A Structural Modeling Flow for Switched Electrical Networks

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Abstract:

A structural flow for modeling of switched electrical circuits by means of an example a switched DC/DC converter will be presented. The modeling flow has a minimal amount of hand calculations and is suitable for script implementation in suitable mathematical computer software such as Matlab or similar.

Switched Hamiltonian Differential Algebraic Equation models and Differential Equation models are obtained which are suitable for analysis in the time domain. Linear models, which can be transformed to the frequency domain for analysis, are obtained from the Hamiltonian models. The duty cycle will be included in the input vector enabling the full system analysis considering the system transfer functions audio susceptibility, output impedance, and control, i.e., the duty cycle to output voltage.

1 Introduction

For full understanding and control over the modeling process of electrical circuits, analytical calculations by hand are in many cases the best alternative. Is the schematic simple, just to use Kirchhoff's laws, the constitute equations for the components, and some algebra are enough. The result is a model on a Differential Equation form, DE-form. The more complex the schematics get the more the algebra gets messy and error prone. Non-linear switched networks add to the complexity and have to be handled with care. As an alternative for analytical calculations by hand, there are different software for modeling and simulation available, such as Spice or Simulink. Using software for modeling and possibly numerical deriving of transfer functions does not offer full insight and control of the modeling process.

We suggest using a structural modeling technique, Hamiltonian modeling based on graph theory and passivity techniques [1], [2], [3]. Modeling of switched networks as complementarity systems have been addressed in [4]. These techniques have been used in [5] for the design and proof of stability of a nonlinear Hamiltonian observer applied to a switched network. The chosen graph theoretical method of network analysis does not use any transformation of certain network configurations to fit the analysis that gives advantages compared to the commonly used nod analysis. We believe that Hamiltonian modeling gives better understanding of the circuit and the linear algebra is suitable for script implementations in your choice of mathematical computer program. The resulting model is a mix of differential and algebraic equations, i.e., on a Differential Algebraic Equation DAE-form. Transformations to a Hamiltonian DE-form and traditional state space ABCD-form are also shown. The switched models are only suitable for analysis in the time domain. For frequency domain analysis approximate models has to be used. The models are made time continues by state space averaging which was already introduced in [6], and the duty cycle can be included in the input vector as in [7]. The time continuous model can be linearized in the operation point. This linearized model can be transformed into the frequency domain for analysis. This makes it possible to investigate simultaneously the systems different transfer functions, the audio susceptibility, i.e., input voltage to output voltage, the output impedance, and the control, i.e., the duty cycle, to output voltage. The singular values can also be used for robustness analysis.

2 Application Example

We will use a standard buck converter, which is a typical example of a switched electrical system, shown in Figure 1.

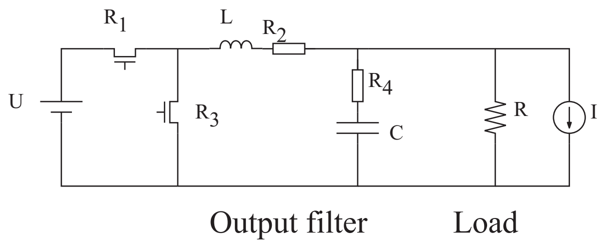

Figure 1. Buck converter with resistive load.

The resistors, R_1, R_3 models the R_{DSon} of the switch-FET and Sync-FET, respectively. When the transistors are off, they are modeled as ideal switch off. Finite non-linear resistances were treated in [5]. However, this is not suitable for modeling of Mosfets, since the ratio in resistance between the off and on state easily exceeds 10^9. Hence, numerically ill-conditioned models will be the result.

3 Structural Modeling of Switched Networks

We will use graph theory in order to obtain structural modeling with relatively small amount of hand made calculations. The result is a Hamiltonian model, which is based on the energy flow in the circuit.

3.1 Graph theoretical concepts

The first step is to draw a spanning-tree, which spans all nodes in the schematic. Definitions of Graph theoretical concepts can be found in [3]. In order to obtain a simple model structure the voltage sources U and as many capacitors C as possible should be tree branches. All current sources I and as many as possible inductors L should be links. Resistors could be placed in the tree branches X or in the links Y. To express this organization of the components in the tree branches and links an index is formulated, that should be maximized.

$$Tree_{index} = N_{tree,C} + N_{link,L} + 2N_{tree,U} + 2N_{link,I}$$

Where, $N_{tree,C}$ is the number of capacitors in tree branches, $N_{Link,L}$ is the number of inductors in links, $N_{tree,U}$ is the number of voltage sources in tree branches, $N_{link,I}$ is the number of current sources in links. Sometimes we have to use an inductor as a tree branch, l, in order to obtain a full spanning tree for all nodes, and use a capacitor as a link, c, in order to exclude cycles when choosing the spanning tree.

