quad r = \sqrt{x^2 + y^2}.$$

We have:

$$X_1(r) = x\frac{\partial r}{\partial x} + y\frac{\partial y}{\partial y} = r,$$

and hence the restriction of X_1 to functions depending only on r has the form

$$X_1 = r\,\frac{\partial}{\partial r}\cdot$$

Therefore the equation $X_1(\tilde{x}) = 1$ from (3.6) is written

$$r\,\frac{d\tilde{x}(r)}{dr} = 1$$

and yields

$$\tilde{x} = \ln r \equiv \ln\sqrt{x^2 + y^2}\,.$$

Now we proceed likewise with Eqs. (3.7). Namely, starting with the equation $X_1(\tilde{y}) = 0$ we conclude that

$$\tilde{y} = \tilde{y}(\lambda), \quad \text{where} \quad \lambda = \frac{y}{x}\cdot$$

The reckoning shows that the restriction of X_2 to functions depending only on λ has the form (see also Section 3.1)

$$X_2 = -(1 + \lambda^2)\,\frac{\partial}{\partial \lambda}\cdot$$

Therefore the equation $X_2(\tilde{y}) = 1$ from (3.7) is written

$$(1 + \lambda^2)\,\frac{d\tilde{y}(\lambda)}{d\lambda} = -1$$

and yields

$$\tilde{y} = -\arctan\lambda \equiv -\arctan\left(\frac{y}{x}\right)\,.$$

Thus, we have the following canonical variables:

$$\tilde{x} = \ln \sqrt{x^2 + y^2}\,, \quad \tilde{y} = -\arctan(y/x)\,. \tag{3.8}$$

The inverse transformation to (3.8) is given by

$$x = \mathrm{e}^{\tilde{x}} \cos\tilde{y}\,, \quad y = -\mathrm{e}^{\tilde{x}} \sin\tilde{y}\,. \tag{3.9}$$

Using Eqs. (3.8) and (3.9) one can readily rewrite the operators (3.1) in the canonical variables and obtain:

$$X_1 = \frac{\partial}{\partial\tilde{x}} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + kw\frac{\partial}{\partial w}\,, \tag{3.10}$$

$$X_2 = \frac{\partial}{\partial\tilde{y}} + \left[v + \mathrm{e}^{\tilde{x}}\,(\alpha\,\cos\tilde{y} - \beta\,\sin\tilde{y})\right]\frac{\partial}{\partial u} - \left[u + \mathrm{e}^{\tilde{x}}\,(\gamma\,\cos\tilde{y} - \delta\,\sin\tilde{y})\right]\frac{\partial}{\partial v}\cdot$$

Now the characteristic system for the equation $X_1(J) = 0$ is written

$$\frac{du}{u} = \frac{dv}{v} = \frac{dw}{kw} = d\tilde{x}$$

and provides the following invariants:

$$\tilde{y}\,, \quad U = u\,\mathrm{e}^{-\tilde{x}}\,, \quad V = v\,\mathrm{e}^{-\tilde{x}}\,, \quad W = w\,\mathrm{e}^{-k\tilde{x}}\,. \tag{3.11}$$

The second operator (3.10) is written in terms of the invariants (3.11) as follows:

$$X_2 = \frac{\partial}{\partial\tilde{y}} + \left[V + \alpha\,\cos\tilde{y} - \beta\,\sin\tilde{y}\right]\frac{\partial}{\partial U} - \left[U + \gamma\,\cos\tilde{y} - \delta\,\sin\tilde{y}\right]\frac{\partial}{\partial V}\cdot \tag{3.12}$$

The characteristic system for the equation $X_2(J) = 0$ reduces to the following simple system of non-homogeneous first-order linear equations:

$$\frac{dU}{d\tilde{y}} = V + \alpha\,\cos\tilde{y} - \beta\,\sin\tilde{y}, \quad \frac{dV}{d\tilde{y}} = -U - \gamma\,\cos\tilde{y} + \delta\,\sin\tilde{y}. \tag{3.13}$$

It can be easily integrated by variation of parameters. I provide the solution when

$$\delta = \alpha, \quad \gamma = -\beta, \tag{3.14}$$

i.e. for the system

$$\frac{dU}{d\tilde{y}} = V + \alpha\,\cos\tilde{y} - \beta\,\sin\tilde{y}, \quad \frac{dV}{d\tilde{y}} = -U + \beta\,\cos\tilde{y} + \alpha\,\sin\tilde{y}. \tag{3.15}$$

The general solution to Eqs. (3.15) has the form

$$\begin{aligned} U &= \left(C_1 + \frac{\beta}{2}\right)\cos\tilde{y} + \left(C_2 + \frac{\alpha}{2}\right)\sin\tilde{y}, \\ V &= \left(C_2 - \frac{\alpha}{2}\right)\cos\tilde{y} - \left(C_1 - \frac{\beta}{2}\right)\sin\tilde{y}. \end{aligned} \tag{3.16}$$

3.3 Summary

Let us summarize our results on invariants for the operators (3.1) in the case (3.14), i.e. for the operators

$$
\begin{aligned}
X_1 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + kw\frac{\partial}{\partial w}\,, \\
X_2 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + (v + \alpha x + \beta y)\frac{\partial}{\partial u} - (u - \beta x + \alpha y)\frac{\partial}{\partial v}\,\cdot
\end{aligned}
\tag{3.17}
$$

Solving Eqs. (3.16) with respect to C_1, C_2 we obtain two first integrals

$$
C_1 = J_1(\tilde{y}, U, V), \quad C_2 = J_2(\tilde{y}, U, V)
$$

of the system (3.15). The functions J_1 and J_2 provide two functionally independent invariants for the operators X_1, X_2. The third invariant J_3 is W from Eqs. (3.11).

Eqs. (3.8) and (3.9) yield:

$$
\mathrm{e}^{\tilde{x}} = \sqrt{x^2 + y^2}\,, \quad \cos\tilde{y} = x\,\mathrm{e}^{-\tilde{x}}\,, \quad \sin\tilde{y} = -y\,\mathrm{e}^{-\tilde{x}}\,. \tag{3.18}
$$

Substituting (3.18) in Eqs. (3.11), (3.16) and setting $C_1 = J_1,\ C_2 = J_2, W = J_3$, we obtain:

$$
\begin{aligned}
u &= \left(J_1 + \frac{\beta}{2}\right)\,x - \left(J_2 + \frac{\alpha}{2}\right)\,y\,, \\
v &= \left(J_1 - \frac{\beta}{2}\right)\,y + \left(J_2 - \frac{\alpha}{2}\right)\,x\,, \\
w &= (x^2 + y^2)^{k/2}\,J_3.
\end{aligned}
\tag{3.19}
$$

Solving Eqs. (3.19) with respect to J_1, J_2, J_3 we finally obtain the following basis o invariants for the operators (3.17):

$$
\begin{aligned}
J_1 &= \frac{1}{x^2 + y^2}\,\left(xu + yv + \alpha xy - \frac{\beta}{2}(x^2 - y^2)\right)\,, \\
J_2 &= \frac{1}{x^2 + y^2}\,\left(xv - yu + \beta xy + \frac{\alpha}{2}(x^2 - y^2)\right)\,, \\
J_3 &= (x^2 + y^2)^{-k/2}\,w.
\end{aligned}
\tag{3.20}
$$

8 March 2009

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GROUP ANALYSIS OF NONLINEAR INTERNAL WAVES IN OCEANS
I: Self-adjointness, conservation laws, invariant solutions

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Abstract. The paper is devoted to the group analysis of equations of motion of two-dimensional uniformly stratified rotating fluids used as a basic model in geophysical fluid dynamics. It is shown that the nonlinear equations in question have a remarkable property to be self-adjoint. This property is crucial for constructing conservation laws provided in the present paper. Invariant solutions are constructed using certain symmetries. The invariant solutions are used for defining internal wave beams..

Keywords: Self-adjointness, Lagrangian, Energy, Invariant solutions, Internal wave beams.

1 Introduction

We will apply Lie group analysis for investigating the system of nonlinear equations

$$\Delta\psi_t - g\rho_x - f v_z = \psi_x \Delta\psi_z - \psi_z \Delta\psi_x\,, \tag{1.1}$$

$$v_t + f\psi_z = \psi_x v_z - \psi_z v_x\,, \tag{1.2}$$

$$\rho_t + \frac{N^2}{g}\psi_x = \psi_x \rho_z - \psi_z \rho_x \tag{1.3}$$

used in geophysical fluid dynamics for investigating internal waves in uniformly stratified incompressible fluids (oceans). In particular, the system (1.1)-(1.3) with $f = 0$ was used in [1] to study two non-unidirectional wave beams propagating and interacting in stratified fluid. An exact solution of the same system, again in the case when

$f = 0$, was employed in [2] for investigating stability of a singe internal plane wave. Weakly nonlinear effects in colliding of internal wave beams were investigated in [3], [4] by using Eqs. (1.1)-(1.3) with $f = 0$. The system (1.1)-(1.3) with $f \neq 0$ was used in [5] to model weakly nonlinear wave interactions governing the time behavior of the oceanic energy spectrum.

In these equations Δ is the two-dimensional Laplacian:

$$\Delta = D_x^2 + D_z^2\,, \quad \text{e.g.} \quad \Delta\psi_t = \frac{\partial^2 \psi_t}{\partial x^2} + \frac{\partial^2 \psi_t}{\partial z^2} \equiv D_t(\Delta\psi)\,,$$

and g, f, N are constants. Namely, g is the gravitational acceleration, f is the Coriolis parameter. The quantity N appears due to the density stratification of a fluid and is constant under the linear stratification hypothesis.

We will show in what follows that the system of equations (1.1)-(1.3) is self-adjoint (in the terminology of [6, 7]) and use this remarkable property of the system for calculating conservation laws associated with symmetry properties of the system (1.1)-(1.3).

In some calculations, e.g. in Sections 4.7, 4.5, 4.8 it is convenient to write Eqs. (1.1)-(1.3) by using the Jacobians $J(\psi, v) = \psi_x v_z - \psi_z v_x$, etc., in the following form:

$$\Delta\psi_t - g\rho_x - f v_z = J(\psi, \Delta\psi), \tag{1.4}$$

$$v_t + f\psi_z = J(\psi, v), \tag{1.5}$$

$$\rho_t + \frac{N^2}{g}\,\psi_x = J(\psi, \rho). \tag{1.6}$$

2 Self-adjointness

2.1 Preliminaries

We will use the terminology and the following definitions from [6, 7] (see also [8]).

Let $x = (x^1, \ldots, x^n)$ be n independent variables, and $u = (u^1, \ldots, u^m)$ be m dependent variables. The partial derivatives of u^α with respect to x^i are denoted by $u_{(1)} = \{u_i^\alpha\}$, $u_{(2)} = \{u_{ij}^\alpha\}$, ... with

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_i(u_j^\alpha) = D_i D_j(u^\alpha), \ldots,$$

where D_i is the operator of *total differentiation* with respect to x^i :

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots\,, \quad i = 1, \ldots, n. \tag{2.1}$$

Even though the operators D_i are given by formal infinite sums, their action $D_i(f)$ is well defined for functions $f(x, u, u_{(1)}, \ldots)$ depending on a finite number of the variables $x, u, u_{(1)}, u_{(2)}, \ldots$ The usual summation convention on repeated indices α and i is assumed in expressions like Eq. (2.1).

The variational derivatives (the *Euler-Lagrange operator*) are defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}, \quad \alpha = 1, \ldots, m, \tag{2.2}$$

where the summation over the repeated indices $i_1 \ldots i_s$ runs from 1 to n.

Definition 2.1. The *adjoint equations* to nonlinear partial differential equations

$$F_\alpha\big(x, u, \ldots, u_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{2.3}$$

are given by (see also [9])

$$F^*_\alpha\big(x, u, \mu, \ldots, u_{(s)}, \mu_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{2.4}$$

where $\mu = (\mu^1, \ldots, \mu^m)$ are new dependent variables, and F^*_α are defined by

$$F^*_\alpha\big(x, u, \mu, \ldots, u_{(s)}, \mu_{(s)}\big) = \frac{\delta(\mu^\beta F_\beta)}{\delta u^\alpha} \cdot \tag{2.5}$$

In the case of linear equations, Definition 2.1 is equivalent to the classical definition of the adjoint equation.

Consider the function

$$\mathcal{L} = \mu^\beta F_\beta\big(x, u, \ldots, u_{(s)}\big) \tag{2.6}$$

involved in (2.5). Eqs. (2.3) and their adjoint equations (2.4) can be obtained from (2.5) by taking the variational derivatives (2.2) with respect to the dependent variables u and the similar variational derivatives with respect to the new dependent variables μ,

$$\frac{\delta}{\delta \mu^\alpha} = \frac{\partial}{\partial \mu^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial \mu^\alpha_{i_1 \cdots i_s}}, \quad \alpha = 1, \ldots, m. \tag{2.7}$$

Namely:

$$\frac{\delta \mathcal{L}}{\delta \mu^\alpha} = F_\alpha\big(x, u, \ldots, u_{(s)}\big), \tag{2.8}$$

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = F^*_\alpha\big(x, u, \mu, \ldots, u_{(s)}, \mu_{(s)}\big). \tag{2.9}$$

This circumstance justifies the following definition.

Definition 2.2. The differential function (2.6) is called a *formal Lagrangian* for the differential equations (2.3). For the sake of brevity, formal Lagrangians are also referred to as Lagrangians.

If the variables u are known, the new variables μ are obtained by solving Eqs. (2.4) which are, according to (2.5), linear partial differential equations (2.4) with respect to μ^α. Using the existing terminology (see, e.g. [10]), we will call μ^α *nonlocal variables*.

Nonlocal variables can be excluded from physical quantities such as conservation laws if Eqs. (2.3) are *self-adjoint* ([6]) or, in general, *quasi-self-adjoint* ([11]) in the following sense.

Definition 2.3. Eqs, (2.3) are said to be *self-adjoint* if the system obtained from the adjoint equations (2.4) by the substitution $\mu = u$:

$$F^*_\alpha\big(x, u, u, \dots, u_{(s)}, u_{(s)}\big) = 0, \quad \alpha = 1, \dots, m, \tag{2.10}$$

is equivalent to the original system (2.3), i.e.

$$F^*_\alpha\big(x, u, u, \dots, u_{(s)}, u_{(s)}\big) = \Phi^\beta_\alpha F_\beta\big(x, u, \dots, u_{(s)}\big), \quad \alpha = 1, \dots, m,$$

with regular (in general, variable) coefficients Φ^β_α.

Definition 2.4. Eqs, (2.3) are said to be quasi-self-adjoint if the system of adjoint equations (2.4) becomes equivalent to the original system (2.3) upon the substitution

$$\mu = h(u) \tag{2.11}$$

with a certain function $h(u)$ such that $h'(u) \neq 0$.

2.2 Adjoint system to Eqs. (1.1)-(1.3)

Let us apply the methods from Section 2.1 to Eqs. (1.1)-(1.3). In this case the formal Lagrangian (2.6) for Eqs. (1.1)-(1.3) is written

$$\begin{aligned} \mathcal{L} = \varphi\big[\Delta\psi_t - g\rho_x - f v_z - \psi_x \Delta\psi_z + \psi_z \Delta\psi_x\big] \\ + \mu\left[v_t + f\psi_z - \psi_x v_z + \psi_z v_x\right] + r\Big[\rho_t + \frac{N^2}{g}\,\psi_x - \psi_x\rho_z + \psi_z\rho_x\Big], \end{aligned} \tag{2.12}$$

where φ, μ and r are new dependent variables. The adjoint equations to Eqs. (1.1)-(1.3) are obtained by taking the variational derivatives of $\mathcal{L}$, namely:

$$\frac{\delta\mathcal{L}}{\delta\psi} = 0, \quad \frac{\delta\mathcal{L}}{\delta v} = 0, \quad \frac{\delta\mathcal{L}}{\delta\rho} = 0, \tag{2.13}$$

where (see (2.2); see also Eqs. (3.6))

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_x\frac{\partial}{\partial v_x} - D_z\frac{\partial}{\partial v_z}\,,$$

$$\frac{\delta}{\delta \rho} = \frac{\partial}{\partial \rho} - D_x\frac{\partial}{\partial \rho_x} - D_z\frac{\partial}{\partial \rho_z}\,,$$

$$\frac{\delta}{\delta \psi} = \frac{\partial}{\partial \psi} - D_x\frac{\partial}{\partial \psi_x} - D_z\frac{\partial}{\partial \psi_z} + D_xD_t\frac{\partial}{\partial \psi_{xt}} + D_zD_t\frac{\partial}{\partial \psi_{zt}} + \cdots\,.$$

Taking into account the special form (2.12) of $\mathcal{L}$, we have:

$$\frac{\delta \mathcal{L}}{\delta \psi} = -D_x \frac{\partial \mathcal{L}}{\partial \psi_x} - D_z \frac{\partial \mathcal{L}}{\partial \psi_z} - (D_x^2 + D_z^2)\Big[D_t \frac{\partial \mathcal{L}}{\partial \Delta\psi_t} + D_x \frac{\partial \mathcal{L}}{\partial \Delta\psi_x} + D_z \frac{\partial \mathcal{L}}{\partial \Delta\psi_z}\Big]$$

$$= D_x\big(\varphi \Delta\psi_z + \mu v_z - \frac{N^2}{g}\, r + r\rho_z\big) - D_z\left(\varphi\Delta\psi_x + f\mu + \mu v_x + r\rho_x\right)$$

$$- (D_x^2 + D_z^2)\Big[D_t(\varphi) + D_x(\varphi\psi_z) - D_z(\varphi\psi_x)\Big]$$

$$= \varphi_x \Delta\psi_z - \varphi_z\Delta\psi_x + \mu_x v_z - \frac{N^2}{g}\, r_x + r_x\rho_z - f\mu_z - \mu_z v_x - r_z\rho_x$$

$$- \Delta\varphi_t + 2\big[\varphi_{xz}\psi_{xx} + \varphi_{zz}\psi_{xz} - \varphi_{xx}\psi_{xz} - \varphi_{xz}\psi_{zz}\big],$$

$$\frac{\delta \mathcal{L}}{\delta v} = -D_t \frac{\partial \mathcal{L}}{\partial v_t} - D_x \frac{\partial \mathcal{L}}{\partial v_x} - D_z \frac{\partial \mathcal{L}}{\partial v_z} = -\mu_t - \mu_x\psi_z + f\varphi_z + \mu_z\psi_x\,,$$

$$\frac{\delta \mathcal{L}}{\delta \rho} = -D_t \frac{\partial \mathcal{L}}{\partial \rho_t} - D_x \frac{\partial \mathcal{L}}{\partial \rho_x} - D_z \frac{\partial \mathcal{L}}{\partial \rho_z} = -r_t + g\varphi_x - r_x\psi_z + r_z\psi_x\,.$$

Hence, the adjoint equations (2.13) can be written as follows:

$$\Delta\varphi_t + \frac{N^2}{g}\, r_x + f\mu_z - \varphi_x\Delta\psi_z + \varphi_z\Delta\psi_x - \Theta = 0, \tag{2.14}$$

$$-\mu_t - \mu_x\psi_z + f\varphi_z + \mu_z\psi_x = 0, \tag{2.15}$$

$$-r_t + g\varphi_x - r_x\psi_z + r_z\psi_x = 0, \tag{2.16}$$

where

$$\Theta = J(\mu, v) + J(r, \rho) + 2\big[\varphi_{xz}\psi_{xx} + \varphi_{zz}\psi_{xz} - \varphi_{xx}\psi_{xz} - \varphi_{xz}\psi_{zz}\big]. \tag{2.17}$$

2.3 Self-adjointness of Eqs. (1.1)-(1.3)

Theorem 2.1. Eqs. (1.1)-(1.3) are quasi-self-adjoint.

Proof. Looking for (2.11) in the form of a general scaling transformation, one can readily obtain that after the transformation

$$\varphi = \psi, \quad \mu = -v, \quad r = -\frac{g^2}{N^2}\,\rho, \tag{2.18}$$

the quantity Θ given by Eq. (2.17) vanishes. Therefore the adjoint equations (2.14)-(2.16) become identical with Eqs. (1.1)-(1.3) after the substitution (2.18). Hence, according to Definition 2.4, Eqs. (1.1)-(1.3) are *quasi-self-adjoint*. Since Eqs. (2.18) are obtained just be simple scaling of the equations $\varphi = \psi, \mu = v, r = \rho$ required for the self-adjointness, we will say that Eqs. (1.1)-(1.3) are *self-adjoint*.

3 Conservation laws

3.1 General discussion of conservation equations

Along with the individual notation $t,\ x,\ z$ for the the independent variables, and $v,\ \rho,\ \psi$ for the dependent variables, we will also use the index notation $x^1 = t,\ x^2 = x,\ x^3 = z$ and $u^1 = v,\ u^2 = \rho,\ u^3 = \psi$, respectively. We will write the conservation laws both in the differential form

$$D_t(C^1) + D_x(C^2) + D_z(C^3) = 0 \tag{3.1}$$

and the integral form

$$\frac{d}{dt}\int\int C^1 dxdz = 0, \tag{3.2}$$

where the double integral in taken over the the (x, z) plane $\mathbb{R}^2$. The equations (3.1) and (3.2) provide a conservation law for Eqs. (1.1)-(1.3) if they hold for the solutions of Eqs. (1.1)-(1.3). The vector $\boldsymbol{C} = (C^1, C^2, C^3)$ satisfying the conservation equation (3.1) is termed a *conserved vector.* Its component C^1 is called the *density* of the conservation law due to Eq. (3.2). The two-dimensional vector (C^2, C^3) defines the *flux* of the conservation law.

The integral form (3.2) of a conservation law follows from the differential form (3.1) provided that the solutions of Eqs. (1.1)-(1.3) vanish or rapidly decrease at the infinity on $\mathbb{R}^2$. Indeed, integrating Eq. (3.1) over an arbitrary region $\Omega \subset \mathbb{R}^2$ we have:

$$\frac{d}{dt}\int\int_\Omega C^1 dxdz = -\int\int_\Omega \left[D_x(C^2) + D_z(C^3)\right] dxdz.$$

According to Green's theorem, the integral on the right-hand side reduces to the integral along the boundary $\partial\Omega$ of Ω :

$$-\int\int_\Omega \left[D_x(C^2) + D_z(C^3)\right] dxdz = \int_{\partial\Omega} C^3 dx - C^2 dz,$$

and hence vanishes as Ω expands and becomes the plane $\mathbb{R}^2$.

Remark 3.1. It is manifest from this discussion that one can ignore in C^1 "divergent type" terms because they do not change the integral in the conservation equation (3.2). Specifically if C^1 evaluated on the solutions of Eqs. (1.1)-(1.3) has the form

$$C^1 = \widetilde{C}^1 + D_x(h^2) + D_z(h^3) \tag{3.3}$$

with some functions h^2, h^3, then the conservation equation (3.1) can be equivalently rewritten in the form (see [12], Paper 1, Section 20.1)

$$D_t(\widetilde{C}^1) + D_x(\widetilde{C}^2) + D_z(\widetilde{C}^3) = 0,$$

where

$$\widetilde{C}^2 = C^2 + D_t(h^2), \quad \widetilde{C}^3 = C^3 + D_t(h^3).$$

Accordingly, we have

$$\int\int C^1 dxdz = \int\int \widetilde{C}^1 dxdz,$$

and hence the integral conservation equation (3.2) provided by the conservation density C^1 of the form (3.3) coincides with that provided by the density $\widetilde{C}^1$.

In particular, if $\widetilde{C}^1 = 0$ the integral in Eq. (3.2) vanishes. This kind of conservation laws are *trivial* from physical point of view. Therefore we single out physically useless conservation laws by the following definition.

Definition 3.1. The conservation law is said to be *trivial* if its density C^1 evaluated on the solutions of Eqs. (1.1)-(1.3) is the divergence,

$$C^1 = D_x(h^2) + D_z(h^3).$$

The following statement ([13], Section 8.4.1; see also [7]) simplifies calculations while dealing with conservation equations.

