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Quasi self-adjoint nonlinear wave equations

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Abstract. In this paper we generalize the classification of self-adjoint second order *linear* partial differential equation to a family of *nonlinear* wave equations with two independent variables. We find a class of quasi self-adjoint nonlinear equations which includes the self-adjoint linear equations as a particular case. The property of a differential equation to be quasi self-adjoint is important, e.g. for constructing conservation laws associated with symmetries of the differential equation.

Keywords: Adjoint equation to nonlinear equations; Lagrangians; Wave equations; Symmetry and conservation laws.

MSC: 58J45, 58J70, 35L65, 35L70

1. Introduction

Let us begin with the linear operator

$$L[u] \equiv a^{ij}(x)D_iD_j(u) + b^i(x)D_i(u) + c(x)u, \quad (1)$$

where $x = (x^1, \dots, x^n)$ are independent variables, u is a scalar dependent variable and D_i are the total differentiations,

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \dots \quad (2)$$

with

$$u_i = \frac{\partial u}{\partial x^i}, \quad u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}, \dots \quad (3)$$

We recall [1] that the adjoint operator $L^*[v]$ to the operator (1) is defined by the equation

$$vL[u] - uL^*[v] = D_i(p^i), \quad (4)$$

where p^i are any functions of $x, u, v, u_{(1)}, v_{(1)}, \dots$. Here v is a new dependent variable, $u_{(1)}, u_{(s)}$ and $v_{(1)}, v_{(s)}$ are the sets of the partial derivatives of the first and s -th orders, e.g. $u_{(1)} = \{u_i\}$.

The definition (4) yields the following formula for the adjoint operator:

$$L^*[v] = D_iD_j(a^{ij}(x)v) - D_i(b^i(x)v) + c(x)v. \quad (5)$$

The operator (1) is said to be self-adjoint if $L^*[v]$ coincides with $L[u]$ after the substitution $v = u$, i.e. if $L^*[u] = L[u]$. It is known that the operator (1) is self-adjoint if and only if

$$b^i(x) = D_j(a^{ij}), \quad i = 1, \dots, n. \quad (6)$$

To the operator (1) we can associate the equation

$$L[u] \equiv a^{ij}(x)D_iD_j(u) + b^i(x)D_i(u) + c(x)u = 0. \quad (7)$$

Then we say that the equation (7) is self-adjoint if the operator (1) is self-adjoint.

In particular, if Eq. (7) has two independent variables, t, x , and is written in the form

$$u_{tt} - u_{xx} + b^1(t, x)u_t + b^2(t, x)u_x + c(t, x)u = 0, \quad (8)$$

the equations (6) show that Eq.(8) is self-adjoint if and only if

$$b^1 = b^2 = 0. \quad (9)$$

Our aim in the present paper is to extend the condition (9) to the following nonlinear wave equations (considered in [5, 6, 7, 8] from the point of view of equivalence transformations):

$$u_{tt} - u_{xx} = f(t, x, u, u_t, u_x). \quad (10)$$

Hyperbolic type second-order nonlinear PDEs in two independent variables are used to describe different types of wave propagation. They are also used in differential

geometry, in various fields of hydrodynamics and gasdynamics, chemical technology, superconductivity, crystal dislocation to mention only a few application areas. Moreover the family of hyperbolic equations considered comprises the well known equations as Liouville, sine/sinh-Gordon, Goursat, d'Alembert and Tzitzeica equations.

In the case of linear differential operators (1), the functional-analytic concept of self-adjoint operators in the Hilbert space is the same as self-adjointness of differential equations defined by Eq. (4). For nonlinear differential equations the situation is completely different, namely, the functional-analytic concept of adjoint equations does not extend to nonlinear equations. On the other hand, in [1] it was shown that the adjoint operator (5) to the linear differential operator (1) can be obtained by an alternative procedure and that the latter procedure can be immediately extended to any nonlinear differential equation. It was also shown in [1] that the adjoint equation to nonlinear equations does not satisfy Eq. (4). Using these recent concepts of adjoint equations to nonlinear equations, self-adjoint nonlinear equations [1] and quasi self-adjoint nonlinear equations [4], we show that the necessary condition for Eq. (10) to be quasi self-adjoint (in particular self-adjoint) is that

$$f = (u_t^2 - u_x^2)A(t, x, u) + u_t B_1(t, x, u) + u_x B_2(t, x, u) + B_3(t, x, u). \quad (11)$$

We also find the sufficient conditions, in terms of the coefficients A , B_1 , B_2 and B_3 , for quasi self-adjointness of Eq. (10).

