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Time-dependent exact solutions of the nonlinear Kompaneets equation

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Abstract. Time-dependent exact solutions of the Kompaneets photon diffusion equation are obtained for several approximations of this equation. One of approximations describes the case when the induced scattering is dominant. In this case the Kompaneets equation has an additional symmetry which is used for constructing some exact solutions as group invariant solutions.

Keywords: Kompaneets equation, Photon diffusion, Symmetries, Exact solutions.

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1. Introduction

1.1. Notation

The Kompaneets equation [1], known also as the photon diffusion equation, is written

$$\frac{\partial n}{\partial t} = \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial n}{\partial x} + n + n^2 \right) \right]. \quad (1)$$

It describes an interaction of free electrons and electromagnetic radiation, specifically, an interaction of a low energy homogeneous photon gas with a rarefied electron gas via the Compton scattering. The model deals with establishing thermodynamic equilibrium in a rarefied plasma. Here n is the density of the photon gas (photon number density), t is time and x is defined by

$$x = \frac{h\nu}{kT}, \quad (2)$$

where h is Planck's constant and ν is the photon frequency so that $h\nu$ denotes the *photon energy*, whereas kT is the *electron temperature* with the standard notation k for Boltzmann's constant. The terms n and n^2 in Eq. (1) correspond to the *spontaneous scattering* (Compton effect) and the *induced scattering*, respectively (see, e.g. [2], [5]). The expanded form of Eq. (1) is

$$n_t = x^2 n_{xx} + (x^2 + 4x + 2x^2 n) n_x + 4x(n + n^2).$$

The problem of finding exact solutions of the Kompaneets equation in an analytical form is of great interest in physics. Though the Kompaneets model has been investigated numerically and analytically in numerous publications (see, e.g. [3], [2], [4], [5], [6], [7], [8], [9]), exact solutions of Eq. (1) were obtained only in the following particular cases.

1.2. Steady-state solutions

Letting in Eq. (1)

$$n = n(x) \quad (3)$$

one obtains the following Riccati equation for the steady-state solutions:

$$\frac{dn}{dx} + n + n^2 = \frac{C}{x^4}, \quad C = \text{const.} \quad (4)$$

Beginning with the original work of Kompaneets [1], the steady-state solutions are discussed in numerous papers.

Consider Eq. (4) with $C = 0$:

$$\frac{dn}{dx} + n + n^2 = 0.$$

Its integration is easy and yields

$$n = \frac{1}{Be^x - 1}, \quad B = \text{const.} \quad (5)$$

Invoking the expression (2) for x we see that (5) with $B = 1$,

$$n = \frac{1}{e^x - 1}, \quad (6)$$

is Plank's equilibrium distribution of photons, whereas the case $B \neq 1$ corresponds to the Bose-Einstein spectrum (see. e.g. [7], [2]). Furthermore, the linearized equation

$$\frac{dn}{dx} + n = 0$$

leads to Wien's distribution

$$n = Ae^{-x}.$$

The solution of Eq. (4) with $C \neq 0$ is described in [8] in terms of the so-called HeunD function.

1.3. Linear approximation

In the case of a low photon number density n the spontaneous scattering is dominant over the induced one. Therefore one can neglect the term n^2 and arrive at the linear approximation [1]

$$\frac{\partial n}{\partial t} = \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial n}{\partial x} + n \right) \right]. \quad (7)$$

The linear approximation leads to significant physical conclusions and is studied in various works. A convenient way for this purpose is to use the integral representation

$$n(x, t) = \int_0^\infty G(x, y, t) n(y, 0) dy$$

of the solution of Eq. (7) in terms of an initial distribution of photons $n(x, 0)$ and Green's function $G(x, y, t)$ defined by the equations

$$\frac{\partial G}{\partial t} = \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial G}{\partial x} + n \right) \right], \quad G(x, y, 0) = \delta(x - y).$$

See, e.g. [4], [6] and the references therein.

1.4. Diffusion approximation

Under certain physical assumptions, e.g. negligibly low frequencies, one can drop the terms n and n^2 . The corresponding equation

$$\frac{\partial n}{\partial t} = \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial n}{\partial x} \right) \right] \quad (8)$$

can be mapped into the usual heat equation by an appropriate change of variables and leads to interesting physical consequences (see [5] and the references therein).

1.5. Limiting case $n^2 \gg n$, $n^2 \gg n_x$

Keeping in the right-hand side of Eq. (1) the quadratic term n^2 only:

$$\frac{\partial n}{\partial t} = \frac{1}{x^2} \frac{\partial(x^4 n^2)}{\partial x}, \quad (9)$$

multiplying by x^2 and denoting $v = x^2 n$, one obtains the first-order equation

$$\frac{\partial v}{\partial t} - 2v \frac{\partial v}{\partial x} = 0. \quad (10)$$

Integrating it by the usual method of characteristics one obtains $x + 2tv = \varphi(v)$, and hence the solution of Eq. (9) in the implicit form:

$$x + 2tx^2 n = \varphi(x^2 n). \quad (11)$$

This approximation leads to an explanation of the formation of “shock waves” in the spectrum similar to the gasdynamic shock waves ([3], [2]).

