ISSN 1652-4934

Archives of ALGA

Volume 5, 2008

ALGA Publications
BLEKINGE INSTITUTE OF TECHNOLOGY
Karlskrona, Sweden

Volume 5, June 2008 **ISSN 1652-4934**

Archives of ALGA

Editor: Nail H. Ibragimov

ALGA Publications

Blekinge Institute of Technology
Karlskrona, Sweden

Information for authors

The manuscript should be prepared in LaTeX using only the standard LaTeX commands. The manuscripts should be sent to nib@bth.se.

1. The author assigns to ALGA the exclusive right to reproduce the contribution in Archives of ALGA and to distribute it both in printed and electronic versions. The latter includes publication at the ALGA website.

2. The author retains the right to republish the contribution in whole or in part in his further research, education and for other purposes without asking written permission from ALGA.

3. The copyright to the whole volume of Archives belongs solely to ALGA.

Contents

V.P. Ermakov

Second order differential equations: Conditions of complete integrability **1**

Editor's preface **1**

§ 1. 2
§ 2. 2
§ 3. 3
§ 4. 4
§ 5. 5
§ 6. 7
§ 7. 8
§ 8. 10
§ 9. 11
§ 10. 12
§ 11. 13
§ 12. 13
§ 13. 14
§ 14. 15
§ 15. 16
§ 16. 17
§ 17. 17
§ 18. 18
§ 19. 21
§ 20. 22
§ 21. 24

Bibliography added by Editor **26**

NAIL H. IBRAGIMOV

Memoir on integration of ordinary differential equations by quadrature 27

1 Riccati equation **28**
1 Introduction 28
2 The linearizable Riccati equations 29
3 Linearization and integration of Equation (2.9) 33
4 Linearization and integration of Equation (2.10) 35

2 Higher-order linear equations reducible to algebraic equation **38**
6 Introduction 38
7 Constant coefficient and Euler's equations from the group standpoint . 40
7.1 Equations with constant coefficients 40
7.2 Euler's equation 42
8 New examples of reducible equations 43
8.1 First example 43
8.2 Second example 46
9 The general result for second-order equations 48
9.1 Main statements 48
9.2 Distinct roots of the characteristic equation 50
9.3 The case of repeated roots 50
10 Examples to Section 9 52
11 Third-order equations 54

3 Using connection between Riccati and second-order linear equations **56**
12 Introduction 56
13 From Riccati to second-order equations 57
13.1 Application to Equation (2.9) 57
13.2 Application to Equation (2.10) 57
14 From second-order to Riccati equations 59
15 Application to Ermakov's equation 61

Bibliography **62**

NAIL H. IBRAGIMOV

Euler's integration method for linear hyperbolic equations and its extension to parabolic equations **63**
1 Introduction 63
2 Euler's method of integration of hyperbolic equations 65
2.1 Standard form of hyperbolic equations 65

2.2 Essence of Euler's method 68
2.3 Equivalence transformations 69
2.4 Reduction to the wave equation 70
3 Parabolic equations 74
3.1 Equivalence transformations 74
3.2 Semi-invariant. Equations reducible to the heat equation . . . 75
3.3 Poisson's solution 77
3.4 Uniqueness class and the general solution therein 78
3.5 Tikhonov's solution 79
3.6 Integration of equations with the vanishing semi-invariant . . 81
4 Application to financial mathematics 85
4.1 Transformations of the Black-Scholes equation 85
4.2 Poisson's form of the solution to the Black-Scholes equation . 86
4.3 Tikhonov's form of the solution to the Black-Scholes equation 88
4.4 Fundamental and other particular solutions 89
4.5 Non-linearization of the Black-Scholes model 91
4.6 Symmetries of the basic equations 92
4.7 Optimal system of one-dimensional subalgebras for Eq. (4.21) 95
4.8 Invariant solutions of Eq. (4.21) 101
4.9 Optimal system of invariant solutions for the Burgers equation 116

Bibliography **119**

NAIL H. IBRAGIMOV

Classical and new results on integrating factors **121**

1 Introduction 121
1.1 Integrating factors for first-order equations 122
1.2 Multipliers for linear differential operators 122
2 Integrating factors for nonlinear higher-order ODEs 127
3 First-order linear equations 130
3.1 Potential of the integrating factor for linear equations 131
4 Second-order linear equations 132
4.1 Determining equation for the potential 132
4.2 Non-homogeneous linear equations 133
4.3 Examples 134
4.4 Integrating factor for a special non-homogeneous equation . . 137
4.5 Integrating factor for the general non-homogeneous equation . 138
4.6 Alternative method for the general non-homogeneous equation 140

Bibliography **142**

RAISA KHAMITOVA

Self-adjointness and quasi-self-adjointness of an equation modelling melt migration through the Earth's mantle. Nonlocal conservation laws **143**

1 Introduction 143
2 Self-adjointness and quasi-self-adjointness 146
3 Conservation laws: General form 148
4 Computation of conservation laws 150
4.1 Translation of time 150
4.2 Translation of the space coordinate z 150
4.3 Dilations 153
5 Conclusion 158

Bibliography **159**

Archives of ALGA
Volume 5, 2008

SECOND ORDER DIFFERENTIAL EQUATIONS: CONDITIONS OF COMPLETE INTEGRABILITY

V.P. ERMAKOV

Translated from Russian by Alexander O. Harin
Edited by Nail H. Ibragimov

[V.P. Ermakov, Second order differential equations. Conditions of complete integrability
Universitetskie Izvestiya, Kiev, Series III, **20**(9), 1880, pp. 1–25.
From: V.P. Ermakov, *Lectures on integration of differential equations* (in Russian)]

Editor's preface

Nowadays, Ermakov's name is associated with the nonlinear equation

$$\frac{d^2z}{dx^2} = M(x)z + \frac{\alpha}{z^3}\,, \quad \alpha = \text{const.}, \tag{1}$$

discussed in this paper, see Eq. (20.2) in § 20. Later, Ermakov's equation (1) with $M(x) = 0$ appeared in Lie's group classification of ordinary differential equations as a representative of one of four classes of second-order equations admitting three infinitesimal symmetries (see [1, 2, 3, 4]; the classification result for second-order equations is summarized in [5]). The current interest to the remarkable properties of Eq. (1) began after 1950s when E. Pinney [6] rediscovered Ermakov's result stating that the solution of Eq. (1) can be expressed via the solution of the corresponding linear equation

$$\frac{d^2y}{dx^2} = M(x)y. \tag{2}$$

A group theoretic generalization of Ermakov's equation is obtained in [7]. Namely, it shown in [7], §3.3.2, that the equation

$$y'' + a(x)y' + b(x)y = c(x)y^{-3} \tag{3}$$

admits a three-dimensional Lie algebra if and only if it has the following form:

$$y'' + a(x)y' + b(x)y = \frac{k}{y^3}\,\mathrm{e}^{-2\int a(x)dx}, \quad k = \text{const.} \tag{4}$$

Note, that the discussion of the nonlinear equation (1) is only an episode in Ermakov's paper. It contains many other remarkable results on integration of second-order differential equations in terms of elementary functions or by quadrature.

N.H. Ibragimov

§ 1.

Linear second order equations with variable coefficients can be completely integrated only in very rare cases. We consider the most important of them.

Let us prove first that if a particular integral of the equation

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

is known then the determination of a complete integral is reduced to quadrature. Let u be a particular integral of this equation:

$$\frac{d^2u}{dx^2} + A\frac{du}{dx} + Bu = 0.$$

Eliminating B from these equations we obtain

$$u\frac{d^2y}{dx^2} - y\frac{d^2u}{dx^2} + A\left(u\frac{dy}{dx} - y\frac{du}{dx}\right) = 0.$$

A first integral of this equation is

$$u\frac{dy}{dx} - y\frac{du}{dx} = Ce^{-\int Adx}.$$

Integrating again we obtain the complete integral:

$$y = Cu\int \exp\left(-\int Adx\right)\frac{dx}{u^2} + Cu.$$

§ 2.

Any linear differential equation of the second order

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

can always be reduced by a variable transformation to the form in which the first derivative is absent. Explicitly putting

$$y = ze^{-\frac{1}{2}\int Adx}$$

leads to

$$\frac{d^2z}{dx^2} = \left(\frac{1}{4}A^2 + \frac{1}{2}\frac{dA}{dx} - B\right)z.$$

Later we will see that this form makes it easy to discover integrability conditions of differential equations.

§ 3.

The majority of differential equations for which it is possible to find integrability conditions reduce to the form:

$$(Ax^2 + Bx + C)\frac{d^2y}{dx^2} + (Dx + E)\frac{dy}{dx} + Fy = 0,$$

where coefficients are some constant numbers. Taking the nth derivative of the equations and putting

$$z = \frac{d^n y}{dx^n},$$

we obtain

$$(Ax^2 + Bx + C)\frac{d^2y}{dx^2} + \{(D + 2An)x + E + Bn\}\frac{dy}{dx}$$
$$+(F - An + Dn + An^2)y = 0.$$

Thus, *if an integral of the first equation is known, we can find an integral of the second equation where n is a whole positive number.*

An integral of the first equation can always be found in the case when $F = 0$,

$$(Ax^2 + Bx + C)\frac{d^2y}{dx^2} + (Dx + E)\frac{dy}{dx} = 0.$$

Designating

$$\int \frac{Dx + E}{Ax^2 + Bx + C}dx = -\varphi(x),$$

we find an integral of the equation in the form:

$$y = \alpha \int e^{\varphi(x)}dx + \beta,$$

where α and β are arbitrary constants. As has been proved, *if n is a whole positive number, then a particular integral of the equation*

$$(Ax^2 + Bx + C)\frac{d^2y}{dx^2} + \{(D + 2An)x + E + Bn\}\frac{dy}{dx}$$
$$+n(D - A + An)y = 0$$

will be expressed by the formula:

$$z = \frac{d^{n-1}}{dx^{n-1}}e^{\varphi(x)}.$$

§ 4.

Let us pass on to a more thorough investigation of particular cases. As has been proved above, the differential equation

$$(x+a)(x+b)\frac{d^2z}{dx^2} + \{(n-\lambda)(x+b) + (n-\mu)(x+a)\}\frac{dz}{dx}$$

$$+n(n-1-\lambda-\mu)z = 0$$

is completely integrable if n is a positive integer. In the present case

$$\varphi(x) = \int \frac{\lambda(x+b)+\mu(x+a)}{(x+a)(x+b)}dx = \lambda\log(x+a) + \mu\log(x+b).$$

Hence, a particular integral of the equation is expressed by the formula:

$$z = \frac{d^{n-1}}{dx^{n-1}}(x+a)^{\lambda}(x+b)^{\mu}.$$

Let us apply the transformation of § 2 to the equation. Putting

$$z = (x+a)^{\frac{\lambda-n}{2}}(x+b)^{\frac{\mu-n}{2}}y,$$

we reduce the equation to the following form:

$$(x+a)(x+b)\frac{d^2y}{dx^2} =$$

$$\left\{\frac{(n-\lambda-1)^2-1}{4}\frac{b-a}{x+a} + \frac{(n-\mu-1)^2-1}{4}\frac{a-b}{x-b} + \frac{(\lambda+\mu+1)^2-1}{4}\right\}y.$$

Comparing this equation with the equation

$$(x+a)(x+b)\frac{d^2x}{dx^2} = \left(\frac{\alpha}{x+a} + \frac{\beta}{x+b} + \gamma\right)y,$$

we obtain three algebraic equations whose solutions are

$$\begin{aligned} n &= \frac{1}{2} \pm \frac{1}{2}\sqrt{1+\frac{4\alpha}{b-a}} \pm \frac{1}{2}\sqrt{1+\frac{4\beta}{a-b}} \pm \frac{1}{2}\sqrt{1+4\gamma}, \\ \lambda &= -\frac{1}{2} \mp \frac{1}{2}\sqrt{1+\frac{4\alpha}{b-a}} \pm \frac{1}{2}\sqrt{1+\frac{4\beta}{a-b}} \pm \frac{1}{2}\sqrt{1+4\gamma}, \\ \mu &= -\frac{1}{2} \pm \frac{1}{2}\sqrt{1+\frac{4\alpha}{b-a}} \mp \frac{1}{2}\sqrt{1+\frac{4\beta}{a-b}} \pm \frac{1}{2}\sqrt{1+4\gamma}. \end{aligned} \tag{4.1}$$

In these equations before each of the roots either upper or lower sign can be taken. Thus, altogether we have eight solutions. This yields the following result.

To find complete integrability conditions of the differential equation

$$\frac{d^2y}{dx^2} = \frac{Ax^2 + Bx + C}{(x+a)^2(x+b)^2}y$$

split the fraction

$$\frac{Ax^2 + Bx + C}{(x+a)(x+b)}$$

into partial fractions:

$$\frac{Ax^2 + Bx + C}{(x+a)(x+b)} = \frac{\alpha}{x+a} + \frac{\beta}{x+b} + \gamma.$$

The equation can be completely integrated if one of the expressions

$$\sqrt{1 + \frac{4\alpha}{b-a}} \pm \sqrt{1 + \frac{4\beta}{a-b}} \pm \sqrt{1 + 4\gamma} \tag{4.2}$$

is an odd integer.

If this condition holds, then a particular integral of the equation is

$$y = (x+a)^{\frac{n-\lambda}{2}}(c+b)^{\frac{n-\mu}{2}}\frac{d^{n-1}}{dx^{n-1}}(x+a)^{\lambda}(x+b)^{\mu},$$

where n, λ and μ are given by the formulae (4.1), in which, naturally, signs should be chosen so that n be a positive integer. If two of the numbers (4.2) are odd integers, then we can find two particular integrals and consequently the complete integral without the use of quadratures.

§ 5.

The conditions of integrability found above are not unique. Let us show that there exist other conditions.

It is easy to verify that the complete integral of the equation

$$(t^2 + b - a)\frac{d^2z}{dt^2} + t\frac{dz}{dt} - \delta^2 z = 0$$

is given by the formula:

$$c\left(t + \sqrt{t^2 + b - a}\right)^{\delta} + c'\left(t - \sqrt{t^2 + b - a}\right)^{\delta}. \tag{5.1}$$

As has been proved in § 3, the nth derivative of this formula is the complete integral of the equation:

$$(t^2+b-a)\frac{d^2z}{dt^2}+(2n+1)t\frac{dz}{dt}+(n^2-\delta^2)z=0.$$

The nth derivative of the expression (5.1) is nothing but the coefficient of u^n in the expansion of the expression

$$\begin{aligned} &c\left\{t+u+\sqrt{(t+u)^2+b-a}\right\}^\delta \\ +&c'\left\{t+u-\sqrt{(t+u)^2+b-a}\right\}^\delta \end{aligned} \tag{5.2}$$

in increasing powers of u.

Changing the variable t in the last differential equation for x, according to the formula $t=\sqrt{x+a}$, we obtain:

$$(x+a)(x+b)\frac{d^2z}{dx^2}+\left\{\frac{1}{2}(x+b)+\left(n+\frac{1}{2}\right)(x+a)\right\}\frac{dz}{dx}+\frac{1}{4}(n^2-\delta^2)z=0.$$

Substituting for t in expression (5.2), we find that the complete integral of the last equation is the coefficient of u^n in the expansion of the expression

$$\begin{aligned} &c\left(\sqrt{x+a}+u+\sqrt{x+b+2u\sqrt{x+a}+u^2}\right)^\delta \\ +&c'\left(\sqrt{x+a}+u-\sqrt{x+b+2u\sqrt{x+a}+u^2}\right)^\delta \end{aligned} \tag{5.3}$$

in increasing powers of u.

Taking the mth derivative of both parts of the last differential equation and putting

$$s=\frac{d^mz}{dx^m}$$

we obtain:

$$(x+a)(x+b)\frac{d^2s}{dx^2}+\left\{\left(m+\frac{1}{2}\right)(x+b)+\left(m+n+\frac{1}{2}\right)(x+a)\right\}\frac{ds}{dx}$$

$$+\frac{1}{4}\left\{(2m+n)^2-\delta^2\right\}s=0.$$

The complete integral of this equation is the mth derivative of the coefficient of u^n in the expansion of expression (5.3) in increasing powers of u.

Let us apply the transformation of § 2 to the last equation. Putting

$$s = (x+a)^{-\frac{m}{2}-\frac{1}{4}}(x+b)^{-\frac{m+n}{2}-\frac{1}{4}}y,$$

we obtain

$$(x+a)(x+b)\frac{d^2y}{dx^2} = \left\{\frac{(2m-1)^2-4}{16}\frac{b-a}{x+a} + \frac{(2m+2n-1)^2-4}{16}\frac{a-b}{x+b} + \frac{\delta^2-1}{4}\right\}y.$$

Comparing this equation with the equation

$$(x+a)(x+b)\frac{d^2y}{dx^2} = \left(\frac{\alpha}{x+a} + \frac{\beta}{x+b} + \gamma\right)y,$$

we obtain

$$m = \frac{1}{2} + \sqrt{1+\frac{4\alpha}{b-a}},\ \ m+n = \frac{1}{2} + \sqrt{1+\frac{4\beta}{a-b}},\ \ \delta = \sqrt{1+4\gamma}.$$

Thus, the last equation is completely integrable if

$$\frac{1}{2} + \sqrt{1+\frac{4\alpha}{b-a}},\ \ \frac{1}{2} + \sqrt{1+\frac{4\beta}{a-b}}$$

are integers. In this case we obtain the complete integral through multiplying

$$(x+a)^{\frac{m}{2}+\frac{1}{4}}(x+b)^{\frac{m-n}{2}+\frac{1}{4}}$$

by the mth derivative of the coefficient of u^n in the expansion of (5.3) in increasing powers of u.

§ 6.

We have found that the equation

$$(x+a)(x+b)\frac{d^2y}{dx^2} = \left(\frac{\alpha}{x+a} + \frac{\beta}{x+b} + \gamma\right)y$$

can be completely integrated if

$$\frac{1}{2} + \sqrt{1+\frac{4\alpha}{b-a}},\ \ \frac{1}{2} + \sqrt{1+\frac{4\beta}{a-b}}$$

are whole numbers. To find further conditions for integrability, let us transform the variables of the equation according to the formulae:

$$x + a = \frac{(b-a)^2}{t+a}, \;\; y = \frac{z}{t+a}.$$

The transformed equation is of the form:

$$(t+a)(t+b)\frac{d^2z}{dt^2} = \left(\frac{\gamma(b-a)}{t+a} + \frac{\beta}{t+b} + \frac{\alpha}{b-a}\right) z.$$

This equation is of the same form as the previous, therefore it is completely integrable if

$$\frac{1}{2} + \sqrt{1+4\gamma}, \;\; \frac{1}{2} + \sqrt{1 + \frac{4\beta}{a-b}}$$

are whole numbers. Subjected to this condition the previous equation is also integrable. In the same way we can prove that the equation is integrable also in the case when

$$\frac{1}{2} + \sqrt{1+4\gamma}, \;\; \frac{1}{2} + \sqrt{1 + \frac{4\alpha}{b-a}}$$

are integers. Thus, we obtain the following result.

The equation

$$(x+a)(x+b)\frac{d^2y}{dx^2} = \left(\frac{\alpha}{x+a} + \frac{\beta}{x+b} + \gamma\right)$$

besides the cases indicated in § 4, is completely integrable if two among the three numbers:

$$\frac{1}{2} + \sqrt{1 + \frac{4\alpha}{b-a}}, \quad \frac{1}{2} + \sqrt{1 + \frac{4\beta}{a-b}}, \quad \frac{1}{2} + \sqrt{1+4\gamma}$$

are whole numbers.

§ 7.

The equation

$$(x+a)(x+b)(x+c)\frac{d^2y}{dx^2} = \left(\frac{\alpha}{x+a} + \frac{\beta}{x+b} + \frac{\gamma}{x+c}\right) y$$

can be transformed to the form of the equation examined before by a transformation of variables. Indeed, transforming this equation according to the formulae:

$$x + c = -\frac{(c-a)(c-b)}{t+a+b-c}, \;\; y = \frac{z}{t+a+b-c},$$

we obtain

$$(t+a)(t+b)\frac{d^2z}{dt^2} = \left(\frac{\alpha}{(c-a)(t+a)} + \frac{\beta}{(c-b)(t+b)} + \frac{\gamma}{(c-a)(c-b)}\right) z.$$

Integrability conditions of this equation can be found according to the rules given in § 4 and § 6. Thus we obtain the following result.

To find integrability conditions of the differential equation

$$\frac{d^2y}{dx^2} = \frac{Ax^2+Bx+C}{(x+a)^2(x+b)^2(x+c)^2}y,$$

split the fraction

$$\frac{Ax^2+Bx+C}{(x+a)(x+b)(x+c)}$$

into partial fractions:

$$\frac{Ax^2+Bx+C}{(x+a)(x+b)(x+c)} = \frac{\alpha}{x+a} + \frac{\beta}{x+b} + \frac{\gamma}{x+c}.$$

The equation is completely integrable if

$$\sqrt{1+\frac{4\alpha}{(a-b)(a-c)}} \pm \sqrt{1+\frac{4\beta}{(b-a)(b-c)}} \pm \sqrt{1+\frac{4\gamma}{(c-a)(c-b)}}$$

is an odd integer. The equation is completely integrable also in the case when two among the following three numbers:

$$\frac{1}{2} + \sqrt{1+\frac{4\alpha}{(a-b)(a-c)}},$$

$$\frac{1}{2} + \sqrt{1+\frac{4\beta}{(b-a)(b-c)}},$$

$$\frac{1}{2} + \sqrt{1+\frac{4\gamma}{(c-a)(c-b)}}$$

are whole numbers.

§ 8.