Lemma 3.1. A function $F(v, \rho, \psi, v_x, v_z, \rho_x, \rho_z, \psi_x, \psi_z, \psi_{xt}, \psi_{zt}, \ldots)$ is the divergence,

$$F = D_x(C^1) + D_z(C^2), \tag{3.4}$$

if and only if satisfies the following equations:

$$\frac{\delta F}{\delta v} = 0, \quad \frac{\delta F}{\delta \rho} = 0, \quad \frac{\delta F}{\delta \psi} = 0. \tag{3.5}$$

Here the variational derivatives act on F as usual (see also Section 2.2):

$$\begin{aligned}
\frac{\delta F}{\delta v} &= \frac{\partial F}{\partial v} - D_x\left(\frac{\partial F}{\partial v_x}\right) - D_z\left(\frac{\partial F}{\partial v_z}\right), \\
\frac{\delta F}{\delta \rho} &= \frac{\partial F}{\partial \rho} - D_x\left(\frac{\partial F}{\partial \rho_x}\right) - D_z\left(\frac{\partial F}{\partial \rho_z}\right), \\
\frac{\delta F}{\delta \psi} &= \frac{\partial F}{\partial \psi} - D_x\left(\frac{\partial F}{\partial \psi_x}\right) - D_z\left(\frac{\partial F}{\partial \psi_z}\right) + D_x D_t\left(\frac{\partial F}{\partial \psi_{xt}}\right) + D_z D_t\left(\frac{\partial F}{\partial \psi_{zt}}\right) + \cdots .
\end{aligned} \tag{3.6}$$

Corollary 3.1. A function C^1 is the density of a conservation law (3.1) if and only if the function

$$F = D_t(C^1)\Big|_{(1.1)-(1.3)} \tag{3.7}$$

satisfies Eqs. (3.5). Here $|_{(1.1)-(1.3)}$ means that the quantity $D_t(C^1)$ is evaluated on the solutions of Eqs. (1.1)-(1.3).

In particular, Lemma 3.1 allows one to single out trivial conservation laws as follows.

Corollary 3.2. The conservation law (3.1) is trivial if and only if its density C^1 evaluated on the solutions of Eqs. (1.1)-(1.3), i.e. the quantity

$$C^1_* = C^1\big|_{(1.1)-(1.3)} \tag{3.8}$$

satisfies Eqs. (3.5),

$$\frac{\delta C^1_*}{\delta v} = 0, \quad \frac{\delta C^1_*}{\delta \rho} = 0, \quad \frac{\delta C^1_*}{\delta \psi} = 0, \tag{3.9}$$

on the solutions of Eqs. (1.1)-(1.3).

3.2 Variational derivatives of expressions with Jacobians

We will use in our calculations the following statement on the behaviour of certain expressions with Jacobians under the action of the variational derivatives (3.6).

Proposition 3.1. The following equations hold:

$$\frac{\delta J(\psi, v)}{\delta v} = 0, \qquad \frac{\delta J(\psi, v)}{\delta \psi} = 0, \tag{3.10}$$

$$\frac{\delta [vJ(\psi, v)]}{\delta v} = 0, \quad \frac{\delta [vJ(\psi, v)]}{\delta \psi} = 0, \tag{3.11}$$

$$\frac{\delta [\rho J(\psi, \rho)]}{\delta \rho} = 0, \quad \frac{\delta [\rho J(\psi, \rho)]}{\delta \psi} = 0, \tag{3.12}$$

$$\frac{\delta J(\psi, \Delta\psi)}{\delta \psi} = 0, \quad \frac{\delta [\psi J(\psi, \Delta\psi)]}{\delta \psi} = 0. \tag{3.13}$$

Proof. Let us verify that the first equation (3.10) holds. We have (see (3.6)):

$$\frac{\delta J(\psi, v)}{\delta v} = \frac{\delta (\psi_x v_z - \psi_z v_x)}{\delta v} = -D_z(\psi_x) + D_x(\psi_z) = -\psi_{xz} + \psi_{zx} = 0.$$

Replacing v by ψ one obtains the second equation (3.10). Let us verify now that Eqs. (3.11) are satisfied. We have:

$$\frac{\delta [vJ(\psi, v)]}{\delta v} = \frac{\delta [v(\psi_x v_z - \psi_z v_x)]}{\delta v} = \frac{\partial [v(\psi_x v_z - \psi_z v_x)]}{\partial v} - D_z(v\psi_x) + D_x(v\psi_z)$$

$$= J(\psi, v) - D_z(v\psi_x) + D_x(v\psi_z) = J(\psi, v) - J(\psi, v) - v\psi_{xz} + v\psi zx = 0$$

and

$$\frac{\delta[vJ(\psi, v)]}{\delta\psi} = -D_x(vv_z) + D_z(vv_x) = -v_xv_z - vv_{xz} + v_zv_x + vv_{zx} = 0.$$

Replacing v by ρ one obtains Eqs. (3.12). Eqs. (3.13) are derived likewise even though they involve higher-order derivatives. We have:

$$\frac{\delta J(\psi, \Delta\psi)}{\delta\psi} = \frac{\delta(\psi_x\Delta\psi_z - \psi_z\Delta\psi_x)}{\delta\psi} = \frac{\delta[\psi_x(\psi_{xxz} + \psi_{zzz}) - \psi_z(\psi_{xxx} + \psi_{xzz})]}{\delta\psi}$$

$$= -D_x(\Delta\psi_z) + D_z(\Delta\psi_x) - D_z(D_x^2 + D_z^2)(\psi_x) + D_x(D_x^2 + D_z^2)(\psi_z)$$

$$= -D_x(\Delta\psi_z) + D_z(\Delta\psi_x) - D_z(\Delta\psi_x) + D_x(\Delta\psi_z) = 0.$$

Derivation of the second equation (3.13) requires only a simple modification of the previous calculations. Namely:

$$\frac{\delta[\psi J(\psi, \Delta\psi)]}{\delta\psi} = \frac{\delta[\psi(\psi_x\Delta\psi_z - \psi_z\Delta\psi_x)]}{\delta\psi}$$

$$= \psi_x\Delta\psi_z - \psi_z\Delta\psi_x - D_x(\psi\Delta\psi_z) + D_z(\psi\Delta\psi_x) - \Delta D_z(\psi\psi_x) + \Delta D_x(\psi\psi_z)$$

$$= \psi_x\Delta\psi_z - \psi_z\Delta\psi_x - \psi_x\Delta\psi_z - \psi\Delta\psi_{zx} + \psi_z\Delta\psi_x + \psi\Delta\psi_{xz}$$

$$-\frac{1}{2}\Delta D_z D_x(\psi^2) + \frac{1}{2}\Delta D_x D_z(\psi^2) = 0.$$

3.3 Nonlocal conserved vectors

It has been demonstrated in [8, 7] that for any operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \tag{3.14}$$

admitted by the system (1.1)-(1.3), the quantities

$$C^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial\mathcal{L}}{\partial u_i^\alpha} - D_j\Big(\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha}\Big) + D_jD_k\Big(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\Big)\right] \tag{3.15}$$

$$+ D_j\big(W^\alpha\big)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}^\alpha} - D_k\Big(\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\Big)\right] + D_jD_k\big(W^\alpha\big)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}^\alpha}\right], \quad i = 1, 2, 3,$$

define the components of a conserved vector for Eqs. (1.1)-(1.3) considered together with the adjoint equations (2.14)-(2.16). Here

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \quad \alpha = 1, 2, 3. \tag{3.16}$$

The formula (3.15) is written by taking into account that the Lagrangian (2.12) involves the derivatives up to third order. Moreover, noting that the Lagrangian (2.12) vanishes on the solutions of Eqs. (1.1)-(1.3), we can drop the first term in (3.15) and use the conserved vector in the abbreviated form

$$C^i = W^\alpha \Big[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\Big(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\Big) + D_j D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\Big)\Big] \tag{3.17}$$

$$+ D_j(W^\alpha)\Big[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\Big)\Big] + D_j D_k(W^\alpha)\,\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\,\cdot$$

For computing the conserved vectors (3.17), the Lagrangian (2.12) containing the mixed derivatives should be written in the symmetric form

$$\mathcal{L} = \frac{1}{3}\,\varphi\big[\psi_{txx} + \psi_{xtx} + \psi_{xxt} + \psi_{tzz} + \psi_{ztz} + \psi_{zzt} - 3g\rho_x - 3fv_z$$

$$-\,\psi_x\big(\psi_{zxx} + \psi_{xzx} + \psi_{xxz} + 3\psi_{zzz}\big) + \psi_z\big(3\psi_{xxx} + \psi_{xzz} + \psi_{zxz} + \psi_{zzx}\big)\big] \tag{3.18}$$

$$+\,\mu\,[v_t + f\psi_z - \psi_x v_z + \psi_z v_x] + r\Big[\rho_t + \frac{N^2}{g}\,\psi_x - \psi_x\rho_z + \psi_z\rho_x\Big].$$

Since the Lagrangian $\mathcal{L}$, and hence the components (3.17) of a conserved vector contain the nonlocal variables φ, μ, r, we obtain in this way *nonlocal conserved vectors*.

3.4 Computation of nonlocal conserved vectors

The substitution of (3.18) in (3.17) yields:

$$C^1 = W^1\,\frac{\partial \mathcal{L}}{\partial v_t} + W^2\,\frac{\partial \mathcal{L}}{\partial \rho_t} + W^3\left[D_x^2\Big(\frac{\partial \mathcal{L}}{\partial \psi_{txx}}\Big) + D_z^2\Big(\frac{\partial \mathcal{L}}{\partial \psi_{tzz}}\Big)\right]$$

$$-\left[D_x(W^3)\,D_x\Big(\frac{\partial \mathcal{L}}{\partial \psi_{txx}}\Big) + D_z(W^3)\,D_z\Big(\frac{\partial \mathcal{L}}{\partial \psi_{tzz}}\Big)\right] + D_x^2(W^3)\,\frac{\partial \mathcal{L}}{\partial \psi_{txx}} + D_z^2(W^3)\,\frac{\partial \mathcal{L}}{\partial \psi_{tzz}}\,,$$

or

$$C^1 = W^1\,\mu + W^2\,r + \frac{1}{3}\,W^3\left[D_x^2(\varphi) + D_z^2(\varphi)\right] \tag{3.19}$$

$$-\,\frac{1}{3}\Big[\varphi_x\,D_x(W^3) + \varphi_z\,D_z(W^3)\Big] + \frac{1}{3}\,\varphi\Big[D_x^2(W^3) + D_z^2(W^3)\Big].$$

Furthermore, using the same procedure, we obtain:

$$
\begin{aligned}
C^2 = {} & W^1\,\frac{\partial \mathcal{L}}{\partial v_x} + W^2\,\frac{\partial \mathcal{L}}{\partial \rho_x} + W^3\left[\frac{\partial \mathcal{L}}{\partial \psi_x} + D_x^2\Big(\frac{\partial \mathcal{L}}{\partial \psi_{xxx}}\Big) + D_z^2\Big(\frac{\partial \mathcal{L}}{\partial \psi_{xzz}}\Big)\right.\\
& \left. + D_t D_x\Big(\frac{\partial \mathcal{L}}{\partial \psi_{xtx}} + \frac{\partial \mathcal{L}}{\partial \psi_{xxt}}\Big) + D_x D_z\Big(\frac{\partial \mathcal{L}}{\partial \psi_{xxz}} + \frac{\partial \mathcal{L}}{\partial \psi_{xzx}}\Big)\right] - D_x(W^3)\,D_x\Big(\frac{\partial \mathcal{L}}{\partial \psi_{xxx}}\Big)\\
& - D_t(W^3)\,D_x\Big(\frac{\partial \mathcal{L}}{\partial \psi_{xtx}}\Big) - D_x(W^3)\,D_t\Big(\frac{\partial \mathcal{L}}{\partial \psi_{xxt}}\Big) - D_z(W^3)\,D_z\Big(\frac{\partial \mathcal{L}}{\partial \psi_{xzz}}\Big)\\
& - D_z(W^3)\,D_x\Big(\frac{\partial \mathcal{L}}{\partial \psi_{xzx}}\Big) - D_x(W^3)\,D_z\Big(\frac{\partial \mathcal{L}}{\partial \psi_{xxz}}\Big) + D_x^2(W^3)\,\frac{\partial \mathcal{L}}{\partial \psi_{xxx}}\\
& + D_z^2(W^3)\,\frac{\partial \mathcal{L}}{\partial \psi_{xzz}} + D_t D_x(W^3)\Big(\frac{\partial \mathcal{L}}{\partial \psi_{xtx}} + \frac{\partial \mathcal{L}}{\partial \psi_{xxt}}\Big) + D_x D_z(W^3)\Big(\frac{\partial \mathcal{L}}{\partial \psi_{xxz}} + \frac{\partial \mathcal{L}}{\partial \psi_{xzx}}\Big),
\end{aligned}
$$

or

$$
\begin{aligned}
C^2 = {} & W^1\,\mu\psi_z + W^2\,(r\psi_z - g\varphi) + W^3\,\big[-\Delta\psi_z - \mu v_z + \frac{N^2}{g}\,r - r\rho_z \qquad (3.20)\\
& + D_x^2(\varphi\psi_z) + \frac{1}{3}\,D_z^2(\varphi\psi_z) + \frac{2}{3}\,\varphi_{xt} - \frac{2}{3}\,D_x D_z(\varphi\psi_x)\big]\\
& - D_x(W^3)\Big[D_x(\varphi\psi_z) + \frac{1}{3}\,\varphi_t - \frac{1}{3}\,D_z(\varphi\psi_x)\Big] - \frac{1}{3}\,D_t(W^3)\varphi_x\\
& - \frac{1}{3}\,D_z(W^3)\big[D_z(\varphi\psi_z) - D_x(\varphi\psi_x)\big] + \Big[D_x^2(W^3) + \frac{1}{3}D_z^2(W^3)\Big]\varphi\psi_z\\
& + \frac{2}{3}\,\varphi D_t D_x(W^3) - \frac{2}{3}\,\varphi\psi_x D_z D_x(W^3).
\end{aligned}
$$

Likewise we get

$$
\begin{aligned}
C^3 = {} & -W^1\,(\mu\psi_x + f\varphi) - W^2\,r\psi_x + W^3\,\big[-\Delta\psi_x + (f + v_x)\mu + r\rho_x \qquad (3.21)\\
& - \frac{1}{3}\,D_x^2(\varphi\psi_x) - D_z^2(\varphi\psi_x) + \frac{1}{3}\,D_z^2(\varphi\psi_z) + \frac{2}{3}\,\varphi_{xt} + \frac{2}{3}\,D_x D_z(\varphi\psi_z)\big]\\
& + \frac{1}{3}\,D_x(W^3)\big[D_x(\varphi\psi_x) - D_z(\varphi\psi_z)\big] - \frac{1}{3}\,D_t(W^3)\varphi_z\\
& - D_z(W^3)\Big[\frac{1}{3}\,\varphi_t - D_z(\varphi\psi_x) + \frac{1}{3}\,D_x(\varphi\psi_z)\Big]\\
& - \Big[\frac{1}{3}D_x^2(W^3) + D_z^2(W^3)\Big]\varphi\psi_x + \frac{2}{3}\,\varphi D_t D_z(W^3) + \frac{2}{3}\,\varphi\psi_z D_z D_x(W^3).
\end{aligned}
$$

3.5 Local conserved vectors

The quantities (3.19)-(3.21) define a nonlocal conserved vector because they contain the nonlocal variables φ, μ, r. In consequence, the conservation equation (3.1) requires not only the basic equations (1.1)-(1.3), but also the adjoint equations (2.14)-(2.16).

However, we can eliminate the nonlocal variables using the self-adjointness of Eqs. (1.1)-(1.3) thus transforming the nonlocal conserved vector into a local one. Namely, we substitute in Eqs. (3.19)-(3.21) the expressions (2.18) for φ, μ, r :

$$\varphi = \psi, \quad \mu = -v, \quad r = -\frac{g^2}{N^2}\,\rho. \tag{2.18}$$

Then the adjoint equations (2.14)-(2.16) are satisfied for any solutions of the basic equations (1.1)-(1.3), and hence the quantities (3.19)-(3.21) satisfy the conservation equation (3.1) on all solutions of Eqs. (1.1)-(1.3).

Let us apply the procedure to C^1. We eliminate the nonlocal variables in (3.19) by substituting there the expressions (2.18) and write C^1 in the following form:

$$C^1 = -v\,W^1 - \frac{g^2}{N^2}\,\rho\,W^2 + \frac{1}{3}\big[W^3\,\Delta\psi - \psi_x\,D_x\big(W^3\big) - \psi_z\,D_z\big(W^3\big) + \psi\Delta W^3\big],$$

where

$$\Delta\psi = D_x^2(\psi) + D_z^2(\psi), \quad \Delta W^3 = D_x^2\big(W^3\big) + D_z^2\big(W^3\big).$$

We further simplify the expression for C^1 by using the identities

$$W^3\,D_x^2(\psi) = D_x\big[W^3 D_x(\psi)\big] - \psi_x D_x\big(W^3\big),$$
$$W^3\,D_z^2(\psi) = D_z\big[W^3 D_z(\psi)\big] - \psi_z D_z\big(W^3\big)$$

and

$$\psi\,D_x^2\big(W^3\big) = D_x\big[\psi D_x\big(W^3\big)\big] - \psi_x D_x\big(W^3\big),$$
$$\psi\,D_z^2\big(W^3\big) = D_z\big[\psi D_z\big(W^3\big)\big] - \psi_z D_z\big(W^3\big).$$

Then we have:

$$C^1 = -v\,W^1 - \frac{g^2}{N^2}\,\rho\,W^2 - \psi_x\,D_x\big(W^3\big) - \psi_z\,D_z\big(W^3\big) + \frac{1}{3}\,\Delta\big(\psi W^3\big). \tag{3.22}$$

Dropping in (3.22) the divergent type term

$$\frac{1}{3}\,\Delta\big(\psi W^3\big) = D_x\left[\frac{1}{3}\,D_x\big(\psi W^3\big)\right] + D_z\left[\frac{1}{3}\,D_z\big(\psi W^3\big)\right]$$

in accordance with Remark 3.1, we finally obtain:

$$C^1 = -v\,W^1 - \frac{g^2}{N^2}\,\rho\,W^2 - \psi_x\,D_x\big(W^3\big) - \psi_z\,D_z\big(W^3\big). \tag{3.23}$$

We will not dwell on a similar modification of the expressions (3.20), (3.21) for the components C^2 and C^3 of conserved vectors. We will see further in Section 4.5 that they can be found by simpler calculations when a density C^1 is known.

4 Utilization of obvious symmetries

4.1 Introduction

Eqs. (1.1)-(1.3) do not contain the dependent and independent variables explicitly and therefore they are invariant with respect to addition of arbitrary constants to all these variables. It means that Eqs. (1.1)-(1.3) admit the one-parameter groups of translations in all variables,

$$\bar{v} = v + a_1, \quad \bar{\rho} = \rho + a_2, \quad \bar{\psi} = \psi + a_3, \quad \bar{t} = t + a_4, \quad \bar{x} = x + a_5, \quad \bar{z} = z + a_6,$$

with the generators

$$X_1 = \frac{\partial}{\partial v}, \quad X_2 = \frac{\partial}{\partial \rho}, \quad X_3 = \frac{\partial}{\partial \psi}, \quad X_4 = \frac{\partial}{\partial t}, \quad X_5 = \frac{\partial}{\partial x}, \quad X_6 = \frac{\partial}{\partial z} \cdot \quad (4.1)$$

One can also find by simple calculations the dilations (scaling transformations)

$$\bar{v} = av, \quad \bar{\rho} = b\rho, \quad \bar{\psi} = c\psi, \quad t = \alpha\bar{t}, \quad x = \beta\bar{x}, \quad z = \beta\bar{z} \quad (4.2)$$

admitted by Eqs. (1.1)-(1.3). These transformations are defined near the identity transformation if the parameters $a, \ldots, \beta$ are positive. The dilations of x and z are taken by the same parameter β in order to keep invariant the Laplacian Δ. Let us find the parameters $a, \ldots, \beta$ from the invariance condition of Eqs. (1.1)-(1.3). The transformations (4.2) change the derivatives involved in Eqs. (1.1)-(1.3) as follows:

$$\begin{aligned}
&\bar{v}_{\bar{t}} = a\alpha v_t, \quad \bar{v}_{\bar{x}} = a\beta v_x, \quad \bar{v}_{\bar{z}} = a\beta v_z,\\
&\bar{\rho}_{\bar{t}} = b\alpha\rho_t, \quad \bar{\rho}_{\bar{x}} = b\beta\rho_x, \quad \bar{\rho}_{\bar{z}} = b\beta\rho_z,\\
&\bar{\psi}_{\bar{t}} = c\alpha\psi_t, \quad \bar{\psi}_{\bar{x}} = c\beta\psi_x, \quad \bar{\psi}_{\bar{z}} = c\beta\psi_z,\\
&\bar{\Delta}\psi_{\bar{t}} = c\alpha\beta^2\Delta\psi_t, \; \bar{\Delta}\psi_{\bar{x}} = c\beta^3\Delta\psi_x, \; \bar{\Delta}\psi_{\bar{z}} = c\beta^3\Delta\psi_z,
\end{aligned} \quad (4.3)$$

where $\bar{\Delta}$ is the Laplacian written in the variables $\bar{x}, \bar{z}$. The invariance of Eqs. (1.1)-(1.3) under the transformations (4.2) means that the following equations are satisfied:

$$\begin{aligned}
&\bar{\Delta}\bar{\psi}_{\bar{t}} - g\bar{\rho}_{\bar{x}} - f\bar{v}_{\bar{z}} - \bar{\psi}_{\bar{x}}\bar{\Delta}\bar{\psi}_{\bar{z}} + \bar{\psi}_{\bar{z}}\bar{\Delta}\bar{\psi}_{\bar{x}} = 0,\\
&\bar{v}_{\bar{t}} + f\bar{\psi}_{\bar{z}} - \bar{\psi}_{\bar{x}}\bar{v}_{\bar{z}} + \bar{\psi}_{\bar{z}}\bar{v}_{\bar{x}} = 0,\\
&\bar{\rho}_{\bar{t}} + \frac{N^2}{g}\bar{\psi}_{\bar{x}} - \bar{\psi}_{\bar{x}}\bar{\rho}_{\bar{z}} + \bar{\psi}_{\bar{z}}\bar{\rho}_{\bar{x}} = 0,
\end{aligned}$$

whenever Eqs.(1.1)-(1.3) hold. Substituting here the expressions (4.3) we have:

$$\begin{aligned}
&\bar{\Delta}\bar{\psi}_{\bar{t}} - g\bar{\rho}_{\bar{x}} - f\bar{v}_{\bar{z}} - \bar{\psi}_{\bar{x}}\bar{\Delta}\bar{\psi}_{\bar{z}} + \bar{\psi}_{\bar{z}}\bar{\Delta}\bar{\psi}_{\bar{x}} = c\alpha\beta^2\Delta\psi_t - b\beta g\rho_x - c^2\beta^4\big(\psi_x\Delta\psi_z - \psi_z\Delta\psi_x\big),\\
&\bar{v}_{\bar{t}} + f\bar{\psi}_{\bar{z}} - \bar{\psi}_{\bar{x}}\bar{v}_{\bar{z}} + \bar{\psi}_{\bar{z}}\bar{v}_{\bar{x}} = a\alpha v_t + c\beta f\psi_z - ac\beta^2\big(\psi_x v_z - \psi_z v_x\big),\\
&\bar{\rho}_{\bar{t}} + \frac{N^2}{g}\bar{\psi}_{\bar{x}} - \bar{\psi}_{\bar{x}}\bar{\rho}_{\bar{z}} + \bar{\psi}_{\bar{z}}\bar{\rho}_{\bar{x}} = b\alpha\rho_t + c\beta\frac{N^2}{g}\psi_x - bc\beta^2\big(\psi_x\rho_z - \psi_z\rho_x\big).
\end{aligned}$$

These equations show that the invariance of Eqs. (1.1)-(1.3) is guaranteed by the following six equations for five undetermined parameter a, b, c, α, β :

$$c\alpha\beta^2 = b\beta = c^2\beta^4, \quad a\alpha = c\beta = ac\beta^2, \quad b\alpha = c\beta = bc\beta^2. \tag{4.4}$$

It can be verified by simple calculations that Eqs. (4.4) yield

$$\alpha = 1, \quad b = a, \quad c = a^2, \quad \beta = \frac{1}{a}\,,$$

where a is an arbitrary parameter. We substitute these values of the parameters in (4.2), denote the positive parameter a by $\mathrm{e}^{\tilde{a}}$, drop the tilde and conclude that Eqs. (1.1)-(1.3) admit the one-parameter non-uniform dilation group

$$\bar{t} = t, \quad \bar{x} = x\mathrm{e}^a, \quad \bar{z} = z\mathrm{e}^a, \quad \bar{v} = v\mathrm{e}^a, \quad \bar{\rho} = \rho\mathrm{e}^a, \quad \bar{\psi} = \psi\mathrm{e}^{2a}$$

with the following generator:

$$X_7 = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + v\frac{\partial}{\partial v} + \rho\frac{\partial}{\partial \rho} + 2\psi\frac{\partial}{\partial \psi}\,\cdot \tag{4.5}$$

We will consider the operators (4.1)-(4.5) as obvious symmetries of Eqs. (1.1)-(1.3) and will compute the local conservation laws provided by these symmetries.