A direction is assigned to each branch that is the same as the direction of the current, shown by arrows in the graph. The direction of the voltage for each component is chosen so the power is positive for passive components. This means that when a voltage or a current source delivers power, the power is negative.

3.2 Model of a general electrical network

A general electrical network can be described by the skew-symmetric equation system

$$\begin{pmatrix} i_{branch} \\ u_{link} \end{pmatrix} = \begin{pmatrix} 0 & -Q_{link} \\ P_{tree} & 0 \end{pmatrix} \cdot \begin{pmatrix} u_{tree} \\ i_{link} \end{pmatrix}$$

where $P_{tree} = -Q_{link}^T$ and

$$i_{tree}^T = \begin{pmatrix} i_U^T & i_C^T & i_l^T & i_X^T \end{pmatrix}$$
$$i_{link}^T = \begin{pmatrix} i_Y^T & i_c^T & i_L^T & i_I^T \end{pmatrix}$$
$$u_{link}^T = \begin{pmatrix} u_Y^T & u_c^T & u_L^T & u_I^T \end{pmatrix}$$
$$u_{tree}^T = \begin{pmatrix} u_U^T & u_C^T & u_l^T & u_X^T \end{pmatrix}$$

$$Q_{link} = \begin{pmatrix} Q_{UY} & Q_{Uc} & Q_{UL} & Q_{UI} \\ Q_{CY} & Q_{Cc} & Q_{CL} & Q_{CI} \\ Q_{lY} & Q_{lc} & Q_{lL} & Q_{lI} \\ Q_{XY} & Q_{Xc} & Q_{XL} & Q_{XI} \end{pmatrix}$$

This is nothing more than a structural method of using Kirchhoff’s current law in all nodes,

$$Q_{cutset} \cdot i = \begin{pmatrix} I_r & Q_{link} \end{pmatrix} \cdot \begin{pmatrix} i_{tree} \\ i_{link} \end{pmatrix} = i_{tree} + Q_{link} \cdot i_{link} = 0 \,,$$

and Kirchhoff’s voltage law in all cycles

$$P_{cycle} \cdot u = \begin{pmatrix} P_{tree} & I_{n-r} \end{pmatrix} \cdot \begin{pmatrix} u_{tree} \\ u_{link} \end{pmatrix} = P_{tree} \cdot u_{tree} + u_{link} = 0$$

We do also need the constitutive equations for the components

$$\frac{dq}{dt} = i_C = C \cdot \frac{du_C}{dt}, \quad \frac{d\varphi}{dt} = u_L = L \cdot \frac{di_L}{dt}, u_{X/Y} = R i_{X/Y}$$

where q is the charge in the capacitor, and φ is the magnetic flux in the inductor.

The use of the introduced maximized index will imply that some sub-matrices are zero, i.e. $Q_{Xc} = Q_{lc} = Q_{lY} = 0$. Since physical networks do not contain cycles with only voltage sources and capacitors or cut-sets with only current sources and inductances, $Q_{Uc} = Q_{lI} = 0$. This yields the simplified Q_{link} matrix.

$$Q_{link} = \begin{pmatrix} Q_{UY} & 0 & Q_{UL} & Q_{UI} \\ Q_{CY} & Q_{Cc} & Q_{CL} & Q_{CI} \\ 0 & 0 & Q_{lL} & 0 \\ Q_{XY} & 0 & Q_{XL} & Q_{XI} \end{pmatrix}$$

3.3 Construction of the Spanning Tree and the Cut-set and Cycle matrices

In the On-phase of our example circuit when the switch FET is conducting and the sync FET is turned-off, the simplified schematic is shown in Figure 2. The tree branches are marked with solid arrows and the links with broken arrows. The arrows direction is directed in the positive direction of the current.