Now let us pass on to the new form of the equation:

$$(x+a)\frac{d^2z}{dx^2}+\{n-\mu-\lambda(x-a)\}\frac{dz}{dx}-\lambda nz=0.$$

This equation, as was shown in § 3, is completely integrable if n is a whole positive number. In the present case

$$\varphi(x)=\int\left(\lambda+\frac{\mu}{x+a}\right)dx=\lambda x+\mu\log(x+a).$$

Therefore a particular integral of the equation is expressed by the formula:

$$z=\frac{d^{n-1}}{dx^{n-1}}(x+a)^{\mu}e^{\lambda x}.$$

Let us apply the transformation of § 2 to the differential equation. Indeed putting

$$z=(x+a)^{\frac{\mu-n}{2}}e^{\frac{\lambda x}{2}}y,$$

yields

$$\frac{d^2y}{dx^2}=\left(\frac{\lambda^2}{4}+\frac{\lambda(n+\mu)}{2(x+a)}+\frac{(n-\mu-1)^2-1}{4(x+a)^2}\right)y.$$

Comparing this equation with the equation

$$\frac{d^2y}{dx^2}=\left(\alpha+\frac{\beta}{x+a}+\frac{\gamma}{(x+a)^2}\right)y,$$

results in

$$\begin{aligned} n&=\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4\gamma}\pm\frac{\beta}{2\sqrt{\alpha}},\\ \mu&=-\frac{1}{2}\mp\frac{1}{2}\sqrt{1+4\gamma}\pm\frac{\beta}{2\sqrt{\alpha}},\\ \lambda&=\pm2\sqrt{\alpha}. \end{aligned} \tag{8.1}$$

Thus we obtain the following result.

The differential equation

$$\frac{d^2y}{dx^2}=\left(\alpha+\frac{\beta}{x+a}+\frac{\gamma}{(x+a)^2}\right)y$$

is completely integrable if

$$\sqrt{1+4\gamma} \pm \frac{\beta}{\sqrt{\alpha}}$$

is a whole positive or negative number. If this condition holds, then a particular integral of the equation is

$$y = (x+a)^{\frac{n-\mu}{2}} e^{-\frac{\lambda x}{2}} \frac{d^{n-1}}{dx^{n-1}} (x+a)^{\mu} e^{\lambda x}.$$

where n, λ and μ are given by (8.1).

§ 9.

The integrability condition found in the previous section is not unique. It does not hold in the case when $\alpha = 0$,

$$\frac{d^2y}{dx^2} = \left(\frac{\beta}{x+a} + \frac{\gamma}{(x+a)^2} \right) y.$$

To find integrability condition of this equation, transform it to the new variables according to the formulae:

$$x + a = \frac{1}{2}(t+a)^2, \ \ y = z\sqrt{t+a}.$$

The transformed equation takes the form:

$$\frac{d^2z}{dt^2} = \left(2\beta + \frac{3+16\gamma}{4(t+a)^2} \right) z.$$

The last equation and, therefore, the previous is completely integrable if $2\sqrt{1+4\gamma}$ is a whole odd number. Thus we obtain the following result.

The differential equation

$$\frac{d^2y}{dx^2} = \left(\alpha + \frac{\beta}{x+a} + \frac{\gamma}{(x+a)^2} \right) y$$

is completely integrable if $\alpha = 0$ *and* $2\sqrt{1+4\gamma}$ *is a whole odd number.*

Besides the two cases indicated, the equation is completely integrable also in the case when $\alpha = \beta = 0$,

$$\frac{d^2y}{dx^2} = \frac{\gamma y}{(x+a)^2}.$$

The complete integral of this equation is expressed by the formula:

$$y = c(x+a)^{\mu} + c'(x+a)^{1-\mu},$$

where

$$\mu = \frac{1+\sqrt{1+4\gamma}}{2}.$$

§ 10.

The integrability condition given in the previous section and the integral of the equation can be found as follows.

It is easy to show that

$$s = ce^{2\delta\sqrt{x+a}} + c'e^{-2\delta\sqrt{x+a}}$$

is a complete integral of the equation

$$(x+a)\frac{d^2 s}{dx^2} + \frac{1}{2}\frac{ds}{dx} - \delta^2 s = 0.$$

Taking the nth derivative of both parts of the equation and putting

$$z = \frac{d^n s}{dx^n}$$

yields

$$(x+a)\frac{d^2 z}{dx^2} + \left(n + \frac{1}{2}\right)\frac{dz}{dx} - \delta^2 z = 0.$$

The integral of this equation is expressed by the formula:

$$z = \frac{d^n}{dx^n}\left(ce^{2\delta\sqrt{x+a}} + c'e^{-2\delta\sqrt{x+a}}\right).$$

Let us apply the transformation of §2 to the last equation. Putting

$$z = (x+a)^{-\frac{n}{2}-\frac{1}{4}} y,$$

we obtain

$$\frac{d^2 y}{dx^2} = \left\{\frac{\delta^2}{x+a} + \frac{(2n-1)^2 - 4}{16(x+a)^2}\right\} y.$$

Comparing this equation with the equation

$$\frac{d^2 y}{dx^2} = \left(\frac{\beta}{x+a} + \frac{\gamma}{(x+a)^2}\right) y,$$

we obtain:

$$\delta = \sqrt{\beta}, \;\; n = \frac{1}{2} + \sqrt{1 + 4\gamma}.$$

If n is a whole number, the integral of the equation is expressed by the formula:

$$y = (x+a)^{\frac{n}{2}+\frac{1}{4}}\frac{d^n}{dx^n}\left(ce^{2\delta\sqrt{x+a}} + c'e^{-2\delta\sqrt{x+a}}\right).$$

§ 11.

The differential equation

$$(x+b)^2\frac{d^2y}{dx^2} = \left(\frac{\alpha}{(x+b)^2} + \frac{\beta}{(x+a)(x+b)} + \frac{\gamma}{(x+a)^2}\right) y$$

can be transformed by a transformation of variables to the form of the equation examined in the three previous sections. Indeed, transforming this equation according to the formulae:

$$x+b = \frac{a-b}{t+a-1}, \;\; y = \frac{z}{t+a-1},$$

we obtain:

$$(a-b)^2\frac{d^2z}{dt^2} = \left(\alpha + \frac{\beta}{t+a} + \frac{\gamma}{(t+a)^2}\right) z.$$

The integrability conditions of this equation can be found according to the rules developed in the three previous sections. Thus we obtain the following result.

The differential equation

$$(x+b)^2\frac{d^2y}{dx^2} = \left(\frac{\alpha}{(x+b)^2} + \frac{\beta}{(x+a)(x+b)} + \frac{\gamma}{(x+a)^2}\right) y$$

is completely integrable in the following three cases: 1) if

$$\sqrt{1+\frac{4\gamma}{(a-b)^2}} \pm \frac{\beta}{(a-b)\sqrt{\alpha}}$$

is an odd integer, 2) if $\alpha = 0$ *and*

$$\frac{1}{2} + \sqrt{1+\frac{4\gamma}{(a-b)^2}}$$

is a whole number and 3) if $\alpha = \beta = 0$.

§ 12.

The differential equation

$$\frac{d^2y}{dx^2} = \left(\frac{\alpha}{(x+a)^4} + \frac{\beta}{(x+a)^3} + \frac{\gamma}{(x+a)^2}\right) y$$

can be transformed by a transformation of variables to the form of the equation examined in §§ 8, 9 and 10. Indeed putting

$$x + a = \frac{1}{t + a}, \quad y = \frac{z}{t + a},$$

we obtain:

$$\frac{d^2 z}{dt^2} = \left(\alpha + \frac{\beta}{t + a} + \frac{\gamma}{(t + a)^2}\right) z.$$

Thus we obtain the following result.

The differential equation

$$\frac{d^2 y}{dt^2} = \left(\frac{\alpha}{(x + a)^4} + \frac{\beta}{(x + a)^3} + \frac{\gamma}{(x + a)^2}\right) y$$

is completely integrable in the following three cases: 1) if

$$\sqrt{1 + 4\gamma} \pm \frac{\beta}{\sqrt{\alpha}}$$

is an odd integer, 2) if $\alpha = 0$ *and* $2\sqrt{1 + 4\gamma}$ *is a whole odd number and 3) if* $\alpha = \beta = 0$.

§ 13.

The first of the integrability conditions, given in the previous section, as well as the integral of the equation, can also be found as follows.

The differential equation

$$(x + a)^2 \frac{d^2 z}{dx^2} + \{(2n - \lambda)(x + a) + \mu\} \frac{dz}{dx} + n(n - 1 - \lambda) z = 0,$$

as was shown in § 3, is completely integrable if n is a whole number. In the present case

$$\varphi(x) = \int \left(\frac{\lambda}{x + a} - \frac{\mu}{(x + a)^2}\right) dx = \lambda \log(x + a) + \frac{\mu}{x + a}.$$

Therefore a particular integral of the equation is expressed by the formula:

$$z = \frac{d^{n-1}}{dx^{n-1}} (x + a)^{\lambda} e^{\frac{\mu}{x+a}}.$$

Applying the transformation of § 2,

$$z = (x + a)^{\frac{\lambda}{2} - n} e^{\frac{\mu}{2(x+a)}} y,$$

we obtain:

$$\frac{d^2y}{dx^2} = \left\{\frac{\mu^2}{4(x+a)^4} + \frac{\mu(2n-\lambda-2)}{2(x+a)^3} + \frac{(\lambda+1)^2-1}{4(x+a)^2}\right\} y.$$

Comparing this equation with the equation

$$\frac{d^2y}{dt^2} = \left(\frac{\alpha}{(x+a)^4} + \frac{\beta}{(x+a)^3} + \frac{\gamma}{(x+a)^2}\right) y,$$

we obtain:

$$n = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\gamma} \pm \frac{\beta}{2\sqrt{\alpha}},$$

$$\lambda = -1 \pm \sqrt{1+4\gamma}, \quad \mu = \pm 2\sqrt{\alpha}.$$

If n is found to be a positive integer, a particular integral of the equation can be expressed by the formula:

$$y = (x+a)^{n-\frac{\lambda}{2}} e^{-\frac{\mu}{2(x+a)}} \frac{d^{n-1}}{dx^{n-1}} (x+a)^{\lambda} e^{\frac{\mu}{x+a}}.$$

§ 14.

The differential equation

$$\frac{d^2y}{dx^2} = (\alpha x^2 + \beta x + \gamma) y$$

can be transformed by a transformation of variables to a particular form of the equation examined in § 8. Indeed, putting

$$x + \frac{\beta}{2\alpha} = \sqrt{t+a}, \quad y = z(t+a)^{-\frac{1}{4}},$$

we obtain

$$\frac{d^2z}{dt^2} = \left\{\frac{\alpha}{4} + \frac{4\alpha\gamma - \beta^2}{16\alpha(t+a)} - \frac{3}{16(t+a)^2}\right\} z.$$

Thus, as has been proved in § 8, we obtain the following result. *The differential equation*

$$\frac{d^2y}{dx^2} = (\alpha x^2 + \beta x + \gamma) y$$

is completely integrable if

$$\frac{1}{2} \pm \frac{4\alpha\gamma - \beta}{8\alpha\sqrt{\alpha}}$$

is an odd integer.

§ 15.

The integrability condition given in the previous section, as well as the integral of the equation, can be found as follows.

The differential equation

$$\frac{d^2z}{dx^2} - (2\lambda + \mu)\frac{dz}{dx} - 2\lambda n z = 0,$$

as has been shown in § 3, is completely integrable if n is a positive integer. In the present case

$$\varphi(x) = \int (2\lambda x + \mu)dx = \lambda x^2 + \mu x.$$

Therefore a particular integral of the equation is expressed by the formula:

$$z = \frac{d^{n-1}}{dx^{n-1}} e^{\lambda x^2 + \mu x}.$$

Applying the transformation of § 2, viz

$$z = e^{\frac{1}{2}(2\lambda x + \mu)} y,$$

we obtain

$$\frac{d^2y}{dx^2} = \left(\lambda^2 x^2 + \lambda\mu x + \frac{\mu^2}{4} - \lambda + 2\lambda n\right) y.$$

Comparing this equation with the equation

$$\frac{d^2y}{dx^2} = (\alpha x^2 + \beta x + \gamma)y,$$

we find that:

$$\lambda = \pm\sqrt{\alpha}, \quad \mu = \pm\frac{\beta}{\sqrt{\alpha}},$$

$$n = \frac{1}{2} \pm \frac{4\alpha\gamma - \beta^2}{8\alpha\sqrt{\alpha}}.$$

If the value n is found to be a positive integer, then the integral of the equation is given by the formula:

$$y = e^{-\frac{1}{2}(2\lambda x + \mu)} \frac{d^{n-1}}{dx^{n-1}} e^{\lambda x^2 + \mu x}.$$

§ 16.

The differential equation

$$\frac{d^2y}{dx^2} = \left(\frac{\alpha}{(x+a)^6} + \frac{\beta}{(x+a)^5} + \frac{\gamma}{(x+a)^4}\right) y$$

can be transformed by a transformation of variables to the equation examined in the previous two sections. Indeed, transforming this equation according to the formulae:

$$x + a = \frac{1}{t}, \quad y = \frac{z}{t},$$

we obtain:

$$\frac{d^2z}{dt^2} = (\alpha t^2 + \beta t + \gamma) z.$$

Taking into account what has been proved in the previous section, we obtain the following result.

The differential equation

$$\frac{d^2y}{dx^2} = \left(\frac{\alpha}{(x+a)^6} + \frac{\beta}{(x+a)^5} + \frac{\gamma}{(x+a)^4}\right) y$$

con be completely integrated if

$$\frac{1}{2} \pm \frac{4\alpha\gamma - \beta^2}{8\alpha\sqrt{\alpha}}$$

is an integer.

§ 17.

All the differential equations we have examined can be expressed by the one general formula:

$$\frac{d^2y}{dx^2} = \frac{Ax^2 + Bx + c}{(Dx^3 + Ex^2 + Fx + G)^2} y.$$

The methods of integration of this equation and the integrability conditions essentially depend on the roots of the equation:

$$Dx^3 + Ex^2 + Fx + G = 0. \tag{17.1}$$

Let us indicate those sections where we examined the different particular cases.

1) The case of different roots of Equation (17.1) has been examined in § 7.

2) The case of two equal roots of Equation (17.1) has been examined in § 11.

3) The case of three equal roots of Equation (17.1) has been examined in § 16.

4) The case when $D = 0$ and the roots of equation

$$Ex^2 + Fx + G = 0 \tag{17.2}$$

are different has been examined in § 4 and § 6.

5) The case when $D = 0$ and the roots of equation (17.2) are equal has been examined in § 12 and § 13.

6) The case when $D = E = 0$ has been examined in § 8, § 9 and § 10.

7) The case when $D = E = F = 0$ has been examined in § 14 and § 15.

We can transform the differential equation

$$(Ex^2 + Fx + G)\frac{d^2y}{dx^2} + (Hx + K)\frac{dy}{dx} + Ly = 0.$$

to the standard form by applying the transformation of § 2,

$$y = ze^{\int -\frac{1}{2}\frac{(Hx+K)dx}{Ex^2+Fx+G}}.$$

The equation becomes:

$$\frac{d^2z}{dx^2} = \frac{Ax^2 + Bx + C}{(Ex^2 + Fx + G)^2}z\ ;$$

which is a particular case of the previous equation with $D = 0$.

§ 18.

There are many differential equations which can be reduced to the equations examined above by a transformation of variables. We consider some of them.

The differential equation

$$\frac{d^2y}{dx^2} = \frac{ae^{2x} + be^x + c}{(\alpha e^x + \beta)^2}y$$

is transformed to the form

$$\frac{d^2z}{dt^2} = \left\{-\frac{1}{4t^2} + \frac{at^2 + bt + c}{t^2(\alpha t + \beta)}\right\}z$$

by the transformation

$$x = \log t, \quad y = \frac{z}{\sqrt{t}}$$

The integrability conditions of this equation can be found according to the rules developed in §§ 4, 6, 8, 9, 10, 12 and 13.

The differential equation

$$x^2\frac{d^2y}{dx^2} = \frac{a(\log x)^2 + b\log x + c}{(\alpha\log x + \beta)^2}y$$

is transformed to the form:

$$\frac{d^z}{dt^2} = \left(\frac{1}{4} + \frac{at^2 + bt + c}{(\alpha t + \beta)^2}\right)z.$$

by the transformation

$$x = e^t, \quad y = ze^{\frac{1}{2}t}$$

The integrability conditions of this equation can be found according to the rules developed in §§ 8, 9, 10, 14 and 15.

The differential equation

$$\cos^2 x\frac{d^2y}{dx^2} = (a\sin^2 x + b\sin x + c)y$$

is transformed to the form:

$$(t^2-1)^2\frac{d^2z}{dt^2} = \left\{\left(a - \frac{1}{4}\right)t^2 + bt + c - \frac{1}{2}\right\}z.$$

by the transformation

$$\sin x = t, \quad y = z(1-t^2)^{-\frac{1}{4}}$$

According to the rule developed in § 4 this equation, and therefore the previous, is completely integrable if

$$\sqrt{\frac{1}{4} + a - b + c} \pm \sqrt{\frac{1}{4} + a + b + c} \pm \sqrt{a}$$

is an odd integer. According to the rule developed in § 6 this equation is completely integrable also in the case when two of the three numbers

$$\frac{1}{2} + \sqrt{\frac{1}{4} + a - b + c}, \quad \frac{1}{2} + \sqrt{\frac{1}{4} + a + b + c}, \quad \frac{1}{2} + \sqrt{\alpha}$$

are integers.

The differential equation

$$\frac{d^2y}{dx^2} = (a\tan^2 x + b\tan x + c)y$$

is transformed to the form

$$(t^2+1)^2\frac{d^2z}{dt^2} = (at^2+bt+c+1)z.$$

by the transformation

$$\tan x = t, \quad y = z(t^2+1)^{-\frac{1}{2}}$$

According to the rule developed in § 4 this equation is completely integrable if

$$\sqrt{1+4a} \pm \sqrt{a-c+b\sqrt{-1}} \pm \sqrt{a-c-b\sqrt{-1}}$$

is an odd integer.

Pfaff's equation:

$$(ax^\delta+b)x^2\frac{d^2y}{dx^2} + (cx^\delta+e)x\frac{dy}{dx} + (fx^\delta+g)y = 0$$

can be transformed to the form

$$(ax^\delta+b)^2x^2\frac{d^2z}{dx^2} = (\alpha x^{2\delta}+\beta x^\delta+\gamma)z.$$

by using the transformation from § 2. Under the transformation

$$x = t^{\frac{1}{\delta}}, \quad z = st^{\frac{1}{2\delta}-\frac{1}{2}}$$

this equation becomes

$$\delta^2\frac{d^2s}{dt^2} = \left\{\frac{1-\delta^2}{4t^2} + \frac{\alpha t^2+\beta t+\gamma}{t^2(at+b)^2}\right\}s.$$

The integrability conditions of this equation can be found according to the rules developed in §§ 4, 6, 8, 9, 10, 12 and 13. Putting $a=\beta=\gamma=0$, $b=1$ in the second equation, we obtain:

$$\frac{d^2z}{dx^2} = \alpha x^{2\delta+2}z.$$

This equation is known as Riccati's equation. Using the transformation indicated above it becomes:

$$\delta^2\frac{d^2s}{dt^2} = \left(\alpha + \frac{1-\delta^2}{4t^2}\right)s.$$

According to the rule developed in § 8 we find that Riccati's equation is completely integrable if $\frac{1}{\delta}$ is an odd integer.

§ 19.

Some nonlinear differential equations of the first and second order are reduced to linear form by a transformation of the dependent variable. So the equation

$$\frac{dy}{dx} = y^2 + Ay + B \tag{19.1}$$

reduces to the linear second order equation

$$\frac{d^2z}{dx^2} - A\frac{dz}{dx} + Bz = 0$$

under the transformation

$$y = -\frac{1}{z}\frac{dz}{dx}$$

The more general equation

$$\frac{dy}{dx} = Ay^2 + By + C \tag{19.2}$$

by a transformation of the dependent variable can be reduced to (19.1). Indeed, putting

$$y = \frac{z}{A}$$

we obtain

$$\frac{dz}{dx} = z^2 + \frac{1}{A}\left(B + \frac{dA}{dx}\right)z + AC.$$

This equation, as has been shown above, can be reduced to a linear second order equation.

The differential equation

$$y\frac{d^2y}{dx^2} + A\left(\frac{dy}{dx}\right)^2 + By\frac{dy}{dx} + Cy^2 = 0 \tag{19.3}$$

by a transformation of the dependent variable can be transformed to the form of (19.2). Indeed, putting

$$y = e^{-\int z dx} \tag{19.4}$$

we obtain

$$\frac{dz}{dx} = (A+1)z^2 - Bz + C.$$

This equation, as has been proved above, can be reduced to a linear second order equation.

The particular case of Equation (19.3) when $A = -1$, viz

$$y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + By\frac{dy}{dx} + Cy^2 = 0,$$

deserves special attention. This equation can always be completely integrated since it reduces to the linear first order equation

$$\frac{dz}{dx} + Bz = C$$

by the transformation (19.4).

§ 20.

Some nonlinear differential equations are related to linear second order differential equations. E. g. *if an integral of the equation*

$$\frac{d^2y}{dx^2} = My \tag{20.1}$$

is known, we can find an integral of the equation

$$\frac{d^2z}{dx^2} = Mz + \frac{\alpha}{z^3}, \tag{20.2}$$

where α is a constant. Eliminating M from these equations we obtain:

$$\frac{d}{dx}\left(y\frac{dz}{dx} - z\frac{dy}{dx}\right) = \frac{\alpha y}{z^3}.$$

Multiplying both sides by

$$2\left(y\frac{dz}{dx} - z\frac{dy}{dx}\right)$$

the last equation becomes

$$\frac{d}{dx}\left(y\frac{dz}{dx} - z\frac{dy}{dx}\right)^2 = -\frac{2\alpha y}{z}\frac{d}{dx}\left(\frac{y}{z}\right).$$

Multiplying by dx and taking integrals of both parts we obtain:

$$\left(y\frac{dz}{dx} - z\frac{dy}{dx}\right)^2 = C - \frac{\alpha y^2}{z^2}. \tag{20.3}$$

If y_1 and y_2 are two particular integrals of Equation (20.1), then substituting them for y in the last equation we obtain two first integrals of Equation (20.2):

$$\left(y_1\frac{dz}{dx} - z\frac{dy_1}{dx}\right)^2 = C_1 - \frac{\alpha y_1^2}{z^2},$$

$$\left(y_2\frac{dz}{dx} - z\frac{dy_2}{dx}\right)^2 = C_2 - \frac{\alpha y_2^2}{z^2}.$$

Eliminating dz/dx from these two equations,we obtain the complete integral of Equation (20.2).