Finally as an application, we use a new theorem, proved in [2, 3], in order to get conservation laws for a wave equation. This theorem allows to find for any differential equation with known Lie, Lie-Bäcklund or nonlocal symmetries the associated conservation laws independently on existence of classical Lagrangians.

2. Adjoint equation to (10)

According to [1] (see also [3]) the adjoint equation to (10) has the form

$$\frac{\delta \mathcal{L}}{\delta u} = 0 \quad (12)$$

where $\mathcal{L}$ is the *formal Lagrangian*,

$$\mathcal{L} = v[u_{tt} - u_{xx} - f(t, x, u, u_t, u_x)], \quad (13)$$

and

$$\frac{\delta \mathcal{L}}{\delta u} = \frac{\partial \mathcal{L}}{\partial u} - D_t \left(\frac{\partial \mathcal{L}}{\partial u_t} \right) - D_x \left(\frac{\partial \mathcal{L}}{\partial u_x} \right) + D_t^2 \left(\frac{\partial \mathcal{L}}{\partial u_{tt}} \right) + D_x^2 \left(\frac{\partial \mathcal{L}}{\partial u_{xx}} \right). \quad (14)$$

Applying Eqs. (12), (13), (14) we obtain the following adjoint equation

$$\begin{aligned} &v_{tt} - v_{xx} + v u_{tt} f_{u_t u_t} + v u_{xx} f_{u_x u_x} + 2v u_{tx} f_{u_t u_x} + v u_t f_{u u_t} \\ &+ v_t f_{u_t} + v u_x f_{u u_x} + v_x f_{u_x} + v f_{t u_t} + v f_{x u_x} - v f_u = 0. \end{aligned} \quad (15)$$

3. Conditions for quasi self-adjointness

Recall that a nonlinear differential equation is said to be quasi self-adjoint if its adjoint equation becomes equivalent to the original equation after a substitution [4]

$$v = \phi(u), \quad \phi(u) \neq 0. \quad (16)$$

Using the differential consequences of (16),

$$v_t = \phi' u_t, \quad v_x = \phi' u_x, \quad v_{tt} = \phi'' u_t^2 + \phi' u_{tt}, \quad v_{xx} = \phi'' u_x^2 + \phi' u_{xx}, \quad (17)$$

the condition for the quasi self-adjointness is written as follows:

$$\begin{aligned} &(\phi' + \phi f_{u_t u_t}) u_{tt} - (\phi' - \phi f_{u_x u_x}) u_{xx} + 2\phi u_{tx} f_{u_t u_x} + \phi''(u_t^2 - u_x^2) + \phi'(u_t f_{u_t} + u_x f_{u_x}) \\ &+ \phi(u_t f_{u u_t} + u_x f_{u u_x} + f_{t u_t} + f_{x u_x} - f_u) = \lambda[u_{tt} - u_{xx} - f], \end{aligned} \quad (18)$$

where λ is an undetermined coefficient. Since f does not depend on the second derivatives, u_{tt} , u_{xx} and u_{tx} , Eq. (18) splits in to the following system

$$\phi' + \phi f_{u_t u_t} = \lambda, \quad \phi' - \phi f_{u_x u_x} = \lambda, \quad (19)$$

$$f_{u_t u_x} = 0, \quad (20)$$

$$\phi''(u_t^2 - u_x^2) + \phi'(u_t f_{u_t} + u_x f_{u_x}) + \phi(u_t f_{u u_t} + u_x f_{u u_x} + f_{t u_t} + f_{x u_x} - f_u) = -\lambda f. \quad (21)$$

From (19) we obtain

$$f_{u_t u_t} = -f_{u_x u_x}. \quad (22)$$

Taking into account Eq. (20) we are able to get the following form for f :

$$f = (u_t^2 - u_x^2)A(t, x, u) + u_t B_1(t, x, u) + u_x B_2(t, x, u) + B_3(t, x, u). \quad (23)$$

Then we have demonstrated the following statement.