4.2 Translation of v

For the operator X_1 from (4.1) Eqs. (3.16) yield

$$W^1 = 1, \quad W^2 = 0, \quad W^3 = 0.$$

Substituting these expressions in Eq. (3.23) we obtain

$$C^1 = -v.$$

In this case the equations (3.20) and (3.21) are also simple. They are written

$$C^2 = u\psi_z, \quad C^3 = -(u\psi_x + f\varphi)$$

and upon using Eqs. (2.18) yield:

$$C^2 = -v\psi_z, \quad C^3 = v\psi_x - f\psi.$$

Since any conserved vector is defined up to multiplication by an arbitrary constant, we change the sign of $C^1,\ C^2,\ C^3$ and obtain the following conserved vector:

$$C^1 = v, \quad C^2 = v\psi_z, \quad C^3 = f\psi - v\psi_x. \tag{4.6}$$

We have:

$$D_t(C^1) + D_x(C^2) + D_z(C^3) = v_t + v_x\psi_z + f\psi_z - v_z\psi_x.$$

Hence, the conservation equation (3.1) coincides with Eq. (1.2).

4.3 Translation of ρ

For the operator X_2 from (4.1) Eqs. (3.16) yield

$$W^1 = 0, \quad W^2 = 1, \quad W^3 = 0.$$

Substituting these expressions in Eq. (3.23) we obtain

$$C^1 = -\frac{g^2}{N^2}\,\rho.$$

Furthermore, Eqs. (3.20), (3.21) and Eqs. (2.18) yield:

$$C^2 = -g\psi - \frac{g^2}{N^2}\,\rho\,\psi_z, \quad C^3 = \frac{g^2}{N^2}\,\rho\,\psi_x.$$

Multiplying $C^1,\ C^2,\ C^3$ by $-N^2/g^2$ we arrive at the following conserved vector:

$$C^1 = \rho, \quad C^2 = \frac{N^2}{g}\,\psi + \rho\,\psi_z, \quad C^3 = -\rho\,\psi_x. \tag{4.7}$$

One can readily verify that the conservation equation (3.1) for the vector (4.7) is also satisfied. Namely, it coincides with Eq. (1.3).

4.4 Translation of ψ

For the operator X_3 from (4.1) Eqs. (3.16) yield

$$W^1 = 0, \quad W^2 = 0, \quad W^3 = 1.$$

Substituting these expressions in Eq. (3.23) we obtain

$$C^1 = 0.$$

Hence, the invariance of Eqs. (1.1)-(1.3) under the translation of ψ furnishes only a trivial conservation law (see Definition 3.1).

4.5 Derivation of the flux of conserved vectors with known densities

We will show here how to find the components C^2 and C^3 of the conserved vector (4.6) without using Eqs. (3.20), (3.21), provided that we know the conserved density $C^1 = v$.

Let us first verify that $C^1 = v$ satisfies Corollary 3.1. In this case $D_t(C^1) = v_t$, and hence Eq. (3.7) yields

$$F = D_t(C^1)\Big|_{(1.1)-(1.3)} = J(\psi, v) - f\psi_z. \tag{4.8}$$

Using Proposition 3.1 we see that Eqs. (3.5) are satisfied:

$$\frac{\delta F}{\delta v} = \frac{\delta F}{\delta \rho} = 0, \quad \frac{\delta F}{\delta \psi} = D_z(f) = 0. \tag{4.9}$$

Therefore Corollary 3.1 guarantees that F defined by Eq. (4.8) satisfies Eq. (3.4):

$$\psi_x v_z - \psi_z v_x - f\psi_z = D_x(H^1) + D_z(H^2) \tag{4.10}$$

with certain functions $H^1,\ H^2$.

In order to find $H^1,\ H^2$, we write

$$\psi_x v_z - f\psi_z = D_z(v\psi_x - f\psi) - v\psi_{xz}, \quad -\psi_z v_x = D_x(-v\psi_z) + v\psi_{zx}$$

and obtain:

$$\psi_x v_z - \psi_z v_x - f\psi_z = D_x(-v\psi_z) + D_z(v\psi_x - f\psi).$$

Thus, $H^1 = -v\psi_z,\ H^2 = v\psi_x - f\psi$. Denoting $C^2 = -H^1,\ C^3 = -H^2$, i.e.

$$C^1 = v\psi_z, \quad C^2 = f\psi - v\psi_x,$$

and invoking Eq. (4.8), we write Eq. (4.10) in the form

$$D_t(C^1)\Big|_{(1.1)-(1.3)} + D_x(C^1) + D_z(C^2) = 0.$$

This is precisely the conservation equation (3.1) for the vector (4.6). Thus, we have obtained the components $C^2,\ C^3$ of the conserved vector (4.6) without using Eqs. (3.20), (3.21).

The components $C^2,\ C^3$ of the conserved vector (4.7) can be derived likewise.

4.6 Translation of x

For the operator X_5 from (4.1) Eqs. (3.16) yield

$$W^1 = -v_x, \quad W^2 = -\rho_x, \quad W^3 = -\psi_x.$$

Substituting these expressions in Eq. (3.23) we obtain

$$C^1 = vv_x + \frac{g^2}{N^2}\,\rho\rho_x + \psi_x\psi_{xx} + \psi_z\psi_{xz} = D_x\left(\frac{1}{2}v^2 + \frac{1}{2}\frac{g^2}{N^2}\,\rho^2 + \frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_z^2\right).$$

Hence, the invariance of Eqs. (1.1)-(1.3) under the translation of x furnishes only a trivial conservation law (see Definition 3.1). Similar calculations show that the invariance under the translation of z provides also a trivial conservation law.

4.7 Time translation

For the operator X_4 from (4.1) Eqs. (3.16) yield

$$W^1 = -v_t, \quad W^2 = -\rho_t, \quad W^3 = -\psi_t.$$

Substituting these expressions in Eq. (3.23) we obtain

$$C^1 = vv_t + \frac{g^2}{N^2}\,\rho\rho_t + \psi_x\psi_{xt} + \psi_z\psi_{zt}. \tag{4.11}$$

Changing the last two terms of C^1 by using the identity

$$\begin{aligned}\psi_x\psi_{xt} + \psi_z\psi_{zt} &= D_x(\psi\psi_{xt}) - \psi\psi_{xxt} + D_z(\psi\psi_{zt}) - \psi\psi_{zzt}\\ &= D_x(\psi\psi_{xt}) + D_z(\psi\psi_{zt}) - \psi\Delta\psi_t\end{aligned} \tag{4.12}$$

and dropping the divergent type terms, we rewrite C^1 given by Eq. (4.11) in the form

$$C^1 = vv_t + \frac{g^2}{N^2}\,\rho\rho_t - \psi\Delta\psi_t. \tag{4.13}$$

Let us clarify if the conservation law with the density (4.13) is trivial or non-trivial. According to Definition 3.1, we have to evaluate the density (4.13) on the solutions of Eqs. (1.1)-(1.3). In this case it is convenient to use Eqs. (1.1)-(1.3) in the form (1.4)-(1.6) and replace Eq. (3.8) by

$$C^1_* = C^1\big|_{(1.4)-(1.6)}.$$

Then we have

$$C^1_* = \Big\{vJ(\psi, v) + \frac{g^2}{N^2}\,\rho J(\psi,\rho) - \psi J(\psi,\Delta\psi)\Big\} - fD_z(v\psi) - gD_x(\rho\psi)$$

and Corollary 3.2 shows that the conservation law is trivial. Indeed, the last two terms of C^1_* have the divergent form. The expression in braces for C^1_* satisfies Eqs. (3.9) according to Proposition 3.1, and hence it also has the divergent form.

Thus, the invariance of Eqs. (1.1)-(1.3) under the time translation furnishes only a trivial conservation law.

4.8 Use of the dilation. Conservation of energy

Consider the generator (4.5) of the dilation group,

$$X_7 = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + v\frac{\partial}{\partial v} + \rho\frac{\partial}{\partial \rho} + 2\psi\frac{\partial}{\partial \psi}\,\cdot$$

In this case the quantities (3.16) have the form

$$W^1 = v - xv_x - zv_z, \quad W^2 = \rho - x\rho_x - z\rho_z, \quad W^3 = 2\psi - x\psi_x - z\psi_z. \quad (4.14)$$

The substitution of (4.14) in (3.23) yields:

$$C^1 = -v^2 + xvv_x + zvv_z + \frac{g^2}{N^2}\left(-\rho^2 + x\rho\rho_x + z\rho\rho_z\right)$$
$$-\psi_x^2 + x\psi_x\psi_{xx} + z\psi_x\psi_{xz} - \psi_z^2 + x\psi_z\psi_{xz} + z\psi_z\psi_{zz}. \quad (4.15)$$

We modify (4.15) by using the identities

$$xvv_x + zvv_z = \frac{1}{2}D_x\left(xv^2\right) + \frac{1}{2}D_z\left(zv^2\right) - v^2,$$
$$x\rho\rho_x + z\rho\rho_z = \frac{1}{2}D_x\left(x\rho^2\right) + \frac{1}{2}D_z\left(z\rho^2\right) - \rho^2,$$
$$x\psi_x\psi_{xx} + x\psi_z\psi_{xz} = \frac{1}{2}D_x\left[x\left(\psi_x^2 + \psi_x^2\right)\right] - \frac{1}{2}\left(\psi_x^2 + \psi_x^2\right),$$
$$z\psi_x\psi_{xz} + z\psi_z\psi_{zz} = \frac{1}{2}D_z\left[z\left(\psi_x^2 + \psi_x^2\right)\right] - \frac{1}{2}\left(\psi_x^2 + \psi_x^2\right).$$

Substituting these in (4.15) and dropping the divergent type terms we have:

$$C^1 = -2\left(v^2 + \frac{g^2}{N^2}\rho^2 + |\nabla\psi|^2\right),$$

where

$$|\nabla\psi|^2 = \psi_x^2 + \psi_z^2.$$

Dividing C^1 by the inessential coefficient (-2) we finally obtain the following conservation law in the integral form (3.2):

$$\frac{d}{dt}\int\int\left[v^2 + \frac{g^2}{N^2}\rho^2 + |\nabla\psi|^2\right]dxdz = 0. \quad (4.16)$$

Eq. (4.16) represents the conservation of the *energy with the density*

$$E = v^2 + \frac{g^2}{N^2}\rho^2 + |\nabla\psi|^2. \quad (4.17)$$

Let us find the components C^2 and C^3 of this conservation law written in the differential form (3.1). We will use the procedure suggested in Section 4.5. Let us first verify that E defined by Eq. (4.17) satisfies Corollary 3.1 for densities of conservation laws. We have

$$D_t(E) = 2\left(vv_t + \frac{g^2}{N^2}\rho\rho_t + \psi_x\psi_{xt} + \psi_z\psi_{zt}\right). \quad (4.18)$$

Since the expression in the brackets in Eq. (4.18) is identical with (4.11) it can be rewritten in the form (4.13), and hence satisfies Eqs. (3.5). Corollary 3.1 guarantees that E is the density of a conservation law. It is manifest from Eq. (4.17) that this conservation law is non-trivial.

According to Corollary 3.1, $D_t(E)$ defined by Eq. (4.18) and evaluated on the solutions of Eqs. (1.1)-(1.3) satisfies Eq. (3.4),

$$D_t(E)\Big|_{(1.1)-(1.3)} = D_x(H^1) + D_z(H^2), \tag{4.19}$$

with certain functions $H^1,\ H^2$. In order to find $H^1,\ H^2$, we use Eq. (4.12),

$$2(\psi_x\psi_{xt} + \psi_z\psi_{zt}) = D_x(2\psi\psi_{xt}) + D_z(2\psi\psi_{zt}) - 2\psi\Delta\psi_t, \tag{4.20}$$

and write:

$$\begin{aligned} 2vv_t &= 2\psi_x v v_z - \psi_z v v_x - 2fv\psi_z \\ &= D_z(v^2\psi_x) - D_x(v^2\psi_z) - 2fD_z(v\psi) + 2f\psi v_z, \end{aligned} \tag{4.21}$$

$$\begin{aligned} 2\,\frac{g^2}{N^2}\,\rho\rho_t &= 2\,\frac{g^2}{N^2}\Big(\psi_x\rho\rho_z - \psi_z\rho\rho_x\Big) - 2g\rho\psi_x \\ &= \frac{g^2}{N^2}\Big[D_z(\rho^2\psi_x) - D_x(\rho^2\psi_z)\Big] - 2gD_x(\rho\psi) + 2g\psi\rho_x, \end{aligned} \tag{4.22}$$

$$\begin{aligned} -2\psi\Delta\psi_t &= -2g\psi\rho_x - 2f\psi v_z - 2\psi\psi_x\Delta\psi_z + 2\psi\psi_z\Delta\psi_x \\ &= -2g\psi\rho_x - 2f\psi v_z - D_x\big(\psi^2\Delta\psi_z\big) + D_z\big(\psi^2\Delta\psi_x\big). \end{aligned} \tag{4.23}$$

Substituting the expressions (4.21), (4.22) and (4.20), (4.23) in the right-hand side of Eq. (4.18), we arrive at Eq. (4.19) with

$$\begin{aligned} H^1 &= -v^2\psi_z - \frac{g^2}{N^2}\,\rho^2\psi_z - 2g\rho\psi + 2\psi\psi_{xt} - \psi^2\Delta\psi_z, \\ H^2 &= v^2\psi_x + \frac{g^2}{N^2}\,\rho^2\psi_x - 2fv\psi + 2\psi\psi_{zt} + \psi^2\Delta\psi_x. \end{aligned}$$

Thus, denoting $C^2 = -H^1,\ C^3 = -H^2$ we arrive at the following differential form (3.1) of the conservation of energy for Eqs. (1.1)-(1.3):

$$D_t(E) + D_x(C^2) + D_x(C^3) = 0 \tag{4.24}$$

with the density E given by Eq. (4.17) and the flux given by the equations

$$\begin{aligned} C^2 &= 2g\rho\psi + v^2\psi_z + \frac{g^2}{N^2}\,\rho^2\psi_z - 2\psi\psi_{xt} + \psi^2\Delta\psi_z, \\ C^3 &= 2fv\psi - v^2\psi_x - \frac{g^2}{N^2}\,\rho^2\psi_x - 2\psi\psi_{zt} - \psi^2\Delta\psi_x. \end{aligned} \tag{4.25}$$

5 Invariant solutions

5.1 Invariant solution based on translation and dilation

Let us find the invariant solution based on the following two operators:

$$
\begin{aligned}
mX_5 - kX_6 &= m\frac{\partial}{\partial x} - k\frac{\partial}{\partial z} \quad (m, k = \text{const.}),\\
X_7 &= x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + v\frac{\partial}{\partial v} + \rho\frac{\partial}{\partial \rho} + 2\psi\frac{\partial}{\partial \psi}\,\cdot
\end{aligned}
\tag{5.1}
$$

We first find their invariants $J(t, x, z, v, \rho, \psi)$ by solving the equations

$$
(mX_5 - kX_6)J = 0, \quad X_7 J = 0. \tag{5.2}
$$

The characteristic equation $kdx + mdz = 0$ the first equation (5.2) yields that the operator $mX_2 - kX_3$ has, along with t, v, ρ, ψ, the following invariant:

$$
\lambda = kx + mz. \tag{5.3}
$$

Therefore we have to find the invariants $J(t, \lambda, v, \rho, \psi)$ for the operator X_4. To this end, we write the action of X_4 on the variables $t, \lambda, v, \rho, \psi$ by the standard formula

$$
X_7 = X_7(\lambda)\frac{\partial}{\partial \lambda} + v\frac{\partial}{\partial v} + \rho\frac{\partial}{\partial \rho} + 2\psi\frac{\partial}{\partial \psi}
$$

and obtain

$$
X_7 = \lambda\frac{\partial}{\partial \lambda} + v\frac{\partial}{\partial v} + \rho\frac{\partial}{\partial \rho} + 2\psi\frac{\partial}{\partial \psi}\,\cdot \tag{5.4}
$$

To solve the equation $X_7 J(t, \lambda, v, \rho, \psi) = 0$ for the invariants, we calculate the first integrals for the characteristic system

$$
\frac{d\lambda}{\lambda} = \frac{dv}{v} = \frac{d\rho}{\rho} = \frac{d\psi}{2\psi}
$$

and see that a basis of invariants for the operators (5.1) is given by

$$
t, \quad V = \frac{v}{\lambda}, \quad R = \frac{\rho}{\lambda}, \quad \phi = \frac{\psi}{\lambda^2}\,\cdot
$$

Accordingly, we assign the invariants V, R, ϕ to be functions of the invariant t and arrive at the following general form of the candidates for the invariant solutions:

$$
v = \lambda V(t), \quad \rho = \lambda R(t), \quad \psi = \lambda^2 \phi(t), \quad \lambda = kx + mz. \tag{5.5}
$$

In order to find the functions $V(t), R(t), \phi(t),$ we have to substitute the expressions (5.5) in Eqs. (1.1)- (1.3).

We have:

$$
\begin{aligned}
&\psi_t = \lambda^2 \phi'(t), \quad \psi_x = 2k\lambda\phi(t), \quad \psi_z = 2m\lambda\phi(t),\\
&\nabla^2\psi_t = 2(k^2+m^2)\phi'(t), \quad \nabla^2\psi_x = 0, \quad \nabla^2\psi_z = 0,\\
&\psi_x v_z = 2km\lambda\,\phi(t)\,V(t), \quad \psi_z v_x = 2km\lambda\,\phi(t)\,V(t),\\
&\psi_x \rho_z = 2km\lambda\,\phi(t)\,R(t), \quad \psi_z \rho_x = 2km\lambda\,\phi(t)\,R(t).
\end{aligned}
$$

Therefore Eqs. (1.1)- (1.3) yield the following system of first-order linear ordinary differential equations:

$$
\begin{aligned}
2\big(k^2+m^2\big)\phi' - gkR - fmV = 0,\\
\lambda V' + 2fm\lambda\phi = 0,\\
\lambda R' + 2\frac{k\lambda}{g}\,N^2\phi = 0,
\end{aligned}
$$

or

$$
\phi' = \frac{1}{2\big(k^2+m^2\big)}\big(gkR + fmV\big), \tag{5.6}
$$

$$
V' = -2fm\,\phi, \tag{5.7}
$$

$$
R' = -2\frac{k}{g}\,N^2\phi\,. \tag{5.8}
$$

Let us integrate Eqs. (5.6)-(5.8). Differentiating Eq. (5.6) and using Eqs. (5.7)-(5.8), we obtain

$$
\phi'' + \omega^2\,\phi = 0, \tag{5.9}
$$

where

$$
\omega^2 = \frac{k^2N^2 + m^2f^2}{k^2+m^2}\,. \tag{5.10}
$$

The general solution of Eq. (5.9) is given by

$$
\phi(t) = C_1\cos(\omega t) + C_2\sin(\omega t), \quad C_1, C_2 = \text{const.} \tag{5.11}
$$

Substituting (5.11) in Eqs. (5.7)-(5.8) and integrating, we obtain

$$
\begin{aligned}
V &= C_3 - \frac{2fm}{\omega}\Big[C_1\sin(\omega t) - C_2\cos(\omega t)\Big],\\
R &= C_4 - \frac{2k}{g\omega}\,N^2\Big[C_1\sin(\omega t) - C_2\cos(\omega t)\Big].
\end{aligned}
$$

To determine the constants C_3 and C_4, we substitute in Eq. (5.6) the above expressions for V, R and the expression (5.11) for ϕ and obtain

$$fmC_3 + gkC_4 = 0.$$

Thus, the solution to Eqs. (5.6)-(5.8) has the following form:

$$\phi(t) = C_1\cos(\omega t) + C_2\sin(\omega t), \tag{5.12}$$

$$V(t) = \frac{2fm}{\omega}\Big[C_2\cos(\omega t) - C_1\sin(\omega t)\Big] + C_3, \tag{5.13}$$

$$R(t) = \frac{2k}{g\,\omega}\,N^2\Big[C_2\cos(\omega t) - C_1\sin(\omega t)\Big] - \frac{fm}{gk}\,C_3. \tag{5.14}$$

Finally, substituting (5.12)-(5.14) in (5.5), we arrive at the following solution to the system (1.1)-(1.3):

$$\rho = \frac{2k}{g\,\omega}\,N^2\Big[C_2\cos(\omega t) - C_1\sin(\omega t)\Big]\lambda - \frac{fm}{gk}\,C_3\lambda, \tag{5.15}$$

$$v = \frac{2fm}{\omega}\Big[C_2\cos(\omega t) - C_1\sin(\omega t)\Big]\lambda + C_3\lambda, \tag{5.16}$$

$$\psi = \big[C_1\cos(\omega t) + C_2\sin(\omega t)\big]\lambda^2, \tag{5.17}$$

where λ is given by (5.3), ω is defined by Eq. (5.10) and C_1, C_2, C_3 are arbitrary constants.

5.2 Generalized invariant solution and wave beams

It is natural to generalize the candidates (5.5) for the invariant solutions and look for particular solutions of the system (1.1)-(1.3) in the following form of separated variables:

$$v = F(\lambda)V(t), \quad \rho = \alpha(\lambda)R(t), \quad \psi = \beta(\lambda)\phi(t), \quad \lambda = kx + mz. \tag{5.18}$$

The reckoning shows that then the right-hand sides of Eqs. (1.1)-(1.3) vanish and Eqs. (1.1)-(1.3) become:

$$\big(k^2 + m^2\big)\beta''(\lambda)\phi'(t) - gk\alpha'(\lambda)R(t) - fmF'(\lambda)V(t) = 0, \tag{5.19}$$

$$F(\lambda)V'(t) + fm\beta'(\lambda)\phi(t) = 0, \tag{5.20}$$

$$\alpha(\lambda)R'(t) + \frac{kN^2}{g}\,\beta'(\lambda)\phi(t) = 0. \tag{5.21}$$

Differentiating Eq. (5.19) with respect to t, using Eqs. (5.20)-(5.21) and dividing by β', we obtain

$$\left(k^2+m^2\right)\frac{\beta''}{\beta'}\,\phi''+\left(N^2k^2\frac{\alpha\,'}{\alpha}+f^2m^2\frac{F'}{F}\right)\phi=0.$$

Assuming that the ratios β''/β', $\alpha\,'/\alpha$, F'/F are proportional with constant coefficients and have one and the same sign, we arrive at an equation of the form (5.9). For example, letting

$$\frac{\beta''}{\beta'}=\frac{\alpha\,'}{\alpha}=\frac{F'}{F}\,, \tag{5.22}$$

we obtain Eq. (5.9). Then, according to (5.11), we can set in (5.18) $\phi(t)=\cos(\omega t)$ and $\phi(t)=\sin(\omega t)$, i.e.

$$\psi=A(\lambda)\cos(\omega t)\quad\text{and}\quad\psi=B(\lambda)\sin(\omega t). \tag{5.23}$$

For each function ψ given by (5.23) we determine the functions $V(t),\ R(t)$ using Eqs. (5.20), (5.21), (5.21), then take the linear combinations of the resulting functions and arrive at the following form of the "generalized invariant solution" (5.18):

$$\psi=A(\lambda)\cos(\omega t)+B(\lambda)\sin(\omega t), \tag{5.24}$$

$$v=\frac{fm}{\omega}\left[B'(\lambda)\cos(\omega t)-A'(\lambda)\sin(\omega t)\right]+F(\lambda), \tag{5.25}$$

$$\rho=\frac{kN^2}{g\omega}\left[B'(\lambda)\cos(\omega t)-A'(\lambda)\sin(\omega t)\right]+H(\lambda), \tag{5.26}$$

where ω is given by Eq. (5.10).