The complete integral of Equation (20.2) can also be obtained as follows. Determining dx from Equation (20.3) we obtain:

$$dx = \frac{ydz - zdy}{\sqrt{C - \alpha\frac{y^2}{z^2}}}.$$

Dividing both sides by y^2 we can transform this to the form:

$$\frac{dx}{y^2} = \frac{\frac{z}{y}d\left(\frac{z}{y}\right)}{\sqrt{C\frac{z^2}{y^2} - \alpha}}.$$

Multiplying by C and integrating both sides we obtain:

$$C\int\frac{dx}{y^2} + C = \sqrt{C\frac{z^2}{y^2} - \alpha}.$$

This is the complete integral of Equation (20.2). Instead of y it is sufficient to take any particular integral of Equation (20.1).

Conversely, *if a particular integral of Equation (20.2) is known, we can find the complete integral of Equation (20.1).* Since it is sufficient to find particular integrals of Equation (20.1), we can put $C = 0$ in Equation (20.3). Thus we obtain:

$$y\frac{dz}{dx} - z\frac{dy}{dx} = \pm\frac{y}{z}\sqrt{-\alpha}.$$

Separating variables leads to

$$\frac{dy}{y} = \frac{dz}{z} \pm \frac{dx\sqrt{-\alpha}}{z^2}.$$

Taking integrals of both sides we obtain:

$$\log y = \log z \pm \sqrt{-\alpha}\int\frac{dx}{z^2},$$

which yields

$$y = z \exp\left(\pm\sqrt{-\alpha}\int\frac{dx}{z^2}\right).$$

Taking the upper and the lower sign we obtain two particular integrals of Eq. (20.1).

§ 21.

The theorem proved above can be generalized as follows.

If p is some known function of x and f is some other given function, then the complete integral of the equation

$$p\frac{d^2y}{dx^2} - y\frac{d^2p}{dx^2} = \frac{1}{p^2} f\left(\frac{y}{p}\right)$$

can be found by means of quadratures.

Multiplying the equation by

$$2\left(p\frac{dy}{dx} - y\frac{dp}{dx}\right)dx$$

leads to the following form

$$d\left(p\frac{dy}{dx} - y\frac{dp}{dx}\right)^2 = 2f\left(\frac{y}{p}\right) d\left(\frac{y}{p}\right).$$

Taking integrals of both sides and setting

$$\varphi(z) = 2\int f(z)dz$$

we obtain

$$\left(p\frac{dy}{dx} - y\frac{dp}{dx}\right)^2 = \varphi\left(\frac{y}{p}\right) + C.$$

This is the expression for a first integral of the equation. Determining dx from this equation, we obtain:

$$dx = \frac{pdy - ydp}{\sqrt{\varphi\left(\frac{y}{p}\right) + C}}.$$

Dividing by p^2 and taking integrals of both parts we obtain:

$$\int\frac{dx}{p^2} + C = \int\frac{d\left(\frac{y}{p}\right)}{\sqrt{\varphi\left(\frac{y}{p}\right) + C}}.$$

This is the complete integral of the equation.

In the particular case when $p = x$, the differential equation takes the form

$$x^3 \frac{d^2y}{dx^2} = f\left(\frac{y}{x}\right).$$

Bibliography

[1] S. Lie, "Klassifikation und Integration von gewönlichen Differentialglei-chungen zwischen $x, y,$ die eine Gruppe von Transformationen gestatten. I," *Archiv for Matematik og Naturvidenskab (*Abbr. *Arch. for Math.)*, vol. 8, Heft 2-3, pp. 187–248, 1883. Reprinted in Lie's Ges. Abhandl., vol. 5, 1924, paper X, pp. 240–281.

[2] S. Lie, "Klassifikation und Integration von gewönlichen Differentialglei-chungen zwischen $x, y,$ die eine Gruppe von Transformationen gestatten. II," *Archiv for Matematik og Naturvidenskab (*Abbr. *Arch. for Math.)*, vol. 8, Heft 3, pp. 249–288, 1883. Reprinted in Lie's Ges. Abhandl., vol. 5, 1924, paper XI, pp. 249–310.

[3] S. Lie, "Klassifikation und Integration von gewönlichen Differentialglei-chungen zwischen $x, y,$ die eine Gruppe von Transformationen gestatten. III," *Archiv for Matematik og Naturvidenskab (*Abbr. *Arch. for Math.)*, vol. 8, Heft 4, pp. 371–458, 1883. Reprinted in Lie's Ges. Abhandl., vol. 5, 1924, paper XIV, pp. 362–427.

[4] S. Lie, "Klassifikation und Integration von gewönlichen Differentialgleichungen zwischen $x, y,$ die eine Gruppe von Transformationen gestatten. IV," *Archiv for Matematik og Naturvidenskab (*Abbr. *Arch. for Math.)*, vol. 9, Heft 4, pp. 431–448, 1884. Reprinted in Lie's Ges. Abhandl., vol. 5, 1924, paper XVI, pp. 432–446.

[5] S. Lie, "Die infinitesimalen Berührungstransformationen der Mechanik," *Leipzig. Ber.*, 1889. Reprinted in Ges. Abhandl., Bd. 6, 1927, paper VI.

[6] E. Pinney, "The nonlinear differential equation $y'' + p(x)y + Cy^{-3} = 0$," *Proc. Am. Math. Soc.*, vol. 1, No. 5, p. 681, 1950.

[7] N. H. Ibragimov, "Group analysis of ordinary differential equations and the invariance principle in mathematical physics (for the 150th anniversary of Sophus Lie)," *Uspekhi Mat. Nauk*, vol. 47, No. 4, pp. 83–144, 1992. English transl., Russian Math. Surveys, 47:2 (1992), 89-156. Reprinted in: N.H. Ibragimov, *Selected Works,* Vol. I, ALGA Publications, Karlskrona. 2006, Paper 21.

Archives of ALGA
Volume 5, 2008

MEMOIR ON INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS BY QUADRATURE

NAIL H. IBRAGIMOV
Department of Mathematics and Science,
Research Centre ALGA: Advances in Lie Group Analysis,
Blekinge Institute of Technology,
SE-371 79 Karlskrona, Sweden

Abstract. The Riccati equations reducible to first-order linear equations by an appropriate change the dependent variable are singled out. All these equations are integrable by quadrature.

A wide class of linear ordinary differential equations reducible to algebraic equations is found. It depends on two arbitrary functions. The method for solving all these equations is given. The new class contains the constant coefficient equations and Euler's equations as particular cases.

Keywords: Linearizable Riccati equations, Higher-order linear equations reducible to algebraic equations, Generalization of Euler's equations.

The old-fashioned title of the present paper indicates that it is dedicated to quite old topics. Namely, it deals with a problem on integration by quadrature of Riccati equations investigated, in terms of elementary functions, by Francesco Riccati and Daniel Bernoulli some 280 years ago for the special Riccati equations (see, e.g. [1])

$$y' = ay^2 + bx^\alpha, \quad a, b, \alpha = \text{const.},$$

and with an integration of higher-order linear equations by reducing them to algebraic equations. The later property was discovered by Leonard Euler in the 1740s for the constant coefficient equations

$$y^{(n)} + A_1 y^{(n-1)} + \cdots + A_{n-1} y' + A_n y = 0, \quad A_1, \ldots, A_n = \text{const.},$$

as well as for the equations of the form

$$x^n y^{(n)} + A_1 x^{n-1} y^{(n-1)} + \cdots + A_{n-1} x y' + A_n y = 0, \quad A_1, \ldots, A_n = \text{const.},$$

known as Euler's equations.

It will be shown in what follows that these classical results can be extended to wide classes of equations.

Chapter 1

Riccati equation

1 Introduction

Consider the special Riccati equation

$$y' = ay^2 + bx^\alpha, \quad a, b, \alpha = \text{const.} \tag{1.1}$$

If $\alpha = 0$, Eq. (1.1) is integrable by the separation of variables:

$$\frac{dy}{ay^2 + b} = dx.$$

Another easily integrable case is $\alpha = -2$. Then the change of the dependent variable

$$z = \frac{1}{y}$$

maps Eq. (1.1) to the homogeneous equation

$$\frac{dz}{dx} = -\left[a - b\left(\frac{z}{x}\right)^2\right]$$

which is integrable by quadrature.

F. Riccati and D. Bernoulli noted independently that Eq. (1.1) can be transformed to the case $\alpha = 0$, and hence integrable by quadrature in terms of elementary functions if α takes the values from the following two series:

$$\begin{aligned} \alpha &= -4,\ -\frac{8}{3},\ -\frac{12}{5},\ -\frac{16}{7}, \ldots\ ; \\ \alpha &= -\frac{4}{3},\ -\frac{8}{5},\ -\frac{12}{7},\ -\frac{16}{9}, \ldots\ . \end{aligned} \tag{1.2}$$

The series (1.2) are given by the formula

$$\alpha = -\frac{4k}{2k-1} \quad \text{with} \quad k = \pm 1, \ \pm 2, \ldots . \tag{1.3}$$

It is manifest from (1.3) that both series in (1.2) have the limit $\alpha = -2$. For a derivation of the transformations mapping Eq. (1.1) with α having the form (1.3) to an integrable form, see [1], Chapter 1, §6.

J. Liouville showed in 1841 that the solution to the special Riccati equation (1.1) is integrable by quadrature in terms of elementary functions only if α has the form (1.3).

It is well known that the general Riccati equation

$$y' = P(x) + Q(x)y + R(x)y^2$$

can be rewritten as a linear second-order equation. But this kind of linearization by *raising the order* does not solve the integration problem. Therefore, I will investigate a possibility of linearization of the Riccati equations *without raising the order* and will show that all Riccati equations of this type can be integrated by quadrature.

2 The linearizable Riccati equations

The following theorem is closely related to the theory of nonlinear superpositions discussed in [2], Section 6.7.

Theorem 2.1. The first-order ordinary differential equation

$$y' = f(x, y) \tag{2.1}$$

can be reduced to a linear first-order equation

$$\frac{dz}{dx} = p(x) + q(x)z \tag{2.2}$$

by a change of the dependent variable y,

$$z = z(y), \tag{2.3}$$

if and only if Eq. (2.1) can be written in the form

$$y' = T_1(x)\xi_1(y) + T_2(x)\xi_2(y) \tag{2.4}$$

such that the operators

$$X_1 = \xi_1(y)\frac{\partial}{\partial y}, \quad X_2 = \xi_2(y)\frac{\partial}{\partial y} \tag{2.5}$$

span a two-dimensional (or a one-dimensional if X_1 and X_2 are linearly dependent) Lie algebra called in [3] the VGL (Vessiot-Guldberg-Lie) algebra.

Proof. Let Eq. (2.1) be linearizable. Then we can assume that it is already reduced by a certain change of the dependent variable

$$z = \zeta(y) \tag{2.6}$$

to a linear equation (2.2),

$$\frac{dz}{dx} = p(x) + q(x)z.$$

The VGL algebra associated with Eq. (2.2) is two-dimensional and is spanned by the operators

$$\overline{X}_1 = \frac{\partial}{\partial z}\,, \quad \overline{X}_2 = z\frac{\partial}{\partial z}\,\cdot \tag{2.7}$$

The form of Eq. (2.4) and the algebra property $[X_1, X_2] = \alpha X_1 + \beta X_2$ remain unaltered under any change (2.3) of the dependent variable. Therefore, rewriting Eq. (2.2) in the original variable $y = \zeta^{-1}(z)$ obtained by the inverse transformation to (2.6), we arrive at an equation of the form (2.4) for which the operators (2.5) span a two-dimensional Lie algebra. Since the equation obtained from Eq. (2.2) by the inverse transformation to (2.6) is the original equation (2.1) we have proved the "only if" part of the theorem.

Let us prove now the "if" part of the theorem. Namely, we have to demonstrate that any equation of the form (2.4) such that the operators (2.5) span a two-dimensional Lie algebra, is linearizable. If the operators (2.5) are linearly dependent, then $\xi_2(x) = \gamma\xi_1(x),\ \gamma = \text{const.}$, and hence Eq. (2.4) has the form

$$y' = \big[T_1(x) + \gamma T_2(x)\big]\xi_1(y).$$

It can be reduced to the linear equation

$$z' = T_1(x) + \gamma T_2(x).$$

upon introducing a canonical variable z for

$$X_1 = \xi_1(y)\frac{\partial}{\partial y}$$

by solving the equation $X_1(z) = 1.$

Suppose now that the operators (2.5) are linearly independent. It is clear that Eq. (2.9) will be linearized if one transforms the operators (2.5) to the form (2.7). One can assume that the first operator (2.5) has been already written, in a proper variable z, in the form of the first operator $\overline{X}_1$ given in (2.7):

$$X_1 = \frac{\partial}{\partial z}\,\cdot$$

Let the second operator (2.5) be written in the variable z as follows:

$$X_2 = f(z)\frac{\partial}{\partial z}\,\cdot$$

We have

$$[X_1, X_2] = f'(z)\frac{\partial}{\partial z}$$

and the requirement $[X_1, X_2] = \alpha X_1 + \beta X_2$ that X_1, X_2 span a Lie algebra L_2 yields the differential equation

$$f' = \alpha + \beta f,$$

where not both α and β vanish because otherwise the operators X_1 and X_2 will be linearly dependent. hence $f'(x) \neq 0$. Solving the above differential equation, we obtain

$$\begin{aligned} f &= \alpha z + C \quad \text{if} \quad \beta = 0,\\ f &= C\,\mathrm{e}^{\beta z} - \frac{\alpha}{\beta} \quad \text{if} \quad \beta \neq 0. \end{aligned}$$

$$f = ax + C \quad \Longrightarrow \quad X_2 = ax\frac{\mathrm{d}}{\mathrm{d}x} + CX_1, \quad \text{if} \quad b = 0,$$

$$f = C\mathrm{e}^{bx} - \frac{a}{b} \quad \Longrightarrow \quad X_2 = C\mathrm{e}^{bx}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{a}{b}X_1, \quad \text{if} \quad b \neq 0.$$

In the first case we have

$$X_2 = \alpha z\frac{\partial}{\partial z} + CX_1,$$

and hence a basis of L_2 is provided by (2.7). In the second case we have

$$X_2 = C\,\mathrm{e}^{\beta z}\,\frac{\partial}{\partial z} - \frac{\alpha}{\beta}\,X_1,$$

and one can take basis operators, by assigning βz as new z, in the form

$$X_1 = \frac{\partial}{\partial z}\,, \quad X_2 = \mathrm{e}^{z}\,\frac{\partial}{\partial z}\,\cdot$$

Finally, substituting $\overline{z} = \mathrm{e}^{-z}$ we arrive at the basis (2.7), thus completing the proof of the theorem.

The following theorem characterizes all Riccati equations that can be reduced to first-order linear equations by changing the dependent variable (see [4], Russian ed., Theorem 4.3; [3], Section 11.2.5 and Note [11.4]; see also Theorem 3.2.2 in [2]).

Theorem 2.2. The following two conditions 1° and 2° are equivalent and provide the necessary and sufficient conditions for the Riccati equation

$$y' = P(x) + Q(x)y + R(x)y^2 \tag{2.8}$$

to be linearizable by a change of the dependent variable (2.3), $z = z(y)$.

1°. Eq. (2.8) has a constant solution $y = c$ including $c = \infty$.

2°. Eq. (2.8) has either the form

$$y' = Q(x)y + R(x)y^2 \tag{2.9}$$

with any functions $Q(x)$ and $R(x)$, or the form

$$y' = P(x) + Q(x)y + k[Q(x) - kP(x)]y^2 \tag{2.10}$$

with any functions $P(x),\ Q(x)$ and any constant k.

Proof. The conditions 1° and 2° are equivalent. Indeed, let Eq. (2.8) have a constant solution $y = c$. Then

$$0 = P(x) + Q(x)c + R(x)c^2. \tag{2.11}$$

If $c = 0$, Eq. (2.11) yields $P(x) = 0$, and hence Eq. (2.8) has the form (2.9). If $c \neq 0$, Eq. (2.11) yields

$$R(x) = -\frac{1}{c^2}\big[P(x) + cQ(x)\big] = -\frac{1}{c}\big[Q(x) + \frac{1}{c}\,P(x)\big].$$

Denoting $k = -1/c$ we get

$$R(x) = k\big[Q(x) - kP(x)\big].$$

Hence Eq. (2.8) has the form (2.10). Thus, we have proved that $1^\circ \Rightarrow 2^\circ$.

Conversely, let Eq. (2.8) satisfy the condition 2°. It is manifest that Eq. (2.9) has the constant solution $y = 0$. Furthermore, one can verify that Eq. (2.10) has the constant solution $y = -1/k$. This proves that $2^\circ \Rightarrow 1^\circ$.

Let us turn to the necessary and sufficient conditions for linearization. The operators

$$X_1 = y\frac{\partial}{\partial y}\,, \quad X_2 = y^2\frac{\partial}{\partial y}$$

associated with Eq. (2.9) have the commutator $[X_1, X_2] = X_2$, and hence span a two-dimensional Lie algebra. Furthermore, the operators

$$X_1 = (1 - k^2y^2)\frac{\partial}{\partial y}\,, \quad X_2 = (y + ky^2)\frac{\partial}{\partial y}$$

associated with Eq. (2.10) have the commutator $[X_1, X_2] = X_1 + 2kX_2$. Hence, they also span a two-dimensional Lie algebra. Therefore, according to Theorem 2.1, the condition 2° is sufficient for linearization.

Note that the equation $y' = P(x) + Q(x)y$ can be regarded as a particular case of Eq. (2.10) with $k = 0$. Since Eq. (2.10) has the constant solution $y = -1/k$, we conclude that the linear equation $y' = P(x) + Q(x)y$ has the constant solution $y = \infty$. We conclude that any linearizable equation has a constant solution because the change of the dependent variable $z = z(y)$ maps a constant solution into a constant solution of the transformed equation. Hence, the condition 1° is necessary for linearization. This completes the proof of the theorem due to the equivalence of the conditions 1° and 2°.

Now we will use Theorem 2.2 for integrating the linearizable Riccati equations. We will find linearizing transformations for the equations (2.9) and (2.10). We will assume that k in Eq. (2.10) is a real number.

3 Linearization and integration of Equation (2.9)

Invoking Eqs. (6.7.5), (6.7.6) from [2], replacing there t and x^i by x and y, respectively, and identifying $T_1(t)$ and $T_2(t)$ with $Q(x)$ and $R(x)$, respectively, we see that the VGL (Vessiot-Guldberg-Lie) algebra associated with Eq. (2.9) is a two-dimensional algebra L_2 spanned by the operators

$$X_1 = y\frac{\partial}{\partial y}\,, \quad X_2 = y^2\frac{\partial}{\partial y}\,\cdot$$

Their commutator is $[X_1, X_2] = X_2$. Introducing the new basis

$$X_1' = X_2, \quad X_2' = -X_1$$

i.e. taking

$$X_1' = y^2\frac{\partial}{\partial y}\,, \quad X_2' = -y\frac{\partial}{\partial y} \tag{3.1}$$

we have a basis in L_2 satisfying the commutator relation

$$[X_1', X_2'] = X_1'. \tag{3.2}$$

Let us find a change of the dependent variable, $z = z(y)$, such that Eq. (2.9) becomes a linear equation (2.2),

$$\frac{dz}{dx} = p(x) + q(x)z.$$

The VGL algebra of this equation is spanned by the operators (2.7),

$$\overline{X}_1 = \frac{\partial}{\partial z}\,, \quad \overline{X}_2 = z\frac{\partial}{\partial z}$$

whose commutator has the same form as Eq. (3.2), i.e. $[\overline{X}_1, \overline{X}_2] = \overline{X}_1$. Consequently, the linearizing transformation $z = z(y)$ is determined by the equations $X_1'(z) = 1,\ X_2'(z) = z$, or

$$y^2 \frac{dz}{dy} = 1, \quad -y\frac{dz}{dy} = z. \tag{3.3}$$

Integrating the first equation (3.3) we obtain

$$z = -\frac{1}{y} + A.$$

Substituting in the second equation (3.3) we get $A = 0$. Thus, the linearizing transformation is

$$z = -\frac{1}{y}\,\cdot \tag{3.4}$$

In this variable, Eq. (2.9) becomes the linear equation

$$z' = R(x) - Q(x)z, \tag{3.5}$$

whence

$$z = \Big[C + \int R(x)\mathrm{e}^{\int Q(x)dx}\,dx\Big]\,\mathrm{e}^{-\int Q(x)dx}, \quad C = \mathrm{const}.$$

Substituting in Eq. (3.4) we finally arrive at the solution to Eq. (2.9):

$$y = -\Big[C + \int R(x)\mathrm{e}^{\int Q(x)dx}\,dx\Big]^{-1}\mathrm{e}^{\int Q(x)dx}. \tag{3.6}$$

Remark 3.1. In Section 13.1 the solution (3.6) is obtained by an alternative method.

Example 3.1. Consider the equation

$$y' = \frac{y}{x} + R(x)y^2.$$

By Eq. (3.6) yields its solution

$$y = \frac{x}{C - \int xR(x)dx}\,\cdot$$

Example 3.2. The equation

$$y' = \frac{y}{x} + 3xy^2$$

is a particular case of the previous equation. Its solution is

$$y = \frac{x}{C - x^2}\,\cdot$$

Example 3.3. The equation

$$y' = y + \frac{1}{x}\, y^2$$

has the solution

$$y = \frac{\mathrm{e}^x}{C + \mathrm{Ei}(x)}\,, \quad \text{where} \quad \mathrm{Ei}(x) = -\int\limits_{-\infty}^{x} \frac{\mathrm{e}^t}{t}\, dt.$$

4 Linearization and integration of Equation (2.10)

Now we identify the coefficients $P(x)$ and $Q(x)$ of Eq. (2.10),

$$y' = P(x) + Q(x)y + k[Q(x) - kP(x)]y^2, \tag{2.10}$$

with the coefficients $T_1(t)$ and $T_2(t)$ of Eqs. (6.7.5), (6.7.6) from [2] and associated with Eq. (2.10) the VGL algebra spanned by the operators

$$X_1 = (1 - k^2y^2)\frac{\partial}{\partial y}\,, \quad X_2 = (y + ky^2)\frac{\partial}{\partial y}\,\cdot$$

Their commutator is $[X_1, X_2] = X_1 + 2kX_2$. We assume that $k \neq 0$, since otherwise Eq. (2.10) is already linear. Therefore, setting

$$X_1' = X_1 + 2kX_2, \quad X_2' = -X_1/(2k)$$

we obtain the new basis

$$X_1' = (1 + ky)^2\frac{\partial}{\partial y}\,, \quad X_2' = \frac{1}{2k}\left(k^2y^2 - 1\right)\frac{\partial}{\partial y}\,, \tag{4.1}$$

satisfying the commutator relation (3.2), $[X_1', X_2'] = X_1'$.