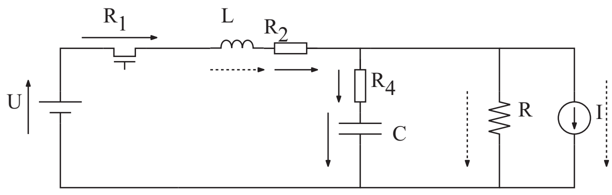

Figure 2. On-phase schematic and spanning tree.

We redraw the graph with the chosen tree as shown in Figure 3.

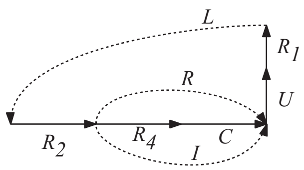

Figure 3. On-phase graph with chosen spanning tree.

The next step is to assign the cut-set matrix Q and cycle matrix P. The cut-set matrices is derived by cutting the spanning tree into two by cutting one tree branch at the time, and investigate which links are connected between the two different trees, as illustrated in Figure 4. The direction of the branch determines what becomes Tree 1 and 2. Tree 2 is connected to the positive direction of the branch that was cut. The links directed towards Tree 2, will be assigned the value $+1$, and links directed towards Tree 1 will be assigned the value -1. The links inside each tree will be assigned the value 0.

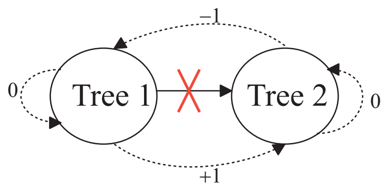

Figure 4. Cut of a branch.

In Figure 5 we cut off branch R_4. In this case will the link L directed from Tree 2 to Tree 1 obtain the value -1. The link current source I and the link resistance R will obtain the value $+1$.

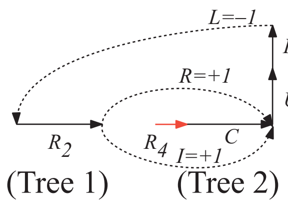

Figure 5. On-phase with R_4 branch cut.

This procedure is repeated for each tree branch and the values are assigned to the cut-set matrix. The following order of tree branches and links is recommended, U, C, l, X, Y, L, c, I . In this example, we do not have any inductors in tree branches or capacitors in links, c, l . This yields the following structure in the cut-set matrix. In Table 1, below the tree branches, U, C, X an identity matrix is obtained and below the links Y, L, I is the Q_{on} matrix.

Table 1 Q matrix in table format in On-phase

$\begin{pmatrix} I & Q_{on} \\ P_{on} & I \end{pmatrix} =$

Tree Branches(U, C, X)					Links(Y, L, I)			
U	C	R_1	R_2	R_4	R	L	I	
1	0	0	0	0	0	−1	0	U
0	1	0	0	0	1	−1	1	C
0	0	1	0	0	0	−1	0	R_1
0	0	0	1	0	0	−1	0	R_2
0	0	0	0	1	1	−1	1	R_4
								R
		P_{on}				I		L
								I

(Rows U, C, R_1, R_2, R_4: Tree Branches; rows R, L, I: Links)

The next step is to derive the cycle matrix P_{on} . We make a cycle for each link with the order Y, L, I Tree branches in the cycle and their sign are defined as in Figure 6. The direction of the link define the direction of the cycle, tree branches with the same direction yields the value $+1$, tree branches with opposite directions yields the value -1, and tree branches outside the cycle yields value 0 .

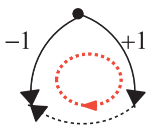

Figure 6. Definition of signs in a cycle.

As an example, the cycle with the link L is shown in Figure 7. The following values $+1,-1,+1,-1,-1$ are obtained for the tree branches *U*, *C*, R_1, R_2, R_4, respectively.

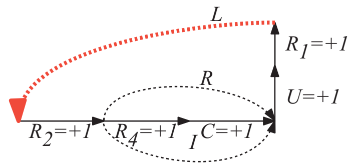

Figure 7. Cycle with link L .

The total P matrix is shown in the lower part of the large Table 2.