Theorem 1 *The necessary condition for Eq. (10) to be quasi self-adjoint is that the function f has the form (23).*

Moreover, by substituting (23) in Eq. (21), eliminating λ by using the first equation (19), and invoking that ϕ depends only on u and A , B_1 , B_2 , B_3 do not depend on u_t , u_x , we obtain

$$(\phi' + \phi A)_u + 2A(\phi' + \phi A) = 0, \quad (24)$$

$$(\phi' + \phi A)B_1 + \phi A_t = 0, \quad (\phi' + \phi A)B_2 - \phi A_x = 0, \quad (25)$$

$$(\phi' + \phi A)B_3 + (B_{1t} + B_{2x} + AB_3 - B_{3u})\phi = 0. \quad (26)$$

We formulate this result as follows.

Theorem 2 *The necessary and sufficient condition for Eq. (10) to be quasi self-adjoint is given by the four equations (24)-(26) for the coefficients A , B_1 , B_2 , B_3 of the function (23) and for the substitution (16).*

4. Classes of quasi self-adjoint equations

In this section we give the solutions of conditions (24)-(26) in two cases:

- (i) $A = A(u)$;
- (ii) $\phi = \text{const.}$

4.1. The case $A = A(u)$

Here we single out all quasi self-adjoint equations of the form

$$u_{tt} - u_{xx} = (u_t^2 - u_x^2)A(u) + u_t B_1(t, x, u) + u_x B_2(t, x, u) + B_3(t, x, u). \quad (27)$$

Eqs. (25) are written

$$(\phi' + \phi A)B_1 = 0, \quad (\phi' + \phi A)B_2 = 0$$

and lead to the following two subcases:

$$\phi' + \phi A = 0 \quad (28)$$

and

$$B_1 = B_2 = 0. \quad (29)$$

- (i) In the subcase (28) Eqs. (24)-(25) are satisfied, whereas Eq. (26) takes the simple form $B_{1t} + B_{2x} + AB_3 - B_{3u} = 0$ and yields

$$B_3 = e^{\int A du} \left[F(t, x) + \int e^{-\int A du} (B_{1t} + B_{2x}) du \right], \quad (30)$$

where $F(t, x)$ is an arbitrary function. Furthermore, we obtain from Eq. (28) that the substitution (16) is given by the function

$$\phi = \phi_0 e^{-\int A du}, \quad \phi_0 = \text{const.} \quad (31)$$

- (ii) Consider now the subcase (29).

- (a) At beginning we assume that $B_3 \neq 0$. Then Eq. (26) yields

$$\frac{\phi'}{\phi} - \frac{B_{3u}}{B_3} + 2A = 0, \quad (32)$$

whence

$$B_3 = G(u) e^{-F(t, x)} \quad (33)$$

where $F(t, x)$ is an arbitrary function, whereas the function $G(u)$ is linked to the functions A and ϕ by the relation

$$\phi(u) = G(u) e^{-2 \int A du}. \quad (34)$$

Substituting this form of ϕ in Eq. (24) we obtain

$$G'' - A'G - AG' = 0. \quad (35)$$

Then

$$G(u) = \left(c_1 + c_2 \int e^{-\int A du} du \right) e^{\int A du}, \quad (36)$$

where c_1 and c_2 are arbitrary constants. Thus, the class of equations

$$u_{tt} - u_{xx} = A(u)(u_t^2 - u_x^2) + \left(c_1 + c_2 \int e^{-\int A du} du\right) e^{\int A du} e^{-F(t,x)} \quad (37)$$

is quasi self-adjoint.