The transformation $z = z(y)$ of the operators (4.1) to the form (2.7) is determined by the equations

$$X_1'(z) = 1, \quad X_2'(z) = z,$$

or

$$(1 + ky)^2\frac{dz}{dy} = 1, \quad \frac{1}{2k}\left(k^2y^2 - 1\right)\frac{dz}{dy} = z. \tag{4.2}$$

Integrating the first equation (4.2) we obtain

$$z = A - \frac{1}{k(1 + ky)}\,\cdot$$

Substituting in the second equation (4.2) we get $A = 1/(2k)$. Thus, the linearizing transformation is

$$z = \frac{ky-1}{2k(ky+1)}\,\cdot \tag{4.3}$$

Eq. (4.3) yields:

$$y = \frac{1+2kz}{k(1-2kz)}\,\cdot \tag{4.4}$$

Substituting (4.4) in Eq. (2.10) we arrive at the linear equation

$$z' = \frac{1}{2k}\,Q(x) + \big[Q(x) - 2kP(x)\big]\,z, \tag{4.5}$$

whence

$$z = \frac{1}{2k}\left[C + \int Q(x)\mathrm{e}^{\int[2kP(x)-Q(x)]dx}\,dx\right]\,\mathrm{e}^{\int[Q(x)-2kP(x)]dx}. \tag{4.6}$$

Substituting (4.6) in (4.4) we will obtain the solution to Eq. (2.10).

Example 4.1. Let us integrate the equation

$$y' = x + 2xy + xy^2. \tag{4.7}$$

Here

$$P(x) = x, \quad Q(x) = 2x, \quad k = 1.$$

Therefore the linearized equation (4.5) is written $z' = x$. Integrating it and denoting the constant of integration by $C/2$, we obtain

$$z = \frac{1}{2}\,(C + x^2)$$

in accordance with Eq. (4.6). Substituting the above z in Eq. (4.4) we obtain the following solution to Eq. (4.7):

$$y = \frac{1+C+x^2}{1-C-x^2}\,\cdot \tag{4.8}$$

Remark 4.1. Eq. (4.7) can also be integrated by separating the variables.

Example 4.2. The equation

$$y' = x^2 + (x + x^2)\,y + \frac{1}{4}\big(2x + x^2\big)\,y^2 \tag{4.9}$$

has the form (2.10) with

$$P(x) = x^2, \quad Q(x) = x + x^2, \quad k = \frac{1}{2}\,\cdot$$

The linearized equation (4.5) is written

$$z' = x + x^2 + xz$$

and has the solution

$$z = \left(C + \int (x + x^2)\mathrm{e}^{-x^2/2}\,dx\right)\mathrm{e}^{x^2/2}.$$

Substitution in Eq. (4.4) yields

$$y = 2\frac{1+z}{1-z}\,.$$

Hence, the solution to Eq. (4.9) is given by

$$y = 2\frac{1 + \left(C + \int (x + x^2)\mathrm{e}^{-x^2/2}\,dx\right)\mathrm{e}^{x^2/2}}{1 - \left(C + \int (x + x^2)\mathrm{e}^{-x^2/2}\,dx\right)\mathrm{e}^{x^2/2}}\,. \tag{4.10}$$

Chapter 2

Higher-order linear equations reducible to algebraic equation

6 Introduction

The linear ordinary differential equations with constant coefficients

$$y^{(n)} + A_1 y^{(n-1)} + \cdots + A_{n-1} y' + A_n y = 0, \quad A_1, \ldots, A_n = \text{const.}, \tag{6.1}$$

and Euler's equations

$$x^n y^{(n)} + A_1 x^{n-1} y^{(n-1)} + \cdots + A_{n-1} x y' + A_n y = 0, \quad A_1, \ldots, A_n = \text{const.}, \tag{6.2}$$

are discussed practically in all textbooks on differential equations. They are useful in applications. The most remarkable property is that both equations (6.1) and (6.2) are *reducible to algebraic equations.* Namely, their fundamental systems of solutions, and hence the general solutions can be obtained by solving algebraic equations. Then one can integrate the corresponding non-homogeneous linear equations by using the method of variation of parameters. It is significant to understand the nature of the reducibility and to extend the class of linear equations reducible to algebraic equations.

In the present paper we will investigate this problem and find a wide class of linear ordinary differential equations that are reducible to algebraic equations. The new class depends on two arbitrary functions of x and contains the equations (6.1) and (6.2) as particular cases. The method for solving all these equations is given in Sections 9 and 11, and illustrated by examples in Section 10.

The main statement for the second-order equations is as follows (Section 9).

Theorem 6.1. The linear second-order equations

$$P(x)y'' + Q(x)y' + R(x)y = F(x)$$

whose general solution can be obtained by solving algebraic equations and by quadratures, have the form

$$\phi^2 y'' + \big(A + \phi' - 2\sigma\big)\phi\, y' + \big(B - A\sigma + \sigma^2 - \phi\sigma'\big)y = F(x), \tag{6.3}$$

where $\phi = \phi(x),\ \sigma = \sigma(x)$ and $F(x)$ are arbitrary (smooth) functions, and $A, B = \text{const}$. The homogeneous equation (6.3),

$$\phi^2 y'' + \big(A + \phi' - 2\sigma\big)\phi\, y' + \big(B - A\sigma + \sigma^2 - \phi\sigma'\big)y = 0, \tag{6.4}$$

has the solutions of the form

$$y = \mathrm{e}^{\int \frac{\sigma(x)+\lambda}{\phi(x)}\, dx}, \tag{6.5}$$

where λ satisfies the *characteristic equation*

$$\lambda^2 + A\lambda + B = 0. \tag{6.6}$$

If the characteristic equation (6.6) has distinct real roots $\lambda_1 \neq \lambda_2$, the general solution to Eq. (6.4) is given by

$$y(x) = K_1\, \mathrm{e}^{\int \frac{\sigma(x)+\lambda_1}{\phi(x)}\, dx} + K_2\, \mathrm{e}^{\int \frac{\sigma(x)+\lambda_2}{\phi(x)}\, dx}, \quad K_1, K_2 = \text{const}. \tag{6.7}$$

In the case of complex roots, $\lambda_1 = \gamma + i\theta,\ \lambda_2 = \gamma - i\theta$, the general solution to Eq. (6.4) is given by

$$y(x) = \left[K_1 \cos\left(\theta \int \frac{dx}{\phi(x)}\right) + K_2 \sin\left(\theta \int \frac{dx}{\phi(x)}\right)\right] \mathrm{e}^{\int \frac{\sigma(x)+\gamma}{\phi(x)}\, dx}. \tag{6.8}$$

If the characteristic equation (6.6) has equal roots $\lambda_1 = \lambda_2$, the general solution to Eq. (6.4) is given by

$$y = \left[K_1 + K_2 \int \frac{dx}{\phi(x)}\right] \mathrm{e}^{\int \frac{\sigma(x)+\lambda_1}{\phi(x)}\, dx}, \quad K_1, K_2 = \text{const}. \tag{6.9}$$

The general solution of the non-homogeneous equation (6.3) can be obtained by the method of variation of parameters.

Remark 6.1. The equations with constant coefficients and Euler's equation are the simplest representatives of Eqs. (6.4). Namely, setting $\phi(x) = 1,\ \sigma(x) = 0$ we obtain the second-order equation with constant coefficients

$$y'' + Ay' + By = 0,$$

and Eq. (6.5) yields the well-known formula

$$y = \mathrm{e}^{\lambda x},$$

where λ is determined by the characteristic equation (6.5).

If we set $\phi(x) = x,\ \sigma(x) = 0$, we obtain the second-order Euler's equation (6.2) written in the form

$$x^2 y'' + (A+1)xy' + By = 0.$$

Then Eq. (6.5) yields the particular solutions for Euler's equations:

$$y = x^\lambda,$$

where λ is determined again by the characteristic equation (6.5). For details, see Section 10. For other functions $\phi(x)$ and $\sigma(x) = 0$, Eqs. (6.5) are new.

7 Constant coefficient and Euler's equations from the group standpoint

For the sake of simplicity, we will consider in this section second-order equations

$$y'' + f(x)y' + g(x)y = 0. \tag{7.1}$$

7.1 Equations with constant coefficients

The usual approach to equations with constant coefficients

$$y'' + Ay' + By = 0, \quad A, B = \text{const}. \tag{7.2}$$

Eq. (7.2) is invariant under the one-parameter groups of translations in x and dilations in y, since it does not involve the independent variable x explicitly (the coefficients A and B are constant) and is homogeneous in the dependent variable y. In other words, Eq. (7.2) admits the generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = y\frac{\partial}{\partial y} \tag{7.3}$$

of the translations in x and dilations in y. We use them as follows [3].

Let us find the invariant solution for $X = X_1 + \lambda X_2$, i.e.

$$X = \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y} \quad \lambda = \text{const}. \tag{7.4}$$

The characteristic equation

$$\frac{dy}{y} = \lambda \frac{dx}{x}$$

of the equation

$$X(J) \equiv \frac{\partial J}{\partial x} + \lambda y \frac{\partial J}{\partial y} = 0$$

for the invariants $J(x, y)$ yields one functionally independent invariant

$$J = y\,\mathrm{e}^{-\lambda x}.$$

According to the general theory, the invariant solution is given by $J = C$ with an arbitrary constant C. Thus, the general form of the invariant solutions for the operator (7.4) is

$$y = C\mathrm{e}^{\lambda x}, \quad C = \text{const.}$$

Since Eq. (7.2) is homogeneous one can set $C = 1$ and obtain *Euler's substitution*:

$$y = \mathrm{e}^{\lambda x}. \tag{7.5}$$

As well known, the substitution (7.5) reduces Eq. (7.2) to the quadratic equation (*characteristic equation*)

$$\lambda^2 + A\lambda + B = 0. \tag{7.6}$$

If Eq. (7.6) has two distinct roots, $\lambda_1 \neq \lambda_2$, then Eq. (7.5) provides two linearly independent solutions

$$y_1 = \mathrm{e}^{\lambda_1 x}, \quad y_2 = \mathrm{e}^{\lambda_2 x},$$

and hence, a fundamental set of solutions. If the roots are real, the general solution to Eq. (7.2) is

$$y(x) = K_1\mathrm{e}^{\lambda_1 x} + K_2\mathrm{e}^{\lambda_2 x}, \quad K_1, K_2 = \text{const.} \tag{7.7}$$

If the roots are complex, $\lambda_1 = \gamma + i\theta,\ \lambda_2 = \gamma - i\theta$, the general solution to Eq. (7.2) is given by

$$y(x) = [K_1 \cos(\theta x) + K_2 \sin(\theta x)]\,\mathrm{e}^{\gamma x}, \quad K_1, K_2 = \text{const.} \tag{7.8}$$

In the case of equal roots $\lambda_1 = \lambda_2$, standard texts in differential equations make a guess, without motivation, that the general solution has the form

$$y(x) = (K_1 + K_2\,x)\,\mathrm{e}^{\lambda_1 x}, \quad K_1, K_2 = \text{const.} \tag{7.9}$$

The motivation is given in [3], Section 13.2.2, and states the following.

Lemma 7.1. Eq. (7.2) can be mapped to the equation $z'' = 0$ by a linear change of the dependent variable

$$y = \sigma(x)\,z, \quad \sigma(x) \neq 0, \tag{7.10}$$

if and only if the characteristic equation (7.6) has equal roots. Specifically, if $\lambda_1 = \lambda_2$, the substitution

$$y = z\,\mathrm{e}^{\lambda_1 x} \tag{7.11}$$

carries Eq. (7.2) to the equation $z'' = 0$. Substituting in (7.11) the solution $z = K_1 + K_2\,x$ of the equation $z'' = 0$, we obtain the general solution (7.9) to Eq. (7.2) whose coefficients satisfy the condition of equal roots for Eq. (7.6):

$$A^2 - 4B = 0. \tag{7.12}$$

Proof. The reckoning shows that after the substitution (7.10) Eq. (7.1) becomes (see, e.g. [2], Section 3.3.2)

$$z'' + I(x)z = 0,$$

where

$$I(x) = g(x) - \frac{1}{4}\,f^2(x) - \frac{1}{2}\,f'(x)$$

and the function $\sigma(x)$ in the transformation (7.10) has the form

$$\sigma(x) = \mathrm{e}^{-\frac{1}{2}\int f(x)dx}. \tag{7.13}$$

Hence, Eq. (7.1) is carried into equation $z'' = 0$ if and only if $I(x) = 0$, i.e.

$$f^2(x) + 2f'(x) - 4g(x) = 0. \tag{7.14}$$

In the case of Eq. (7.2), the condition (7.14) is identical with Eq. (7.12), and the function $\sigma(x)$ given by Eq. (7.13) becomes

$$\sigma(x) = \mathrm{e}^{-\frac{A}{2}\,x}. \tag{7.15}$$

Furthermore, under the condition Eq. (7.12), the repeated root of Eq. (7.6) is $\lambda_1 = -A/2$. Therefore Eq. (7.15) can be written

$$\sigma(x) = \mathrm{e}^{\lambda_1\,x}$$

and we arrive at the substitution (7.11), and hence at the solution (7.9), thus proving the lemma.

7.2 Euler's equation

Consider Euler's equation

$$x^2\,y'' + A\,x\,y' + B\,y = 0, \quad A, B = \mathrm{const.} \tag{7.16}$$

It is double homogeneous (see [2], Section 6.6.1), i.e. admits the dilation groups in x and in y with the generators

$$X_1 = x\frac{\partial}{\partial x}\,, \quad X_2 = y\frac{\partial}{\partial y}\,\cdot \tag{7.17}$$

We proceed as in Section 7.1 and find the invariant solutions for the linear combination $X = X_1 + \lambda X_2$:

$$X = x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}\,, \quad \lambda = \text{const.} \tag{7.18}$$

The characteristic equation

$$\frac{dy}{y} = \lambda\,\frac{dx}{x}$$

of the equation $X(J) = 0$ for the invariants $J(x, y)$ yields the invariant

$$J = y\,x^{-\lambda}$$

for the operator (7.18). The invariant solutions are given by $J = C$, whence

$$y = Cx^{\lambda}, \quad C = \text{const.}$$

Due to the homogeneity of Eq. (7.2) we can set $C = 1$ and obtain

$$y = x^{\lambda}. \tag{7.19}$$

Differentiating and multiplying by x, we have:

$$xy' = \lambda\,x^{\lambda}, \quad x^2y'' = \lambda(\lambda - 1)x^{\lambda}.$$

Substituting in Eq. (7.18) and dividing by the common factor Cx^{λ} we obtain the following *characteristic equation* for Euler's equation (7.16):

$$\lambda^2 + (A - 1)\,\lambda + B = 0. \tag{7.20}$$

Remark 7.1. According to Eqs. (7.6), (7.20), the characteristic equation for Euler's equation written in the form

$$x^2\,y'' + (A + 1)\,x\,y' + B\,y = 0 \tag{7.21}$$

is identical with the characteristic equation (7.6) for Eq. (7.2) with constant coefficients.

8 New examples of reducible equations

8.1 First example

Let us find the linear second-order equations (7.1),

$$y'' + f(x)y' + g(x)y = 0, \tag{7.1}$$

admitting the operator

$$X_1 = x^\alpha \frac{\partial}{\partial x}\,, \tag{8.1}$$

where α is any real-valued parameter. Taking the second prolongation of X_1,

$$X_1 = x^\alpha \frac{\partial}{\partial x} - \alpha x^{\alpha-1} y' \frac{\partial}{\partial y'} - \left[\alpha(\alpha-1)x^{\alpha-2}y' + 2x^{\alpha-1}y''\right] \frac{\partial}{\partial y''}\,,$$

we write the invariance condition of Eq. (7.1),

$$X_1(y'' + f(x)y' + g(x)y)\Big|_{(7.1)} = 0,$$

and obtain:

$$x^{\alpha-2}[x^2 f' + \alpha x f - \alpha(\alpha-1)]y' + x^{\alpha-1}[xg' + 2\alpha g]y = 0. \tag{8.2}$$

Since Eq. (8.2) should be satisfied identically in the variables x, y, y', it splits into two equations:

$$x^2 f' + \alpha x f - \alpha(\alpha-1) = 0, \quad xg' + 2\alpha g = 0. \tag{8.3}$$

Solving the first-order linear differential equations (8.3) for the unknown functions $f(x)$ and $g(x)$, we obtain:

$$f(x) = \frac{\alpha}{x} + Ax^{-\alpha}, \quad g(x) = Bx^{-2\alpha}, \quad A, B = \text{const.}$$

Thus, we arrive at the following linear equation admitting the operator (8.1):

$$x^{2\alpha}y'' + \left(Ax^\alpha + \alpha x^{2\alpha-1}\right)y' + By = 0, \quad A, B = \text{const.} \tag{8.4}$$

Remark 8.1. When $\alpha = 0$, Eq. (8.4) yields the equation (7.2) with constant coefficients. When $\alpha = 1$, Eq. (8.4), upon setting $A + 1$ as a new coefficient A, coincides with Euler's equation (7.16).

Since Eq. (8.4) is linear homogeneous, it admits, along with (8.1), the operator

$$X_2 = y\frac{\partial}{\partial y}\cdot$$

Now we proceed as in Section 7 and find the invariant solutions for the linear combination $X = X_1 + \lambda X_2$:

$$X = x^\alpha \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}\,, \quad \lambda = \text{const.}$$

Let $\alpha \neq 1$. The characteristic equation

$$\frac{dy}{y} = \lambda \frac{dx}{x^\alpha}$$

of the equation $X(J) = 0$ for the invariants $J(x, y)$ yields the invariant

$$J = y\, \mathrm{e}^{\frac{\lambda}{\alpha-1}\, x^{1-\alpha}}.$$

Hence, the invariant solutions are obtained by setting $J = C$, whence letting $C = 1$ we have

$$y = \mathrm{e}^{\frac{\lambda}{1-\alpha}\, x^{1-\alpha}}. \tag{8.5}$$

Differentiating we have:

$$y' = \lambda\, x^{-\alpha}\, \mathrm{e}^{\frac{\lambda}{1-\alpha}\, x^{1-\alpha}}, \quad y'' = \left[\lambda^2 x^{-2\alpha} - \lambda\alpha x^{-\alpha-1}\right] \mathrm{e}^{\frac{\lambda}{1-\alpha}\, x^{1-\alpha}}.$$

Substituting in Eq. (8.4) and dividing by the non-vanishing factor $\mathrm{e}^{\frac{\lambda}{1-\alpha}\, x^{1-\alpha}}$, we obtain the following *characteristic equation* for Eq. (8.4):

$$\lambda^2 + A\,\lambda + B = 0. \tag{8.6}$$

Eq. (8.6) is identical with the characteristic equation (7.6) for the equation (7.2) with constant coefficients.

Remark 8.2. Eq. (8.5) contains Euler's substitution (7.5) for the equation (7.2) with constant coefficients as a particular case $\alpha = 0$.

Remark 8.3. Using the statement that Eq. (7.1) is mapped to the equation $z'' = 0$ if and only if the function

$$I(x) = g(x) - \frac{1}{4}\, f^2(x) - \frac{1}{2}\, f'(x)$$

vanishes (see Lemma 7.1), one can verify that Eq. (8.5) is equivalent by function to the equation $z'' = 0$ if and only if

$$A^2 - 4B = 0 \quad \text{and} \quad \alpha = 0 \text{ or } \alpha = 2. \tag{8.7}$$

The first equation in (8.7) means that the characteristic equation (8.6) has a repeated root, and hence there is only one solution of the form (8.5). Then, using the reasoning of Lemma 7.1 we can show that the general solution to Eq. (8.4) with $A^2 - 4B = 0,\ \alpha = 2$, i.e. of the equation

$$y'' + \left(\frac{A}{x^2} + \frac{2}{x}\right) y' + \frac{B}{x^4}\, y = 0, \quad A^2 - 4B = 0,$$

is given by

$$y = \left(K_1 + \frac{K_2}{x}\right) \mathrm{e}^{-\frac{\lambda}{x}}, \tag{8.8}$$

where K_1, K_2 are arbitrary constants and λ is the repeated root of the characteristic equation (8.6). For a more general statement, see Section 10.

8.2 Second example

Let us find the linear second-order equations (7.1) admitting the projective group with the generator

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} \cdot \tag{8.9}$$

Taking the second prolongation of the operator (8.9),

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} + (y - xy') \frac{\partial}{\partial y'} - 3xy'' \frac{\partial}{\partial y''} \,,$$

and writing the invariance condition of Eq. (7.1),

$$X_1 (y'' + f(x)y' + g(x)y) \Big|_{(7.1)} = 0,$$

we obtain:

$$x(xf' + 2f)y' + (x^2 g' + 4xg + f)y = 0. \tag{8.10}$$

Eqs. (8.10) yield:

$$f(x) = \frac{A}{x^2} \,, \quad g(x) = \frac{B}{x^4} - \frac{A}{x^3} \,, \quad A, B = \text{const.}$$

Thus, we arrive at the following linear equation admitting the operator (8.9):

$$x^4 y'' + Ax^2 y' + (B - Ax)y = 0, \quad A, B = \text{const.} \tag{8.11}$$

Eq. (8.11) is linear homogeneous, and hence admits, along with (8.9), the operator

$$X_2 = y \frac{\partial}{\partial y} \cdot$$

Now we proceed as in Section 7 and find the invariant solutions for the linear combination $X = X_1 + \lambda X_2$:

$$X = x^2 \frac{\partial}{\partial x} + (x + \lambda) y \frac{\partial}{\partial y} \,, \quad \lambda = \text{const.}$$

Rewriting the characteristic equation of the equation $X(J) = 0$ for the invariants $J(x, y)$ in the form

$$\frac{dy}{y} = \frac{x + \lambda}{x^2} dx$$

we obtain the invariant

$$J = \frac{y}{x} \, \mathrm{e}^{\frac{\lambda}{x}}.$$

Setting $J = C$ and letting $C = 1$ we obtain the following form of the invariant solutions:

$$y = x\,\mathrm{e}^{-\frac{\lambda}{x}}. \tag{8.12}$$

Substituting (8.12) in Eq. (8.11) we reduce the differential equation (8.11) to the algebraic equation

$$\lambda^2 + A\,\lambda + B = 0$$

which is identical with the characteristic equation (7.6) for the equation (7.2) with constant coefficients.