Table 2 *P*, *Q* matrices in table format in On-phase

U	C	R_1	R_2	R_4	R	L	I	
1	0	0	0	0	0	−1	0	U
0	1	0	0	0	1	−1	1	C
0	0	1	0	0	0	−1	0	R_1
0	0	0	1	0	0	−1	0	R_2
0	0	0	0	1	1	−1	1	R_4
0	−1	0	0	−1	1	0	0	R
1	1	1	1	1	0	1	0	L
0	−1	0	0	−1	0	0	1	I

$= \begin{pmatrix} I & Q_{on} \\ P_{on} & I \end{pmatrix}$

Since, $P_{on} = -Q_{on}^T$ it is easy to check if we have performed the work properly.

$$\begin{pmatrix} 0 & -1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & -1 \end{pmatrix} \equiv -\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}^T$$

3.3.1 Off-Phase

The same work is performed for the off-phase, the schematic with spanning tree are shown in Figure 8. Notice, that *U*, R_1 is not used in the tree since the current is zero in these components during this phase.

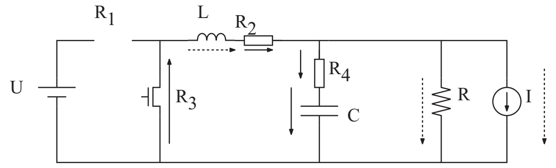

Figure 8. Schematic in the Off-phase

The new spanning tree is redrawn in Figure 9.

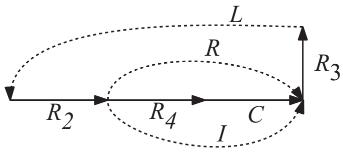

Figure 9. Spanning tree in the off-phase.

The following Q_{off}, and P_{off} matrices are obtained. Note, that the R_1 in the on-matrix and the R_3 in the off-matrix have the same position.

Table 3 *P*, *Q* Matrices in table format in Off-phase

C	R_3	R_2	R_4	R	L	I	
1	0	0	0	1	−1	1	C
0	1	0	0	0	−1	0	R_3
0	0	1	0	0	−1	0	R_2
0	0	0	1	1	−1	1	R_4
−1	0	0	−1	1	0	0	R
1	1	1	1	0	1	0	L
−1	0	0	−1	0	0	1	I

$= \begin{pmatrix} I & Q_{off} \\ P_{off} & I \end{pmatrix}$

4 Hamiltonian Modeling

Now when the schematics structure is captured in the *Q*-, and *P*-matrices it is time to use them.

4.1 Differential Algebraic Equation Model

As an electro engineer, you probably are used to use the inductor currents and capacitor voltages as states. However, in Hamiltonian modeling we go back to the physics, and use the physical quantities, q the charge in the capacitors and the magnetic flux φ in the inductors. This is however, not a big issue since the inductor flux and current and the capacitor charge and voltage are related as

$$q = Cu_C, \quad \varphi = Li_L .$$

Hence, the difference is only scaling of the states. The inputs to our system are the voltage and current sources and an output w is defined. The Hamiltonian differential algebraic model is hence described by

$$\tilde{I}\begin{pmatrix}\dot{x}\\ z\end{pmatrix} = JM^{-1}\begin{pmatrix}x\\ z\end{pmatrix} + Bu, \quad w = B^T M^{-1}\begin{pmatrix}x\\ z\end{pmatrix} + J_D u \ ,$$

where

$$x = \begin{pmatrix}q\\ \varphi\end{pmatrix}, z = \begin{pmatrix}i_X\\ Y\, i_Y\end{pmatrix}, u = \begin{pmatrix}u_U\\ i_I\end{pmatrix}, w = \begin{pmatrix}-i_U\\ -u_I\end{pmatrix},$$

and

$$J = \begin{pmatrix}J_x & -N\\ N^T & J_z\end{pmatrix}, J_x = \begin{pmatrix}0 & -Q_{CL}\\ Q_{CL}^T & 0\end{pmatrix}, J_z = \begin{pmatrix}0 & -Q_{XY}\\ Q_{XY}^T & 0\end{pmatrix}, N = \begin{pmatrix}0 & Q_{CY}\\ -Q_{XL}^T & 0\end{pmatrix}$$

$$B = \begin{pmatrix}B_x\\ B_z\end{pmatrix}, B_x = \begin{pmatrix}0 & -Q_{CI}\\ Q_{UL}^T & 0\end{pmatrix}, B_z = \begin{pmatrix}0 & -Q_{XI}\\ Q_{UY}^T & 0\end{pmatrix}, J_D = \begin{pmatrix}0 & Q_{UI}\\ -Q_{UI}^T & 0\end{pmatrix}$$

$$M = \begin{pmatrix}M_x & 0\\ 0 & M_z\end{pmatrix}, \ M_x = \begin{pmatrix}C & 0\\ 0 & L\end{pmatrix}, \ M_z = \begin{pmatrix}X^{-1} & 0\\ 0 & Y\end{pmatrix}.$$