2. Symmetries and exact solutions

2.1. Preliminaries

Let G be a one-parameter group of transformations

$$\bar{t} = f(t, x, a), \quad \bar{x} = g(t, x, a), \quad \bar{u} = h(t, x, u, a),$$

depending on a continuous parameter a and let

$$\bar{t} \approx t + a\tau(t, x), \quad \bar{x} \approx x + a\xi(t, x), \quad \bar{u} \approx u + a\eta(t, x, u)$$

be its infinitesimal transformations. The differential operator

$$X = \tau(t, x) \frac{\partial}{\partial t} + \xi(t, x) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}$$

is called the generator of the group G . If a differential equation

$$u_t = F(t, x, u, u_x, u_{xx}) \quad (12)$$

has the same form when it is rewritten in the new variables $\bar{t}, \bar{x}, \bar{u}$, i.e. it is written

$$\bar{u}_{\bar{t}} = F(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{x}\bar{x}}),$$

then G is called a *symmetry group* (or admitted group) for Eq. (12). The group G transforms any solution of Eq. (12) into a solution of the same equation. A solution that is mapped by G into itself is known as an invariant solution with respect to the group G . The invariant solutions for a group with a generator X are obtained as follows (see, e.g. [10]). One calculates two *invariants* $J_1 = \lambda(t, x)$ and $J_2 = \Phi(t, x, u)$ of the group G by solving the equation

$$X(J) \equiv \tau(t, x) \frac{\partial J}{\partial t} + \xi(t, x) \frac{\partial J}{\partial x} + \eta(t, x, u) \frac{\partial J}{\partial u} = 0,$$

or its characteristic system:

$$\frac{dt}{\tau(t, x)} = \frac{dx}{\xi(t, x)} = \frac{du}{\eta(t, x, u)}.$$

Then letting

$$\Phi = \phi(\lambda), \quad (13)$$

solving this equation for u and substituting the resulting expression for u in Eq. (12), one obtains an ordinary differential equation for the unknown function $\phi(\lambda)$.

2.2. Invariant solutions of Eq. (1)

Eq. (1) admits only the time translation group $\bar{t} = t + a$ with the generator

$$X_1 = \frac{\partial}{\partial t}. \quad (14)$$

The invariants of this group are $\lambda = x$ and $\Phi = n$. The corresponding Eq. (13) for the invariant solution yields the steady-state solution $n = n(x)$ considered in Section 1.2.

2.3. Invariant solutions of Eq. (9)

Eq. (9) has two additional symmetries, namely the scaling one-parameter groups

$$\bar{t} = t e^a, \quad \bar{x} = x, \quad \bar{n} = n e^{-a}$$

and

$$\bar{t} = t e^b, \quad \bar{x} = x e^{-b}, \quad \bar{n} = n$$

with the respective generators X_2 and X_3 defined by

$$X_2 = t \frac{\partial}{\partial t} - n \frac{\partial}{\partial n}, \quad X_3 = t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x}. \quad (15)$$

The invariant variables for X_2 are $\lambda = x$ and $\Phi = tn$. The corresponding invariant form (13) of the solution yields

$$n = \frac{\phi(x)}{t}. \quad (16)$$

Substituting (16) in Eq. (9),

$$n_t - 4xn^2 - 2x^2nn_x = 0, \quad (17)$$

we obtain the first-order linear equation

$$2x^2\phi' + 4x\phi + 1 = 0,$$

whence

$$\phi(x) = \frac{p}{x^2} - \frac{1}{2x}, \quad p = \text{const.}$$

Thus, the invariant solution is

$$n = \frac{1}{t} \left(\frac{p}{x^2} - \frac{1}{2x} \right). \quad (18)$$

One can construct the invariant solutions for any linear combination of the operators (14)-(15). But we can ignore the operator (14) since it yields only the trivial change of time $t' = t + \text{const.}$ Therefore one can consider the symmetry $aX_2 + bX_3$. Let us find the invariant solution, e.g. for the operator

$$X_2 - X_3 = x \frac{\partial}{\partial x} - n \frac{\partial}{\partial n},$$

In this case the invariant variables are $\lambda = t$, $\Phi = xn$, and (13) yields the following form of the invariant solution:

$$n = \frac{\phi(t)}{x}. \quad (19)$$

The substitution in Eq. (17) gives $\phi' = 2\phi^2$, whence

$$\phi(t) = \frac{1}{q - 2t}, \quad q = \text{const.}$$

Hence, the invariant solution is

$$n = \frac{1}{x(q - 2t)}. \quad (20)$$

Let us find the invariant solution using the operator X_3 from (15). The invariant variables are $\lambda = tx$, $\Phi = n$, and (13) yields the following form of the invariant solution:

$$n = \phi(\lambda), \quad \lambda = tx. \quad (21)$$

The substitution in Eq. (17) gives the nonlinear equation

$$(2\lambda\phi - 1)\phi' + 4\phi^2 = 0. \quad (22)$$

It can be integrated by noting that any ordinary differential equation of the form

$$(kxy + l)y' + my^2 = 0$$

with the independent variable x , the dependent variable y and arbitrary constants k, l, m is invariant under the scaling transformation $\bar{x} = xe^a$, $\bar{y} = ye^{-a}$. Hence, Eq. (22) is invariant under the scaling group with the generator

$$Y = \lambda \frac{\partial}{\partial \lambda} - \phi \frac{\partial}{\partial \phi}.$$

Therefore Eq. (22) can be integrated by introducing the *canonical variables* z, w defined by the equations $Y(z) = 1$, $Y(w) = 0$. Taking the simple solutions of these equations,

$$z = \ln \lambda, \quad w = \lambda\phi,$$

as new independent and dependent variables, we rewrite Eq. (22) in the form

$$\frac{dw}{dz} + \frac{(2w + 1)w}{2w - 1} = 0.$$

Its integration yields

$$\frac{(2w+1)^2}{w} = Ce^{-z}, \quad C > 0.$$

Substituting here the expressions for z, w we obtain the quadratic equation

$$(2\lambda\phi + 1)^2 = C\phi.$$

Solving it and invoking the definition (21) of λ and ϕ we obtain the invariant solution

$$n = \frac{C - 4tx \pm \sqrt{C(C - 8tx)}}{8(tx)^2}, \quad C - 8tx \geq 0. \quad (23)$$

Finally, let us take the general symmetry $aX_2 + bX_3$. We will assume $b \neq 0$ because the invariant solution for X_2 has been discussed above. Since any symmetry can be considered up to a constant factor, we can reduce the problem, e.g. to the case $b = -1$ and consider $aX_2 - X_3$. This yields the one-parameter set of invariant solutions

$$n = x^{-a}\phi(\lambda), \quad \lambda = tx^{1-a},$$

where $\phi(\lambda)$ is determined by the equations similar to (22),

$$[2(1-a)\lambda\phi - 1]\phi' + 4\phi^2 = 0,$$

and can be integrated as above.

2.4. Linear equation

The linear approximation (7) has two symmetries,

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = u \frac{\partial}{\partial u}.$$

Their linear combination $X_1 + \alpha X_2$ has the invariants $\lambda = x$, $\Phi = ne^{-\alpha t}$. Eq. (13) yields $n = \phi(x)e^{\alpha t}$. Substituting this in Eq. (7) we obtain:

$$x^2\phi'' + (4x + x^2)\phi' + (4x - \alpha)\phi = 0. \quad (24)$$

Eq. (24) is a double confluent Riemann equation and can be solved in terms of confluent hypergeometric series, or by quadratures for special values of the parameter (see [11]).

2.5. The case with dominating induced scattering

If $n^2 \gg n$ Eq. (1) becomes

$$\frac{\partial n}{\partial t} = \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial n}{\partial x} + n^2 \right) \right], \quad (25)$$

or in the expanded form

$$n_t = x^2 n_{xx} + (4x + 2x^2 n) n_x + 4x n^2. \quad (26)$$

Eq. (26) has two symmetries,

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = x \frac{\partial}{\partial x} - n \frac{\partial}{\partial n}. \quad (27)$$

Let us find the invariant solution provided by the operator X_2 from (27). The general form of this solution is given by (19), $n = \phi(t)/x$. Substituting it in Eq. (26) we obtain $\phi' = 2(\phi^2 - \phi)$, whence $\phi(t) = (1 - Ce^{2t})^{-1}$. Thus, the invariant solution is

$$n = \frac{1}{x(1 - Ce^{2t})}. \quad (28)$$

Let us consider the invariant solution provided by the linear combination $X_1 - \alpha X_2$ of the operators (27). The invariants are $\lambda = xe^{\alpha t}$, $\Phi = xn$ and Eq. (13) is written:

$$n = \frac{\phi(\lambda)}{x}, \quad \lambda = xe^{\alpha t}. \quad (29)$$

Substituting (29) in Eq. (26) we obtain the nonlinear ordinary differential equation

$$\lambda \phi'' + (2\phi + 2 - \alpha)\phi' + \frac{2}{\lambda}(\phi^2 - \phi) = 0. \quad (30)$$

3. Conclusion

Very few exact solutions of the nonlinear Kompaneets equation were known in the literature. In this work, new exact time-dependent solutions are obtained using Lie group analysis. These solutions are given in the closed form by Eqs. (18), (20), (23), (28) and in terms of ODE by Eqs. (29), (30). Exact solutions are convenient for data analysis and allow to determine the spectrum directly, without solving the nonlinear partial differential equation numerically. This is a topic for further investigation.

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