Example 8.1. Solve the equation

$$y'' + \frac{\omega^2}{x^4} = 0, \quad \omega = \text{const.} \tag{8.13}$$

This is an equation of the form (8.11) with $A = 0,\ B = \omega^2$. The algebraic equation (7.6) yields $\lambda_1 = -i\omega,\ \lambda_2 = i\omega$, and hence we have two independent invariant solutions (8.12):

$$y_1 = x\,\mathrm{e}^{i\frac{\omega}{x}}, \quad y_2 = x\,\mathrm{e}^{-i\frac{\omega}{x}}.$$

Taking their real and imaginary parts, just like in the case of constant coefficient equations, we obtain the following fundamental system of solutions:

$$y_1 = x\,\cos\left(\frac{\omega}{x}\right), \quad y_2 = x\,\sin\left(\frac{\omega}{x}\right). \tag{8.14}$$

Hence, the general solution to Eq. (8.12) is given by

$$y = x\left[C_1\cos\left(\frac{\omega}{x}\right) + C_2\sin\left(\frac{\omega}{x}\right)\right].$$

We can also solve the non-homogeneous equation

$$y'' + \frac{\omega^2}{x^4} = F(x), \tag{8.15}$$

e.g. by the method of *variation of parameters*, and obtain

$$y = x\left[C_1\cos\left(\frac{\omega}{x}\right) + C_2\sin\left(\frac{\omega}{x}\right)\right] \tag{8.16}$$
$$+\,\frac{x}{\omega}\left[\cos\left(\frac{\omega}{x}\right)\int xF(x)\sin\left(\frac{\omega}{x}\right)dx - \sin\left(\frac{\omega}{x}\right)\int xF(x)\cos\left(\frac{\omega}{x}\right)dx\right].$$

9 The general result for second-order equations

9.1 Main statements

Note that the operators given in Eqs. (7.3), (7.17), (8.1), (8.9) are particular cases of the generator

$$X_1 = \phi(x)\frac{\partial}{\partial x} + \sigma(x)y\frac{\partial}{\partial y} \tag{9.1}$$

of the general equivalence group of all linear ordinary differential equations. We will find now all linear second-order equations (7.1) admitting the operator (9.1) with any *fixed* functions $\phi(x)$ and $\sigma(x)$.

Taking the second prolongation of the operator (9.1),

$$X_1 = \phi\frac{\partial}{\partial x} + \sigma y\frac{\partial}{\partial y} + \big[\sigma' y + (\sigma - \phi')y'\big]\frac{\partial}{\partial y'} + \big[\sigma'' y + (2\sigma' - \phi'')y' + (\sigma - 2\phi')y''\big]\frac{\partial}{\partial y''}\,,$$

and writing the invariance condition of Eq. (7.1),

$$X_1(y'' + f(x)y' + g(x)y)\Big|_{(7.1)} = 0,$$

we obtain:

$$(\phi f' + f\phi' + 2\sigma' - \phi'')y' + (\phi g' + 2\phi' g + f\sigma' + \sigma'')y = 0.$$

It follows:

$$\begin{aligned} &\phi f' + f\phi' \ + 2\sigma' - \phi'' = 0,\\ &\phi g' + 2\phi' g + f\sigma' + \sigma'' = 0. \end{aligned} \tag{9.2}$$

The first equation (9.2) is written

$$(\phi f)' = (\phi' - 2\sigma)'$$

and yields:

$$f(x) = \frac{1}{\phi}\big[A + \phi' - 2\sigma\big], \quad A = \text{const.} \tag{9.3}$$

Substituting this in the second equation (9.2), we obtain the following non-homogeneous linear first-order equation for determining $g(x)$:

$$\phi g' + 2\phi' g = -\sigma'' - \frac{\sigma'}{\phi}\big[A + \phi' - 2\sigma\big]. \tag{9.4}$$

The homogeneous equation

$$\phi(x)g' + 2\phi'(x)g = 0$$

with a given function $\phi(x)$ yields

$$g = \frac{C}{\phi^2(x)}\,\cdot$$

By variation of the parameter C, we set

$$g = \frac{u(x)}{\phi^2(x)}\,,$$

substitute it in Eq. (9.4) and obtain:

$$u' = -A\sigma' + 2\sigma\sigma' - \phi'\sigma' - \phi\sigma'' \equiv -(A\sigma)' + (\sigma^2)' - (\phi\sigma')',$$

whence

$$u = B - A\sigma + \sigma^2 - \phi\sigma', \quad B = \text{const.}$$

Therefore,

$$g = \frac{1}{\phi^2(x)}\Big[B - A\sigma + \sigma^2 - \phi\sigma'\Big]. \tag{9.5}$$

Thus, we have arrived at the following result.

Theorem 9.1. The homogeneous linear second-order equations (7.1) admitting the operator (9.1) with any given functions $\phi = \phi(x)$ and $\sigma = \sigma(x)$ have the form

$$\phi^2\, y'' + \big(A + \phi' - 2\sigma\big)\phi\, y' + \big(B - A\sigma + \sigma^2 - \phi\sigma'\big)y = 0. \tag{9.6}$$

Now we use the homogeneity of Eq. (9.6) characterized by the generator

$$X_2 = y\frac{\partial}{\partial y}\,\cdot$$

Namely, we look for the invariant solutions with respect to the linear combination $X = X_1 + \lambda X_2$:

$$X = \phi(x)\frac{\partial}{\partial x} + (\sigma(x) + \lambda)y\frac{\partial}{\partial y}\,, \quad \lambda = \text{const.},$$

and arrive at the following statement reducing the problem of integration of the differential equation (9.6) to solution of the quadratic equation, namely, the characteristic equation as in the case of equations with constant coefficients.

Theorem 9.2. Eq. (9.6) has the invariant solutions of the form

$$y = \mathrm{e}^{\int \frac{\sigma(x)+\lambda}{\phi(x)}\,dx}, \tag{9.7}$$

where λ satisfies the *characteristic equation*

$$\lambda^2 + A\lambda + B = 0. \tag{9.8}$$

Proof. We solve the equation $X(J) = 0$ for the invariants $J(x, y)$, i.e. integrate the equation

$$\frac{dy}{y} = \frac{\sigma(x) + \lambda}{\phi(x)} dx$$

and obtain the invariant

$$J = y\,\mathrm{e}^{-\int \frac{\sigma(x)+\lambda}{\phi(x)}\,dx}.$$

Setting $J = C$ and letting $C = 1$ we obtain Eq. (9.7) for the invariant solutions. Thus, we have:

$$y = \mathrm{e}^{\int \frac{\sigma(x)+\lambda}{\phi(x)}\,dx}, \quad y' = \frac{\sigma + \lambda}{\phi}\,\mathrm{e}^{\int \frac{\sigma(x)+\lambda}{\phi(x)}\,dx} \tag{9.9}$$

$$y'' = \frac{1}{\phi^2}\left[(\sigma + \lambda)^2 - (\sigma + \lambda)\phi' + \phi\sigma'\right]\mathrm{e}^{\int \frac{\sigma(x)+\lambda}{\phi(x)}\,dx}.$$

Substituting (9.9) in Eq. (9.6) we obtain Eq. (9.8), thus completing the proof.

9.2 Distinct roots of the characteristic equation

It is manifest that if the characteristic equation (9.8) has distinct real roots $\lambda_1 \neq \lambda_2$, the general solution to Eq. (9.6) is given by

$$y(x) = K_1\,\mathrm{e}^{\int \frac{\sigma(x)+\lambda_1}{\phi(x)}\,dx} + K_2\,\mathrm{e}^{\int \frac{\sigma(x)+\lambda_2}{\phi(x)}\,dx}, \quad K_1, K_2 = \mathrm{const.} \tag{9.10}$$

In the case of complex roots, $\lambda_1 = \gamma + i\theta,\ \lambda_2 = \gamma - i\theta$, the general solution to Eq. (9.6) is given by

$$y(x) = \left[K_1\,\cos\left(\theta \int \frac{dx}{\phi(x)}\right) + K_2\,\sin\left(\theta \int \frac{dx}{\phi(x)}\right)\right]\mathrm{e}^{\int \frac{\sigma(x)+\gamma}{\phi(x)}\,dx}. \tag{9.11}$$

9.3 The case of repeated roots

Theorem 9.3. If the characteristic equation (9.8) has equal roots $\lambda_1 = \lambda_2$, the general solution to Eq. (9.6) is given by

$$y = \left[K_1 + K_2 \int \frac{dx}{\phi(x)}\right]\mathrm{e}^{\int \frac{\sigma(x)+\lambda_1}{\phi(x)}\,dx}, \quad K_1, K_2 = \mathrm{const.} \tag{9.12}$$

Proof. Let us new variables t and z defined by the linear first-order equations

$$X_1(t) \equiv \phi(x)\frac{\partial t}{\partial x} + \sigma(x)\,y\,\frac{\partial t}{\partial y} = 1, \quad X_2(t) \equiv y\,\frac{\partial t}{\partial y} = 0 \tag{9.13}$$

and

$$X_1(z) \equiv \phi(x)\,\frac{\partial z}{\partial x} + \sigma(x)\,y\,\frac{\partial z}{\partial y} = 0, \quad X_2(z) \equiv y\,\frac{\partial z}{\partial y} = z, \tag{9.14}$$

respectively. Eqs. (9.13) are easily solved and yield

$$t = \int \frac{dx}{\phi(x)}\,\cdot \tag{9.15}$$

Integration of the second equation (9.14) with respect to y gives

$$z = v(x)y.$$

Substituting this in the first equation (9.14) we obtain

$$\phi(x)\,\frac{dv}{dx} + \sigma(x)\,v = 0, \quad \text{whence} \quad v = \mathrm{e}^{-\int \frac{\sigma(x)}{\phi(x)}\,dx}.$$

Thus,

$$z = y\,\mathrm{e}^{-\int \frac{\sigma(x)}{\phi(x)}\,dx}. \tag{9.16}$$

The passage to the new variables (9.15), (9.16) converts the operator X_1 given by (9.1) to the translation generator without changing the form of the dilation generator X_2. In other words, upon introducing the new independent and dependent variables t and z given by (9.15) and (9.16), respectively, we arrive at the operators (7.3). Hence, in the new variables, Eq. (9.16) becomes an equation with constant coefficients. Invoking that the equations (9.6) and (7.2) have Eq. (9.8) as their common characteristic equation, we use Lemma 7.1 and write

$$z = (K_1 + K_2\,t)\,\mathrm{e}^{\lambda_1 t}.$$

Substituting this in Eq. (9.16) and replacing t and z by their expressions (9.15) and (9.16), respectively, and solving for y, we finally arrive at Eq. (9.12).

Remark 9.1. We can easily solve the non-homogeneous equation Eq. (9.6):

$$\phi^2\,y'' + \big(A + \phi' - 2\sigma\big)\phi\,y' + \big(B - A\sigma + \sigma^2 - \phi\sigma'\big)y = F(x). \tag{9.17}$$

Namely, we rewrite Eq. (9.6) in the form

$$y'' + a(x)\,y' + b(x)\,y = P(x)$$

and employ the representation of the general solution (see, e.g. [2], Section 3.3.4)

$$y = K_1 y_1(x) + K_2 y_2(x) - y_1(x)\int \frac{y_2(x)P(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)P(x)}{W(x)}\,dx \tag{9.18}$$

furnished by the method of variation of parameters. Here

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

is the Wronskian of a fundamental system of solutions $y_1(x),\ y_2(x)$ for the homogeneous equation

$$y'' + a(x)\,y' + b(x)\,y = 0.$$

10 Examples to Section 9

Euler's substitution (7.5) as well as the solutions (7.19), (8.5) and (8.12) are encapsulated in Eq. (9.7). We will consider now these and several other examples.

Example 10.1. Let us take $\phi(x) = 1,\ \sigma(x) = 0$. Then Eqs. (9.6), (9.7) and (9.12) coincide with Eqs. (7.2), (7.5) and (7.9), respectively. Eq. (9.11) becomes (7.8).

Example 10.2. Let us take $\phi(x) = x,\ \sigma(x) = 0$. Then Eq. (9.6) becomes Euler's equation written in the form (7.21), Eq. (9.7) yields Eq. (7.19) for invariant solutions, whereas Eq. (9.12) provides the general solution

$$y(x) = (K_1 + K_2 \ln x)\, x^{\lambda_1} \quad K_1, K_2 = \text{const.}, \tag{10.1}$$

to Euler's equation (7.21) whose characteristic equation (7.6) has equal roots. Eq. (9.11) leads to the following solution for complex roots $\lambda_1 = \gamma + i\theta, \lambda_2 = \gamma - i\theta$:

$$y(x) = [K_1 \cos(\theta \ln x) + K_2 \sin(\theta \ln x)]\, x^{\gamma}. \tag{10.2}$$

Example 10.3. Let us take $\phi(x) = x^{\alpha},\ \sigma(x) = 0$. Then Eqs. (9.6) and (9.7) coincide with Eqs. (8.4) and (8.5), respectively, whereas Eq. (9.12) provides the general solution

$$y(x) = \big(K_1 + K_2\, x^{1-\alpha}\big) \mathrm{e}^{\frac{\lambda_1}{1-\alpha}\, x^{1-\alpha}}, \quad K_1, K_2 = \text{const.}, \tag{10.3}$$

to Eq. (8.4) whose characteristic equation (8.6) has equal roots $\lambda_1 = \lambda_2$. Eq. (10.3) extends the solution (8.8) to all equations (8.4) with with the coefficients A, B, C satisfying the condition (7.12) of equal roots for the characteristic equation (8.6).

Example 10.4. Let us take $\phi(x) = 1 + x^2,\ \sigma(x) = x$. Then Eq. (9.6) becomes

$$\big(1 + x^2\big)^2 y'' + \big(1 + x^2\big) A\, y' + (B - Ax - 1) = 0. \tag{10.4}$$

Working out the integral in Eq. (9.7),

$$\int \frac{\sigma(x) + \lambda}{\phi(x)}\, dx = \int \frac{x}{1 + x^2}\, dx + \int \frac{\lambda}{1 + x^2}\, dx = \ln \sqrt{1 + x^2} + \lambda \arctan x,$$

we obtain the following expression for the invariant solutions:

$$y = \sqrt{1 + x^2}\, \mathrm{e}^{\lambda \arctan x}, \tag{10.5}$$

where λ satisfies the characteristic equation (9.8):

$$\lambda^2 + A\lambda + B = 0. \tag{10.6}$$

If the characteristic equation (10.6) has distinct real roots, $\lambda_1 \neq \lambda_2$, the general solution to Eq. (10.4) is given by

$$y(x) = \sqrt{1+x^2}\ \left[K_1\,\mathrm{e}^{\lambda_1 \arctan x} + K_2\,\mathrm{e}^{\lambda_2 \arctan x}\right]. \tag{10.7}$$

In the case of complex roots, $\lambda_1 = \gamma + i\theta,\ \lambda_2 = \gamma - i\theta$, the general solution to Eq. (10.4) is given by

$$y(x) = \left[K_1\,\cos\left(\theta\,\arctan x\right) + K_2\,\sin\left(\theta\,\arctan x\right)\right]\sqrt{1+x^2}\ \mathrm{e}^{\lambda \arctan x}. \tag{10.8}$$

Finally, if the characteristic equation (10.6) has equal roots $\lambda_1 = \lambda_2$, the general solution to Eq. (10.4) is given by

$$y(x) = \big(K_1 + K_2\,\arctan x\big)\sqrt{1+x^2}\ \mathrm{e}^{\lambda \arctan x}. \tag{10.9}$$

Example 10.5. Consider Eq. (10.4) with $A = 0,\ B = \omega^2$. Then, according to Example 10.4, the equation

$$\left(1+x^2\right)^2 y'' + (\omega^2 - 1)y = 0 \tag{10.10}$$

has the following general solution:

$$y(x) = \left[K_1\,\cos\left(\omega\,\arctan x\right) + K_2\,\sin\left(\omega\,\arctan x\right)\right]\sqrt{1+x^2}\,. \tag{10.11}$$

Example 10.6. Let us solve the non-homogeneous equation

$$\left(1+x^2\right)^2 y'' + (\omega^2 - 1)y = F(x). \tag{10.12}$$

Example 10.5 provides the fundamental system of solutions

$$y_1 = \sqrt{1+x^2}\ \cos\left(\omega\,\arctan x\right), \quad y_2 = \sqrt{1+x^2}\ \sin\left(\omega\,\arctan x\right)$$

for the homogeneous equation (10.10). We have:

$$y_1' = \frac{1}{\sqrt{1+x^2}}\ \left[x\,\cos\left(\omega\,\arctan x\right) - \omega\sin\left(\omega\,\arctan x\right)\right],$$

$$y_2' = \frac{1}{\sqrt{1+x^2}}\ \left[x\,\sin\left(\omega\,\arctan x\right) + \omega\cos\left(\omega\,\arctan x\right)\right].$$

Hence the Wronskian is $W[y_1, y_2] = y_1 y_2' - y_2 y_1' = \omega$. Now we rewrite Eq. (10.12), in accordance with Remark 9.1, in the form

$$y'' + \frac{\omega^2 - 1}{(1+x^2)^2}\,y = \frac{F(x)}{(1+x^2)^2}\,, \tag{10.13}$$

use Eq. (9.18) and obtain the following general solution to Eq. (10.12):

$$\begin{aligned} y(x) = \sqrt{1+x^2}\,\Big[&K_1\,\cos\left(\omega\,\arctan x\right) + K_2\,\sin\left(\omega\,\arctan x\right) \\ &- \frac{1}{\omega}\,\cos\left(\omega\,\arctan x\right)\int \frac{F(x)}{(1+x^2)^{3/2}}\ \sin\left(\omega\,\arctan x\right)dx \\ &+ \frac{1}{\omega}\,\sin\left(\omega\,\arctan x\right)\int \frac{F(x)}{(1+x^2)^{3/2}}\ \cos\left(\omega\,\arctan x\right)dx\Big]. \end{aligned} \tag{10.14}$$

11 Third-order equations

The previous results can be extended to higher-order linear ordinary differential equations. I will discuss here the third-order equations

$$y''' + f(x)y'' + g(x)y' + h(x)y = 0. \tag{11.1}$$

Theorem 11.1. The homogeneous linear third-order equations (11.1) admitting the operator (9.1) with any given $\phi = \phi(x)$ and $\sigma = \sigma(x)$ have the form

$$\begin{aligned}
&\phi^3 y''' + \big[A + 3(\phi' - \sigma)\big]\phi^2 y'' \qquad (11.2)\\
&+ \big[B + A\phi' - 2A\sigma + \phi\phi'' + (\phi')^2 - 3(\phi\sigma)' + 3\sigma^2\big]\phi\, y'\\
&+ \big[C - B\sigma + A\sigma^2 - A\phi\sigma' - \sigma^3 - \phi^2\sigma'' - \phi\phi'\sigma' + 3\phi\sigma\sigma'\big]\, y = 0.
\end{aligned}$$

Proof. We take the third prolongation of the operator (9.1):

$$\begin{aligned}
X_1 = \phi\frac{\partial}{\partial x} + \sigma y\frac{\partial}{\partial y} &+ \big[\sigma' y + (\sigma - \phi')y'\big]\frac{\partial}{\partial y'}\\
&+ \big[\sigma'' y + (2\sigma' - \phi'')y' + (\sigma - 2\phi')y''\big]\frac{\partial}{\partial y''}\\
&+ \big[\sigma''' y + (3\sigma'' - \phi''')y' + 3(\sigma' - \phi'')y'' + (\sigma - 3\phi')y'''\big]\frac{\partial}{\partial y'''}\,,
\end{aligned}$$

and write the invariance condition of Eq. (11.1):

$$X_1(y''' + f(x)y'' + g(x)y' + h(x)y)\Big|_{(11.1)} = 0.$$

We annul the coefficients for y'', y' and y of the left-hand side of the above equation and split it into the following three equations:

$$\phi f' + \phi' f + 3(\sigma' - \phi'') = 0, \tag{11.3}$$

$$\phi g' + 2\phi' g + (2\sigma' - \phi'')f - \phi''' + 3\sigma'' = 0, \tag{11.4}$$

$$\phi h' + 3\phi' h + \sigma'' f + \sigma' g + \sigma''' = 0. \tag{11.5}$$

Eq. (11.3) is written

$$(\phi f)' = 3(\phi' - \sigma)'$$

and yields:

$$f = \frac{1}{\phi}\big[A + 3(\phi' - \sigma)\big], \quad A = \text{const}. \tag{11.6}$$

We substitute (11.6) in Eq. (11.4), integrate the resulting non-homogeneous linear first-order equation for g and obtain:

$$g = \frac{1}{\phi^2}\left[B + A\phi' - 2A\sigma + \phi\phi'' + (\phi')^2 - 3(\phi\sigma)' + 3\sigma^2\right], \tag{11.7}$$

where B is an arbitrary constant. Now we substitute (11.6), (11.7) in Eq. (11.5), integrate the resulting first-order equation for h and obtain:

$$h = \frac{1}{\phi^3}\left[C - B\sigma + A\sigma^2 - A\phi\sigma' - \sigma^3 - \phi^2\sigma'' - \phi\phi'\sigma' + 3\phi\sigma\sigma'\right], \tag{11.8}$$

where C is an arbitrary constant. Finally, substituting (11.6), (11.7) and (11.8) in Eq. (11.1), we arrive at Eq. (11.2).

Theorem 11.2. Eq. (11.2) has the invariant solutions of the form (9.7),

$$y = \mathrm{e}^{\int \frac{\sigma(x)+\lambda}{\phi(x)}\,dx}, \tag{9.7}$$

where λ satisfies the *characteristic equation*

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0. \tag{11.9}$$

Proof. Adding to Eqs. (9.9) the expression for the third derivative y''' and substituting in Eq. (11.2) we obtain Eq. (11.9).