The defined output w, makes it possible to calculate the delivered power to the circuit as $w^T u$. This is used in [5], but we will not use it here. The matrices J, B are given by the structure and originate from the *Q*- or *P*-matrix. A matrix is Q_{nm} defined by the sub matrix of Q_{link} with the rows indexed *n* and columns indexed *m*. As an example sees Table 3, where $Q_{CL} = [-1]$.The component values are placed in the *M*-matrices, where C, L, X, Y are diagonal block matrices with the respective component values. In our example the *M*-matrices equals

$$M_{xOn/Off} = \begin{pmatrix}C & 0\\ 0 & L\end{pmatrix},$$

$$M_{zOn} = \begin{pmatrix}1/R_1 & 0 & 0 & 0\\ 0 & 1/R_2 & 0 & 0\\ 0 & 0 & 1/R_4 & 0\\ 0 & 0 & 0 & R\end{pmatrix}, \quad M_{zOff} = \begin{pmatrix}1/R_3 & 0 & 0 & 0\\ 0 & 1/R_2 & 0 & 0\\ 0 & 0 & 1/R_4 & 0\\ 0 & 0 & 0 & R\end{pmatrix}$$

4.2 Transformation to DE-form

The model in DAE form tells us how the energy flows and dissipates in the system. However, in order to simulate the system the algebraic equations have to be eliminated. This is performed by solving the algebraic equations and replaces them in the DAE equation system, which yields a pure Differential Equation system, DE-system.

$$\begin{pmatrix}i_X\\ u_Y\end{pmatrix} = \begin{pmatrix}0 & -Q_{XY}\\ Q_{XY}^T & 0\end{pmatrix}\begin{pmatrix}X^{-1} & 0\\ 0 & Y\end{pmatrix}^{-1}\begin{pmatrix}i_X\\ u_Y\end{pmatrix} + \\ + \begin{pmatrix}0 & -Q_{XL}\\ Q_{CY}^T & 0\end{pmatrix}\begin{pmatrix}C^{-1}q_X\\ L^{-1}\varphi\end{pmatrix} + \begin{pmatrix}0 & -Q_{XI}\\ Q_{UY}^T & 0\end{pmatrix}\begin{pmatrix}u_U\\ i_I\end{pmatrix}$$

By solving the equation, we obtain the tree branch voltages Xi_X and link currents i_Y in the resistive components.

$$\begin{pmatrix}Xi_X\\ i_Y\end{pmatrix} = \begin{pmatrix}X^{-1} & Q_{XY}\\ Q_{XY}^T & Y\end{pmatrix}^{-1}\left[\begin{pmatrix}0 & -Q_{XL}\\ Q_{CY}^T & 0\end{pmatrix}\begin{pmatrix}C^{-1}q_X\\ L^{-1}\varphi\end{pmatrix} + \begin{pmatrix}0 & -Q_{XI}\\ Q_{UY}^T & 0\end{pmatrix}\begin{pmatrix}u_U\\ i_I\end{pmatrix}\right]$$

The full Hamiltonian DE-model can be described as

$$\dot{x} = (J_A - R_A)M_x^{-1}x + (F - G)u$$
$$w = (F + G)^T M_x^{-1}x + (J_D + R_D)u$$

where the different block matrices are defined as,

$$J_A = \begin{pmatrix}0 & -Q_{CY}Z^T Q_{XL} - Q_{CL}\\ Q_{XL}^T Z Q_{CY}^T + Q_{CL}^T & 0\end{pmatrix}, \quad R_A = \begin{pmatrix}Q_{CY}R_{22}Q_{CY}^T & 0\\ 0 & Q_{XL}^T R_{11} Q_{XL}\end{pmatrix}$$

$$Z = -XQ_{XY}R_{22}, \ R_{22} = \left(Y + Q_{XY}^T X Q_{XY}\right)^{-1}, \quad R_{11} = \left(X^{-1} + Q_{XY}Y^{-1}Q_{XY}^T\right)^{-1}$$