The reckoning shows that the functions (5.24)-(5.26) with arbitrary $A(\lambda),\ B(\lambda)$ solve Eqs. (1.1)-(1.3) provided that $F(\lambda),\ H(\lambda)$ satisfy the following equation:

$$gkH'(\lambda)+fmF'(\lambda)=0. \tag{5.27}$$

One can readily verify that the invariant solution (5.15)-(5.17), which is a particular case of (5.24)-(5.26), obeys the condition (5.27).

5.3 Energy of the generalized invariant solution

If we substitute in (4.17) the generalized invariant solution (see Eqs. (5.24)-(5.26))

$$\psi=A(\lambda)\cos(\omega t)+B(\lambda)\sin(\omega t), \tag{5.28}$$

$$v=\frac{fm}{\omega}\left[B'(\lambda)\cos(\omega t)-A'(\lambda)\sin(\omega t)\right], \tag{5.29}$$

$$\rho=\frac{kN^2}{g\omega}\left[B'(\lambda)\cos(\omega t)-A'(\lambda)\sin(\omega t)\right], \tag{5.30}$$

where

$$\lambda = kx + mz, \quad \omega^2 = \frac{k^2N^2 + m^2f^2}{k^2 + m^2},$$

we obtain:

$$E = (k^2 + m^2)\big[A'^{\,2}(\lambda) + B'^{\,2}(\lambda)\big].$$

Invoking that any conserved vector is defined up to multiplication by an arbitrary constant, we divide the above expression for E by $(k^2 + m^2)$ and obtain the following energy:

$$E = A'^{\,2}(\lambda) + B'^{\,2}(\lambda). \tag{5.31}$$

Since the energy density (5.31) depends only on $\lambda = kx + mz$, it is constant along the straight line

$$kx + mz = \text{const.} \tag{5.32}$$

Accordingly, the "local energy" (5.31) has one and the same value at points (x_0, z_0) and (x_1, z_1) provided that

$$kx_0 + mz_0 = kx_1 + mz_1. \tag{5.33}$$

The energy density (5.31) describes the local behavior of the solutions. Therefore it is significant to understand its distribution on the (x, z) plane. Suppose that the functions $A(\lambda),\ B(\lambda)$ and their derivatives rapidly decrease as $\eta \to \infty$. If we take, as an example, the functions

$$A(\lambda) = \frac{a}{1 + \lambda^2}, \quad B(\lambda) = \frac{a\lambda}{1 + \lambda^2}, \tag{5.34}$$

where a is a positive constant, then the energy density (5.31) of the wave beams has the form

$$E = \frac{a^2}{(1 + \lambda^2)^2}\,\cdot$$

Hence, the energy is localized along the straight line (5.33). Therefore we can define a *wave beam through a point* (x_0, z_0) *as the totality of the points* (x_1, z_1) *satisfying Eq.* (5.33), i.e. identify it with the straight line (5.33).

6 February 2009

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GROUP ANALYSIS OF NONLINEAR INTERNAL WAVES IN OCEANS

II: The symmetries and rotationally invariant solution

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Abstract. The maximal group of Lie point symmetries of a system of nonlinear equations used in geophysical fluid dynamics is presented. The Lie algebra of this group is infinite-dimensional and involves three arbitrary functions of time. The invariant solution under the rotation and dilation is constructed. Qualitative analysis of the invariant solution is provided and the energy of this solution is presented.

Keywords: Geophysical fluid dynamics, Symmetries, Infinite Lie algebra, Invariant solution.

1 Introduction

This is a continuation of the paper [1]. We present here the Lie algebra of the maximal group of Lie point symmetries for system nonlinear equations

$$\Delta\psi_t - g\rho_x - fv_z = \psi_x\Delta\psi_z - \psi_z\Delta\psi_x\,, \tag{1.1}$$

$$v_t + f\psi_z = \psi_x v_z - \psi_z v_x\,, \tag{1.2}$$

$$\rho_t + \frac{N^2}{g}\,\psi_x = \psi_x\rho_z - \psi_z\rho_x \tag{1.3}$$

used in geophysical fluid dynamics, e.g. for investigating internal waves in uniformly stratified incompressible fluids (oceans). Here g, f, N are constants and Δ is the two-dimensional Laplacian:

$$\Delta = D_x^2 + D_z^2 \cdot$$

2 Symmetries

2.1 General case

The point symmetries of Eqs. (1.1)-(1.3) have been computed with the help of DIMSYM 2.3 package. The maximal admitted Lie point transformation group is infinite for arbitrary constants f and N. If $f \neq 0$, the group is generated by the infinite-dimensional Lie algebra spanned by the following operators:

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial v}, \quad X_2 = \frac{\partial}{\partial \rho}, \quad X_3 = a(t)\frac{\partial}{\partial \psi}, \quad X_4 = \frac{\partial}{\partial t},\\
&X_5 = b(t)\left[\frac{\partial}{\partial x} - f\,\frac{\partial}{\partial v}\right] + b'(t)z\,\frac{\partial}{\partial \psi},\\
&X_6 = c(t)\left[\frac{\partial}{\partial z} + \frac{N^2}{g}\,\frac{\partial}{\partial \rho}\right] - c'(t)x\,\frac{\partial}{\partial \psi},\\
&X_7 = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + v\frac{\partial}{\partial v} + \rho\frac{\partial}{\partial \rho} + 2\psi\frac{\partial}{\partial \psi},\\
&X_8 = t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 2z\frac{\partial}{\partial z} + 3\psi\frac{\partial}{\partial \psi} - 2fx\frac{\partial}{\partial v} + 2\frac{N^2}{g}\,z\frac{\partial}{\partial \rho},\\
&X_9 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} - \frac{1}{f}\left[g\rho + (f^2 - N^2)z\right]\frac{\partial}{\partial v} + \frac{1}{g}\left[fv + (f^2 - N^2)x\right]\frac{\partial}{\partial \rho}\cdot
\end{aligned} \tag{2.1}$$

Here $a(t)$, $b(t)$ and $c(t)$ are arbitrary functions of time t.

Remark 2.1. The presence of the arbitrary functions $a(t), b(t), c(t)$ in the symmetry Lie algebra is a characteristic property of incompressible fluids ([2], see also [3]). Namely, the operator X_3 generates the group transformation $\bar{\psi} = \psi + \varepsilon_3\, a(t)$ of the stream function ψ, where ε_3 is the group parameter. The invariance of fluid flows under this transformation is quite obvious because the velocity vector $(\psi_z, v, -\psi_x)$ is invariant under this transformation. The operators X_5, X_6 express the invariance under the generalization $\bar{x} = x + \varepsilon_5\, b(t),\ \bar{z} = z + \varepsilon_6\, c(t)$ of the coordinate translations and the Galilean transformations. They provide a *generalized relativity principle* for the Euler equations in terms of conservation laws (see [4], Section 25.3).

2.2 The case $f = 0$

In order to include the special case $f = 0$, we multiply the operator X_9 by the constant f and consider the operator

$$X_9' = f\Big[z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\Big] - \big[g\rho + (f^2 - N^2)z\big]\frac{\partial}{\partial v} + \frac{f}{g}\left[fv + (f^2 - N^2)x\right]\frac{\partial}{\partial \rho}\,.$$

Then we let $f = 0$ and obtain the operator

$$X_9' = -\big[g\rho - N^2\,z\big]\frac{\partial}{\partial v}$$

admitted by Eqs. (1.1)-(1.3) with $f = 0$. The solution of the determining equations shows that X_9' is a particular case of a more general symmetry involving an arbitrary function of two variables. Namely, the system (1.1)-(1.3) with $f = 0$ admits the infinite-dimensional Lie algebra spanned by the following operators:

$$\begin{aligned}
&X_1 = h(v, g\rho - N^2\,z)\,\frac{\partial}{\partial v}\,, \quad X_2 = \frac{\partial}{\partial \rho}\,, \quad X_3 = a(t)\frac{\partial}{\partial \psi}\,, \quad X_4 = \frac{\partial}{\partial t}\,,\\
&X_5 = b(t)\frac{\partial}{\partial x} + b'(t)z\,\frac{\partial}{\partial \psi}\,,\\
&X_6 = c(t)\left[\frac{\partial}{\partial z} + \frac{N^2}{g}\,\frac{\partial}{\partial \rho}\right] - c'(t)x\,\frac{\partial}{\partial \psi}\,,\\
&X_7 = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + v\frac{\partial}{\partial v} + \rho\frac{\partial}{\partial \rho} + 2\psi\frac{\partial}{\partial \psi}\,,\\
&X_8 = t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 2z\frac{\partial}{\partial z} + 3\psi\frac{\partial}{\partial \psi} + 2\frac{N^2}{g}\,z\frac{\partial}{\partial \rho}\,,
\end{aligned} \tag{2.2}$$

where $h(v, g\rho - N^2\,z)$ is an arbitrary function of two variables. The operator X_1 in (2.1) is obtained from the operator X_1 in (2.2) by taking $h = 1$.

3 Invariant solution based on rotations and dilations

3.1 The invariants

We will investigate here the invariant solutions with respect to the dilations and rotations with the generators X_7 and X_9. Let us introduce the notation

$$v_* = fv, \quad u = g\rho, \quad \alpha = f^2 - N^2 \tag{3.1}$$

and write the operators $X_7,\ X_9$ in the form

$$
\begin{aligned}
X_7 &= x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} + v_*\frac{\partial}{\partial v_*} + 2\psi\frac{\partial}{\partial \psi}\,,\\
X_9 &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + \big(v_* + \alpha\, x\big)\frac{\partial}{\partial u} - \big(u + \alpha\, z\big)\frac{\partial}{\partial v_*}\,\cdot
\end{aligned}
\tag{3.2}
$$

The operators (3.2) coincide with the operators (3.17) from [5],

$$
X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + kw\frac{\partial}{\partial w}\,,
$$

$$
X_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + (v + \alpha x + \beta y)\frac{\partial}{\partial u} - (u - \beta x + \alpha y)\frac{\partial}{\partial v}\,,
$$

with $k = 2$ and $\beta = 0$ upon identifying v with v_* and y with z. Hence, a basis of invariants for the operators (3.2) contains the time t and the invariants (3.20) from [5] which have now the form

$$
\begin{aligned}
J_1 &= \frac{1}{x^2+z^2}\ (xu + zv_* + \alpha xz)\ ,\\
J_2 &= \frac{1}{x^2+y^2}\ \Big(xv_* - zu + \frac{\alpha}{2}(x^2 - z^2)\Big)\ ,\\
J_3 &= \frac{\psi}{x^2+y^2}\,\cdot
\end{aligned}
$$

It is more convenient for our purposes to use, instead of these invariants, the equivalent equations (3.19) from [5] which are written now as follows:

$$
\begin{aligned}
u &= J_1\, x - \Big(J_2 + \frac{\alpha}{2}\Big)\ z\,,\\
v_* &= J_1\, z + \Big(J_2 - \frac{\alpha}{2}\Big)\ x\,,\\
\psi &= (x^2 + z^2)\, J_3.
\end{aligned}
\tag{3.3}
$$

3.2 Candidates for the invariant solution

Knowledge of a symmetry algebra allows one to obtain particular exact solutions to differential equations in question. These kind of solutions were considered by S. Lie [6]. They are known today as group invariant solutions (briefly *invariant solutions*) and widely used in the modern literature, particularly in investigating nonlinear differential equations.

The general form of regular invariant solutions is obtained from Eqs. (3.3) by setting

$$J_1 = R(t), \quad J_2 = V(t), \quad J_3 = \phi(t)$$

with undetermined functions $R(t),\ V(t),\ \phi(t)$. Invoking the notation (3.1) we arrive at the following general form of candidates for the invariant solution with respect to the dilations and rotations with the generators X_7 and X_9 from (2.1):

$$\begin{aligned} v &= \frac{1}{f}\left[R(t)\,z + V(t)\,x + \frac{N^2 - f^2}{2}\,x\right], \\ \rho &= \frac{1}{g}\left[R(t)\,x - V(t)\,z + \frac{N^2 - f^2}{2}\,z\right], \\ \psi &= (x^2 + z^2)\,\phi(t). \end{aligned} \tag{3.4}$$

Remark 3.1. Solving the Lie equations for the operator X_9 from (3.2) and using the notation (3.1), one can verify that the operator X_9 from (2.1),

$$X_9 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} - \frac{1}{f}\left[g\rho + (f^2 - N^2)z\right]\frac{\partial}{\partial v} + \frac{1}{g}\left[fv + (f^2 - N^2)x\right]\frac{\partial}{\partial \rho},$$

generates the following one-parameter transformation group with the parameter ε :

$$\begin{aligned} &\bar{x} = x\cos\varepsilon + z\sin\varepsilon, \quad \bar{z} = z\cos\varepsilon - x\sin\varepsilon, \\ &g\bar{\rho} = g\rho\cos\varepsilon + fv\sin\varepsilon - (N^2 - f^2)\,x\sin\varepsilon, \\ &f\bar{v} = fv\cos\varepsilon - g\rho\sin\varepsilon + (N^2 - f^2)\,z\sin\varepsilon, \\ &\bar{t} = t, \quad \bar{\psi} = \psi. \end{aligned} \tag{3.5}$$

One can verify by inspection that the transformations (3.5) leave invariant Eqs. (3.4):

$$\begin{aligned} \bar{v} &= \frac{1}{f}\left[R(t)\,\bar{z} + V(t)\,\bar{x} + \frac{N^2 - f^2}{2}\,\bar{x}\right], \\ \bar{\rho} &= \frac{1}{g}\left[R(t)\,\bar{x} - V(t)\,\bar{z} + \frac{N^2 - f^2}{2}\,\bar{z}\right], \\ \bar{\psi} &= (\bar{x}^2 + \bar{z}^2)\,\phi(t). \end{aligned}$$

3.3 Construction of the invariant solution

It remains to determine the functions $R(t),\ V(t),\ \phi(t)$ by substituting the expressions (3.4) for $\rho,\ v,\ \psi$ in Eqs. (1.1)-(1.3).

Differentiating (3.4) we obtain:

$$v_t = \frac{1}{f}\left[R'\,z + V'\,x\right], \quad v_x = \frac{1}{f}\left[\frac{N^2 - f^2}{2} + V\right], \quad v_z = \frac{1}{f}\,R,$$

$$\rho_t = \frac{1}{g}\left[R'\,x - V'\,z\right], \quad \rho_x = \frac{1}{g}\,R, \quad \rho_z = \frac{1}{g}\left[\frac{N^2 - f^2}{2} - V\right], \tag{3.6}$$

$$\psi_t = (x^2 + z^2)\,\phi', \quad \psi_x = 2x\,\phi, \quad \psi_z = 2z\,\phi, \quad \Delta\psi_t = 4\phi'.$$

Substitution of (3.6) in Eqs. (1.1)-(1.3) yields:

$$2\phi' - R = 0, \tag{3.7}$$

$$[V' - 2R\phi]\,x + [R' + 2V\phi + (N^2 + f^2)\phi]\,z = 0, \tag{3.8}$$

$$[R' + 2V\phi + (N^2 + f^2)\phi]\,x - [V' - 2R\phi]\,z = 0. \tag{3.9}$$

Since V, R, ϕ depend only on$t$, Eq. (3.8) implies that

$$V' - 2R\phi = 0 \tag{3.10}$$

and

$$R' + 2V\phi + (N^2 + f^2)\phi = 0. \tag{3.11}$$

Eq. (3.9) is satisfied due to Eqs. (3.10), (3.11). Hence, Eqs. (1.1)-(1.3) are reduced to Eqs. (3.7), (3.10), (3.11).

Let us write Eq. (3.7) in the form

$$R = 2\phi'. \tag{3.12}$$

Substitution of the expression for R into Eq. (3.10) yields $V' = 4\phi\phi'$, whence upon integration

$$V = 2\phi^2 + A, \quad A = \text{const}. \tag{3.13}$$

Finally, substituting Eqs. (3.12) and (3.13) in Eq. (3.11) we obtain the following nonlinear second-order ordinary differential equation for $\phi(t)$:

$$\phi'' + 2\phi^3 + \left(A + \frac{f^2 + N^2}{2}\right)\phi = 0. \tag{3.14}$$

Thus, we have arrived at the following result.

Theorem 3.1. The solutions of the system (1.1)-(1.3) that are invariant with respect to the dilations and rotations with the generators X_7 and X_9 from (2.1) are given by

$$
\begin{aligned}
v &= \frac{1}{f}\left[\left(2\phi^2(t) + A + \frac{N^2 - f^2}{2}\right) x + 2\phi'(t) z\right], \\
\rho &= \frac{1}{g}\left[2\phi'(t) x - \left(2\phi^2(t) + A - \frac{N^2 - f^2}{2}\right) z\right], \\
\psi &= (x^2 + z^2)\,\phi(t),
\end{aligned}
\tag{3.15}
$$

where $\phi(t)$ is defined by the differential equation (3.14) and A is an arbitrary constant.

3.4 Qualitative analysis of the invariant solution

One can integrate Eq. (3.14) once, e.g., upon multiplying by $2\phi'$ and obtain

$$
\phi'^2 + \phi^4 + \left(A + \frac{f^2 + N^2}{2}\right)\phi^2 = \text{const.} \tag{3.16}
$$

We will analyze the behavior of the solutions to Eq. (3.16) under the assumption that the expression in the parentheses is a non-negative constant which we denote by K :

$$
K = A + \frac{f^2 + N^2}{2}, \quad K \geq 0, \tag{3.17}
$$

and write Eq. (3.16) in the form

$$
\phi'^2 + \phi^4 + K\phi^2 = B^2, \quad B = \text{const.}, \tag{3.18}
$$

or solving for ϕ' :

$$
\phi' = \pm\sqrt{B^2 - \phi^4 - K\phi^2}\,. \tag{3.19}
$$

Note that $\phi(t) = 0$ solves Eq. (3.14). Let us turn to Eq. (3.19). When ϕ is small, i.e. close to the trivial solution $\phi(t) = 0$, then

$$
B^2 - \phi^4 - K\phi^2 \approx B^2
$$

and hence ϕ' is close to the constant value

$$
\phi' \approx \pm B.
$$

When $\phi(t)$ varies according to Eq. (3.14), then $|\phi'|$ decreases since

$$
B^2 - \phi^4 - K\phi^2 < B^2
$$

when $\phi \neq 0$. We obtain $\phi' = 0$ when $\phi(t) = C_*$, where

$$C_*^2 = \frac{-K + \sqrt{B^2 + K^2}}{2} \cdot \tag{3.20}$$

If $|\phi| > |C_*|$, then $B^2 - \phi^4 - K\phi^2 < 0$, and hence Eq. (3.19) does not have a solution. We have arrived at the following significant results.

Theorem 3.2. Provided that the condition (3.17) holds, the solutions of Eq. (3.19) are bounded oscillating functions $\phi(t)$ satisfying the condition

$$-C_* \leq \phi(t) \leq C_*, \tag{3.21}$$

where C_* is the positive constant defined by Eq. (3.20). In this notation, the invariant solution (3.15) is written as follows:

$$\begin{aligned}
v &= \frac{1}{f}\left[\left(2\phi^2(t) + K - f^2\right)x + 2\phi'(t)z\right], \\
\rho &= \frac{1}{g}\left[2\phi'(t)x - \left(2\phi^2(t) + K - N^2\right)z\right], \\
\psi &= \left(x^2 + z^2\right)\phi(t).
\end{aligned} \tag{3.22}$$

Remark 3.2. The invariance of the solution (3.15) with respect to rotations (rotational symmetry) means that it has the same values on any circle

$$x^2 + z^2 = r^2$$

with a given radius r. The invariance under dilations means that we can obtain the solution at any circle just by stretching the radius r. According to Theorem 3.2, this solution is given by bounded oscillating functions.

4 Energy of the rotationally symmetric solution

The conservation of energy for Eqs. (1.1)-(1.3) has the form [1]

$$\frac{d}{dt}\int\int\left[v^2 + \frac{g^2}{N^2}\rho^2 + |\nabla\psi|^2\right]dxdz = 0. \tag{4.1}$$

Hence, the *energy density* is

$$E = v^2 + \frac{g^2}{N^2}\rho^2 + |\nabla\psi|^2. \tag{4.2}$$

For the rotationally invariant solution (3.22) we have

$$|\nabla\psi|^2 = 4(x^2 + z^2)\,\phi^2(t). \tag{4.3}$$

Substituting the expression (4.3) and the expressions (3.22) of v and ρ in Eq. (4.2) we obtain the following energy density for the invariant solution (3.22):

$$\begin{aligned} E =& 4\left(\frac{1}{f^2} - \frac{1}{N^2}\right)(x^2 - z^2)\big[\phi^2(t) + K\big]\phi^2(t) + \left(f - \frac{K}{f}\right)^2 x^2 \\ &+ \left(N - \frac{K}{N}\right)^2 z^2 + 4\left(\frac{1}{f^2} - \frac{1}{N^2}\right) xz\,\big[2\phi^2(t) + K\big]\phi'(t). \end{aligned} \tag{4.4}$$

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GROUP ANALYSIS OF NONLINEAR INTERNAL WAVES IN OCEANS
III: Additional conservation laws

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Abstract. Using the maximal Lie algebra of point symmetries of a system of nonlinear equations used in geophysical fluid dynamics, two conservation laws are found in addition to the conservation of energy.

Keywords: Geophysical fluid dynamics, Symmetries, Conservation laws.

1 Introduction

The maximal group of Lie point symmetries of the system

$$\Delta\psi_t - g\rho_x - fv_z = \psi_x\Delta\psi_z - \psi_z\Delta\psi_x\,, \tag{1.1}$$

$$v_t + f\psi_z = \psi_x v_z - \psi_z v_x\,, \tag{1.2}$$

$$\rho_t + \frac{N^2}{g}\psi_x = \psi_x\rho_z - \psi_z\rho_x \tag{1.3}$$

has been presented in [1]. It is generated by the infinite-dimensional Lie algebra spanned by the following operators:

$$
\begin{aligned}
& X_1=\frac{\partial}{\partial v}, \quad X_2=\frac{\partial}{\partial \rho}, \quad X_3=a(t) \frac{\partial}{\partial \psi}, \quad X_4=\frac{\partial}{\partial t}, \\
& X_5=b(t)\left[\frac{\partial}{\partial x}-f \frac{\partial}{\partial v}\right]+b^{\prime}(t) z \frac{\partial}{\partial \psi}, \\
& X_6=c(t)\left[\frac{\partial}{\partial z}+\frac{N^2}{g} \frac{\partial}{\partial \rho}\right]-c^{\prime}(t) x \frac{\partial}{\partial \psi}, \\
& X_7=x \frac{\partial}{\partial x}+z \frac{\partial}{\partial z}+v \frac{\partial}{\partial v}+\rho \frac{\partial}{\partial \rho}+2 \psi \frac{\partial}{\partial \psi}, \\
& X_8=t \frac{\partial}{\partial t}+2 x \frac{\partial}{\partial x}+2 z \frac{\partial}{\partial z}+3 \psi \frac{\partial}{\partial \psi}-2 f x \frac{\partial}{\partial v}+2 \frac{N^2}{g} z \frac{\partial}{\partial \rho}, \\
& X_9=z \frac{\partial}{\partial x}-x \frac{\partial}{\partial z}-\frac{1}{f}\left[g \rho+\left(f^2-N^2\right) z\right] \frac{\partial}{\partial v}+\frac{1}{g}\left[f v+\left(f^2-N^2\right) x\right] \frac{\partial}{\partial \rho} \cdot
\end{aligned} \tag{1.4}
$$

2 Conservation law provided by the semi-dilation

Consider the operator X_8 from (1.4),

$$
X_8=t \frac{\partial}{\partial t}+2 x \frac{\partial}{\partial x}+2 z \frac{\partial}{\partial z}+3 \psi \frac{\partial}{\partial \psi}-2 f x \frac{\partial}{\partial v}+2 \frac{N^2}{g} z \frac{\partial}{\partial \rho} \tag{2.1}
$$

It generates the following one-parameter transformation group with the parameter ε :

$$
\begin{aligned}
& \bar{t}=t \mathrm{e}^{\varepsilon}, \quad \bar{x}=x \mathrm{e}^{2 \varepsilon}, \quad \bar{z}=z \mathrm{e}^{2 \varepsilon}, \quad \bar{\psi}=\psi \mathrm{e}^{3 \varepsilon}, \\
& \bar{v}=v+f x\left(1-\mathrm{e}^{2 \varepsilon}\right), \quad \bar{\rho}=\rho-\frac{N^2}{g}\left(1-\mathrm{e}^{2 \varepsilon}\right) .
\end{aligned} \tag{2.2}
$$

Since some of variables, namely t, x, z and ψ are subjected to dilations while two other variables transform otherwise, we call (2.2) the *semi-dilation group*. Let us construct the conserved vector provided by this group.