Chapter 3

Using connection between Riccati and second-order linear equations

12 Introduction

Recall that the Riccati equation

$$y' = P(x) + Q(x)y + R(x)y^2, \quad R(x) \neq 0, \tag{12.1}$$

is mapped by the substitution

$$y = -\frac{1}{R(x)}\,\frac{u'}{u} \tag{12.2}$$

to the linear second-order equation

$$u'' + f(x)u' + g(x)u = 0 \tag{12.3}$$

with the coefficients

$$f(x) = -\left[Q(x) + \frac{R'(x)}{R(x)}\right], \quad g(x) = P(x)R(x). \tag{12.4}$$

Indeed, we have:

$$y' = -\frac{1}{R}\,\frac{u''}{u} + \frac{R'}{R^2}\,\frac{u'}{u} + \frac{1}{R}\,\frac{u'^2}{u^2}$$

and

$$P + Qy + Ry^2 = P - \frac{Q}{R}\,\frac{u'}{u} + \frac{1}{R}\,\frac{u'^2}{u^2}\,.$$

Substituting these expressions in Eq. (12.1) and multiplying by $-Ru$ we obtain the equation

$$u'' - \frac{R'}{R}\,u' = Qu' - PRu,$$

i.e. Eq. (12.3) with the coefficients (12.4).

13 From Riccati to second-order equations

13.1 Application to Equation (2.9)

Applying Eqs. (12.3)-(12.4) to Eq. (2.9),

$$y' = Q(x)y + R(x)y^2, \tag{2.9}$$

we obtain the following second-order linear equation:

$$u'' = \left(Q + \frac{R'}{R}\right)u'. \tag{13.1}$$

The integration yields:

$$\ln u' = \int \left(Q + \frac{R'}{R}\right)dx + \ln C_1 = \int Q dx + \ln R + \ln C_1.$$

Hence

$$u' = C_1 R(x) \mathrm{e}^{\int Q(x)dx} \tag{13.2}$$

and

$$u = C_1 \int R(x) \mathrm{e}^{\int Q(x)dx}\, dx + C_2. \tag{13.3}$$

Substituting (13.2) and (13.3) in Eq. (12.2) and denoting $C = C_2/C_1$, we arrive at the solution (3.6) to Eq. (2.9):

$$y = -\frac{\mathrm{e}^{\int Q(x)dx}}{C + \int R(x)\mathrm{e}^{\int Q(x)dx}\, dx}\,. \tag{3.6}$$

13.2 Application to Equation (2.10)

The following examples clarify how to use the linearizable equations (2.10),

$$y' = P(x) + Q(x)y + k[Q(x) - kP(x)]y^2, \quad k = \text{const.}, \tag{2.10}$$

for integrating the corresponding second-order linear equations (12.3).

Example 13.1. If we apply Eqs. (12.3)-(12.4) to Eq. (4.7) from Example 4.1,

$$y' = x + 2xy + xy^2,$$

we obtain the following second-order linear equation:

$$u'' - \left(2x + \frac{1}{x}\right)u' + x^2 u = 0. \tag{13.4}$$

Let us integrate this equation. Writing Eq. (12.2) in the form

$$\frac{u'}{u} = -R(x)y,$$

substituting here $R(x) = x$ and the expression (4.8) for y, we get

$$\frac{u'}{u} = -x\frac{1 + C + x^2}{1 - C - x^2}\,\cdot$$

Writing this equation in the form

$$\frac{d\ln u}{dx} = x + \frac{2x}{x^2 + C - 1}$$

and integrating we obtain the following general solution to Eq. (13.4):

$$u = K(x^2 + C - 1)\mathrm{e}^{x^2/2}, \quad C, K = \text{const.} \tag{13.5}$$

Example 13.2. If we apply Eqs. (12.3)-(12.4) to Eq. (4.9) from Example 4.2,

$$y' = x^2 + (x + x^2)\,y + \frac{1}{4}\big(2x + x^2\big)\,y^2,$$

we obtain the following second-order linear equation:

$$u'' - (1 + x)\left(x + \frac{2}{2x + x^2}\right)u' + \frac{1}{4}x^2\big(2x + x^2\big)u = 0. \tag{13.6}$$

Let us integrate this equation. Writing Eq. (12.2) in the form

$$\frac{u'}{u} = -R(x)y,$$

substituting here

$$R(x) = \frac{1}{4}\big(2x + x^2\big)$$

and the expression (4.10) for y, we get

$$\frac{u'}{u} = -\frac{1}{2}\big(2x + x^2\big)\frac{1 + \Big(C + \int (x + x^2)\mathrm{e}^{-x^2/2}\,dx\Big)\,\mathrm{e}^{x^2/2}}{1 - \Big(C + \int (x + x^2)\mathrm{e}^{-x^2/2}\,dx\Big)\,\mathrm{e}^{x^2/2}}\,\cdot$$

The integration yields $\ln|u| = \ln|K| + \phi(x)$, where

$$\phi(x) = -\frac{1}{2}\int \big(2x + x^2\big)\frac{1 + \Big(C + \int (x + x^2)\mathrm{e}^{-x^2/2}\,dx\Big)\,\mathrm{e}^{x^2/2}}{1 - \Big(C + \int (x + x^2)\mathrm{e}^{-x^2/2}\,dx\Big)\,\mathrm{e}^{x^2/2}}\,dx.$$

Hence, the general solution to Eq. (13.6) has the form

$$u = K\,\mathrm{e}^{\phi(x)}, \tag{13.7}$$

where $\phi(x)$ is given above and K is an arbitrary constant.

Applying Eqs. (12.3)-(12.4) to Eq. (2.10) and using the solution procedure for Eq. (2.10) described in Section 4, we obtain the following general result.

Theorem 13.1. The general solution of the second-order linear equation

$$u'' - \left[Q(x)x + \frac{Q'(x) - kP'(x)}{Q(x) - kP(x)}\right]u' + k\big[P(x)Q(x) - kP^2(x)\big]u = 0 \qquad (13.8)$$

with an arbitrary constant k and two arbitrary functions $P(x)$ and $Q(x)$ can be obtained by quadratures.

14 From second-order to Riccati equations

It is manifest from Eqs. (12.4) that *two* coefficients $f(x)$ and $g(x)$ of a given second-order equation (12.3) do not uniquely determine *three* coefficients $P(x), Q(x), R(x)$ of the corresponding Riccati equation (12.1). Namely, if we know solutions of an equation (12.3), we can solve by using the formula (12.2) an infinite set of the Riccati equations

$$y' = R(x)y^2 - \left[f(x) + \frac{R'(x)}{R(x)}\right]y + \frac{g(x)}{R(x)} \qquad (14.1)$$

with an arbitrary function $R(x) \neq 0$.

Example 14.1. Consider the following equation with constant coefficients:

$$u'' + u = 0. \qquad (14.2)$$

Here $f = 0,\ g = 1$. Hence, the corresponding Riccati equation (14.1) has the form

$$y' = R(x)\,y^2 - \frac{R'(x)}{R(x)}\,y + \frac{1}{R(x)}\,\cdot \qquad (14.3)$$

Substituting the general solution

$$u = C_1 \cos x + C_2 \sin x$$

of Eq. (14.2) in (12.2) we obtain the following solution to Eq. (14.3):

$$y = \frac{1}{R(x)}\,\frac{C_1 \sin x - C_2 \cos x}{C_1 \cos x + C_2 \sin x}\,\cdot$$

If $C_2 \neq 0$ we denote $K = C_2/C_1$ and writhe the solution in the form

$$y = \frac{1}{R(x)}\,\frac{\sin x - K \cos x}{\cos x + K \sin x}$$

or, upon dividing the numerator and denominator by $\cos x$,

$$y = \frac{1}{R(x)}\,\frac{\mathrm{tg}x - K}{1 + K\mathrm{tg}x}\,, \quad K = \mathrm{const}. \tag{14.4}$$

If $C_2 = 0$ the solution becomes

$$y = -\frac{\mathrm{ctg}x}{R(x)}$$

which can be obtained from (14.4) by letting $K \to \infty$. Thus, the general solution to the Riccati equation (14.5) is given by (14.4) where $-\infty \leq K \leq +\infty$.

In particular, taking in (14.5), (14.4) $R(x) = \mathrm{e}^x$ we conclude that the equation

$$y' = \mathrm{e}^x\, y^2 - y + \mathrm{e}^{-x} \tag{14.5}$$

has the general solution

$$y = \frac{\mathrm{tg}x - K}{1 + K\mathrm{tg}x}\,\mathrm{e}^{-x}, \quad -\infty \leq K \leq +\infty. \tag{14.6}$$

Sometimes it is convenient to use a restricted correspondence between second-order and Riccati equations by writing Eqs. (12.3)-(12.4) corresponding to the Riccati equation (12.1) in the following form:

$$R(x)\,u'' - \big[R'(x) + Q(x)R(x)\big]u' + P(x)R^2(x)\,u = 0. \tag{14.7}$$

Example 14.2. Consider Euler's equation written in the form (7.21):

$$x^2\,u'' + (A+1)\,x\,u' + B\,u = 0, \quad A, B = \mathrm{const}. \tag{14.8}$$

Comparing the equations (14.7) and (14.8) we take $R(x) = x^2$ and obtain

$$Q(x) = -\frac{A+3}{x} \qquad P(x) = \frac{B}{x^4}\,\cdot$$

Thus, we have arrived at the following Riccati equation:

$$y' = x^2y^2 - \frac{A+3}{x}\,y + \frac{B}{x^4}\,\cdot \tag{14.9}$$

We know that the solutions of Eq. (14.8) have the form $u = x^\lambda$. Substitution in (12.2) yields the following form of solutions to Eq. (14.9):

$$y = -\frac{\lambda}{x^3}\,\cdot \tag{14.10}$$

Substituting (14.10) in Eq. (14.9) we obtain again the characteristic equation (7.6):

$$\lambda^2 + A\lambda + B = 0. \tag{14.11}$$

However, Eqs. (14.10), (14.11) provide only two particular solutions to Eq. (14.9). In order to find the general solution of the *nonlinear* equation (14.9), we have to construct the general solution $u(x)$ of the *linear* equation (14.8) using the *superposition principle* and substitute $u = u(x)$ in (12.2).

Using the results of Section 9 on integrability of Eq. (9.6),

$$\phi^2 y'' + \big(A + \phi' - 2\sigma\big)\phi\, y' + \big(B - A\sigma + \sigma^2 - \phi\sigma'\big)y = 0,$$

we can formulate the following general result.

Theorem 14.1. The Riccati equation

$$y' = R(x)\, y^2 - \left[\frac{A + \phi\,' - 2\sigma}{\phi} + \frac{R'}{R}\right] y + \frac{B - A\sigma + \sigma^2 - \phi\sigma\,'}{R\phi^2} \tag{14.12}$$

with three arbitrary functions $R(x),\ \phi(x),\ \sigma(x)$ and two arbitrary constants $A,\ B$ is integrable by quadratures.

15 Application to Ermakov's equation

The above results on integration of linear equations

$$u'' + a(x)u' + b(x)u = 0 \tag{15.1}$$

can be combined with Ermakov's method for solving nonlinear equations of the following form (see Editor's preface to Ermakov's paper in this volume):

$$u'' + a(x)u' + b(x)u = \frac{\alpha}{u^3}\, \mathrm{e}^{-2\int a(x)dx}, \quad \alpha = \text{const.} \tag{15.2}$$

Example 15.1. Using the solution (13.5) of Eq. (13.4) and applying Ermakov's method one can solve the following nonlinear equation:

$$u'' - \left(2x + \frac{1}{x}\right)u' + x^2 u = \alpha x^2 \mathrm{e}^{2x^2}\, u^{-3}. \tag{15.3}$$

Example 15.2. The nonlinear equation (15.2) associated with the integrable equation (9.6) has the form

$$\phi^2 u'' + \big(A + \phi' - 2\sigma\big)\phi\, u' + \big(B - A\sigma + \sigma^2 - \phi\sigma'\big)u = \frac{\alpha}{u^3}\, \mathrm{e}^{2\int(2\sigma - A)/\phi\, dx}, \tag{15.4}$$

where ϕ and σ are arbitrary functions of x, and A is an arbitrary constant. Eq. (15.4) is integrable by quadratures.

Bibliography

[1] V. V. Stepanov, *Course of differential equations*. Moscow: State Publisher of Phys.-Math. Literature, 1958. Russian, 7th ed.

[2] N. H. Ibragimov, *A practical course in differential equations and mathemtical modelling*. Karlskrona: ALGA Publications, 3rd ed., 2006.

[3] N. H. Ibragimov, *Elementary Lie group analysis and ordinary differential equations*. Chichester: John Wiley & Sons, 1999.

[4] N. H. Ibragimov, *Primer of group analysis*. Moscow: Znanie, No. 8, 1989. (Russian). Revised edition in English: *Introduction to modern group analysis*, Tau, Ufa, 2000. Available also in Swedish: *Modern grouppanalys: En inledning till Lies lösningsmetoder av ickelinjära differentialekvationer,* Studentlitteratur, Lund, 2002.

Archives of ALGA
Volume 5, 2008

EULER'S INTEGRATION METHOD FOR LINEAR HYPERBOLIC EQUATIONS AND ITS EXTENSION TO PARABOLIC EQUATIONS

NAIL H. IBRAGIMOV
Department of Mathematics and Science,
Research Centre ALGA: Advances in Lie Group Analysis,
Blekinge Institute of Technology,
SE-371 79 Karlskrona, Sweden

Abstract. The first significant results towards the general integration theory for hyperbolic equations with two independent variables were obtained by Leonard Euler. He generalized d'Alembert's solution to a wide class of linear hyperbolic equations with two independent variables and introduced the quantities that were rediscovered by Laplace and known today as the Laplace invariants. In the present paper, I give an overview of Euler's method for hyperbolic equations and then extend Euler's method to parabolic equations. The new method, based on the invariant of parabolic equations, allows one to identify all linear parabolic equations reducible to the heat equation and find their general solution. The method is illustrated by the Black-Scholes equation for which the general solution and the solution of an arbitrary Cauchy problem are provided.

Keywords: Laplace invariants, Semi-invariant for parabolic equations, Reducible parabolic equations, Non-linearization of Black-Scholes model, Optimal system of invariant solutions.

1 Introduction

When I studied partial differential equations at university, one of amazing facts for me was that one could integrate the wave equation

$$u_{tt} - u_{xx} = 0 \tag{1.1}$$

and obtain explicitly its general solution

$$u = f(x+t) + g(x-t),$$

while the general solution is not given in university texts for two other ubiquitous equations: the heat equation

$$v_t - v_{xx} = 0 \tag{1.2}$$

and the Laplace equation

$$u_{xx} + u_{yy} = 0. \tag{1.3}$$

Later it become clear that this was partially due to a significant difference between the three classical equations in terms of characteristics. Namely, the wave equation has two families of real characteristics and can be written in the characteristic variables $\xi = x + t,\ \eta = x - t$ in the *factorable* form:

$$u_{\xi\eta} \equiv D_\eta D_\xi u = 0. \tag{1.4}$$

Therefore it can be solved by consecutive integration of two first-order linear ordinary differential equations:

$$D_\eta v = 0, \quad D_\xi u = v.$$

The integration can also be done by writing Eq. (1.1) in the factorized form:

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u = 0.$$

A similar factorization to first-order differential operators is impossible for the heat and Laplace operators since the heat equation has only one family of characteristics whereas the Laplace equation has no real characteristics. Of course, one can factorize the Laplace operator by introducing the complex variable $\zeta = iy$, but this trick is just a conversion of the problem on integration to an equivalent problem on extracting the real part of the complex solution

$$u = f(x + iy) + g(x - iy).$$

Let us consider the hyperbolic equations written, using the characteristic variables, in the standard form

$$u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0. \tag{1.5}$$

Can we factorize the left-hand side of every equation (1.5), i.e. write it as the product of two first-order linear differential operators? Leonard Euler [1] (see also [2], Introduction, and the references therein) showed that, in general, the answer to this question is negative. Furthermore, he gave the necessary and sufficient condition for Eq. (1.5) to be factorable and formulated the result in terms of the quantities

$$h = a_x + ab - c, \quad k = b_y + ab - c. \tag{1.6}$$

Namely, he demonstrated that Eq. (1.5) is factorable if and only if at least one of the quantities (1.6) vanishes. The solution of the factorized equation (1.5) is obtained by the consecutive integration of two first-order ordinary differential equations. If $h = k = 0$, Eq. (1.5) is reducible to the wave equation (1.4).

Later, the quantities (1.6) were rediscovered by Laplace [3] in connection with what is called today "Laplace's cascade method" and became known in the literature as the *Laplace invariants.*

In order to understand if Euler's method strongly requires *two families of characteristics* or it can be extended to parabolic equations using only *one family of characteristics*, I first investigated in [4] the question on existence of invariants of parabolic equations similar the Laplace invariants.

The present paper (see also [5]) is a continuation of the work [4] and gives an affirmative answer to the question on possibility of an extension of Euler's method to the parabolic equations. First of all, the linear parabolic equations with two independent variables are transformed to the *standard form*

$$u_t - u_{xx} + a(t,x)u_x + c(t,x)u = 0 \tag{1.7}$$

by an appropriate change of coordinates. Then the condition of reducibility of Eq. (1.7) to the heat equation (1.2) is obtained in terms of the invariant

$$K = aa_x - a_{xx} + a_t + 2c_x \tag{1.8}$$

of Eq. (1.7) found in [4]. Namely, it is shown that Eq. (1.7) can be mapped to the heat equation (1.2) by an appropriate change of the dependent variable if and only if the invariant (1.8) vanishes, i.e. $K = 0$.

The method developed here allows one to derive an explicit formula for the general solution of a wide class of parabolic equations. In particular, the general solution of the Black-Scholes equation is obtained and used for the solution of the Cauchy problem.

2 Euler's method of integration of hyperbolic equations

2.1 Standard form of hyperbolic equations

The general form of the homogeneous linear second-order partial differential equations with two independent variables, x and y, is

$$A\,u_{xx} + 2B\,u_{xy} + C\,u_{yy} + a\,u_x + b\,u_y + c\,u = 0, \tag{2.1}$$

where $A = A(x,y), \ldots, c = c(x,y)$ are prescribed functions. The terms with the second derivatives,

$$A\,u_{xx} + 2B\,u_{xy} + C\,u_{yy}, \tag{2.2}$$

compose the *principal part* of Equation (2.1).

The crucial step in studying Eq. (2.1) is the reduction of its principal part (2.2) to the so-called *standard form* by a change of variables

$$\xi = \varphi(x,y), \quad \eta = \psi(x,y). \tag{2.3}$$

Let us obtain the standard forms of the principal parts for Eq. (2.1). The change of variables (2.3) leads to the following transformation of derivatives:

$$
\begin{aligned}
& u_x = \varphi_x u_\xi + \psi_x u_\eta, \qquad u_y = \varphi_y u_\xi + \psi_y u_\eta,\\
& u_{xx} = \varphi_x^2 u_{\xi\xi} + 2\varphi_x\psi_x u_{\xi\eta} + \psi_x^2 u_{\eta\eta} + \varphi_{xx} u_\xi + \psi_{xx} u_\eta,\\
& u_{yy} = \varphi_y^2 u_{\xi\xi} + 2\varphi_y\psi_y u_{\xi\eta} + \psi_y^2 u_{\eta\eta} + \varphi_{yy} u_\xi + \psi_{yy} u_\eta,\\
& u_{xy} = \varphi_x\varphi_y u_{\xi\xi} + (\varphi_x\psi_y + \varphi_y\psi_x) u_{\xi\eta} + \psi_x\psi_y u_{\eta\eta} + \varphi_{xy} u_\xi + \psi_{xy} u_\eta.
\end{aligned}
\tag{2.4}
$$

Substituting the expressions (2.4) in (2.1) we see that Eq. (2.1) is written in the new variables as follows:

$$
\widetilde{A}\, u_{\xi\xi} + 2\widetilde{B}\, u_{\xi\eta} + \widetilde{C}\, u_{\eta\eta} + \tilde{a} u_\xi + \tilde{b} u_\eta + \tilde{c} u = 0, \tag{2.5}
$$

where

$$
\begin{aligned}
& \widetilde{A} = A\varphi_x^2 + 2B\varphi_x\varphi_y + C\varphi_y^2,\\
& \widetilde{B} = A\varphi_x\psi_x + B(\varphi_x\psi_y + \varphi_y\psi_x) + C\varphi_y\psi_y,\\
& \widetilde{C} = A\psi_x^2 + 2B\psi_x\psi_y + C\psi_y^2\\
& \tilde{a} = A\varphi_{xx} + 2B\varphi_{xy} + C\varphi_{yy} + a\varphi_x + b\varphi_y,\\
& \tilde{b} = A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + a\psi_x + b\psi_y,\\
& \tilde{c} = c.
\end{aligned}
\tag{2.6}
$$

The principal part of Eq. (2.5) is

$$
\widetilde{A}\, u_{\xi\xi} + 2\widetilde{B}\, u_{\xi\eta} + \widetilde{C}\, u_{\eta\eta}. \tag{2.7}
$$

It is manifest from (2.6) that the principal part (2.7) will have only one term, $2\widetilde{B}\, u_{\xi\eta}$, if we choose for $\varphi(x,y)$ and $\psi(x,y)$ two functionally independent solutions, $\omega_1 = \varphi(x,y)$ and $\omega_2 = \psi(x,y)$, of the equation

$$
A\,\omega_x^2 + 2B\,\omega_x\omega_y + C\,\omega_y^2 = 0 \tag{2.8}
$$

known as the *characteristic equation* for Eq. (2.1). Recall that for hyperbolic equations (2.1) the characteristic equation (2.8) has precisely two functionally independent solutions.