$$F = \begin{pmatrix}0 & -Q_{CY}Z^T Q_{XI} - Q_{CI}\\ Q_{XL}^T Z Q_{UY}^T + Q_{UL}^T & 0\end{pmatrix}, \quad G = \begin{pmatrix}Q_{CY}R_{22}Q_{UY}^T & 0\\ 0 & Q_{XL}^T R_{11} Q_{XI}\end{pmatrix}$$

$$J_D = \begin{pmatrix}0 & -Q_{UY}Z^T Q_{XI} - Q_{UI}\\ Q_{XI}^T Z Q_{UY}^T + Q_{UI}^T & 0\end{pmatrix}, \quad R_D = \begin{pmatrix}Q_{UY}R_{22}Q_{UY}^T & 0\\ 0 & Q_{XI}^T R_{11} Q_{XI}\end{pmatrix}$$

i.e., not as simple and structured as in the DAE model. However, this model can with preference be simulated and used for accurate analysis in the time domain.

4.3 Traditional State Space *ABCD*-model

The Hamiltonian models use the charge and flux as states. In electrical circuits we are often more interested in the currents and voltages. Hence, a change of variables solves this. The new states are

$$\xi = M_x^{-1}x$$

This yields the well-known state space *ABCD*-form

$$\dot{\xi} = A\xi + Bu$$
$$y = C\xi + Eu$$

However, the *D* is replaced with *E* since *D* in power electronics is already occupied by the duty cycle. Henceforth, the variables *C*, *Y*, *U*, will change meaning and are not component or source values. The *A*, and *B* matrices are obtained from the Hamiltonian DE-form by the following assignment.

$$A = M_x^{-1}(J_A - R_A)$$
$$B = M_x^{-1}(F - G)$$

By means of proper choice of *C* and *E* matrices any branch voltage or current can be modeled as an output, *y*. As an example if all voltages for all resistors, *X* and *Y* should be the output, *y*, is shown below. We multiply the link currents i_Y with resistors *Y* that yields the link voltages.

$$C = \begin{pmatrix}1 & 0\\ 0 & Y\end{pmatrix}\begin{pmatrix}X^{-1} & Q_{XY}\\ Q_{XY}^T & Y\end{pmatrix}^{-1}\begin{pmatrix}0 & -Q_{XL}\\ -Q_{CY}^T & 0\end{pmatrix}$$

$$E = \begin{pmatrix}1 & 0\\ 0 & Y\end{pmatrix}\begin{pmatrix}X^{-1} & Q_{XY}\\ Q_{XY}^T & Y\end{pmatrix}^{-1}\begin{pmatrix}0 & -Q_{XI}\\ -Q_{UY}^T & 0\end{pmatrix}$$

5 Linearization of the Switched Model

For analysis and design in the frequency domain, a linear model is required. A linear model can be obtained from the switched *ABCD*-model. First, a time invariant model can be obtained by state-space averaging, second, the linear model is obtained by linearization in the operation point. We define the deviations from the operation point as.

$$d = D + \hat{d}(t), \quad \xi(t) = \Xi + \hat{\xi}(t), \quad u(t) = U + \hat{u}(t), \quad y(t) = Y + \hat{y}(t)$$

i.e., the small-signals $\hat{d}(t), \hat{\xi}(t), \hat{u}(t), \hat{y}(t)$ are the deviations from the DC-values D, Ξ, U, Y, respectively.

5.1 State Space Time Averaged Model

The state space-time averaging is an averaging of the different state space models for each phase with the respect to the duty cycle *D*, [6].

$$A_{DC} = DA_{on} + (1-D)A_{off} \quad C_{DC} = DC_{on} + (1-D)C_{off}$$
$$B_{DC} = DB_{on} + (1-D)B_{off} \quad E_{DC} = DE_{on} + (1-D)E_{off}$$

These matrices can be used for calculation of the operation point. It is easy to solve for the operation point since $\dot{\xi} = 0$ in the operation point.

$$\begin{matrix} 0 = \dot{\xi} = A_{DC}\Xi + B_{DC}U \\ Y = C_{DC}\Xi + E_{DC}U \end{matrix} \Rightarrow \begin{matrix} \Xi = -A_{DC}^{-1}B_{DC}U \\ Y = -\left(C_{DC}A_{DC}^{-1}B_{DC} - E_{DC}\right)U \end{matrix}$$

5.2 Linearization in the Operation Point

The small-signal model is obtained in this section, where we also include the duty-cycle in the input vector. This is not usually done in subject area of power electronics [6], but is customary in traditional control theory. The control and disturbance signals are placed in the input vector, in our case *d*, V_{in} , and I_{inj} , this was done in [7].