- (b) Consider now the subcase (29) by assuming that $B_3 = 0$. Eqs. (25)-(26) are satisfied for arbitrary $A = A(u)$, whereas the substitution (16) is defined by Eq. (24). Namely,

$$\phi = \left(c_1 + c_2 \int e^{-\int A du} du\right) e^{-\int A du}. \quad (38)$$

4.2. Degenerate case $\phi = \text{const.}$

In this case we consider $\phi = \phi_0$ and Eqs. (24)-(26) become

$$A_u + 2A^2 = 0, \quad (39)$$

$$AB_1 + A_t = 0, \quad AB_2 - A_x = 0, \quad (40)$$

$$2AB_3 + B_{1t} + B_{2x} - B_{3u} = 0. \quad (41)$$

Solving (39) we get

$$A = \frac{1}{2u + a(t, x)}, \quad (42)$$

where $a(t, x)$ is an arbitrary function. Substituting (42) in (40) we obtain

$$B_1 = \frac{a_t}{2u + a(t, x)}, \quad B_2 = -\frac{a_x}{2u + a(t, x)} \quad (43)$$

and substituting (42) and (43) in (41) we get

$$B_{3u} = \frac{2}{2u + a} B_3 + \frac{a_{tt} - a_{xx}}{2u + a} - \frac{a_t^2 - a_x^2}{(2u + a)^2}. \quad (44)$$

The solution of (44) is

$$B_3 = (2u + a) \left(b(t, x) - \frac{a_{tt} - a_{xx}}{2(2u + a)} + \frac{a_t^2 - a_x^2}{4(2u + a)^2} \right), \quad (45)$$

where $b(t, x)$ is an arbitrary function.

5. Conservation laws

We use the following statement on conservation laws proved in [2, 3].

Theorem 3 *Any symmetry (Lie point, Lie-Bäcklund, nonlocal symmetry)*

$$X = \xi^i(x, u, u_{(1)}, \dots) \frac{\partial}{\partial x^i} + \eta(x, u, u_{(1)}, \dots) \frac{\partial}{\partial u} \quad (46)$$

of an equation

$$F(x, u, u_{(1)}, \dots, u_{(s)}) = 0, \quad (47)$$

with n independent variables $x = (x^1, \dots, x^n)$ and the dependent variable u is inherited by the adjoint equation. Specifically the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \eta_* \frac{\partial}{\partial v}, \quad (48)$$

with an appropriately chosen coefficient η_* is admitted by the system consisting of Eq. (47) and its adjoint equation

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \dots, u_{(s)}, v_{(s)}) \equiv \frac{\delta(vF)}{\delta u} = 0. \quad (49)$$

Furthermore, the combined system (47), (49) has the conservation law $D_i(C^i) = 0$, where

$$\begin{aligned} C^i = & \xi^i \mathcal{L} + W \left[\frac{\partial \mathcal{L}}{\partial u_i} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}} \right) - \dots \right] \\ & + D_j(W) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}} \right) + \dots \right] + D_j D_k(W) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}} - \dots \right] + \dots \end{aligned} \quad (50)$$

with

$$W = \eta - \xi^i u_i. \quad (51)$$

If Eq. (47) is quasi self-adjoint, we can exclude from the conserved vector (50) the “non-physical” variable v by using the substitution (16). Then (50) will provide a conserved vector for Eq. (47).

As an application we consider the equation (see, e.g. [6])

$$u_{tt} - u_{xx} - (u_t^2 - u_x^2) + uu_t - uu_x = 0. \quad (52)$$

It has the form (27) with $A = 1$, $B_1 = -u$, $B_2 = u$ and $B_3 = 0$. Eq. (52) is quasi self-adjoint and the substitution (16) is given by

$$v = \phi(u) \equiv e^{-u}. \quad (53)$$

The formal Lagrangian for Eq. (52) is

$$\mathcal{L} = v \left[u_{tt} - u_{xx} - (u_t^2 - u_x^2) + uu_t - uu_x \right]. \quad (54)$$

Eqs. (50) give the conservation law

$$D_t(C^1) + D_x(C^2) = 0 \quad (55)$$

with

$$C^i = \xi^i \mathcal{L} + W \left[\frac{\partial \mathcal{L}}{\partial u_i} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}} \right) \right] + D_j(W) \frac{\partial \mathcal{L}}{\partial u_{ij}}. \quad (56)$$