If $\omega(x,y)$ is any solution of Eq. (2.1), the curves

$$
\omega(x,y) = \text{const.} \tag{2.9}
$$

are called *characteristics* of Eq. (2.1). In order to find the characteristics, we set

$$\frac{\omega_x}{\omega_y} = \lambda \tag{2.10}$$

and rewrite the characteristic equation (2.8) in the form

$$A(x,y)\lambda^2 + 2B(x,y)\lambda + C(x,y) = 0. \tag{2.11}$$

For hyperbolic equations $B^2 - AC > 0$ and the quadratic equation (2.11) has two distinct real roots, $\lambda_1(x,y)$ and $\lambda_2(x,y)$ given by

$$\lambda_1(x,y) = \frac{-B + \sqrt{B^2 - AC}}{A}, \quad \lambda_2(x,y) = \frac{-B - \sqrt{B^2 - AC}}{A}. \tag{2.12}$$

Substituting them in (2.10), we see that the characteristic equation (2.8) splits into two different linear first-order partial differential equations:

$$\frac{\partial \omega}{\partial x} - \lambda_1 \frac{\partial \omega}{\partial y} = 0, \quad \frac{\partial \omega}{\partial x} - \lambda_2 \frac{\partial \omega}{\partial y} = 0. \tag{2.13}$$

The characteristic systems for the equations (2.13) are

$$\frac{dx}{1} + \frac{dy}{\lambda_1(x,y)} = 0, \quad \frac{dx}{1} + \frac{dy}{\lambda_2(x,y)} = 0. \tag{2.14}$$

Each equation (2.14) has one independent first integral, $\varphi(x,y) = \text{const.}$ and $\psi(x,y) = \text{const.}$ for the first and the second equation (2.14), respectively. Accordingly, the functions $\varphi(x,y)$ and $\psi(x,y)$ satisfy the first and the second equation (2.13), respectively:

$$\frac{\partial \varphi}{\partial x} - \lambda_1 \frac{\partial \varphi}{\partial y} = 0, \quad \frac{\partial \psi}{\partial x} - \lambda_2 \frac{\partial \psi}{\partial y} = 0, \tag{2.15}$$

and hence they are functionally independent. Thus, they provide two functionally independent solutions of the characteristic equation (2.11) and therefore one can take them as the right-hand sides in the change of variables (2.3). The new variables

$$\xi = \omega_1(x,y), \quad \eta = \omega_2(x,y), \tag{2.16}$$

where $\omega_1(x,y)$ and $\omega_2(x,y)$ are two functionally independent solutions of the characteristic equation are termed the *characteristic variables.* Thus, in the characteristic variables Eq. (2.5) becomes

$$2\widetilde{B}\, u_{\xi\eta} + \tilde{a}u_\xi + \tilde{b}u_\eta + \tilde{c}u = 0.$$

Dividing it by $2\widetilde{B}$, skipping the tilde and denoting ξ and η by x and y, respectively, we arrive at the following *standard form* of the hyperbolic equations:

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0. \tag{2.17}$$

2.2 Essence of Euler's method

We owe to Leonard Euler [1] the first significant results in integration theory of hyperbolic equations. He generalized d'Alembert's solution to a wide class of Eqs. (2.17). He introduced the quantities[1]

$$h = a_x + ab - c, \quad k = b_y + ab - c \tag{2.18}$$

and showed that Eq. (2.17) is factorable if and only if at least one of the quantities h and k vanishes (see [1]; see also [2], Introduction, and the references therein). The solution of the factorized equation (2.17) is obtained by the consecutive integration of two first-order ordinary differential equations.

Euler's method consists in the following. Consider Eq. (2.17) with $h = 0$. Then this equation is factorable in the form

$$\left(\frac{\partial}{\partial x} + b\right)\left(\frac{\partial u}{\partial y} + au\right) = 0. \tag{2.19}$$

Setting

$$v = u_y + a\,u \tag{2.20}$$

one can rewrite Equation (2.19) as a first-order equation

$$v_x + b\,v = 0$$

and integrate it to obtain:

$$v = Q(y)\mathrm{e}^{-\int b(x,y)dx}. \tag{2.21}$$

Now we substitute (2.21) in (2.20), integrate the resulting non-homogeneous linear equation

$$u_y + au = Q(y)\mathrm{e}^{-\int b(x,y)dx} \tag{2.22}$$

with respect to y and obtain the following general solution of Eq. (2.17) with $h = 0$:

$$u = \Big[P(x) + \int Q(y)\mathrm{e}^{\int a\,dy - b\,dx}dy\Big]\mathrm{e}^{-\int a\,dy}, \tag{2.23}$$

where $P(x)$ and $Q(y)$ are arbitrary functions.

Likewise, if $k = 0$, Eq. (2.17) is factorable in the form

$$\left(\frac{\partial}{\partial y} + a\right)\left(\frac{\partial u}{\partial x} + b\,u\right) = 0. \tag{2.24}$$

[1]The quantities (2.18) were rediscovered by Laplace [3] in connection with what is called today "Laplace's cascade method" and became known in the literature as the *Laplace invariants.*

In this case, we replace the substitution (2.20) by

$$w = u_x + bu \tag{2.25}$$

Now we repeat the calculations made in the case $h = 0$ and obtain the following general solution of Eq. (2.17) with $k = 0$:

$$u = \left[Q(y) + \int P(x) e^{\int b\,dx - a\,dy} dx\right] e^{-\int b\,dx}. \tag{2.26}$$

2.3 Equivalence transformations

In particular, Euler's method allows one to identify those equations (2.17) that can be reduced to the wave equation by a change of variables, and hence, solved by d'Alembert's method. We will single out all such equations in the next section. Here we discuss the most general form of the changes of variables preserving the linearity and homogeneity of hyperbolic equations as well as their standard form (2.17). These changes of variables are termed *equivalence transformations.* They are well known and have the form

$$\widetilde{x} = f(x), \quad \widetilde{y} = g(y), \quad v = \sigma(x, y)\, u, \tag{2.27}$$

where $f'(x) \neq 0,\ g'(y) \neq 0$, and $\sigma(x, y) \neq 0$. Here u and v are regarded as functions of x, y and $\widetilde{x}, \widetilde{y}$, respectively. The equations (2.17) related by an equivalence transformation (2.27) are said to be *equivalent.*

Let us begin with the restricted equivalence transformations (2.27) by setting $\widetilde{x} = x,\ \widetilde{y} = y$ and find the equations (2.17) reducible to the wave equation by the linear transformation of the dependent variable written in the form

$$v = u\, \mathrm{e}^{\varrho(x,y)}. \tag{2.28}$$

We substitute the expressions

$$\begin{aligned} u &= v\, \mathrm{e}^{-\varrho(x,y)}, \\ u_x &= (v_x - v\varrho_x)\, \mathrm{e}^{-\varrho(x,y)}, \quad u_y = (v_y - v\varrho_y)\, \mathrm{e}^{-\varrho(x,y)}, \\ u_{xy} &= (v_{xy} - v_x\varrho_y - v_y\varrho_x - v\, \varrho_{xy} + v\, \varrho_x\, \varrho_y)\, \mathrm{e}^{-\varrho(x,y)} \end{aligned}$$

in the left-hand side of Eq. (2.17) and obtain:

$$\begin{aligned} & u_{xy} + a\, u_x + b\, u_y + cu \\ &= \big[v_{xy} + (a - \varrho_y)\, v_x + (b - \varrho_x)\, v_y \\ &+ (-\varrho_{xy} + \varrho_x\, \varrho_y - a\varrho_x - b\varrho_y + c)\, v\big] \mathrm{e}^{-\varrho(x,y)}. \end{aligned} \tag{2.29}$$

2.4 Reduction to the wave equation

Eq. (2.29) reduces to the wave equation $v_{xy} = 0$ if and only if the equations

$$a - \varrho_y = 0, \quad b - \varrho_x = 0 \tag{2.30}$$

and

$$\varrho_{xy} - \varrho_x \varrho_y + a\varrho_x + b\varrho_y - c = 0 \tag{2.31}$$

hold. Eqs. (2.30) provide a system of *two* equations for *one* unknown function $\varrho(x, y)$ of two variables. Recall that a system of equations is called an *over-determined system* if it contains more equations than unknown functions to be determined by the system in question. Over-determined systems have solutions only if they satisfy certain *compatibility conditions.*

Thus, the system of equations (2.30) is over-determined. Its compatibility condition is obtained from the equation $\varrho_{xy} = \varrho_{yx}$ and has the form

$$a_x = b_y. \tag{2.32}$$

Eq. (2.31), upon using Eqs. (2.30) and (2.32), is written

$$a_x + ab - c = 0. \tag{2.33}$$

In terms of the quantities h and k defined by (2.18) the conditions (2.32) and (2.33) are written in the following symmetric form:

$$h = 0, \quad k = 0. \tag{2.34}$$

Remark 2.1. The equations (2.34) are invariant under the general equivalence transformation (2.27). In consequence, the change of the independent variables does not provide new equations reducible to the wave equation.

Summing up the above calculations and taking into account Remark 2.1, we arrive at the following result.

Theorem 2.1. Eq. (2.17) is equivalent to the wave equation if and only if Eqs. (2.34) are satisfied. Any equation (2.17) with $h = k = 0$ can be reduced to the wave equation $v_{xy} = 0$ by the linear transformation of the dependent variable:

$$u = v\, \mathrm{e}^{-\varrho(x,y)} \tag{2.35}$$

without changing the independent variables x and y. The function ϱ in (2.35) is obtained by solving the following compatible system:

$$\frac{\partial \varrho}{\partial x} = b(x, y), \quad \frac{\partial \varrho}{\partial y} = a(x, y). \tag{2.36}$$

Theorem 2.1 furnishes us with a practical method for solving a wide class of hyperbolic equations (2.1) by reducing them to the wave equation. In order to apply this method, one has to rewrite Eq. (2.1) in the standard form (2.17) by introducing the characteristic variables (2.16). Then one should calculate the Laplace invariants (2.18). If $h = k = 0$, one can find $\varrho(x, y)$ by solving the equations (2.36) and reduce the equation in question to the wave equation $v_{xy} = 0$ by the transformation (2.35). Finally, substituting

$$v = f(x) + g(y)$$

into (2.35) one will obtain the solution in the characteristic variables:

$$u = [f(x) + g(y)]\,\mathrm{e}^{-\varrho(x,y)}. \tag{2.37}$$

Example 2.1. Let us illustrate the method by the equation (see [6], Section 5.3.2))

$$\frac{u_{xx}}{x^2} - \frac{u_{yy}}{y^2} + 3\left(\frac{u_x}{x^3} - \frac{u_y}{y^3}\right) = 0. \tag{2.38}$$

Here Eq. (2.8) for the characteristics has the form

$$\left(\frac{\omega_x}{x}\right)^2 - \left(\frac{\omega_y}{y}\right)^2 = \left(\frac{\omega_x}{x} - \frac{\omega_y}{y}\right)\left(\frac{\omega_x}{x} + \frac{\omega_y}{y}\right) = 0.$$

It splits into two equations:

$$\frac{\omega_x}{x} + \frac{\omega_y}{y} = 0, \quad \frac{\omega_x}{x} - \frac{\omega_y}{y} = 0.$$

They have the following first integrals:

$$x^2 - y^2 = \text{const.}, \quad x^2 + y^2 = \text{const.}$$

Hence, the characteristic variables (2.16) are defined by

$$\xi = x^2 - y^2, \quad \eta = x^2 + y^2. \tag{2.39}$$

We have (cf. Eqs. (2.4)):

$$\begin{aligned}
u_x &= u_\xi \cdot \xi_x + u_\eta \cdot \eta_x = 2x(u_\xi + u_\eta),\\
u_y &= u_\xi \cdot \xi_y + u_\eta \cdot \eta_y = 2y(u_\eta - u_\xi),\\
u_{xx} &= 2(u_\xi + u_\eta) + 4x^2[(u_\xi + u_\eta)_\xi + (u_\xi + u_\eta)_\eta]\\
&= 2(u_\xi + u_\eta) + 4x^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}),\\
u_{yy} &= 2(u_\eta - u_\xi) + 4y^2[(u_\eta - u_\xi)_\eta - (u_\eta - u_\xi)_\xi]\\
&= 2(u_\eta - u_\xi) + 4y^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}).
\end{aligned}$$

Therefore, Eq. (2.38) takes the following form:

$$u_{\xi\eta} + \frac{x^2+y^2}{2x^2y^2}\,u_\xi - \frac{x^2-y^2}{2x^2y^2}\,u_\eta = 0.$$

Invoking Eqs. (2.39) and noting that

$$\eta^2 - \xi^2 = 4x^2y^2, \tag{2.40}$$

we ultimately arrive at the following standard form (2.17) of Eq. (2.38):

$$u_{\xi\eta} + \frac{2\eta}{\eta^2-\xi^2}\,u_\xi - \frac{2\xi}{\eta^2-\xi^2}\,u_\eta = 0. \tag{2.41}$$

The coefficients of Eq. (2.41) are:

$$a = \frac{2\eta}{\eta^2-\xi^2}\,, \quad b = -\frac{2\xi}{\eta^2-\xi^2}\,, \quad c = 0.$$

We substitute them in (2.18) where we replace x and y by ξ and η, respectively. We have

$$a_\xi = b_\eta = \frac{4\xi\eta}{(\eta^2-\xi^2)^2}$$

and see that $h = k = 0$. Now we solve Eqs. (2.36):

$$\frac{\partial\varrho}{\partial\eta} = \frac{2\eta}{\eta^2-\xi^2}, \quad \frac{\partial\varrho}{\partial\xi} = -\frac{2\xi}{\eta^2-\xi^2}$$

and obtain

$$\varrho = \ln(\eta^2-\xi^2).$$

Hence, the substitution (2.35) is written

$$v = ue^{\ln(\eta^2-\xi^2)} = (\eta^2-\xi^2)\,u. \tag{2.42}$$

It maps Eq. (2.41) to the wave equation

$$v_{\xi\eta} = 0. \tag{2.43}$$

Therefore

$$v(\xi,\eta) = f(\xi) + g(\eta),$$

and (2.41) yields:

$$u(\xi,\eta) = \frac{f(\xi)+h(\eta)}{\eta^2-\xi^2}\,\cdot$$

Returning to the original variables by using Eqs. (2.39) and denoting $F = f/4$ and $H = h/4$, we finally obtain the following general solution to Eq. (2.38):

$$u(x,y) = \frac{F(x^2-y^2)+H(x^2+y^2)}{x^2y^2}\,\cdot \tag{2.44}$$

Remark 2.2. We can integrate Eq. (2.38) by rewriting it in a factorized form in the original variables. Note that the differentiations D_x, D_y in the original variables are related with the differentiations D_ξ, D_η in the new variables (2.3) as follows (see (2.4)):

$$D_x = \xi_x D_\xi + \eta_x D_\eta, \quad D_y = \xi_y D_\xi + \eta_y D_\eta, \tag{2.45}$$

whence

$$D_\xi = \frac{1}{\Omega}\left(\eta_y D_x - \eta_x D_y\right), \quad D_\eta = \frac{1}{\Omega}\left(\xi_x D_y - \xi_y D_x\right) \tag{2.46}$$

with

$$\Omega = \xi_x \eta_y - \xi_y \eta_x.$$

Applying Eqs. (2.46) to the change of variables (2.39) we obtain:

$$D_\xi = \frac{1}{4}\left(\frac{1}{x}D_x - \frac{1}{y}D_y\right), \quad D_\eta = \frac{1}{4}\left(\frac{1}{x}D_x + \frac{1}{y}D_y\right).$$

Therefore, Eq. (2.43) written in the form

$$D_\xi D_\eta(v) = 0$$

and Eqs. (2.42), (2.40) yield the following factorization of Eq. (2.38):

$$L_1 L_2(u) = 0, \tag{2.47}$$

where $L_1,\ L_2$ are the first-order linear differential operators given by

$$L_1 = \frac{1}{x}D_x - \frac{1}{y}D_y, \quad L_2 = \left(\frac{1}{x}D_x + \frac{1}{y}D_y\right)x^2y^2. \tag{2.48}$$

To solve Eq. (2.47), we denote $L_2(u) = w$, write Eq. (2.47) as the the first-order equation

$$L_1(w) \equiv \frac{1}{x}\frac{\partial w}{\partial x} - \frac{1}{y}\frac{\partial w}{\partial y} = 0$$

for w, whence

$$w = \phi(\eta), \quad \eta = x^2 + y^2,$$

on integration. It remains to integrate the non-homogeneous first-order equation $L_2(u) = \phi(\eta)$:

$$\left(\frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{y}\frac{\partial}{\partial y}\right)(x^2y^2u) = \phi(\eta).$$

Denoting $x^2y^2u = v$ for the sake of simplicity, we rewrite the equation in the form

$$\frac{1}{x}\frac{\partial v}{\partial x} + \frac{1}{y}\frac{\partial v}{\partial y} = \phi(\eta).$$

The first equation of the characteristic system

$$xdx = ydy = \frac{dv}{\phi(\eta)}$$

provides the first integral $x^2 - y^2 \equiv \xi = \text{const}$. Substituting $\eta = 2x^2 + \xi$ in the second equation of the characteristic system, $dv = x\phi(\eta)dx$ we have:

$$dv = \phi(\xi + 2x^2)xdx = \frac{1}{4}\,\phi(\xi + 2x^2)d(x\xi + 2x^2),$$

whence

$$v = H(\xi + 2x^2) + F(\xi).$$

Since $\xi + 2x^2 = \eta$ and $v = x^2y^2u$, we finally obtain

$$x^2y^2u = H(\eta) + F(\xi),$$

i.e. the solution (2.44):

$$u = \frac{H(x^2 + y^2) + F(x^2 - y^2)}{x^2y^2}\,.$$

It is manifest from the above calculations that the use of characteristic variables simplifies the integration procedure significantly.

3 Parabolic equations

3.1 Equivalence transformations

The standard form of the parabolic equations is

$$u_t + A(t,x)u_{xx} + a(t,x)u_x + c(t,x)u = 0. \tag{3.1}$$

The equivalence transformations of Eqs. (3.1) comprise the invertible changes of the independent variables of the form

$$\tau = \phi(t), \quad y = \psi(t,x) \tag{3.2}$$

and the linear transformation of the dependent variable

$$v = \sigma(t,x)u, \quad \sigma(t,x) \neq 0, \tag{3.3}$$

where $\phi(t), \psi(t,x)$ and $\sigma(t,x)$ are arbitrary functions.

Under the change of the independent variables (3.2) the derivatives of u undergo the following transformations:

$$u_t = \phi' u_\tau + \psi_t u_y, \quad u_x = \psi_x u_y, \quad u_{xx} = \psi_x^2 u_{yy} + \psi_{xx} u_y,$$

and Eq. (3.1) becomes:

$$\phi' u_\tau + A\psi_x^2 u_{yy} + \big(\psi_t + A\psi_{xx} + a\psi_x\big) u_y + cu = 0.$$

Therefore, by choosing ψ such that $|A|\psi_x^2 = 1$ and taking $\phi = \pm\, t$ in accordance with the sign of A, we can rewrite any parabolic equation in the form

$$u_t - u_{xx} + a(t,x)u_x + c(t,x)u = 0. \tag{3.4}$$

In what follows, we will use the parabolic equations (3.4) and employ their equivalence transformation (3.3) written in the form (cf. Eq. (2.28)):

$$v = u\,\mathrm{e}^{\varrho(t,x)}. \tag{3.5}$$

Solving Eq. (3.5) for u and differentiating, we have:

$$\begin{aligned} u &= v\,\mathrm{e}^{-\varrho(t,x)}, \quad u_t = (v_t - v\varrho_t)\,\mathrm{e}^{-\varrho(t,x)},\\ u_x &= (v_x - v\varrho_x)\,\mathrm{e}^{-\varrho(t,x)},\\ u_{xx} &= \big[v_{xx} - 2v_x\varrho_x + (\varrho_x^2 - \varrho_{xx})v\big]\,\mathrm{e}^{-\varrho(t,x)}. \end{aligned} \tag{3.6}$$

Inserting the expressions (3.6) in the left-hand side of Eq. (3.4), we obtain:

$$\begin{aligned} &u_t - u_{xx} + au_x + cu\\ &= \big[v_t - v_{xx} + (a + 2\varrho_x)\,v_x\\ &+ (\varrho_{xx} - \varrho_x^2 - \varrho_t - a\varrho_x + c)\,v\big]\mathrm{e}^{-\varrho(t,x)}. \end{aligned} \tag{3.7}$$

3.2 Semi-invariant. Equations reducible to the heat equation

Eq. (3.7) shows that Eq. (3.4) can be reduced to the heat equation

$$v_t - v_{xx} = 0, \quad t > 0, \tag{3.8}$$

by an equivalence transformation (3.5) if and only if

$$a + 2\varrho_x = 0, \quad \varrho_{xx} - \varrho_x^2 - \varrho_t - a\varrho_x + c = 0. \tag{3.9}$$

The first equation (3.9) yields

$$\varrho_x = -\frac{1}{2}\,a, \quad \varrho_{xx} = -\frac{1}{2}\,a_x,$$

and hence the second equation (3.9) becomes

$$\frac{1}{4}\,a^2 - \frac{1}{2}\,a_x - \varrho_t + c = 0.$$

Thus, Eqs. (3.9) can be rewritten as the following over-determine system of first-order equations for the unknown function $\varrho(t,x)$:

$$\varrho_x = -\frac{1}{2}\,a, \quad \varrho_t = \frac{1}{4}\,a^2 - \frac{1}{2}\,a_x + c. \tag{3.10}$$

The compatibility condition $\varrho_{xt} = \varrho_{tx}$ for the system (3.10) has the form

$$aa_x - a_{xx} + a_t + 2c_x = 0. \tag{3.11}$$

The left-hand side of Eq. (3.11) is the invariant (more specifically, *semi-invariant*) for the parabolic equations first obtained by the author in 2000 (Preprint) and published in [4]. Namely, it is shown in [4] that the linear parabolic equations

$$u_t + A(t,x)u_{xx} + B(t,x)u_x + C(t,x)u = 0$$

have the following invariant with respect to the equivalence transformation (3.5):

$$\begin{aligned} K = \frac{1}{2}\,B^2 A_x + \Big(A_t + AA_{xx} - A_x^2\Big)B \\ + (AA_x - AB)B_x - AB_t - A^2 B_{xx} + 2A^2 C_x. \end{aligned}$$

Since Eq. (3.4) corresponds to $A = -1,\ B = a,\ C = c$, the above semi-invariant is written

$$K = aa_x - a_{xx} + a_t + 2c_x. \tag{3.12}$$

Summing up the above calculations, we arrive at the following result.