2.1 Computation of the density of the conservation law

We will use the following formula for computing the density of conservation laws (see Eq. (3.23) in [2])

$$
C^1=-v\, W^1-\frac{g^2}{N^2}\, \rho\, W^2-\psi_x\, D_x\big(W^3\big)-\psi_z\, D_z\big(W^3\big), \tag{2.3}
$$

where (see Eq. (3.16) in [2])

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \quad \alpha = 1, 2, 3. \tag{2.4}$$

These formulas are written using the notation $t,\ x,\ z,\ v,\ \rho,\ \psi$.

In the case of the operator (2.1) the quantities (2.4) are written:

$$\begin{aligned} W^1 &= -2fx - tv_t - 2xv_x - 2zv_z, \\ W^2 &= 2\frac{N^2}{g}\,z - t\rho_t - 2x\rho_x - 2z\rho_z, \\ W^3 &= 3\psi - t\psi_t - 2x\psi_x - 2z\psi_z. \end{aligned} \tag{2.5}$$

Substituting (2.5) in (2.4) we obtain upon simple calculations:

$$\begin{aligned} C^1 = 2(fxv - gz\rho) - |\nabla\psi|^2 + \big(xD_x + zD_z\big)\Big(v^2 + \frac{g^2}{N^2}\,\rho^2\Big) \\ + \big(xD_x + zD_z\big)\big(|\nabla\psi|^2\big) + t\left[vv_t + \frac{g^2}{N^2}\,\rho\rho_t + \psi_x\psi_{xt} + \psi_z\psi_{zt}\right]. \end{aligned} \tag{2.6}$$

We can drop the last term in (2.6) because it can be written in the divergent form upon elimination of $v_t,\ \rho_4$ and ψ_t by using Eqs. (1.1)-(1.3). Indeed, it is shown in [2], Section 4.6, that the expression in the square brackets (cf. Eq. (4.11) in [2]) evaluated on the solutions of Eqs. (1.1)-(1.3) has the divergent form. Multiplication by t does not violate this property. Then we use the identities

$$\begin{aligned} \big(xD_x + zD_z\big)\Big(v^2 + \frac{g^2}{N^2}\,\rho^2\Big) &= -2\Big(v^2 + \frac{g^2}{N^2}\,\rho^2\Big) \\ &+ D_x\Big[x\Big(v^2 + \frac{g^2}{N^2}\,\rho^2\Big)\Big] + D_z\Big[z\Big(v^2 + \frac{g^2}{N^2}\,\rho^2\Big)\Big], \\ \big(xD_x + zD_z\big)\big(|\nabla\psi|^2\big) &= -2\psi|^2 + D_x\big[x\big(|\nabla\psi|^2\big)\big] + D_z\big[z\big(|\nabla\psi|^2\big)\big], \end{aligned}$$

drop the divergent type terms and obtain the following conserved density:

$$C^1 = 2\left(fxv - gz\rho - \frac{1}{2}|\nabla\psi|^2\right) - 2\Big(v^2 + \frac{g^2}{N^2}\,\rho^2 + |\nabla\psi|^2\Big). \tag{2.7}$$

Finally we note that the last term in (2.7) is the energy density (see Eq. (4.17) in [2]). Therefore we eliminate it and conclude that the invariance under the semi-dilation with the generator (2.1) provides the conservation law with the density

$$P = fxv - gz\rho - \frac{1}{2}|\nabla\psi|^2. \tag{2.8}$$

2.2 Conserved vector

Let us find the components $C^2,\ C^3$ of the conserved vector with the density (2.8). We will apply the procedure used in [2], Section 4.7. We have:

$$D_t(P) = fxv_t - gz\rho_t - (\psi_x\psi_{xt} + \psi_z\psi_{zt}).$$

Using Eqs. (1.1)-(1.3), we obtain:

$$\begin{aligned} D_t(P)\Big|_{(1.1)-(1.3)} &= -f^2x\psi_z + fx\psi_x v_z - fx\psi_z v_x + N^2z\psi_x \\ &- gz\psi_x\rho_z + gz\psi_z\rho_x - D_x(\psi\psi_{xt}) - D_z(\psi\psi_{zt}) + \psi\Delta\psi_t. \end{aligned}$$

One can rewrite this equation, using Eq. (4.23) from [2], in the following form:

$$\begin{aligned} D_t(P)\Big|_{(1.1)-(1.3)} &= D_x\big(N^2z\psi + fx\psi v_z - gz\psi\rho_z - \psi\psi_{xt} + \frac{1}{2}\psi^2\Delta\psi_z\big) \\ &- D_z\big(f^2x\psi + fx\psi v_x - gz\psi\rho_x + \psi\psi_{zt} + \frac{1}{2}\psi^2\Delta\psi_x\big). \end{aligned}$$

Thus, the generator (2.1) provides the conservation law

$$D_t(P) + D_x(C^2) + D_z(C^3 = 0$$

with the density P given by (2.8) and the flux given by the equations

$$\begin{aligned} C^2 &= -N^2z\psi - fx\psi v_z + gz\psi\rho_z + \psi\psi_{xt} - \frac{1}{2}\psi^2\Delta\psi_z, \\ C^3 &= f^2x\psi + fx\psi v_x - gz\psi\rho_x + \psi\psi_{zt} + \frac{1}{2}\psi^2\Delta\psi_x. \end{aligned}$$

2.3 Conserved density P of the generalized invariant solution

If we substitute in (2.8) the generalized invariant solution (5.28)-(5.30) from [2],

$$\begin{aligned} \psi &= A(\lambda)\cos(\omega t) + B(\lambda)\sin(\omega t), \\ v &= \frac{fm}{\omega}\big[B'(\lambda)\cos(\omega t) - A'(\lambda)\sin(\omega t)\big], \\ \rho &= \frac{kN^2}{g\omega}\big[B'(\lambda)\cos(\omega t) - A'(\lambda)\sin(\omega t)\big], \end{aligned}$$

we obtain:

$$\begin{aligned} P = \frac{1}{\omega}\,(f^2mx - N^2kz)&\big[B'(\lambda)\cos(\omega t) - A'(\lambda)\sin(\omega t)\big] \\ &- \frac{k^2+m^2}{2}\big[A'(\lambda)\cos(\omega t) + B'(\lambda)\sin(\omega t)\big]^2. \end{aligned}$$

3 Conservation law provided by the rotation

Taking the rotation generator X_9 from (1.4) and proceedings as in Section 2 we obtain the following conserved density:

$$Q = v\rho + fx\rho - \frac{N^2}{g}\, zv. \tag{3.1}$$

Writing Eqs. (1.1)-(1.3) by using the Jacobians $J(\psi, v) = \psi_x v_z - \psi_z v_x$, etc., we have:

$$\begin{aligned} D_t(Q)\Big|_{(1.1)-(1.3)} &= v\left[J(\psi,\rho) - \frac{N^2}{g}\,\psi_x\right] + \rho\left[J(\psi,v) - f\psi_z\right] \\ &+ fx\left[J(\psi,\rho) - \frac{N^2}{g}\,\psi_x\right] - \frac{N^2}{g}\, z\left[J(\psi,v) - f\psi_z\right]. \end{aligned} \tag{3.2}$$

The reckoning shows that

$$\begin{aligned} &vJ(\psi,\rho) + \rho J(\psi,v) = D_z(v\rho\psi_x) - D_x(v\rho\psi_z), \\ &xJ(\psi,\rho) - \rho\psi_z = D_z(x\rho\psi_x) - D_x(x\rho\psi_z), \\ &zJ(\psi,v) + v\psi_x = D_x(zv\psi_z) - D_z(zv\psi_x), \\ &z\psi_z - x\psi_x = D_z(z\psi) - D_x(x\psi). \end{aligned}$$

Substituting these expressions in Eq. (3.2) we conclude that the rotation generator X_9 provides the conservation law

$$D_t(Q) + D_x(C^2) + D_z(C^3 = 0$$

with the density P given by (3.1) and the flux given by the equations

$$\begin{aligned} C^2 &= \left[v\rho + fx\rho - \frac{N^2}{g}\, zv\right]\psi_z + \frac{N^2}{g}\, fz\psi, \\ C^3 &= \left[\frac{N^2}{g}\, zv - v\rho - fx\rho\right]\psi_x - \frac{N^2}{g}\, fx\psi. \end{aligned}$$

4 Summary of conservation laws

It has been demonstrated in [2] that the system of nonlinear equations (1.1)-(1.3) is self-adjoint. This property of the system has been used for deriving local conservation laws applying the method developed in [3] to the infinitesimal symmetries (1.4). Some of the conservation laws associated with these symmetries are *trivial*, i.e. have vanishing densities. But five conservation laws are nontrivial.

The nontrivial conservation laws obtained in [2] and in the present paper are summarized below. For the convenience of the reader, we formulate them both in the integral and differential forms.

4.1 Conservation laws in integral form

$$\frac{d}{dt}\int\!\!\int v\, dxdz = 0. \tag{4.1}$$

$$\frac{d}{dt}\int\!\!\int \rho\, dxdz = 0. \tag{4.2}$$

$$\frac{d}{dt}\int\!\!\int \left[v^2 + \frac{g^2}{N^2}\,\rho^2 + |\nabla\psi|^2\right] dxdz = 0. \tag{4.3}$$

$$\frac{d}{dt}\int\!\!\int \left[fxv - gz\rho - \frac{1}{2}|\nabla\psi|^2\right] dxdz = 0. \tag{4.4}$$

$$\frac{d}{dt}\int\!\!\int \left[v\rho + fx\rho - \frac{N^2}{g}\,zv\right] dxdz = 0. \tag{4.5}$$

4.2 Conservation laws in differential form

$$D_t(v) + D_x(v\psi_z) + D_z(f\psi - v\psi_x) = 0. \tag{4.1$'$}$$

$$D_t(\rho) + D_x\left(\frac{N^2}{g}\,\psi + \rho\,\psi_z\right) + D_z(-\rho\,\psi_x) = 0. \tag{4.2$'$}$$

$$
\begin{aligned}
& D_t\left(v^2+\frac{g^2}{N^2}\,\rho^2+|\nabla\psi|^2\right) \\
& + D_x\left(2g\rho\psi+v^2\psi_z+\frac{g^2}{N^2}\,\rho^2\psi_z-2\psi\psi_{xt}+\psi^2\Delta\psi_z\right) \\
& + D_z\left(2fv\psi-v^2\psi_x-\frac{g^2}{N^2}\,\rho^2\psi_x-2\psi\psi_{zt}-\psi^2\Delta\psi_x\right)=0.
\end{aligned}
\tag{4.3$'$}
$$

$$
\begin{aligned}
& D_t\left(fxv-gz\rho-\frac{1}{2}|\nabla\psi|^2\right) \\
& + D_x\left(-N^2z\psi-fx\psi v_z+gz\psi\rho_z+\psi\psi_{xt}-\frac{1}{2}\psi^2\Delta\psi_z\right) \\
& + D_z\left(f^2x\psi+fx\psi v_x-gz\psi\rho_x+\psi\psi_{zt}+\frac{1}{2}\psi^2\Delta\psi_x\right)=0.
\end{aligned}
\tag{4.4$'$}
$$

$$
\begin{aligned}
& D_t\left(v\rho+fx\rho-\frac{N^2}{g}\,zv\right) \\
& + D_x\left(\Big[v\rho+fx\rho-\frac{N^2}{g}\,zv\Big]\psi_z+\frac{N^2}{g}\,fz\psi\right) \\
& + D_z\left(\Big[\frac{N^2}{g}\,zv-v\rho-fx\rho\Big]\psi_x-\frac{N^2}{g}\,fx\psi\right)=0.
\end{aligned}
\tag{4.5$'$}
$$

The conservation law (4.3) defines the energy of the system. It seems that the conservation laws (4.4) and (4.5), unlike (4.3), do not have direct analogies in mechanics and should be investigated from point of view of their physical significance.

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ALTERNATIVE PRESENTATION OF LAGRANGE'S METHOD OF VARIATION OF PARAMETERS

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Abstract. An alternative approach to Lagrange's method of variation of parameters is presented. Explicit formulas for solutions of arbitrary initial value problems for linear equations of the first, second and third order are provided. These formulas are as simple for practical using as the formula for roots of quadratic equations.

Keywords: Linear equations, Method of variation of parameters, Initial value problem.

1 Introduction

The simplest general method for integrating non-homogeneous linear ordinary differential equations of an arbitrary order is the method of variation of parameters. It was first discovered by Jean Bernoulli for first-order equations in 1697 and later extended by J.L. Lagrange to higher-order equations. Lagrange's approach requires imposition of additional restrictions on varied parameters. I give here an alternative approach.

Theorem 4.1 provides an explicit formula for the solution of an arbitrary initial value problem for any linear first-order equation. Theorems 4.2 and 4.3 provide similar formulas for equations of the second and third order, respectively, provided that fundamental systems of solutions for the corresponding homogeneous equations are known. These formulas are as simple for practical using as the formula for roots of quadratic equations. Namely, the solutions to the Cauchy problem are obtained just by inserting the coefficients of the differential equations in question and the initial data.

Of course, the integrals involved in the solutions may be difficult or, in general, impossible to work out in terms of elementary functions. But this circumstance is unessential. Anyhow, equations whose solutions can be expressed in terms of elementary functions are very rear and do not have a practical value. They appear mostly in textbooks as simple illustrations of various methods.

2 Traditional presentation

2.1 First-order equations

Jean Bernoulli noticed in 1697 that any non-homogeneous first-order linear ordinary differential equation

$$y' + P(x)y = Q(x) \tag{2.1}$$

can be easily integrated by varying the parameter C in the general solution

$$y = C\mathrm{e}^{-\int P dx} \tag{2.2}$$

of the homogeneous equation

$$y' + P(x)y = 0. \tag{2.3}$$

Namely, we replace the constant C in (2.2) by an undetermined function $u(x)$ and look for the solution of the non-homogeneous equation (2.1) in the form

$$y = u(x)\mathrm{e}^{-\int P dx}. \tag{2.4}$$

Substitution of (2.4) in Eq. (2.1) yields

$$u'(x) = Q(x)\,\mathrm{e}^{\int P dx}, \tag{2.5}$$

whence

$$u(x) = \int Q\,\mathrm{e}^{\int P dx} dx + C, \quad C = \text{const.} \tag{2.6}$$

Inserting the expression (2.6) for $u(x)$ in (2.4) we obtain the general solution to the non-homogeneous linear equation (2.1) given by two quadratures:

$$y = C\,\mathrm{e}^{-\int P dx} + \mathrm{e}^{-\int P dx} \int Q(x)\,\mathrm{e}^{\int P dx} dx. \tag{2.7}$$

Remark 2.1. Often the integrals in (2.7) cannot be worked out in terms of elementary functions. This fact does not mean, however, that the method of variation of parameter has disadvantages of its own. It only means that *the general solution of the differential equation in question cannot be expressed in terms of elementary functions*.

2.2 Second-order equations

Some 90 years later Lagrange showed that Bernoulli's method of variation of parameters can be extended to higher-order equations. Lagrange's method of variation of parameters allows one to integrate non-homogeneous linear ordinary differential equations of any order provided that one knows the fundamental set of solutions for the corresponding homogeneous equation. Recall the common way of presentation of this method, e.g., for second-order equations

$$y'' + a(x)y' + b(x)y = f(x). \tag{2.8}$$

Let us assume that we know two linearly independent solutions $y_1 = y_1(x),\ y_2 = y_2(x)$ of the homogeneous equation

$$y'' + a(x)y' + b(x)y = 0. \tag{2.9}$$

Then the general solution to Eq. (2.9) is given by

$$y = C_1\, y_1 + C_2\, y_2, \quad C_1, C_2 = \text{const.} \tag{2.10}$$

The essence of the method of variation of parameters is the same as in the case of first-order equations. Namely, we replace the parameters C_1, C_2 by unknown functions $u_1(x), u_2(x)$ and seek the solution of the non-homogeneous equation (2.8) in the form

$$y = u_1(x)\, y_1 + u_2(x)\, y_2. \tag{2.11}$$

Substituting (2.11) in Eq. (2.8) we will obtain only one equation for two unknown functions $u_1(x)$ and $u_2(x)$. Furthermore, computing the derivative of (2.11),

$$y' = u_1\, y_1' + u_2\, y_2' + y_1\, u_1' + y_2\, u_2', \tag{2.12}$$

we see that y'' will involve second derivatives of u_1, u_2. Therefore, the substitution (2.11) in Eq. (2.8) leads to a single differential equation of the second order for two unknowns u_1 and u_2. The common way to avoid this complication is to impose on u_1, u_2 the following restriction:

$$y_1\, u_1' + y_2\, u_2' = 0. \tag{2.13}$$

Then Eq. (2.12) reduces to

$$y' = u_1\, y_1' + u_2\, y_2' \tag{2.14}$$

and yields:

$$y'' = u_1\, y_1'' + u_2\, y_2'' + y_1'\, u_1' + y_2'\, u_2'. \tag{2.15}$$

Substituting (2.15), (2.14) and (2.11) in Eq. (2.8) and invoking that the functions $y_1(x)$ and $y_2(x)$ solve the homogeneous equation (2.8), we obtain

$$y_1'\, u_1' + y_2'\, u_2' = f(x). \tag{2.16}$$

Since $y_1 = y_1(x),\ y_2 = y_2(x)$ are known functions, (2.13) and (2.16) provide two equations for determining two unknown functions u_1 and u_2 :

$$\begin{aligned} y_1\, u_1' + y_2\, u_2' &= 0, \\ y_1'\, u_1' + y_2'\, u_2' &= f(x). \end{aligned} \tag{2.17}$$

Solving the system (2.17) with respect to $u_1', \; u_2'$:

$$u_1' = -\frac{y_2\, f(x)}{W[y_1, y_2]}\,, \quad u_2' = \frac{y_1\, f(x)}{W[y_1, y_2]}, \tag{2.18}$$

and integrating we obtain

$$u_1 = -\int \frac{y_2\, f(x)}{W[y_1, y_2]}\, dx + C_1, \quad u_2 = \int \frac{y_1\, f(x)}{W[y_1, y_2]}\, dx + C_2, \tag{2.19}$$

where $C_1, \; C_2$ are arbitrary constants, and $W[y_1, y_2]$ is the Wronskian:

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1\, y_2' - y_2\, y_1'. \tag{2.20}$$

Inserting (2.20) in (2.11) we obtain the general solution to Eq. (2.8):

$$y = C_1\, y_1 + C_2\, y_2 - y_1 \int \frac{y_2\, f(x)}{W[y_1, y_2]}\, dx + y_2 \int \frac{y_1\, f(x)}{W[y_1, y_2]}\, dx. \tag{2.21}$$

2.3 Remark

For an extension of the method to higher-order equations, it is useful to write the system (2.17) in the vector form

$$MU = F, \tag{2.22}$$

where M is the 2×2 matrix, and U, F are the column vectors defined as follows:

$$M = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}, \quad U = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}. \tag{2.23}$$

The determinant of M is the Wronskian (2.20),

$$\det M = W[y_1, y_2].$$

Since the solutions $y_1(x), \; y_2(x)$ are linearly independent, we have $W[y_1, y_2] \neq 0$. Hence, the matrix M is invertible and has the following inverse:

$$M^{-1} = \frac{1}{W[y_1, y_2]} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}. \tag{2.24}$$

Therefore the solution to Eq. (2.22) is given by

$$U = M^{-1} F,$$

or

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{W[y_1, y_2]} \begin{pmatrix} -y_2\, f(x) \\ y_1\, f(x) \end{pmatrix}. \tag{2.25}$$

In other words, we have arrived at Eqs. (2.18):

$$u_1' = -\frac{y_2\, f(x)}{W[y_1, y_2]}\,, \quad u_2' = \frac{y_1\, f(x)}{[y_1, y_2]}.$$

Note that for computing the solution (2.25) to Eq. (2.22) we need only the last column in the inverse matrix. Therefore we can write M^{-1} by keeping only the last column:

$$M^{-1} = \frac{1}{W[y_1, y_2]} \begin{pmatrix} \cdots & -y_2 \\ \cdots & y_1 \end{pmatrix}. \tag{2.26}$$

3 Alternative presentation

Imposition of the additional restriction (2.13) often becomes a stumbling block for students who consider it as an artificial trick that one has to remember. For higher-order equations the situation is more complicated. I give here an alternative presentation of the method of variation of parameters which is free from this disadvantage of the traditional presentation of the method.