We obtain the following model

$$\begin{cases} \dot{\hat{\xi}} = A'\hat{\xi}(t) + B'\hat{u}' \\ \hat{y} = C'\hat{\xi}(t) + E'\hat{u}' \end{cases}, \; u' = \begin{pmatrix} u \\ d \end{pmatrix}$$

where the matrices A', B', C', E' are given by the gradients in the operation point. Study the following equations

$$f(\xi(t), u'(t)) = \dot{\xi} = \left[\hat{d}A_{on} + (1-\hat{d})A_{off}\right]\xi + \left[\hat{d}B_{on} + (1-\hat{d})B_{off}\right]u$$

$$g(\xi(t), u'(t)) = y = \left[\hat{d}C_{on} + (1-\hat{d})C_{off}\right]\xi + \left[\hat{d}E_{on} + (1-\hat{d})E_{off}\right]u$$

The partial derivatives becomes

$$A' = \left[\frac{\partial f}{\partial \xi}\right]\Bigg|_{\substack{\xi(t)=\Xi \\ u'(t)=U'}} = \hat{d}A_{on} + (1-\hat{d})A_{off}\Big|_{\substack{\xi(t)=\Xi \\ u'(t)=U'}} = A_{DC}$$

$$B' = \left[\frac{\partial f}{\partial u'}\right]\Bigg|_{\substack{\xi(t)=\Xi \\ u'(t)=U'}} = \begin{bmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial \hat{d}} \end{bmatrix}\Bigg|_{\substack{\xi(t)=\Xi \\ u'(t)=U'}} = \begin{bmatrix} B_{DC} & B_{\hat{d}} \end{bmatrix}$$

$$C' = \left[\frac{\partial g}{\partial \xi}\right]\Bigg|_{\substack{\xi(t)=\Xi \\ u'(t)=U'}} = \hat{d}C_{on} + (1-\hat{d})C_{off}\Big|_{\substack{\xi(t)=\Xi \\ u'(t)=U'}} = C_{DC}$$

$$E' = \left[\frac{\partial g}{\partial u'}\right]\Bigg|_{\substack{\xi(t)=\Xi \\ u'(t)=U'}} = \begin{bmatrix} \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial \hat{d}} \end{bmatrix}\Bigg|_{\substack{\xi(t)=\Xi \\ u'(t)=U'}} = \begin{bmatrix} E_{DC} & E_{\hat{d}} \end{bmatrix}$$

where $\begin{matrix} B_d = (A_{on} - A_{off})\Xi + (B_{on} - B_{off})U \\ E_d = (C_{on} - C_{off})\Xi + (E_{on} - E_{off})U \end{matrix}$.

This model can be transformed to the frequency domain using traditional control theory tools.

5.3 Sampling of the system

In mixed signal systems, e.g., digital controlled switched power converters, we have a mix of systems, which are naturally described using different transforms, i.e., the Laplace- and the z-transform. Hence, you have a choice of domain, s- or z-domain, in which you will do your analysis and design. Working in the z-domain requires a sampled model $H(z)$ of the time continues plant model $H(s)$, or working in the s-domain requires a time continues model $R(s)$ of the time discrete regulator $R(z)$. The dual models or points of view are illustrated in Figure 10. Working in the z-domain has many advantages [8], e.g., the inherent delay in the digital control is simply modeled by a sample delay, and the sampling theorem is naturally built in the model, limiting the frequency range to the Nyquist frequency, $f_s/2$. Where f_s is the switching frequency for the power converter.

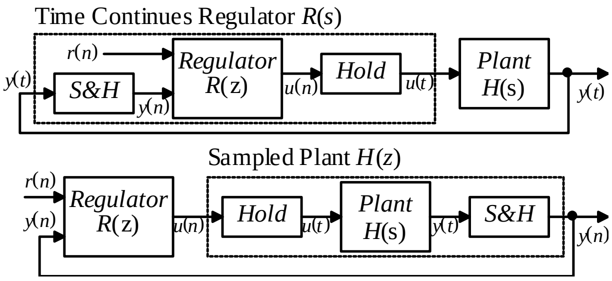

Figure 10. Continues control model versus a sampled plant model.