Considering the following symmetry generator of Eq. (52):

$$X = e^{\frac{t+x}{2}} \frac{\partial}{\partial u} \quad (57)$$

we obtain

$$C^1 = \left(u - u_t + \frac{1}{2} \right) e^{\frac{t+x}{2}-u}, \quad C^2 = \left(-u + u_x - \frac{1}{2} \right) e^{\frac{t+x}{2}-u}. \quad (58)$$

The vector (58) satisfies the conservation equation for Eq. (52) because

$$\begin{aligned} D_t \left(\left(u - u_t + \frac{1}{2} \right) e^{\frac{t+x}{2}-u} \right) + D_x \left(\left(-u + u_x - \frac{1}{2} \right) e^{\frac{t+x}{2}-u} \right) = \\ -e^{\frac{t+x}{2}-u} \left[u_{tt} - u_{xx} - (u_t^2 - u_x^2) + uu_t - uu_x \right]. \end{aligned} \quad (59)$$

6. Conclusions

We have demonstrated that Eq. (10) is quasi self-adjoint if and only if f has the form

$$f = A(t, x, u)(u_t^2 - u_x^2) + B_1(t, x, u)u_t + B_2(t, x, u)u_x + B_3(t, x, u)$$

such that A , B_1 , B_2 , B_3 and the substitution (16) satisfy Eqs. (24)-(26). Using these conditions we obtained the following quasi self-adjoint subclasses of Eq. (10):

(i)

$$\begin{aligned} A &= A(u), \\ B_3 &= e^{\int A du} \left[F(t, x) + \int e^{-\int A du} (B_{1t} + B_{2x}) du \right], \\ \phi &= \phi_0 e^{-\int A du}, \quad \phi_0 = \text{const.} \end{aligned}$$

where $A(u)$, $B_1(t, x, u)$, $B_2(t, x, u)$ and $F(t, x)$ are arbitrary functions.

(ii)

$$\begin{aligned} A &= A(u), \quad B_1 = B_2 = 0, \\ B_3 &= \left(c_1 + c_2 \int e^{-\int A du} du \right) e^{\int A du} e^{-F(t, x)}, \\ \phi &= \left(c_1 + c_2 \int e^{-\int A du} du \right) e^{-\int A du}, \end{aligned}$$

where $A(u)$ and $F(t, x)$ are arbitrary functions, c_1 , c_2 arbitrary constants not both zero.

(iii)

$$\begin{aligned} A &= A(u), \quad B_1 = B_2 = B_3 = 0, \\ \phi &= \left(c_1 + c_2 \int e^{-\int A du} du \right) e^{-\int A du} \end{aligned}$$

where $A(u)$ is an arbitrary function.

(iv)

$$\begin{aligned} A &= \frac{1}{2u + a(t, x)}, \quad B_1 = \frac{a_t}{2u + a(t, x)}, \quad B_2 = -\frac{a_x}{2u + a(t, x)}, \\ B_3 &= (2u + a) \left(b(t, x) - \frac{a_{tt} - a_{xx}}{2(2u + a)} + \frac{a_t^2 - a_x^2}{4(2u + a)^2} \right), \\ \phi &= \phi_0, \end{aligned}$$

where $a(t, x)$, $b(t, x)$ are arbitrary functions and $\phi_0 = \text{const.} \neq 0$.

Remark. If we consider Eq. (10) written in arbitrary coordinates x^1, x^2 in the form

$$\sum_{i,j=1}^2 a^{ij}(x) u_{ij} + h(x, u, u_{(1)}) = 0, \quad (60)$$

where $x = (x^1, x^2)$, then we obtain for h the formula similar to (23) and arrive at the following candidate for the quasi self-adjoint equations:

$$\sum_{i,j=1}^2 a^{ij}(x) [u_{ij} + A(x, u) u_i u_j] + b^i(x, u) u_i + c(x, u) = 0. \quad (61)$$

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