3.1 Second-order equations

Let us rewrite the second-order equation (2.8) as the following non-homogeneous system of two first-order linear equations for two dependent variables y, z :

$$\begin{aligned} &y' = z, \\ &z' + a(x)\, z + b(x)\, y = f(x). \end{aligned} \tag{3.1}$$

Two linearly independent solutions

$$y_1 = y_1(x), \quad y_2 = y_2(x)$$

of the homogeneous equation (2.9), taken together with

$$z_1 = y_1'(x), \quad z_2 = y_2'(x),$$

provide two linearly independent solutions

$$Y_1 = \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \tag{3.2}$$

of the homogeneous system

$$
\begin{aligned}
& y' = z, \\
& z' + a(x)\, z + b(x)\, y = 0.
\end{aligned} \tag{3.3}
$$

The linear combination of (3.2)

$$
Y = C_1 Y_1 + C_2 Y_2 = \begin{pmatrix} C_1\, y_1 + C_2\, y_2 \\ C_1\, z_1 + C_2\, z_2 \end{pmatrix}
$$

furnishes the following general solution to the system (3.3):

$$
\begin{aligned}
y &= C_1\, y_1 + C_2\, y_2, \\
z &= C_1\, z_1 + C_2\, z_2.
\end{aligned} \tag{3.4}
$$

Now we replace the parameters C_1, C_2 in (3.4) by $u_1(x),\ u_2(x)$ and seek the solution to the non-homogeneous system (3.1) in the form

$$
\begin{aligned}
y &= u_1(x)\, y_1 + u_2(x)\, y_2, \\
z &= u_1(x)\, z_1 + u_2(x)\, z_2.
\end{aligned} \tag{3.5}
$$

Substitution of (3.5) in Eqs. (3.1) yields:

$$
\begin{aligned}
& u_1\, y_1' + u_2\, y_2' + u_1'\, y_1 + u_2'\, y_2 = u_1\, z_1 + u_2\, z_2, \\
& u_1\, z_1' + u_2\, z_2' + u_1'\, z_1 + u_2'\, z_2 + a(x)(u_1\, z_1 + u_2\, z_2) \\
& + b(x)(u_1(x)\, y_1 + u_2(x)\, y_2) = f(x),
\end{aligned}
$$

or upon rearranging the terms:

$$
u_1\, y_1' + u_2\, y_2' + u_1'\, y_1 + u_2'\, y_2 = u_1\, z_1 + u_2\, z_2, \tag{3.6}
$$

$$
\begin{aligned}
& [z_1' + a(x)\, z_1 + b(x)\, y_1]u_1 + [z_2' + a(x)\, z_2 + b(x)\, y_2]u_2 \\
& + u_1'\, z_1 + u_2'\, z_2 = f(x).
\end{aligned} \tag{3.7}
$$

Since (y_1, z_1) and (y_2, z_2) solve the homogeneous system (3.3), the terms in brackets in Eq. (3.7) vanish and Eqs. (3.6)-(3.7) are written

$$
\begin{aligned}
& u_1\, y_1' + u_2\, y_2' + u_1'\, y_1 + u_2'\, y_2 = u_1\, y_1' + u_2\, y_2', \\
& u_1'\, z_1 + u_2'\, z_2 = f(x),
\end{aligned}
$$

whence

$$
\begin{aligned}
& y_1\, u_1' + y_2\, u_2' = 0, \\
& y_1'\, u_1' + y_2'\, u_2' = f(x).
\end{aligned} \tag{3.8}
$$

Thus, we have arrived at Eqs. (2.17) without imposing the restriction (2.13) *a priori.*

3.2 Third-order equations

Let us rewrite the third-order equation

$$y''' + a(x)y'' + b(x)y' + c(x)y = f(x) \tag{3.9}$$

as the system of three first-order linear equations for three dependent variables y, z, v :

$$\begin{aligned} &y' = z, \\ &z' = v, \\ &v' + a(x)\, v + b(x)\, z + c(x)\, y = f(x). \end{aligned} \tag{3.10}$$

Three linearly independent solutions

$$y_1 = y_1(x), \quad y_2 = y_2(x), \quad y_3 = y_3(x)$$

of the homogeneous equation

$$y''' + a(x)y'' + b(x)y' + c(x)y = 0 \tag{3.11}$$

taken together with

$$\begin{aligned} z_1 &= y_1'(x), \quad v_1 = y_1''(x), \\ z_2 &= y_2'(x), \quad v_2 = y_2''(x), \\ z_3 &= y_3'(x), \quad v_3 = y_3''(x), \end{aligned}$$

provide three linearly independent solutions

$$Y_1 = \begin{pmatrix} y_1 \\ z_1 \\ v_1 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} y_2 \\ z_2 \\ v_2 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} y_3 \\ z_3 \\ v_3 \end{pmatrix} \tag{3.12}$$

of the homogeneous system

$$\begin{aligned} &y' = z, \\ &z' = v, \\ &v' + a(x)\, v + b(x)\, z + c(x)\, y = 0. \end{aligned} \tag{3.13}$$

The linear combination of (3.12)

$$Y = C_1 Y_1 + C_2 Y_2 + C_3 Y_3 = \begin{pmatrix} C_1\, y_1 + C_2\, y_2 + C_3\, y_3 \\ C_1\, z_1 + C_2\, z_2 + C_3\, z_3 \\ C_1\, v_1 + C_2\, v_2 + C_3\, v_3 \end{pmatrix}$$

furnishes the following general solution to the system (3.13):

$$
\begin{aligned}
y &= C_1\, y_1 + C_2\, y_2 + C_3\, y_3,\\
z &= C_1\, z_1 + C_2\, z_2 + C_3\, z_3,\\
v &= C_1\, v_1 + C_2\, v_2 + C_3\, v_3.
\end{aligned}
\tag{3.14}
$$

Now we follow the procedure used in the case of second-order equations. Namely, we replace the parameters C_1, C_2, C_3 in (3.13) by $u_1(x),\ u_2(x),\ u_3(x)$ and seek the solution to the non-homogeneous system (3.10) in the form

$$
\begin{aligned}
y &= u_1(x)\, y_1 + u_2(x)\, y_2 + u_3(x)\, y_3,\\
z &= u_1(x)\, z_1 + u_2(x)\, z_2 + u_3(x)\, z_3,\\
v &= u_1(x)\, v_1 + u_2(x)\, v_2 + u_3(x)\, v_3.
\end{aligned}
\tag{3.15}
$$

Substituting (3.15) in Eqs. (3.10) and rearranging the terms we obtain:

$$
\begin{aligned}
&\sum_{i=1}^{3} u_i\, y_i' + \sum_{i=1}^{3} y_i u_i' = \sum_{i=1}^{3} u_i\, z_i,\\
&\sum_{i=1}^{3} u_i\, z_i' + \sum_{i=1}^{3} z_i u_i' = \sum_{i=1}^{3} u_i\, v_i,\\
&\sum_{i=1}^{3} \big(v_i' + a v_i + b z_i + c y_i\big)\, u_i + \sum_{i=1}^{3} v_i u_i' = f(x).
\end{aligned}
\tag{3.16}
$$

Since $(y_i, z_i, v_i),\ i = 1, 2, 3$, solve the homogeneous system (3.13), we have

$$
y_i' = z_i, \quad z_i' = v_i, \quad v_i' + a v_i + b z_i + c y_i = 0,
$$

and hence Eqs. (3.16) are written:

$$
\sum_{i=1}^{3} y_i u_i' = 0, \quad \sum_{i=1}^{3} z_i u_i' = 0, \quad \sum_{i=1}^{3} v_i u_i' = f(x).
$$

Thus, we have arrived at the equations

$$
\begin{aligned}
&y_1\, u_1' + y_2\, u_2' + y_3\, u_3' = 0,\\
&y_1'\, u_1' + y_2'\, u_2' + y_3'\, u_3' = 0,\\
&y_1''\, u_1' + y_2''\, u_2' + y_3''\, u_3' = f(x).
\end{aligned}
\tag{3.17}
$$

In the traditional approach the first two equations (3.17) are imposed *a priori.*

One can readily solve the system of linear equations (3.17) for $u_1',\ u_2',\ u_3'$ proceeding as in Section 2.3. Namely, we write the system (3.17) in the vector form

$$MU = F, \tag{3.18}$$

where M is the 3×3 matrix, and U, F are column vectors defined as follows:

$$M = \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix}, \quad U = \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ 0 \\ f(x) \end{pmatrix}. \tag{3.19}$$

The determinant of the matrix M is the Wronskian of $y_1,\ y_2,\ y_3$:

$$\det M = W[y_1, y_2, y_3] = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}. \tag{3.20}$$

Since the solutions $y_1(x),\ y_2(x),\ y_3(x)$ are linearly independent, we have

$$W[y_1, y_2, y_3] \neq 0.$$

Hence, the matrix M is invertible. For our purposes, it suffices to write the inverse matrix in the form (see, e.g., [1], Section 1.1.1)

$$M^{-1} = \frac{1}{W[y_1, y_2, y_3]} \begin{pmatrix} \cdots & W[y_2, y_3] \\ \cdots & W[y_3, y_1] \\ \cdots & W[y_1, y_2] \end{pmatrix}, \tag{3.21}$$

where

$$W[y_i, y_k] = \begin{vmatrix} y_i & y_k \\ y_i' & y_k' \end{vmatrix} = y_i y_k' - y_k y_i', \quad i, k = 1, 2, 3. \tag{3.22}$$

Accordingly, the solution to Eq. (3.18) is given by

$$\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \frac{1}{W[y_1, y_2, y_3]} \begin{pmatrix} W[y_2, y_3] f(x) \\ W[y_3, y_1] f(x) \\ W[y_1, y_2] f(x) \end{pmatrix}. \tag{3.23}$$

In other words,

$$u_1' = \frac{W[y_2, y_3] f(x)}{W[y_1, y_2, y_3]}, \quad u_2' = \frac{W[y_3, y_1] f(x)}{W[y_1, y_2, y_3]}, \quad u_3' = \frac{W[y_1, y_2] f(x)}{W[y_1, y_2, y_3]},$$

whence, upon integration:

$$
\begin{aligned}
u_1 &= \int \frac{W[y_2, y_3] f(x)}{W[y_1, y_2, y_3]}\, dx + C_1,\\
u_2 &= \int \frac{W[y_3, y_1] f(x)}{W[y_1, y_2, y_3]}\, dx + C_2,\\
u_3 &= \int \frac{W[y_1, y_2] f(x)}{W[y_1, y_2, y_3]}\, dx + C_3,
\end{aligned}
\tag{3.24}
$$

Substituting (3.24) in the first equation (3.15) we obtain the general solution of the non-homogeneous equation (3.9):

$$
\begin{aligned}
y = C_1\, y_1 + C_2\, y_2 + C_3\, y_3 + y_1 \int \frac{W[y_2, y_3] f(x)}{W[y_1, y_2, y_3]}\, dx\\
+ y_2 \int \frac{W[y_3, y_1] f(x)}{W[y_1, y_2, y_3]}\, dx + y_3 \int \frac{W[y_1, y_2] f(x)}{W[y_1, y_2, y_3]}\, dx.
\end{aligned}
\tag{3.25}
$$

3.3 Higher-order equations

Consider an nth-order linear equation

$$
y^{(n)} + a_1(x) y^{(n-1)} + \ldots + a_{n-1}(x) y' + a_n(x) y = f(x) \tag{3.26}
$$

with known linearly independent solutions $y_1(x), \ \ldots, \ y_n(x)$ of the homogeneous equation. Proceeding as in Section 3.2, we obtain the equations similar to (3.17):

$$
\begin{aligned}
&y_1\, u_1' \quad + \cdots + \quad y_n\, u_n' \quad = 0,\\
&y_1'\, u_1' \quad + \cdots + \quad y_n'\, u_n' \quad = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&y_1^{(n-1)}\, u_1' + \cdots + y_n^{(n-1)}\, u_n' = f(x).
\end{aligned}
\tag{3.27}
$$

Then we introduce the following matrix M and the vectors U, F:

$$
M = \begin{pmatrix} y_1 & \cdots & y_n\\ y_1' & \cdots & y_n'\\ \cdots & \cdots & \cdots\\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}, \quad U = \begin{pmatrix} u_1'\\ u_2'\\ \cdots\\ u_n' \end{pmatrix}, \quad F = \begin{pmatrix} 0\\ 0\\ \cdots\\ f(x) \end{pmatrix} \tag{3.28}
$$

and write the system (3.27) in the vector form:

$$
MU = F. \tag{3.29}
$$

In order to solve Eq. (3.29), we have to find the inverse matrix to M and write it in the form similar to (3.21).

Recall that the inverse to a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \tag{3.30}$$

with non-vanishing determinant $|A| = \det A$ has the form

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}, \tag{3.31}$$

where A_{ij} is the cofactor to the element a_{ij} of the matrix (3.30).

Applying the formula (3.31) to the matrix M given in (3.27) and invoking that the determinant of the matrix M is the Wronskian

$$W_n[y_1, y_2, \ldots, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \multicolumn{4}{c}{\dotfill} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}, \tag{3.32}$$

we obtain by keeping only the last column (cf. (2.26) and (3.21)):

$$M^{-1} = \frac{1}{W_n[y_1, y_2, \ldots, y_n]} \begin{pmatrix} \cdots & (-1)^{n-1} W_{n-1}[y_2, y_3, \ldots, y_n] \\ \cdots & (-1)^{n-2} W_{n-1}[y_1, y_3, \ldots, y_n] \\ \cdots & \dots\dots\dots\dots \\ \cdots & W_{n-1}[y_1, y_2, \ldots, y_{n-1}] \end{pmatrix}. \tag{3.33}$$

Accordingly, the solution to Eq. (3.29) is given by

$$\begin{pmatrix} u_1' \\ u_2' \\ \cdots \\ u_n' \end{pmatrix} = \frac{1}{W_n[y_1, y_2, \ldots, y_n]} \begin{pmatrix} (-1)^{n-1} W_{n-1}[y_2, y_3, \ldots, y_n]\, f(x) \\ (-1)^{n-1} W_{n-1}[y_1, y_3, \ldots, y_n]\, f(x) \\ \dots\dots\dots\dots \\ W_{n-1}[y_1, y_2, \ldots, y_{n-1}]\, f(x) \end{pmatrix}. \tag{3.34}$$

Integrating (3.34), we obtain $u_i(x),\ i = 1, \ldots, n,$ and hence the general solution

$$y = \sum_{i=1}^{n} u_i(x) y_i(x) \tag{3.35}$$

of the non-homogeneous equation (3.26).

Example 3.1. Let us solve the fourth-order equation

$$\frac{d^4 y}{dx^4} - y = f(x) \tag{3.36}$$

describing the phenomenon of "beating" of driving shafts due to the centrifugal force (see [2], Section 2.3.3) in presence of an external force $f(x)$ such as friction, etc. The general solution of the homogeneous equation

$$y^{(4)} - y = 0$$

is given by

$$y = C_1 \mathrm{e}^x + C_2 \mathrm{e}^{-x} + C_3 \cos x + C_4 \sin x.$$

Hence, four linearly independent solutions of the homogeneous equation are

$$y_1(x) = \mathrm{e}^x, \quad y_2(x) = \mathrm{e}^{-x}, \quad y_3(x) = \cos x, \quad y_4(x) = \sin x. \tag{3.37}$$

In our case Eq. (3.34) is written

$$\begin{pmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{pmatrix} = \frac{1}{W_4[y_1, y_2, y_3, y_4]} \begin{pmatrix} -W_3[y_2, y_3, y_4]\, f(x) \\ W_3[y_1, y_3, y_4]\, f(x) \\ -W_3[y_1, y_2, y_4]\, f(x) \\ W_3[y_1, y_2, y_3]\, f(x) \end{pmatrix}. \tag{3.38}$$

The Wronskian (3.32) of the functions (3.37) has the form

$$W_4[y_1, y_2, y_3, y_4] = \begin{vmatrix} \mathrm{e}^x & \mathrm{e}^{-x} & \cos x & \sin x \\ \mathrm{e}^x & -\mathrm{e}^{-x} & -\sin x & \cos x \\ \mathrm{e}^x & \mathrm{e}^{-x} & -\cos x & -\sin x \\ \mathrm{e}^x & -\mathrm{e}^{-x} & \sin x & -\cos x \end{vmatrix}. \tag{3.39}$$

Working out the determinant (3.39), one obtains

$$W_4[y_1, y_2, y_3, y_4] = -8. \tag{3.40}$$

Let us compute the Wronskians W_3 in (3.38). We have:

$$W_3[y_2,y_3,y_4] = \begin{vmatrix} e^{-x} & \cos x & \sin x \\ -e^{-x} & -\sin x & \cos x \\ e^{-x} & -\cos x & -\sin x \end{vmatrix} = 2\,e^{-x}. \tag{3.41}$$

$$W_3[y_1,y_3,y_4] = \begin{vmatrix} e^{x} & \cos x & \sin x \\ e^{x} & -\sin x & \cos x \\ e^{x} & -\cos x & -\sin x \end{vmatrix} = 2\,e^{x}. \tag{3.42}$$

$$W_3[y_1,y_2,y_4] = \begin{vmatrix} e^{x} & e^{-x} & \sin x \\ e^{x} & -e^{-x} & \cos x \\ e^{x} & e^{-x} & -\sin x \end{vmatrix} = 4\,\sin x. \tag{3.43}$$

$$W_3[y_1,y_2,y_3] = \begin{vmatrix} e^{x} & e^{-x} & \cos x \\ e^{x} & -e^{-x} & -\sin x \\ e^{x} & e^{-x} & -\cos x \end{vmatrix} = 4\,\cos x. \tag{3.44}$$

Substituting (3.40)-(3.44) in (3.38) an integrating, we have:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} C_1 + \frac{1}{4}\int f(x)\,e^{-x}\,dx \\ C_2 - \frac{1}{4}\int f(x)\,e^{x}\,dx \\ C_3 + \frac{1}{2}\int f(x)\,\sin x\,dx \\ C_4 - \frac{1}{2}\int f(x)\,\cos x\,dx \end{pmatrix}.$$

Now the formula (3.35) gives the following general solution to Eq. (3.36):

$$y = y_*(x) + C_1 e^{x} + C_2 e^{-x} + C_3 \cos x + C_4 \sin x,$$

where $C_1, \ldots, C_4$ are arbitrary parameters and $y_*(x)$ is a particular solution of Eq. (3.36) defined by

$$\begin{aligned} y_*(x) &= \frac{1}{4}\,e^{x}\int f(x)\,e^{-x}\,dx - \frac{1}{4}\,e^{-x}\int f(x)\,e^{x}\,dx \\ &+ \frac{\cos x}{2}\int f(x)\,\sin x\,dx - \frac{\sin x}{2}\int f(x)\,\cos x\,dx. \end{aligned} \tag{3.45}$$

Let us verify that the function $y_*(x)$ solves Eq. (3.36). The differentiation yields:

$$y_*'(x) = \frac{1}{4}\,\mathrm{e}^x \int f(x)\,\mathrm{e}^{-x}\,dx + \frac{1}{4}\,f(x) + \frac{1}{4}\,\mathrm{e}^{-x} \int f(x)\,\mathrm{e}^{x}\,dx - \frac{1}{4}\,f(x)$$
$$- \frac{\sin x}{2} \int f(x)\ \sin x\,dx + \frac{1}{2}\,f(x)\cos x \sin x$$
$$- \frac{\cos x}{2} \int f(x)\ \cos x\,dx - \frac{1}{2}\,f(x)\sin x \cos x.$$

Hence,

$$y_*'(x) = \frac{1}{4}\,\mathrm{e}^x \int f(x)\,\mathrm{e}^{-x}\,dx + \frac{1}{4}\,\mathrm{e}^{-x} \int f(x)\,\mathrm{e}^{x}\,dx$$
$$- \frac{\sin x}{2} \int f(x)\ \sin x\,dx - \frac{\cos x}{2} \int f(x)\ \cos x\,dx.$$

Differentiating further, we obtain likewise:

$$y_*''(x) = \frac{1}{4}\,\mathrm{e}^x \int f(x)\,\mathrm{e}^{-x}\,dx - \frac{1}{4}\,\mathrm{e}^{-x} \int f(x)\,\mathrm{e}^{x}\,dx$$
$$- \frac{\cos x}{2} \int f(x)\ \sin x\,dx + \frac{\sin x}{2} \int f(x)\ \cos x\,dx,$$

$$y_*'''(x) = \frac{1}{4}\,\mathrm{e}^x \int f(x)\,\mathrm{e}^{-x}\,dx + \frac{1}{4}\,\mathrm{e}^{-x} \int f(x)\,\mathrm{e}^{x}\,dx$$
$$+ \frac{\sin x}{2} \int f(x)\ \sin x\,dx + \frac{\cos x}{2} \int f(x)\ \cos x\,dx$$

and finally

$$y_*^{(4)}(x) = \frac{1}{4}\,\mathrm{e}^x \int f(x)\,\mathrm{e}^{-x}\,dx - \frac{1}{4}\,\mathrm{e}^{-x} \int f(x)\,\mathrm{e}^{x}\,dx$$
$$+ \frac{\cos x}{2} \int f(x)\ \sin x\,dx - \frac{\sin x}{2} \int f(x)\ \cos x\,dx + f(x)$$
$$y_*(x) + f(x).$$

Hence, Eq. (3.36) is satisfied: $y_*^{(4)}(x) - y_*(x) = f(x)$.

4 Solution of initial value problems

The integral representations of the general solutions obtained by the method of variation of parameters are very convenient for solving arbitrary initial value problems (Cauchy's problems). I will illustrate the statement using the solutions considered in the previous sections.

4.1 First-order equations

Let us solve an arbitrary initial value problem for Eq. (2.1) at $x = x_0$:

$$y' + P(x)y = Q(x), \quad y(x_0) = Y_0, \tag{4.1}$$

where x_0 and Y_0 are arbitrary constants. We will use the integral representation (2.7) of the general solution:

$$y = C\,\mathrm{e}^{-\int P dx} + \mathrm{e}^{-\int P dx} \int Q(x)\,\mathrm{e}^{\int P dx} dx, \tag{2.7}$$

Recall that the function $\mathrm{e}^{-\int P dx}$ represents any solution of the homogeneous equations. We will chose for our convenience one of them, $y_1(x)$, which equals to 1 at $x = x_0$, namely

$$y_1(x) = \mathrm{e}^{-\int_{x_0}^{x} P(\xi) d\xi}. \tag{4.2}$$

Then Eq. (2.7) is written

$$y = C\,y_1(x) + y_1(x) \int \frac{Q(\xi)}{y_1(\xi)}\,d\xi, \tag{4.3}$$

where, according to (4.2),

$$\frac{1}{y_1(\xi)} = \mathrm{e}^{\int_{x_0}^{\xi} P(\eta) d\eta}. \tag{4.4}$$

The second term in (4.3) is an unspecified particular solution to the non-homogeneous equation. We can take any of them. We will chose the particular solution, $y_*(x)$, which vanishes at $x = x_0$, namely

$$y_*(x) = y_1(x) \int_{x_0}^{x} \frac{Q(\xi)}{y_1(\xi)}\,d\xi. \tag{4.5}$$

Thus the general solution (2.7) of the non-homogeneous equation is written in the form

$$y = C\,y_1(x) + y_*(x), \tag{4.6}$$

where the particular solution $y_1(x)$ of the homogeneous equation satisfies the initial condition $y_1(x_0) = 1$ and the particular solution $y_*(x)$ of the non-homogeneous equation satisfies the initial condition $y_*(x_0) = 0$. Now we substitute (4.6) in the initial condition of the problem (4.1) and obtain $C = Y_0$.

Finally, substituting in Eq. (4.6) $C = Y_0$ and invoking Eqs. (4.2), (4.4), (4.5), we obtain the following result.

Theorem 4.1. The solution to the initial value problem (4.1) with arbitrary x_0 and Y_0 has the form

$$y(x) = Y_0 \, \mathrm{e}^{-\int_{x_0}^{x} P(\xi)d\xi} + \mathrm{e}^{-\int_{x_0}^{x} P(\xi)d\xi} \int_{x_0}^{x} Q(\xi)\mathrm{e}^{\int_{x_0}^{\xi} P(\eta)d\eta} \, d\xi. \tag{4.7}$$

4.2 Second-order equations

Let us solve an arbitrary initial value problem for Eq. (2.8) at $x = x_0$:

$$y'' + a(x)y' + b(x)y = f(x), \quad y(x_0) = Y_0, \quad y'(x_0) = Y_1, \tag{4.8}$$

where $x_0,\ Y_0$ and Y_1 are arbitrary constants.

Following the discussion of the initial value problem for the first-order equations, we will specify the integral representation (2.21) of the general solution as follows:

$$y(x) = C_1 \, y_1(x) + C_2 \, y_2(x) + y_*(x). \tag{4.9}$$

Here $y_*(x)$ is a particular solution of the non-homogeneous equation (2.8) defined by

$$y_*(x) = -y_1(x) \int_{x_0}^{x} \frac{y_2(\xi) \, f(\xi)}{W[y_1, y_2](\xi)} \, d\xi + y_2(x) \int_{x_0}^{x} \frac{y_1(\xi) \, f(\xi)}{W[y_1, y_2](\xi)} \, d\xi, \tag{4.10}$$

where $W[y_1, y_2](\xi) = y_1(\xi)y_2'(\xi) - y_2(\xi)y_1'(\xi)$.

Lemma 4.1. The particular solution (4.10) satisfies the following initial conditions:

$$y_*(x_0) = 0, \quad y\,'_*(x_0) = 0. \tag{4.11}$$

Proof. It is obvious from the definition (4.10) of $y_*(x)$ that the first equation (4.11) is satisfied. Let us verify the second equation. We have by differentiation:

$$y\,'_*(x) = -y\,'_1(x) \int_{x_0}^{x} \frac{y_2(\xi) \, f(\xi)}{W[y_1, y_2](\xi)} \, d\xi + y\,'_2(x) \int_{x_0}^{x} \frac{y_1(\xi) \, f(\xi)}{W[y_1, y_2](\xi)} \, d\xi$$

$$- \, y_1(x) \frac{y_2(x) \, f(x)}{W[y_1, y_2](x)} + y_2(x) \frac{y_1(x) \, f(x)}{W[y_1, y_2](x)} \, .$$

Hence,

$$y'_*(x) = -y'_1(x) \int_{x_0}^{x} \frac{y_2(\xi)\, f(\xi)}{W[y_1, y_2](\xi)}\, d\xi + y'_2(x) \int_{x_0}^{x} \frac{y_1(\xi)\, f(\xi)}{W[y_1, y_2](\xi)}\, d\xi. \tag{4.12}$$

It is manifest from (4.12) that $y'_*(x_0) = 0$. This completes the proof.