Assuming a piecewise constant input signal over the sampling interval, T. i.e., $u(t) = u(nT)$. The sampled state-space model of the plant becomes

$$x(nT+T) = Fx(nT) + Gu(nT)$$
$$y(nT) = C'x(nT) + E'u(nT)$$

Where the *F*, *G* matrices are given by

$$F = e^{A't} \quad G = \int_0^T e^{A't}B'dt.$$

This model can be transformed into the frequency domain using the z-transform yielding the system transfer function

$$H(z) = C'(zI - F)^{-1}G + E'$$

6 Another Modeling Example

In order to show the efficiency of this structural modeling flow, a more complex model of the buck model is used and is shown in Figure 11. This model includes, internal resistance in the voltage supply R_0, an input filter R_1, C_1, and a more complex load, which consists of a wire resistance R_6, two parallel load capacitors with different ESR, C_3, R_7, and C_4, R_8.

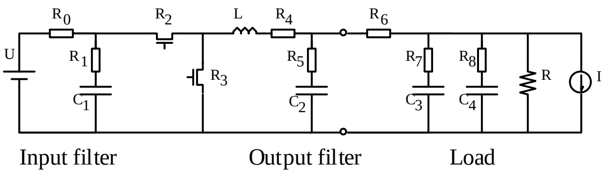

Figure 11. A more complex model of the buck converter

One of many possible graphs with maximum tree index for the On-phase is shown in Figure 12.

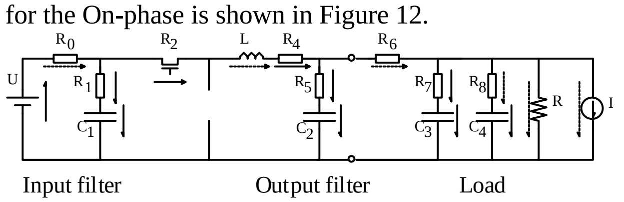

Figure 12. The directed graph in the On-phase.

The corresponding cut-set and cycle matrices is shown in Table 4. The rest of the linear algebra work is already performed ones and for all in a script. Only the component values have to be determined. Hence, with some experience a new model can be obtained within a few minutes almost no matter how complicated the schematic is.

7 Conclusion

We have shown a structural flow for modeling of switched electrical circuits, by means of an example using a switched DC/DC converter. The modeling flow has a minimal amount of hand calculations and is suitable for script implementation in mathematical computer software. Switched Hamiltonian Differential Algebraic Equation models and Differential Equation models were obtained, which are suitable for analysis in the time domain. Linear models, which can be transformed to the frequency domain for analysis, were obtained from the Hamiltonian models. The duty cycle was included in the input vector enabling the full system analysis considering audio susceptibility, output impedance, and control to output voltage analysis. Finally, a sampled system model was obtained which enables open and closed loop analysis in the z-domain.

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Table 4 The cycle and cut-set matrices for the complex model in the On-phase

$$\begin{pmatrix} I & Q_{on} \\ P_{on} & I \end{pmatrix} =$$

		Tree Branches (U, C, X)										*Links*(Y, L, I)					
		U	C_1	C_2	C_3	C_4	R_1	R_2	R_4	R_5	R_7	R_0	R_6	R_8	R	L	I
Tree Branches	U	I										−1	0	0	0	0	0
	C_1											−1	0	0	0	1	0
	C_2											0	1	0	0	−1	0
	C_3											0	−1	1	1	0	1
	C_4											0	0	−1	0	0	0
	R_1											−1	0	0	0	1	0
	R_2											0	0	0	0	−1	0
	R_4											0	0	0	0	−1	0
	R_5											0	1	0	0	−1	0
	R_7											0	−1	1	1	0	1
Links	R_0	1	1	0	0	0	1	0	0	0	0	I					
	R_6	0	0	−1	1	0	0	0	0	−1	1						
	R_8	0	0	0	−1	1	0	0	0	0	−1						
	R	0	0	0	−1	0	0	0	0	0	−1						
	L	0	−1	1	0	0	−1	1	1	1	0						
	I	0	0	0	−1	0	0	0	0	0	−1						