Theorem 4.2. The solution to the initial value problem (4.8) with arbitrary $x_0,\ Y_0$ and Y_1 has the form

$$\begin{aligned} y(x) &= C_1\, y_1(x) + C_2\, y_2(x) \\ &\quad - y_1(x) \int_{x_0}^{x} \frac{y_2(\xi)\, f(\xi)}{W[y_1, y_2](\xi)}\, d\xi + y_2(x) \int_{x_0}^{x} \frac{y_1(\xi)\, f(\xi)}{W[y_1, y_2](\xi)}\, d\xi, \end{aligned} \tag{4.13}$$

where C_1, C_2 are determined by solving the system of linear algebraic equations

$$\begin{aligned} C_1\, y_1(x_0) + C_2\, y_2(x_0) &= Y_0, \\ C_1\, y'_1(x_0) + C_2\, y'_2(x_0) &= Y_1. \end{aligned} \tag{4.14}$$

Proof. We substitute the function $y(x)$ defined by (4.9) and its derivative

$$y'(x) = C_1\, y'_1(x) + C_2\, y'_2(x) + y'_*(x)$$

in the initial conditions (4.8), use Eqs. (4.11) and see that the initial conditions lead to Eqs. (4.14). This proves the theorem because the function (4.13) solves the differential equation in the problem (4.8).

Remark 4.1. The system (4.14) can be solved at once, giving

$$\begin{aligned} C_1 &= \frac{Y_0\, y'_2(x_0) - Y_1\, y_2(x_0)}{W[y_1, y_2](x_0)}\,, \\ C_2 &= \frac{Y_1\, y_1(x_0) - Y_0\, y'_1(x_0)}{W[y_1, y_2](x_0)}\,. \end{aligned} \tag{4.15}$$

4.3 Third-order equations

Let us solve an arbitrary initial value problem for Eq. (3.9) at $x = x_0$:

$$\begin{aligned} & y''' + a(x)y'' + b(x)y' + c(x)y = f(x), \\ & y(x_0) = Y_0, \quad y'(x_0) = Y_1, \quad y''(x_0) = Y_2, \end{aligned} \tag{4.16}$$

where $x_0,\ Y_0,\ Y_1$ and Y_2 are arbitrary constants.

Proceeding as in Section 4.2, we rewrite the formula (3.25) for the general solution in the form

$$y(x) = C_1\, y_1(x) + C_2\, y_2(x) + C_3\, y_3(x) + y_*(x), \tag{4.17}$$

where $y_*(x)$ is a particular solution of the non-homogeneous equation (3.9) defined by

$$y_*(x) = y_1(x) \int_{x_0}^{x} \frac{W[y_2, y_3](\xi)\, f(\xi)}{W[y_1, y_2, y_3](\xi)}\, d\xi$$

$$+ y_2(x) \int_{x_0}^{x} \frac{W[y_3, y_1](\xi)\, f(\xi)}{W[y_1, y_2, y_3](\xi)}\, d\xi + y_3(x) \int_{x_0}^{x} \frac{W[y_1, y_2](\xi)\, f(\xi)}{W[y_1, y_2, y_3](\xi)}\, d\xi. \tag{4.18}$$

Lemma 4.2. The particular solution (4.18) satisfies the following initial conditions:

$$y_*(x_0) = 0, \quad y'_*(x_0) = 0, \quad y''_*(x_0) = 0. \tag{4.19}$$

Proof. It is obvious from the definition (4.18) of $y_*(x)$ that the first equation (4.19) is satisfied. In order to verify the second equation (4.19), we differentiate (4.18):

$$y'_*(x) = y'_1(x) \int_{x_0}^{x} \frac{W[y_2, y_3](\xi)\, f(\xi)}{W[y_1, y_2, y_3](\xi)}\, d\xi + y_1(x) \frac{W[y_2, y_3](x)\, f(x)}{W[y_1, y_2, y_3](x)}$$

$$+ y'_2(x) \int_{x_0}^{x} \frac{W[y_3, y_1](\xi)\, f(\xi)}{W[y_1, y_2, y_3](\xi)}\, d\xi + y_2(x) \frac{W[y_3, y_1](x)\, f(x)}{W[y_1, y_2, y_3](x)}$$

$$+ y'_3(x) \int_{x_0}^{x} \frac{W[y_1, y_2](\xi)\, f(\xi)}{W[y_1, y_2, y_3](\xi)}\, d\xi + y_3(x) \frac{W[y_1, y_2](x)\, f(x)}{W[y_1, y_2, y_3](x)}\,.$$

The reckoning shows that

$$y_1(x)\, W[y_2, y_3](x) + y_2(x)\, W[y_3, y_1](x) + y_3(x)\, W[y_1, y_2](x) = 0.$$

Therefore

$$y'_*(x) = y'_1(x) \int_{x_0}^{x} \frac{W[y_2, y_3](\xi)\, f(\xi)}{W[y_1, y_2, y_3](\xi)}\, d\xi + y'_2(x) \int_{x_0}^{x} \frac{W[y_3, y_1](\xi)\, f(\xi)}{W[y_1, y_2, y_3](\xi)}\, d\xi$$

$$+ y'_3(x) \int_{x_0}^{x} \frac{W[y_1, y_2](\xi)\, f(\xi)}{W[y_1, y_2, y_3](\xi)}\, d\xi. \tag{4.20}$$

Another differentiation yields:

$$y''_*(x) = y''_1(x) \int_{x_0}^{x} \frac{W[y_2, y_3](\xi)\, f(\xi)}{W[y_1, y_2, y_3](\xi)}\, d\xi + y'_1(x) \frac{W[y_2, y_3](x)\, f(x)}{W[y_1, y_2, y_3](x)}$$

$$+ y''_2(x) \int_{x_0}^{x} \frac{W[y_3, y_1](\xi)\, f(\xi)}{W[y_1, y_2, y_3](\xi)}\, d\xi + y'_2(x) \frac{W[y_3, y_1](x)\, f(x)}{W[y_1, y_2, y_3](x)}$$

$$+ y''_3(x) \int_{x_0}^{x} \frac{W[y_1, y_2](\xi)\, f(\xi)}{W[y_1, y_2, y_3](\xi)}\, d\xi + y'_3(x) \frac{W[y_1, y_2](x)\, f(x)}{W[y_1, y_2, y_3](x)}\,.$$

The reckoning shows that

$$y_1'(x)\,W[y_2,y_3](x) + y_2'(x)\,W[y_3,y_1](x) + y_3'(x)\,W[y_1,y_2](x) = 0.$$

Therefore

$$y_*''(x) = y_1''(x)\int_{x_0}^{x}\frac{W[y_2,y_3](\xi)\,f(\xi)}{W[y_1,y_2,y_3](\xi)}\,d\xi + y_2''(x)\int_{x_0}^{x}\frac{W[y_3,y_1](\xi)\,f(\xi)}{W[y_1,y_2,y_3](\xi)}\,d\xi$$
$$+\,y_3''(x)\int_{x_0}^{x}\frac{W[y_1,y_2](\xi)\,f(\xi)}{W[y_1,y_2,y_3](\xi)}\,d\xi. \tag{4.21}$$

Eqs. (4.20), (4.21) yield that $y_*'(x_0) = 0,\ y_*''(x_0) = 0$. This completes the proof.

Theorem 4.3. The solution to the initial value problem (4.16) with arbitrary $x_0,\ Y_0,\ Y_1$ and Y_3 has the form

$$y(x) = C_1\,y_1(x) + C_2\,y_2(x) + C_3\,y_3(x) + y_1(x)\int_{x_0}^{x}\frac{W[y_2,y_3](\xi)\,f(\xi)}{W[y_1,y_2,y_3](\xi)}\,d\xi$$
$$+\,y_2(x)\int_{x_0}^{x}\frac{W[y_3,y_1](\xi)\,f(\xi)}{W[y_1,y_2,y_3](\xi)}\,d\xi + y_3(x)\int_{x_0}^{x}\frac{W[y_1,y_2](\xi)\,f(\xi)}{W[y_1,y_2,y_3](\xi)}\,d\xi, \tag{4.22}$$

where C_1, C_2 are determined by solving the system of linear algebraic equations

$$\begin{aligned} &C_1\,y_1(x_0) + C_2\,y_2(x_0) + C_3\,y_3(x_0) = Y_0,\\ &C_1\,y_1'(x_0) + C_2\,y_2'(x_0) + C_3\,y_3'(x_0) = Y_1,\\ &C_1\,y_1''(x_0) + C_2\,y_2''(x_0) + C_3\,y_3''(x_0) = Y_2. \end{aligned} \tag{4.23}$$

The proof is similar to the proof of Theorem 4.2.

Higher-order linear equations can be treated likewise.

5 Examples

Theorem 4.1 provides the simple formula (4.7) for the solution of an arbitrary initial value problem for any linear first-order equation. This formula is as simple for the practical use as the formula for roots of quadratic equations. The solution to the Cauchy problem (4.1) is obtained just by inserting the coefficients $P(x),\ Q(x)$ of the differential equations in question and the initial data x_0, Y_0.

Theorems 4.2 and 4.3 play the same role for second-order and third-order equations, respectively, provided that fundamental systems of solutions for the corresponding homogeneous equations are known.

Of course, the integrals in (4.1), (4.13) and (4.22) may be difficult or impossible (this is the general case) to work out in terms of elementary functions. But this circumstance is unessential. Anyhow, equations whose solutions can be expressed in terms of elementary functions are very rear and do not have a practical value. They appear mostly in textbooks as simple illustrations of various methods.

5.1 First-order equations

Example 5.1. Let us solve the Cauchy problem

$$y' - 2xy = x^3, \quad y(1) = Y_0. \tag{5.1}$$

Here

$$P(x) = -2x, \quad Q(x) = x^3, \quad x_0 = 1,$$

and the solution formula (4.7) is written:

$$y(x) = Y_0\, \mathrm{e}^{\int_1^x (2\xi)d\xi} + \mathrm{e}^{\int_1^x (2\xi)d\xi} \int_1^x \xi^3 \mathrm{e}^{-\int_1^\xi (2\eta)d\eta}\, d\xi,$$

or

$$y(x) = Y_0\, \mathrm{e}^{x^2-1} + \mathrm{e}^{x^2-1} \int_1^x \xi^3 \mathrm{e}^{1-\xi^2}\, d\xi. \tag{5.2}$$

One can leave the solution in the integral form (5.2). But in this particular case the integral can be easily worked out:

$$\int_1^x \xi^3 \mathrm{e}^{1-\xi^2} d\xi = \frac{\mathrm{e}}{2} \int_1^x \xi^2 \mathrm{e}^{-\xi^2} d(\xi^2) = -\frac{1}{2}\left(1+\xi^2\right) \mathrm{e}^{1-\xi^2}\Big|_1^x$$

$$= 1 - \frac{1}{2}\,(1+x^2)\, \mathrm{e}^{1-x^2}. \tag{5.3}$$

Substituting (5.3) in (5.2) we obtain the solution to the problem (5.1) in elementary functions:

$$y(x) = (1+Y_0)\, \mathrm{e}^{x^2-1} - \frac{1}{2}\left(1+x^2\right). \tag{5.4}$$

Example 5.2. Let us solve the Cauchy problem

$$y' - y\cos x = x, \quad y(0) = Y_0. \tag{5.5}$$

Here

$$P(x) = -\cos x, \quad Q(x) = x, \quad x_0 = 0,$$

and the solution formula (4.7) is written:

$$y(x) = Y_0\, \mathrm{e}^{\int_0^x \cos\xi d\xi} + \mathrm{e}^{\int_0^x \cos\xi d\xi} \int_0^x \xi\, \mathrm{e}^{-\int_0^\xi \cos\eta d\eta}\, d\xi.$$

Since

$$e^{\int_0^x \cos\xi d\xi} = e^{\sin x}$$

we obtain the solution to the problem (5.5) containing one quadrature:

$$y(x) = Y_0\, e^{\sin x} + e^{\sin x} \int_0^x \xi\, e^{-\sin\xi} d\xi. \tag{5.6}$$

5.2 Second-order equations

Example 5.3. Consider the Cauchy problem

$$\begin{aligned} &y'' + y = f(x), \\ &y(x_0) = Y_0\,, \quad y'(x_0) = Y_1\,. \end{aligned} \tag{5.7}$$

The functions

$$y_1(x) = \cos x, \quad y_2(x) = \sin x \tag{5.8}$$

provide a fundamental system of solutions for the homogeneous equation $y'' + y = 0$ and have the Wronskian

$$W[y_1, y_2] = 1. \tag{5.9}$$

Furthermore, Eqs. (4.15) yield

$$\begin{aligned} C_1 &= Y_0\, \cos x_0 - Y_1\, \sin x_0\,, \\ C_2 &= Y_1\, \cos x_0 + Y_0\, \sin x_0\,. \end{aligned}$$

Therefore

$$\begin{aligned} &C_1 y_1(x) + C_2 y_2(x) \\ &= Y_0\, \cos x_0 \cos x - Y_1\, \sin x_0 \cos x + Y_1\, \cos x_0 \sin x + Y_0\, \sin x_0 \sin x \\ &= Y_0 \cos(x - x_0) + Y_1 \sin(x - x_0)\,. \end{aligned} \tag{5.10}$$

Substituting (5.8), (5.9) and (5.10) in the formula (4.13) we obtain the following solution to the problem (5.7):

$$\begin{aligned} y(x) = {} & Y_0 \cos(x - x_0) + Y_1 \sin(x - x_0) \\ & - \cos x \int_{x_0}^x f(\xi) \sin\xi\, d\xi + \sin x \int_{x_0}^x f(\xi) \cos\xi\, d\xi\,. \end{aligned} \tag{5.11}$$

Exercise 5.1. Solve the Cauchy problem

$$\begin{aligned} & y'' + y = x^n, \quad (n = 1, 2, \ldots), \\ & y(x_0) = Y_0\,, \quad y'(x_0) = Y_1\,. \end{aligned} \tag{5.12}$$

Solution. Eq. (5.11) provides the following integral representation of solution to the problem (5.12):

$$\begin{aligned} y(x) = & \, Y_0 \cos(x - x_0) + Y_1 \sin(x - x_0) \\ & - \cos x \int_{x_0}^{x} \xi^n \sin \xi \, d\xi + \sin x \int_{x_0}^{x} \xi^n \cos \xi \, d\xi\,. \end{aligned} \tag{5.13}$$

The solution (5.13) can be written in terms of elementary functions by using the well-known integrals

$$\begin{aligned} & \int \sin x \, dx = -\cos x, \qquad \int \cos x \, dx = \sin x, \\ & \int x \sin x \, dx = \sin x - x \cos x, \qquad \int x \cos x \, dx = \cos x + x \sin x \end{aligned}$$

and the recursion formulae

$$\begin{aligned} & \int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx, \\ & \int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx. \end{aligned}$$

Exercise 5.2. Solve the Cauchy problem

$$\begin{aligned} & y'' + y = \frac{1}{x + 1} \quad (x \geq 0), \\ & y(0) = Y_0\,, \quad y'(0) = Y_1\,. \end{aligned} \tag{5.14}$$

Solution. Substituting $x_0 = 0$ and

$$f(x) = \frac{1}{x + 1}$$

in Eq. (5.11) we obtain the following solution to the problem (5.14):

$$y(x) = Y_0 \cos x + Y_1 \sin x - \cos x \int_0^x \frac{\sin \xi}{\xi + 1} \, d\xi + \sin x \int_0^x \frac{\cos \xi}{\xi + 1} \, d\xi\,. \tag{5.15}$$

The integrals in (5.15) cannot be worked out in terms of elementary functions. Nevertheless, one can readily verify that the function (5.15) satisfies the initial conditions and the differential equation of the problem (5.14). The condition $y(0) = Y_0$ is obviously satisfied. The differentiation of (5.15) yields:

$$\begin{aligned} y'(x) = &- Y_0 \sin x + Y_1 \cos x + \sin x \int_0^x \frac{\sin \xi}{\xi + 1}\, d\xi \\ &- \frac{\cos x \sin x}{x+1} + \cos x \int_0^x \frac{\cos \xi}{\xi+1}\, d\xi + \frac{\sin x \cos x}{x+1} \\ = &-Y_0 \sin x + Y_1 \cos x + \sin x \int_0^x \frac{\sin \xi}{\xi+1}\, d\xi + \cos x \int_0^x \frac{\cos \xi}{\xi+1}\, d\xi. \end{aligned}$$

It is manifest from the above expression for $y'(x)$ that the condition $y'(0) = Y_1$ is also satisfied. Another differentiation yields:

$$\begin{aligned} y''(x) = &- Y_0 \cos x - Y_1 \sin x + \cos x \int_0^x \frac{\sin \xi}{\xi+1}\, d\xi \\ &+ \frac{\sin^2 x}{x+1} - \sin x \int_0^x \frac{\cos \xi}{\xi+1}\, d\xi + \frac{\cos^2 x}{x+1} \\ = &-Y_0 \cos x - Y_1 \sin x + \cos x \int_0^x \frac{\sin \xi}{\xi+1}\, d\xi - \sin x \int_0^x \frac{\cos \xi}{\xi+1}\, d\xi + \frac{1}{x+1} \\ = &- y + \frac{1}{x+1} \cdot \end{aligned}$$

Hence, the differential equation (5.14) is satisfied.

Example 5.4. Consider the Cauchy problem

$$\begin{aligned} &x^2 y'' + 3xy' + y = \frac{1}{x} \quad (x \geq 1), \\ &y(1) = Y_0\,, \quad y'(1) = Y_1\,. \end{aligned} \tag{5.16}$$

Solving the homogeneous equation, i.e. the Euler equation

$$x^2 y'' + 3xy' + y = 0,$$

we obtain the following fundamental system of solutions:

$$y_1(x) = \frac{1}{x}\,, \quad y_2(x) = \frac{\ln x}{x} \cdot \tag{5.17}$$

Their Wronskian is

$$W[y_1, y_2](x) = \frac{1}{x^3}\cdot \tag{5.18}$$

Eqs. (4.14) yield

$$C_1 = Y_0, \quad C_2 = Y_0 + Y_1. \tag{5.19}$$

Now we write the differential equation of the problem (5.16) in the form (4.8),

$$y'' + \frac{3}{x}\, y' + \frac{1}{x^2}\, y = \frac{1}{x^3}\,, \tag{5.20}$$

apply the formula (4.13) and, invoking Eqs. (5.17), (5.18), (5.19), obtain:

$$y(x) = Y_0\, \frac{1}{x} + (Y_0 + Y_1)\, \frac{\ln x}{x} - \frac{1}{x}\int_1^x \frac{\ln \xi}{\xi}\, d\xi + \frac{\ln x}{x}\int_1^x \frac{1}{\xi}\, d\xi. \tag{5.21}$$

The integrals can be worked out at once, giving

$$\int_1^x \frac{\ln \xi}{\xi}\, d\xi = \frac{1}{2}\big(\ln \xi\big)^2\Big|_1^x = \big(\ln x\big)^2,$$

$$\int_1^x \frac{1}{\xi}\, d\xi = \ln \xi\Big|_1^x = \ln x.$$

Substituting these expression in (5.21) we obtain the solution to the problem (5.16) in elementary functions:

$$y(x) = \frac{1}{x}\left[Y_0 + (Y_0 + Y_1)\ln x + \frac{1}{2}\big(\ln x\big)^2\right]. \tag{5.22}$$

It is useful to verify by direct substitution that the function (5.22) satisfies the differential equation and the initial conditions of the problem (5.16).

Example 5.5. Let us solve the following Cauchy problem:

$$\begin{aligned} &y'' - y'\, \cos x + y\, \sin x = f(x),\\ &y(0) = Y_0\,, \quad y'(0) = Y_1\,. \end{aligned} \tag{5.23}$$

First, we will find two linearly independent solutions (a fundamental system of solutions) for the homogeneous equation

$$y'' - y'\, \cos x + y\, \sin x = 0, \tag{5.24}$$

noting that its order can be reduced. Indeed, it can be written in the form

$$(y' - y\cos x)' = 0$$

and integrated, giving

$$y' - y\cos x = K_1\,. \tag{5.25}$$

One can easily integrate the first-order equation (5.25) and find the general solution

$$y = K_2 \mathrm{e}^{\sin x} + K_1 \mathrm{e}^{\sin x}\int \mathrm{e}^{-\sin x} dx \tag{5.26}$$

to Eq. (5.24) containing two arbitrary constants K_1, K_2. Hence, one can take for a fundamental system of solutions for Eq. (5.24) the following the functions:

$$y_1(x) = \mathrm{e}^{\sin x}, \quad y_2(x) = \mathrm{e}^{\sin x}\int_0^x \mathrm{e}^{-\sin\xi} d\xi. \tag{5.27}$$

Let us find the Wronskian of the functions (5.27). We have:

$$y_1'(x) = \cos x\,\mathrm{e}^{\sin x}, \quad y_2'(x) = 1 + \cos x\,\mathrm{e}^{\sin x}\int_0^x \mathrm{e}^{-\sin\xi} d\xi,$$

and hence

$$W[y_1, y_2](x) = \mathrm{e}^{\sin x}. \tag{5.28}$$

Eqs. (4.14) yield

$$C_1 = Y_0, \quad C_2 = Y_1 - Y_0.$$

Substituting now (5.27) and (5.28) in (4.13) we obtain the following solution to the problem (5.23):

$$\begin{aligned} y(x) = {} & Y_1\ \mathrm{e}^{\sin x} + (Y_1 - Y_0)\,\mathrm{e}^{\sin x}\int_0^x \mathrm{e}^{-\sin\xi} d\xi \\ & - \mathrm{e}^{\sin x}\int_0^x f(\xi)\Big[\int_0^\xi \mathrm{e}^{-\sin\eta} d\eta\Big] d\xi + \mathrm{e}^{\sin x}\int_0^x \mathrm{e}^{-\sin\xi} d\xi \int_0^x f(\xi)\, d\xi. \end{aligned} \tag{5.29}$$

5.3 Third-order equations

Example 5.6. Let us solve the following Cauchy problem:

$$\begin{aligned} & y''' - y'' + y' - y = f(x), \\ & y(0) = Y_0\,, \quad y'(0) = Y_1\,, \quad y''(0) = Y_2\,. \end{aligned} \tag{5.30}$$

The characteristic polynomial for the homogeneous homogeneous equation

$$y''' - y'' + y' - y = 0 \tag{5.31}$$

is written

$$\lambda^3 - \lambda^2 + \lambda - 1 = (\lambda - 1)(\lambda^2 + 1)$$

and has the roots $\lambda_1 = 1,\ \lambda_2 = i,\ \lambda_3 = -i$. Consequently, a fundamental system of solutions for Eq. (5.31) is provided by the functions

$$y_1(x) = \mathrm{e}^x, \quad y_2(x) = \cos x, \quad y_2(x) = \sin x. \tag{5.32}$$

Let us find the Wronskians (3.20) and (3.22). The reckoning yields

$$W[y_1, y_2, y_3](x) = \begin{vmatrix} \mathrm{e}^x & \cos x & \sin x \\ \mathrm{e}^x & -\sin x & \cos x \\ \mathrm{e}^x & -\cos x & -\sin x \end{vmatrix} = 2\,\mathrm{e}^x \tag{5.33}$$

and

$$\begin{aligned} &W[y_2, y_3](x) = 1, \\ &W[y_3, y_1](x) = \mathrm{e}^x(\sin x - \cos x), \\ &W[y_1, y_2](x) = -\mathrm{e}^x(\sin x + \cos x). \end{aligned} \tag{5.34}$$

In the problem (5.30) we have x_0 and Eqs. (4.23), (5.32) yield:

$$C_1 + C_2 = Y_0, \quad C_1 + C_3 = Y_1, \quad C_1 - C_2 = Y_2,$$

whence

$$C_1 = \frac{1}{2}\,(Y_0 + Y_2), \quad C_2 = \frac{1}{2}\,(Y_0 - Y_2), \quad C_3 = \frac{1}{2}\,(2Y_1 - Y_0 - Y_2). \tag{5.35}$$

Substituting (5.32)-(5.35) in (4.22) we obtain the following integral representation of the solution to the problem (5.30):

$$\begin{aligned} y(x) = \frac{1}{2}\Big[&(Y_0 + Y_2)\,\mathrm{e}^x + (Y_0 - Y_2)\,\cos x + (2Y_1 - Y_0 - Y_2)\,\sin x \\ &+ \mathrm{e}^x \int_0^x \mathrm{e}^{-\xi}\, f(\xi)\, d\xi + \cos x \int_0^x (\sin\xi - \cos\xi)\, f(\xi)\, d\xi \\ &- \sin x \int_0^x (\sin\xi + \cos\xi)\, f(\xi)\, d\xi\Big]. \end{aligned} \tag{5.36}$$

Let us verify that (5.36) solves our problem. The first initial condition, $y(0) = Y_0$, is obviously satisfied. We differentiate (5.36):

$$\begin{aligned} y'(x) = \frac{1}{2}\Big[&(Y_0 + Y_2)\,\mathrm{e}^x - (Y_0 - Y_2)\,\sin x + (2Y_1 - Y_0 - Y_2)\,\cos x \\ &+ \mathrm{e}^x \int_0^x \mathrm{e}^{-\xi}\, f(\xi)\, d\xi - \sin x \int_0^x (\sin\xi - \cos\xi)\, f(\xi)\, d\xi \\ &- \cos x \int_0^x (\sin\xi + \cos\xi)\, f(\xi)\, d\xi + f(x) \\ &+ \cos x(\sin x - \cos x)\, f(x) - \sin x(\sin x + \cos x)\, f(x)\Big], \end{aligned}$$

whence

$$y'(x) = \frac{1}{2}\bigg[(Y_0+Y_2)\,\mathrm{e}^x - (Y_0-Y_2)\,\sin x + (2Y_1-Y_0-Y_2)\,\cos x + \mathrm{e}^x\int_0^x \mathrm{e}^{-\xi} f(\xi)\,d\xi - \sin x\int_0^x (\sin\xi-\cos\xi)\,f(\xi)\,d\xi - \cos x\int_0^x (\sin\xi+\cos\xi)\,f(\xi)\,d\xi\bigg]. \tag{5.37}$$

It is obvious now that $y'(0) = Y_1$. We differentiate (5.37) again and proceed as above to obtain:

$$y''(x) = \frac{1}{2}\bigg[(Y_0+Y_2)\,\mathrm{e}^x - (Y_0-Y_2)\,\cos x - (2Y_1-Y_0-Y_2)\,\sin x + \mathrm{e}^x\int_0^x \mathrm{e}^{-\xi} f(\xi)\,d\xi - \cos x\int_0^x (\sin\xi-\cos\xi)\,f(\xi)\,d\xi + \sin x\int_0^x (\sin\xi+\cos\xi)\,f(\xi)\,d\xi\bigg]. \tag{5.38}$$

One can easily see that $y''(0) = Y_2$. We differentiate again and obtain:

$$y'''(x) = \frac{1}{2}\bigg[(Y_0+Y_2)\,\mathrm{e}^x + (Y_0-Y_2)\,\sin x - (2Y_1-Y_0-Y_2)\,\cos x + \mathrm{e}^x\int_0^x \mathrm{e}^{-\xi} f(\xi)\,d\xi + \sin x\int_0^x (\sin\xi-\cos\xi)\,f(\xi)\,d\xi + \cos x\int_0^x (\sin\xi+\cos\xi)\,f(\xi)\,d\xi\bigg] + f(x). \tag{5.39}$$

It follows from (5.36)-(5.39) that the differential equation (5.30) is satisfied.

For particular types of function $f(x)$ the problem (5.30) may have the solution given by elementary functions. For instance, for the problem (5.30) with $f(x) = x$ one can easily work out the integrals in (5.36) and obtain

$$y(x) = -(1+x) + \frac{1}{2}\bigg[(Y_0+Y_2+1)\,\mathrm{e}^x + (Y_0-Y_2+1)\,\cos x + (2Y_1-Y_0-Y_2+1)\,\sin x\bigg].$$

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APPLICATION OF GROUP ANALYSIS TO LIQUID METAL SYSTEMS

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Abstract. A nonlinear equation arising in metallurgical applications of Magnetohydrodynamics is discussed. Lie group analysis reveals two exceptional values of the exponent playing a significant role in the model. Self-adjointness and first integrals are investigated.

Keywords: Liquid metals, Metallurgy, Group analysis, Boundary-layer equation.

1 Introduction

The growing interest to theoretical investigations on applications of Magnetohydrodynamics in the metallurgical industry in the 1970s have been motivated by possibilities of using alternating magnetic fields in the processing of liquid metals when the high quality of the product is required. For example, high frequency external magnetic fields are widely used in the casting process in the steel industry in order to control a flow of liquid metals and to generate internal stirring within the liquid phase. This allows one to reach the required homogeneity of solidifying metals by eliminating blowholes usually caused by escaping gases.

The reader can find a good discussion of the process from the physical point of view in [1] (see also [2], Sections 1 and 4 on electromagnetic shaping and stirring). The mathematical model suggested by H.K. Moffatt in [1] for describing the "skin effects" in a thin surface layer of liquid metals near a sharp corner is thoroughly investigated in the recent paper [3] from the point of view of existence of solutions. In both papers, [1] and [3], the value $m = -1/2$ of the exponent $m,$ playing the significant part in the main equation (3.1), appears as a "critical value".

The present paper is devoted to the group analysis of Moffatt's model. The analysis reveals two exceptional values, $m = -1/2$ and $m = 3,$ of the exponent $m.$ Namely, it is shown that Eq. (3.1) is self-adjoint if $m = -1/2;$ in this case a first integral is found for Eq. (3.1). The second exceptional case $m = 3$ singles out the equation having more symmetries than for all other values of $m.$

2 Preliminaries

2.1 The Prandtl boundary-layer equations

The system of boundary layer equations for a planar steady flow of liquid with a constant density ρ and a constant coefficient of the kinematic viscosity ν has the form

$$
\begin{aligned}
&uu_x + vu_y = \nu u_{yy} - \frac{1}{\rho}\, p_x,\\
&p_y = 0, \quad u_x + v_y = 0.
\end{aligned}
\tag{2.1}
$$

It is known as the Prandtl equation. The flow is parallel to a flat plate and is directed along the x axis in the Cartesian coordinates (x, y).

2.2 Invariance principle for boundary value problems

We will use a general principle for tackling boundary and/or initial value problems for equations having certain symmetries. It was formulated in [4] (see also [5]) and called an *invariance principle.* This principle states that if a differential equation has a symmetry group G and if the initial (boundary) data, including the initial manifold, are invariant under a subgroup $H \subset G$ then one should seek the solution to the problem among H-invariant solutions of the differential equation in question. The invariance principle is applicable both to linear and nonlinear equations.

2.3 Adjoint equation to nonlinear equations and conserved quantities associated with symmetries

These concepts have been employed in [6], Sections 2 and 3. We will apply them to a third-order ordinary differential equation

$$
f''' + F(\eta, f, f', f'') = 0, \tag{2.2}
$$

where f and η are a dependent and independent variables, respectively, f' is the first derivative of f with respect to η, etc. The adjoint equation to Eq. (2.2) is defined by

$$
\frac{\delta \mathcal{L}}{\delta f} = 0 \tag{2.3}
$$

with

$$
\mathcal{L} = z\big[f''' + F(\eta, f, f', f'')\big], \tag{2.4}
$$

where z is a new dependent variable, and $\delta/\delta f$ is the variational derivative:

$$
\frac{\delta}{\delta f} = \frac{\partial}{\partial f} - D\frac{\partial}{\partial f'} + D^2\frac{\partial}{\partial f''} - D^3\frac{\partial}{\partial f'''}\,\cdot \tag{2.5}
$$

Here D is the total derivative with respect to η :

$$D = \frac{\partial}{\partial \eta} + f' \frac{\partial}{\partial f} + f'' \frac{\partial}{\partial f'} + f''' \frac{\partial}{\partial f''} + \cdots .$$

In particular, $D^3(z) = z'''$, and the left-hand side of the adjoint equation (2.3) is

$$\frac{\delta}{\delta f}\{z[f''' + F(\eta, f, f', f'')]\} = -z''' + z\frac{\partial F}{\partial f} - D\Big(z\frac{\partial F}{\partial f'}\Big) + D^2\Big(z\frac{\partial F}{\partial f''}\Big). \quad (2.6)$$

Eq. (2.2) is self-adjoint if the adjoint equation (2.3) becomes equivalent with Eq. (2.2) after the substitution $z = f$. If the equivalence occurs upon the substitution $z = h(f)$ with a certain invertible function $h(f)$, then Eq. (2.2) is said to be quasi-self-adjoint.

If an operator

$$X = \xi(\eta, f)\frac{\partial}{\partial \eta} + \Gamma(\eta, f)\frac{\partial}{\partial f} \quad (2.7)$$

is admitted by Eq. (2.2), i.e. if X is an infinitesimal symmetry for this equation, then

$$\begin{aligned} C = W \Big[\frac{\partial \mathcal{L}}{\partial f'} - D\Big(\frac{\partial \mathcal{L}}{\partial f''}\Big) + D^2\Big(\frac{\partial \mathcal{L}}{\partial f'''}\Big)\Big] \\ + D(W)\Big[\frac{\partial \mathcal{L}}{\partial f''} - D\Big(\frac{\partial \mathcal{L}}{\partial f'''}\Big)\Big] + D^2(W)\frac{\partial \mathcal{L}}{\partial f'''}, \quad W = \Gamma - \xi f', \end{aligned} \quad (2.8)$$

is a conserved quantity for Eq. (2.2) considered together with the adjoint equation (2.5). It is necessary to involve the adjoint equation (2.5) because the quantity (2.8) contains the variable z and its derivatives. However, if Eq. (2.2) is self-adjoint or quasi-self-adjoint, then one can eliminate z from Eq. (2.8) by substituting $z = f$ or $z = h(f)$, respectively, and obtain conserved quantity (first integral) for Eq. (2.2) considered without its adjoint equation.

3 Internal stirring of liquid metals by magnetic fields

3.1 Boundary-layer description of high Reynolds number flows

To describe the "skin effects" in a thin surface layer of high Reynolds number flows of liquid metals placed in a high frequency magnetic field, Moffatt [1] suggested to use the boundary layer equation

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = \nu\psi_{yyy} \quad (3.1)$$

for the stream function $\psi(x, y)$. He assumes that there is no pressure gradient outside the boundary layer and obtains the boundary conditions

$$\psi = 0, \quad \psi_y = Ax^m \quad \text{on} \quad y = 0, \quad (3.2)$$

$$\psi_y \to 0 \quad \text{as} \quad y \to \infty. \quad (3.3)$$

Here $\nu,\ A,\ m = \mathrm{const}$. The physical meaning of the constants A and m shows that the following conditions hold:

$$A > 0 \quad \text{when} \quad m + 1 > 0, \tag{3.4}$$

$$A < 0 \quad \text{when} \quad m + 1 < 0. \tag{3.5}$$

Remark 3.1. Eq. (3.1) is equivalent to the system (2.1) if p_x is negligible compare with the other terms in the first equation in (2.1). Namely, upon introducing the stream function $\psi(x, y)$ defined by the equations $u = \psi_y,\ v = \psi_x$, the third equation in (2.1) is satisfied identically, and the first equation in (2.1) takes the form (3.1).

3.2 Moffatt's solution

Referring to "standard similarity arguments of boundary-layer theory" presented in [7], Section 5.9, Moffatt states that upon letting

$$\psi = \left(\nu |A|\, x^{m+1}\right)^{1/2} f(\eta) \tag{3.6}$$

with

$$\eta = \left(\nu^{-1} |A|\, x^{m+1}\right)^{1/2} y \tag{3.7}$$

the partial differential equation (3.1) reduces to the third-order ordinary differential equation

$$f''' + \frac{m+1}{2} f f'' - m f'^2 = 0 \tag{3.8}$$

The boundary conditions (3.2)-(3.3) yield:

$$f(0) = 0, \quad f'(\infty) = 0 \tag{3.9}$$

and

$$f'(0) = \begin{cases} +1 & \text{in case} \quad (3.4), \\ -1 & \text{in case} \quad (3.5). \end{cases} \tag{3.10}$$

It is mentioned in [1], page 186, that

$$m = -\frac{1}{2} \tag{3.11}$$

is a critical value of the exponent m for existence of a solution to the problem (3.8)-(3.9). One can find there also an interesting discussion of a physical significance of this critical value of m.

4 Application of the invariance principle

4.1 Scaling symmetries of the boundary-layer equation

Let us apply the invariance principle (see Section 2.2) to the initial value problem (3.1)-(3.2). First we have to find appropriate symmetries of the differential equation (3.1). It is manifest that Eq. (3.1) is invariant under translations of x, y and ψ. But these translations change the initial data (3.2) and cannot be used in the invariance principle. Therefore we will look for another type of symmetries that can be easily found, namely scaling symmetries (called also dilations or similarity transformations). Taking them in the form

$$\bar{x} = a\,x, \quad \bar{y} = b\,y, \quad \bar{\psi} = c\,\psi \tag{4.1}$$

and writing the invariance condition for Eq. (3.1):

$$\bar{\psi}_{\bar{y}}\,\bar{\psi}_{\bar{x}\bar{y}} - \bar{\psi}_{\bar{x}}\,\bar{\psi}_{\bar{y}\bar{y}} - \nu\bar{\psi}_{\bar{y}\bar{y}\bar{y}} = \lambda\big[\psi_y\psi_{xy} - \psi_x\psi_{yy} - \nu\psi_{yyy}\big],$$

where λ is an undetermined constant coefficient, we obtain the following equations:

$$\frac{c^2}{ab^2} = \frac{c}{b^3} = \lambda.$$

Hence, the dilation (4.1) is admitted by Eq. (3.1) if the three parameters a, b, c are connected by one equation

$$\frac{c}{a} = \frac{1}{b}\,\cdot$$

It means that Eq. (3.1) admits the group G of dilations (4.1) where two parameters, e.g. a, b, are arbitrary, and the third parameter c is determined by the equation

$$c = \frac{a}{b}\,\cdot$$

The generators of this two-parameter group G span the Lie algebra L_2 with the basis

$$X_1 = x\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial \psi}\,, \quad X_2 = y\frac{\partial}{\partial y} - \psi\frac{\partial}{\partial \psi}\,\cdot \tag{4.2}$$

4.2 Operator admitted by the initial data

Let us find the subgroup $H \subset G$ leaving invariant the initial data (3.2). We write the generator of H in the form $X = \alpha X_1 + \beta X_2$:

$$X = \alpha x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y} + (\alpha - \beta)\psi\frac{\partial}{\partial \psi} \tag{4.3}$$

and determine the constant coefficients α and β from the invariance conditions of the initial data (3.2). It is manifest that the invariance test for the initial manifold $y = 0$,

$$X(y)\Big|_{y=0} = 0,$$

is satisfied. Hence the invariance of the initial data (3.2) is guaranteed by the equations

$$\widetilde{X}(\psi)\Big|_{(3.2)} = 0, \quad \widetilde{X}(\psi_y - Ax^m)\Big|_{(3.2)} = 0, \tag{4.4}$$

where

$$\widetilde{X} = \alpha x \frac{\partial}{\partial x} + (\alpha - \beta)\psi \frac{\partial}{\partial \psi} + (\alpha - 2\beta)\psi_y \frac{\partial}{\partial \psi_y} \tag{4.5}$$

is the restriction on $y = 0$ of the prolongation of the operator (4.3) to ψ_y. We have:

$$\widetilde{X}(\psi) = (\alpha - \beta)\psi, \quad \widetilde{X}(\psi_y - Ax^m) = (\alpha - 2\beta)\psi_y - m\alpha A x^m. \tag{4.6}$$

It follows from (4.6) that the first equation (4.4) is satisfied for any values of the parameters α, β. The second equation (4.4) is written

$$(\alpha - 2\beta - m\alpha)Ax^m = 0,$$

whence $\alpha - 2\beta - m\alpha = 0$. Letting $\alpha = 2$, we obtain $\beta = 1 - m$ and arrive at the following generator (4.3):

$$X = 2x \frac{\partial}{\partial x} + (1 - m)\, y \frac{\partial}{\partial y} + (m + 1)\, \psi \frac{\partial}{\partial \psi} \cdot \tag{4.7}$$

4.3 Derivation of Moffatt's solution

We find two functionally independent invariants of the subgroup $H \subset G$ with the generator (4.7) by computing two first integrals the characteristic system

$$\frac{dx}{2x} = \frac{dy}{(1 - m)\, y} = \frac{d\psi}{(m + 1)\, \psi}$$

of the equation $X(J) = 0$. Writing the characteristic system in the form

$$\frac{dy}{y} + \frac{m - 1}{2} \frac{dx}{x} = 0, \quad \frac{d\psi}{\psi} - \frac{m + 1}{2} \frac{dx}{x} = 0$$

we obtain two first integrals

$$y\, x^{(m-1)/2} = \text{const.}, \quad \psi\, x^{-(m+1)/2} = \text{const.}$$

The left-hand sides of these first integrals can be multiplied by any non-vanishing constants, k, l^{-1}, and provide two independent invariants denoted here by η and J :

$$\eta = ky\,x^{(m-1)/2}, \quad J = l^{-1}\,\psi\,x^{-(m+1)/2}. \tag{4.8}$$

Letting $J = f(\eta)$ we obtain the following form for the invariant solutions:

$$\psi = l\,x^{(m+1)/2}\,f(\eta). \tag{4.9}$$

We begin by computing the derivatives of η defined in (4.8):

$$\eta_x = \frac{k(m-1)}{2}\,y\,x^{(m-3)/2}, \quad \eta_y = k\,x^{(m-1)/2}.$$

The differentiations of (4.9) yield:

$$\psi_x = \frac{l}{2}\,x^{(m-1)/2}\,[(m+1)f + (m-1)\eta f'], \quad \psi_y = kl\,x^m\,f', \tag{4.10}$$

$$\psi_{xy} = klx^{m-1}\left[mf' + \frac{m-1}{2}\,\eta f''\right], \quad \psi_{yy} = k^2 l\,x^{(3m-1)/2}\,f'', \tag{4.11}$$

and

$$\psi_{yyy} = k^3 l\,x^{2m-1}\,f'''. \tag{4.12}$$

Now we substitute the expressions (4.10)-(4.12) in Eq. (3.1). We have:

$$\nu\psi_{yyy} + \psi_x\psi_{yy} - \psi_y\psi_{xy} = k^2 l^2 x^{2m-1}\left[\frac{k\nu}{l}\,f''' + \frac{m+1}{2}\,ff'' - mf'^2\right]. \tag{4.13}$$

Since k and l are arbitrary constants, we will take

$$l = k\nu \tag{4.14}$$

and reduce Eq. (3.1) to the ordinary differential equation (3.8):

$$f''' + \frac{m+1}{2}\,ff'' - mf'^2 = 0. \tag{4.15}$$

Let us turn now to the boundary conditions (3.2) and (3.3). Applying the first equation in (3.2) and the condition (3.3) to (4.9) we obtain Eqs. (3.9):

$$f(0) = 0, \quad f'(\infty) = 0. \tag{4.16}$$

Substituting the expression for ψ_y from (4.10) in the second equation (3.2) and invoking (4.14) we obtain

$$k^2\nu\,x^m\,f'(0) = A\,x^m.$$

This equation implies the equation

$$k^2\nu = |A| \tag{4.17}$$

and Eq. (3.10):

$$f'(0) = \begin{cases} +1 & \text{when} \quad m+1>0, \\ -1 & \text{when} \quad m+1<0. \end{cases} \tag{4.18}$$

Furthermore, Eqs. (4.17), (4.14) yield

$$k = \Big(\nu^{-1}|A|\Big)^{1/2}, \quad l = \Big(\nu|A|\Big)^{1/2}.$$

Finally, invoking (4.9) and (4.8), we conclude that the substitution

$$\psi = \Big(\nu|A|\, x^{m+1}\Big)^{1/2} f(\eta) \tag{4.19}$$

with

$$\eta = \Big(\nu^{-1}|A|\, x^{m-1}\Big)^{1/2} y \tag{4.20}$$

reduces the partial differential equation (3.1) with the boundary conditions (3.2), (3.3) to the ordinary differential equation (4.15) with the boundary conditions (4.16), (4.18).

Remark 4.1. The substitution of (3.6) with η defined by (3.7) and used in [1] and [3] reduces Eq. (3.1) not to the ordinary differential equation (3.8) but yields

$$xf''' + \frac{m+1}{2}\, ff'' - (m+1)f'^2 = 0.$$

This is due to the fact that η defined by (3.7) is not an invariant of the one-parameter group with the generator (4.7).

5 Exceptional values of the exponent m

5.1 Self-adjointness in the case $m = -1/2$

Let us find the adjoint equation to Eq. (4.15). In this case Eq. (2.4) is written

$$\mathcal{L} = z\Big[f''' + \frac{m+1}{2}\, ff'' - mf'^2\Big] \tag{5.1}$$

and Eq. (2.6) yields:

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta f} &= -z''' + \frac{m+1}{2}\, zf'' + 2mD(zf') + \frac{m+1}{2}\, D^2(zf) \\ &= -z''' + \frac{m+1}{2}\, zf'' + 2m(zf'' + z'f') + \frac{m+1}{2}\,(zf'' + 2z'f' + fz'') \\ &= -z''' + (3m+1)zf'' + \frac{m+1}{2}\, fz'' + (3m+1)z'f'. \end{aligned}$$

Changing the sign in Eq. (2.3), we get the following the adjoint equation to Eq. (4.15):

$$z''' - (3m+1)zf'' - \frac{m+1}{2}\,fz'' - (3m+1)f'z' = 0. \tag{5.2}$$

If we let $z = f$, Eq. (5.2) becomes

$$z''' - \Big(3m + 1 + \frac{m+1}{2}\Big)ff'' - (3m+1)f'^2 = 0.$$

Hence, Eq. (4.15) is self-adjoint if the following equations hold:

$$3m + 1 + \frac{m+1}{2} = -\frac{m+1}{2}\,, \quad 3m+1 = m.$$

These two equations are identical and yield $m = -1/2$. Thus, we have proved the following statement leading by another approach to the critical value (3.11) of the exponent m.

Theorem 5.1. Eq. (4.15) is self-adjoint if and only if

$$m = -\frac{1}{2}\,\cdot \tag{5.3}$$

Remark 5.1. The reckoning shows that the substitution $z = h(f)$ does not provide new cases when the adjoint equation (5.2) is equivalent with Eq. (4.15). Hence, there are no quasi-self-adjoint equations (4.15) except the self-adjoint case (5.3).

5.2 First integral

Investigation of the determining equations shows that Eq. (4.15) with an arbitrary exponent m has only the obvious translation and scaling symmetries provided by the following two generators:

$$X_1 = \frac{\partial}{\partial \eta}\,, \quad X_2 = \eta\frac{\partial}{\partial \eta} - f\frac{\partial}{\partial f}\,\cdot \tag{5.4}$$

Using these symmetries and conservation formula (2.8), one can calculate first integrals for the self-adjoint equation (4.15) with $m = -1/2$, i.e. for the equation

$$f''' + \frac{1}{4}\,ff'' + \frac{1}{2}\,f'^2 = 0. \tag{5.5}$$

The reckoning shows that application of the formula (2.8) to the translation generator X_1 yields $C = 0$. Hence, X_1 does not provide a nontrivial first integral. However the operator X_2 does. Namely, it provides the following first integral:

$$2f'^2 - f^2 f' - 4ff'' = C. \tag{5.6}$$

5.3 Additional symmetry in the case $m = 3$

One can demonstrate by inspecting the determining equations that Eq. (4.15) has an additional symmetry in the case

$$m = 3. \tag{5.7}$$

Namely, the equation

$$f''' + 2ff'' - 3f'^2 = 0 \tag{5.8}$$

has, along with (5.4), the following symmetry:

$$X_3 = \eta^2 \frac{\partial}{\partial \eta} + (6 - 2\eta f) \frac{\partial}{\partial f} \cdot \tag{5.9}$$

The operator X_3 generates the projective transformation of η followed by a linear transformation of f. Namely, upon solving the Lie equations, one obtains the following transformation with the group parameter a :

$$\bar{\eta} = \frac{\eta}{1 - a\eta}, \quad \bar{f} = (1 - a\eta)^2 f + 6a(1 - a\eta). \tag{5.10}$$

Thus, the Lie group analysis reveals two exceptional (critical) values of the exponent m, namely, $m = -1/2$ and $m = 3$.

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Archives of ALGA

Brief facts about the centre ALGA: Advances in Lie Group Analysis

ALGA at Blekinge Institute of Technology, Sweden, is an international research and educational centre aimed at producing new knowledge of Lie group analysis of differential equations and enhancing the understanding of the classical results and modern developments.

The main objectives of ALGA are:

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- To advance studies in modern group analysis, differential equations and non-linear mathematical modelling and to implement a database containing all the latest information in this field.

For more information, contact the director of ALGA, Professor N.H. Ibragimov.
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