Theorem 3.1. The parabolic equation (3.4)

$$u_t - u_{xx} + a(t,x)u_x + c(t,x)u = 0$$

can be reduced to the heat equation (3.8)

$$v_t - v_{xx} = 0$$

by an appropriate linear transformation (3.5) of the dependent variable,

$$u = v\,\mathrm{e}^{-\varrho(t,x)}, \tag{3.13}$$

without changing the independent variables t and x if and only if the semi-invariant (3.12) vanishes, $K = 0$. The function ϱ in the transformation (3.5) of Eq. (3.4) to the heat equation is obtained by solving the following compatible system (3.10):

$$\frac{\partial \varrho}{\partial x} = -\frac{1}{2}\,a, \quad \frac{\partial \varrho}{\partial t} = \frac{1}{4}\,a^2 - \frac{1}{2}\,a_x + c.$$

The system (3.10) is compatible (has a solution $\varrho(t,x)$) due to $K = 0$, i.e. Eq. (3.11):

$$aa_x - a_{xx} + a_t + 2c_x = 0.$$

Corollary 3.1. Any Eq. (3.4) with constant coefficients a and c can be reduced to the heat equation. Indeed, the semi-invariant (3.12) vanishes if $a, c = \mathrm{const}$.

Remark 3.1. For hyperbolic equations a similar statement does not valid. For example, the telegraph equation $u_{xy} + u = 0$ cannot be reduced to the wave equation.

Example 3.1. ([7], [4]). The semi-invariant (3.12) of any equation of the form

$$u_t - u_{xx} + c(t)u = 0$$

vanishes. Therefore this equation reduces to the heat equation by the equivalence transformation $u = v\,\mathrm{e}^{\int c(t)dt}$.

Theorem 3.1 furnishes us with a practical method for solving by quadrature a wide class of parabolic equations (3.4) by reducing them to the heat equation. First, let us discuss the known solutions to the heat equation.

3.3 Poisson's solution

Recall that, given any continuous and bounded function $f(x)$, the function $v(t, x)$ defined by Poisson's formula

$$v(t,x) = \frac{1}{2\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz, \quad t > 0, \tag{3.14}$$

solves the heat equation (3.8) and satisfies the initial condition

$$v(0,x) \equiv \lim_{t\to+0} v(t,x) = f(x). \tag{3.15}$$

To make the text self-contained, let us verify that the function $v(t, x)$ defined by (3.14) satisfies the heat equation. The integral (3.14) itself as well as the integrals obtained by differentiating it under the integral sign any number of times converge uniformly due to the presence of the rapidly decreasing factor $\mathrm{e}^{-\frac{(x-z)^2}{4t}}$ when $t > 0$. The integral (3.14) solves the heat equation $v_t = v_{xx}$ because

$$v_t = -\frac{1}{4t\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz + \frac{1}{8t^2\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} (x-z)^2 f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz,$$

$$v_x = -\frac{1}{4t\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} (x-z) f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz,$$

$$v_{xx} = -\frac{1}{4t\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz + \frac{1}{8t^2\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} (x-z)^2 f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz.$$

3.4 Uniqueness class and the general solution therein

The uniqueness of the solution to the Cauchy problem guarantees that (3.14) provides the *general solution* to the heat equation in the half-plane $t > 0$ provided that the solution is bounded.

In general, the solution to the Cauchy problem for the heat equation may not be unique in the class of unbounded functions. It can be shown, however, that the presence of the factor $\mathrm{e}^{-\frac{(x-z)^2}{4t}}$ under the integral sign in (3.14) guarantees the uniqueness of the solution in the class of continuous functions $v(t, x)$ defined on a strip

$$0 \leq t \leq T < +\infty, \ -\infty < x < +\infty$$

and such that $|v(t, x)|$ grows not faster than e^{x^2} as $x \to \infty$. In other words, the following statement holds (see, e.g. [8], Ch. IV).

Theorem 3.2. The solution to the Cauchy problem

$$v_t - v_{xx} = 0 \ (t > 0), \quad v(0, x) \equiv \lim_{t \to +0} v(t, x) = f(x)$$

is unique and is given by Poisson's formula (3.14) in the class of functions $v(t, x)$ satisfying the following condition:

$$\max_{0 \leq t \leq T} \big|v(t, x)\big| \,\mathrm{e}^{-\beta x^2} \to 0 \quad \text{as} \quad |x| \to \infty, \tag{3.16}$$

where β is a certain constant.

Corollary 3.2. Let $\tilde{v}(t, x)$ be a continuous function satisfying the uniqueness condition (3.16). If $\tilde{v}(t, x)$ solves the heat equation for $t > 0$, then it is given by Poisson's formula (3.14) with a certain function $f(x)$.

Proof. Substituting in the integral (3.14) the function $f(x) = \tilde{v}(0, x)$, we obtain a solution $v(t, x)$ of the heat equation. Since the initial values of both solutions $\tilde{v}(t, x)$ and $v(t, x)$ coincide, $v(0, x) = f(x) = \tilde{v}(0, x)$, the uniqueness of the solution to the Cauchy problem shows that $\tilde{v}(t, x) = v(t, x)$, i.e. that the solution $\tilde{v}(t, x)$ is given by the formula (3.14).

According to Corollary 3.2, all continuous solutions of the heat equation (3.8) satisfying the condition (3.16) admit the integral representation (3.14). In other words, *Poisson's formula* (3.14) *provides the general solution of the heat equation in the class of functions satisfying the condition* (3.16).

3.5 Tikhonov's solution

A.N. Tikhonov [9] showed that the condition (3.16) is exact. Namely, he noticed that there are solutions of the heat equation that are not identically zero but vanish at $t = 0$,

$$v(0,x) = 0, \quad -\infty < x < +\infty,$$

and satisfy the following condition with any small $\varepsilon > 0$:

$$\max_{0\le t\le T} \big|v(t,x)\big|\, \mathrm{e}^{-x^{2+\varepsilon}} \to 0 \quad \text{as} \quad |x| \to \infty.$$

These solutions are, in fact, particular cases of the following *Tikhonov's solution*:

$$\begin{aligned} v(t,x) =& F(t) + xF_1(t) + \frac{x^2}{2!}\, F'(t) + \frac{x^3}{3!}\, F_1'(t) + \cdots \\ &+ \frac{x^{2n}}{(2n)!}\, F^{(n)}(t) + \frac{x^{2n+1}}{(2n+1)!}\, F_1^{(n)}(t) + \cdots \end{aligned} \tag{3.17}$$

given in [9]. Here $F(t)$ and $F_1(t)$ are any two C^∞ functions (i.e. differentiable infinitely many times) such that the series (3.17) is uniformly convergent. Let us verify that the series (3.17) solves the heat equation (3.8). We can differentiate each term of the series (3.17) due to its uniform convergence and obtain:

$$\begin{aligned} v_t =& F'(t) + xF_1'(t) + \frac{x^2}{2!}\, F''(t) + \frac{x^3}{3!}\, F_1''(t) + \cdots \\ &+ \frac{x^{2n}}{(2n)!}\, F^{(n+1)}(t) + \frac{x^{2n+1}}{(2n+1)!}\, F_1^{(n+1)}(t) + \cdots\,, \\ v_x =& F_1(t) + xF'(t) + \frac{x^2}{2!}\, F_1'(t) + \frac{x^3}{3!}\, F''(t) + \frac{x^4}{4!}\, F_1''(t) + \cdots \\ &+ \frac{x^{2n+1}}{(2n+1)!}\, F^{(n+1)}(t) + \frac{x^{2n+2}}{(2n+2)!}\, F_1^{(n+1)}(t) + \cdots\,, \\ v_{xx} =& F'(t) + x\, F_1'(t) + \frac{x^2}{2!}\, F''(t) + \frac{x^3}{3!}\, F_1''(t) + \cdots \\ &+ \frac{x^{2n}}{(2n)!}\, F^{(n+1)}(t) + \frac{x^{2n+1}}{(2n+1)!}\, F_1^{(n+1)}(t) + \cdots\,. \end{aligned}$$

Subtracting term by term we obtain $v_t - v_{xx} = 0.$

Tikhonov also showed that any solution $v(t,x)$ of the heat equation defined for all x and $t > 0$ can be represented in the form (3.17). This solution satisfies the conditions

$$v(t,0) = F(t), \quad v_x(t,0) = F_1(t). \tag{3.18}$$

Furthermore, the solution of the heat equation satisfying the conditions (3.18) is unique.

Note that the solution (3.17) is the superposition of two different solutions:

$$v(t,x) = F(t) + \frac{x^2}{2!}\,F'(t) + \frac{x^4}{4!}\,F''(t) + \cdots + \frac{x^{2n}}{(2n)!}\,F^{(n)}(t) + \cdots \tag{3.19}$$

and

$$v(t,x) = xF_1(t) + \frac{x^3}{3!}\,F_1'(t) + \frac{x^5}{5!}\,F_1''(t) + \cdots + \frac{x^{2n+1}}{(2n+1)!}\,F_1^{(n)}(t) + \cdots\,. \tag{3.20}$$

The infinite series representations (3.17), (3.19) and (3.20) of solutions to the heat equation are particularly useful for obtaining approximate solutions to the heat equation (3.8) and to the equivalent equations, e.g. by truncating the infinite series. Tikhonov's series representations are also convenient for obtaining solutions in closed forms, in particular, in terms of elementary functions. One of such cases is obtained by taking for $F(t)$ and $F_1(t)$ any polynomials. Let us consider examples.

Example 3.2. Letting in (3.20)

$$F_1(t) = a + bt + ct^2 + kt^3,$$

we obtain the following polynomial solution:

$$v(t,x) = (a + bt + ct^2 + kt^3)x + \frac{1}{3!}\,\big(b + 2ct + 3kt^2\big)x^3 + \frac{1}{5!}\,\big(2c + 6kt\big)x^5 + \frac{6k}{7!}\,x^7.$$

In particular, taking $a = b = k = 0$ and $c = 60$, we obtain the solution

$$v(t,x) = 60t^2x + 20tx^3 + x^5. \tag{3.21}$$

The solution (3.21) satisfies the initial condition $v(0,x) = x^5$, and hence admits the following integral representation (3.14) when $t > 0$:

$$v(t,x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} z^5\, \mathrm{e}^{-\frac{(x-z)^2}{4t}}\, dz. \tag{3.22}$$

It is manifest that that the explicit form (3.21) of the solution is essentially simpler.

Example 3.3. Setting $F(t) = \mathrm{e}^{-t}$ and $F_1(t) = \mathrm{e}^{-t}$ in (3.19) and (3.20), respectively, we obtain the following two particular solutions:

$$v(t,x) = \mathrm{e}^{-t}\,\cos x, \quad v(t,x) = \mathrm{e}^{-t}\,\sin x.$$

Let us discuss now applications of Tikhonov's solution (3.17) and Poisson's solution (3.14) to the parabolic equations (3.4) with the vanishing semi-invariant (3.12).

3.6 Integration of equations with the vanishing semi-invariant

The solution of parabolic equations Eq. (3.4) with the vanishing semi-invariant (3.12) is given by Eq. (3.13),

$$u = \mathrm{e}^{-\varrho(t,x)}\, v(t,x), \tag{3.13}$$

where the function $\varrho(t,x)$ is obtained by solving Eqs. (3.10), and $v(t,x)$ is the solution of the heat equation written either in Poisson's integral representation (3.14) or in Tikhonov's series representation (3.17). We will consider separately the use of Poisson's and Tikhonov's representations.

3.6.1 Poisson's form of the solution

In this section we will employ the uniqueness condition (3.16). Using Eqs. (3.13), (3.14) as well as Theorem 3.1 and Corollary 3.2, we arrive at the following statement.

Theorem 3.3. Let the semi-invariant (3.12) of Eq. (3.4) vanish. Then any solution to Eq. (3.4) belonging to the class of functions satisfying the condition (cf. (3.16))

$$\max_{0\le t\le T} \left|u(t,x)\,\mathrm{e}^{\varrho(t,x)}\right| \mathrm{e}^{-\beta x^2} \to 0 \quad \text{as} \quad |x| \to \infty, \tag{3.23}$$

where $\varrho(t,x)$ is found by solving the system (3.10), admits the integral representation

$$u(t,x) = \frac{1}{2\sqrt{\pi t}}\,\mathrm{e}^{-\varrho(t,x)} \int_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz, \quad t > 0. \tag{3.24}$$

Thus, Eq. (3.24) furnishes the general solution to Eq. (3.4) with the vanishing semi-invariant K, provided that the condition (3.23) is satisfied.

Example 3.4. The equation

$$u_t - u_{xx} + 2u_x - u = 0 \tag{3.25}$$

has the vanishing semi-invariant (3.12) (see Corollary 3.1). The system (3.10) yields

$$\varrho(t,x) = -x,$$

and hence Eq. (3.13) is written

$$u(t,x) = \mathrm{e}^{x}\, v(t,x). \tag{3.26}$$

Taking one of the simplest solutions to the heat equation, namely $v = x$, we get the solution

$$u(t,x) = x\mathrm{e}^{x}$$

to Eq. (3.25). This particular solution is unbounded. However, it satisfies the condition (3.23), and hence admits the integral representation (3.24) with a certain function $f(z)$. How to find the corresponding function $f(z)$? Let us discuss this question in general.

Poisson's form of the solution is well suited for solving the initial value problem not only for the heat equation but also for all parabolic equations with the vanishing semi-invariant. The result is formulated in the next theorem. In particular, it gives the answer to the questions similar to that formulated in Example 3.4.

Theorem 3.4. Let (3.4) be an equation with the vanishing semi-invariant (3.12). Then the solution to the initial value problem

$$u_t - u_{xx} + a(t,x)u_x + c(t,x)u = 0, \quad u\big|_{t=0} = u_0(x) \tag{3.27}$$

is given by

$$u(t,x) = \frac{1}{2\sqrt{\pi t}}\,\mathrm{e}^{-\varrho(t,x)} \int\limits_{-\infty}^{+\infty} u_0(z)\,\mathrm{e}^{\varrho(0,z)}\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz, \quad t > 0, \tag{3.28}$$

where $\varrho(t,x)$ solves the system (3.10).

Proof. Letting $t \to +0$ in Eq. (3.24), using the initial condition

$$\lim_{t\to+0} u(t,x) = u\big|_{t=0} = u_0(x)$$

and the well-known equation

$$\lim_{t\to+0} \frac{1}{2\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz = f(x), \tag{3.29}$$

we obtain $u_0(x) = \mathrm{e}^{-\varrho(0,x)}\,f(x)$. Hence, the solution to the problem (3.27) is given by Eq. (3.24) with

$$f(z) = u_0(z)\,\mathrm{e}^{\varrho(0,z)},$$

i.e. by Eq. (3.28).

Example 3.5. Let us give the answer to the question put in Example 3.4. Since $\varrho(t,x) = -x$ and the particular solution is given by $u(t,x) = x\mathrm{e}^x$, we have:

$$f(z) = u_0(z)\,\mathrm{e}^{\varrho(0,z)} = z\mathrm{e}^z\mathrm{e}^{-z} = z.$$

Therefore Eq. (3.28) provides the following integral representation (3.24) of the particular solution $u(t,x) = x\mathrm{e}^x$ to the equation $u_t - u_{xx} + 2u_x - u = 0$:

$$u(t,x) \equiv x\mathrm{e}^x = \frac{\mathrm{e}^x}{2\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} z\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz, \quad t > 0.$$

Example 3.6. Consider again Eq. (3.25),

$$u_t - u_{xx} + 2u_x - u = 0.$$

Substituting in Eq. (3.26),

$$u(t,x) = \mathrm{e}^x\, v(t,x),$$

the fundamental solution

$$v(t,x) = \frac{\theta(t)}{2\sqrt{\pi t}}\, \mathrm{e}^{-\frac{x^2}{4t}}$$

of the heat equation, we obtain $u = \mathcal{E}(t,x)$, where

$$\mathcal{E}(t,x) = \frac{\theta(t)}{2\sqrt{\pi t}}\, \mathrm{e}^{x-\frac{x^2}{4t}} \tag{3.30}$$

and $\theta(t)$ is the Heaviside function. The function (3.30) is the fundamental solution for Eq. (3.25). Indeed, writing the left-hand side of Eq. (3.25) in the operator form

$$L(u) = u_t - u_{xx} + 2u_x - u,$$

we see that the linear differential operator L acts on the function (3.30) as follows:

$$L(\mathcal{E}) = \frac{\theta'(t)}{2\sqrt{\pi t}}\, \mathrm{e}^{x-\frac{x^2}{4t}} + \theta(t)\, L\left(\frac{1}{2\sqrt{\pi t}}\, \mathrm{e}^{x-\frac{x^2}{4t}}\right).$$

Invoking that

$$L\left(\frac{1}{2\sqrt{\pi t}}\, \mathrm{e}^{x-\frac{x^2}{4t}}\right), \quad t > 0,$$

and

$$\theta'(t) = \delta(t), \quad F(t,x)\delta(t) = F(0,x)\delta(t), \quad \lim_{t\to+0} \frac{1}{2\sqrt{\pi t}}\, \mathrm{e}^{-\frac{x^2}{4t}} = \delta(x),$$

where δ is Dirac's δ-function, we obtain:

$$L(\mathcal{E}) = \mathrm{e}^x\, \delta(t) \lim_{t\to+0} \frac{1}{2\sqrt{\pi t}}\, \mathrm{e}^{-\frac{x^2}{4t}} = \mathrm{e}^x\, \delta(t)\delta(x).$$

Since $f(x)\delta(x) = f(0)\delta(x)$ for any C^∞-function, and $\delta(t)\delta(x) = \delta(t,x)$, we finally see that the function (3.30) satisfies the definition of the fundamental solution:

$$L(\mathcal{E}) = \delta(t,x).$$

3.6.2 Tikhonov's form of the solution

If we do not require the condition (3.23), we can substitute in Eq. (3.13) Tikhonov's solution (3.17) and arrive at the following statement.

Theorem 3.5. Let the semi-invariant (3.12) of Eq. (3.4) vanish. Then the series

$$\begin{aligned} u(t,x) =& \mathrm{e}^{-\varrho(t,x)} \Big[F(t) + xF_1(t) + \frac{x^2}{2!}\,F'(t) + \frac{x^3}{3!}\,F_1'(t) + \cdots \\ & + \frac{x^{2n}}{(2n)!}\,F^{(n)}(t) + \frac{x^{2n+1}}{(2n+1)!}\,F_1^{(n)}(t) + \cdots \Big], \end{aligned} \tag{3.31}$$

where $\varrho(t,x)$ is determined by Eqs. (3.10), solves the equation (3.4).

Example 3.7. Consider again Eq. (3.25). Since here $\varrho(t,x) = -x$, Eq. (3.31) is written

$$u(t,x) = \mathrm{e}^x \Big[F(t) + xF_1(t) + \frac{x^2}{2!}\,F'(t) + \frac{x^3}{3!}\,F_1'(t) + \cdots \Big].$$

The solution $u(t,x) = x\mathrm{e}^x$ from Example 3.4 corresponds to $F(t) = 0,\ F_1(t) = 1.$

3.6.3 Additional comments

Remark 3.2. Theorems 3.1 and 3.5 can be extended to the parabolic equations

$$\bar{u}_t + \bar{u}_{xx} + \bar{a}(t,x)u_x + \bar{c}(t,x)\bar{u} = 0$$

reducible to the "time reversal heat equation"

$$w_t + w_{xx} = 0$$

by the linear transformation of the dependent variable

$$w = \bar{u}\,\mathrm{e}^{\bar{\varrho}(t,x)},$$

where $\bar{\varrho}$ is obtained by solving the overdetermined system (cf. Eqs. (3.10))

$$\bar{\varrho}_x = \frac{1}{2}\,\bar{a}, \quad \bar{\varrho}_t = \bar{c} - \frac{1}{4}\,\bar{a}^2 - \frac{1}{2}\,\bar{a}_x.$$

The compatibility condition of this system has the form (cf. (3.12))

$$\overline{K} \equiv \bar{a}\bar{a}_x + \bar{a}_{xx} + \bar{a}_t - 2\bar{c}_x = 0.$$

Let us dwell upon the solutions related to Poisson's solution. Since the solution to the "time reversal heat equation" is given by

$$w(t,x) = \frac{1}{2\sqrt{-\pi t}} \int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{\frac{(x-z)^2}{4t}}\,dz, \quad t < 0,$$

the solution of a reducible equation (i.e. when $\overline{K} = 0$) has the form similar to (3.24):

$$\bar{u}(t,x) = \frac{1}{2\sqrt{-\pi t}}\, \mathrm{e}^{-\bar{\varrho}(t,x)} \int\limits_{-\infty}^{+\infty} f(z)\, \mathrm{e}^{\frac{(x-z)^2}{4t}}\, dz, \quad t < 0.$$

Remark 3.3. If a parabolic equation is given in the general form (2.1),

$$A\, u_{xx} + 2B\, u_{xy} + C\, u_{yy} + a\, u_x + b\, u_y + c\, u = 0, \quad A \neq 0,$$

where $A = A(x,y), \ldots, c = c(x,y)$, we rewrite it in the form (3.1) by introducing the characteristic variable t. Specifically, since $B^2 - AC = 0$, the characteristic equation (2.1) reduces to the linear equation

$$A\, \frac{\partial \varphi}{\partial x} + B\, \frac{\partial \varphi}{\partial y} = 0.$$

Taking any solution $\varphi(x,y)$ of this equation and rewriting the equation in question in the variables x and $t = \varphi(x,y)$, we will arrive at an equation of the form (3.1).

4 Application to financial mathematics

4.1 Transformations of the Black-Scholes equation

Consider the Black-Scholes equation

$$u_t + \frac{1}{2} A^2 x^2 u_{xx} + Bxu_x - Cu = 0. \quad A, B, C = \text{const.}, \tag{4.1}$$

Upon the change of variables

$$\tau = t_0 - t, \quad y = \frac{\sqrt{2}}{A}\, (\ln|x| - \ln|x_0|), \tag{4.2}$$

it assumes the form of Eq. (3.4) with constant coefficients:

$$u_\tau - u_{yy} + \left(\frac{A}{\sqrt{2}} - \frac{\sqrt{2}}{A}\, B \right) u_y + Cu = 0. \tag{4.3}$$

Therefore, according to Corollary 3.1, the Black-Scholes equation reduces to the heat equation written in the variables (4.2):

$$v_\tau - v_{yy} = 0. \tag{4.4}$$

The corresponding system (3.10) for determining the function $\varrho(\tau, y)$ is written

$$\frac{\partial \varrho}{\partial y} = \frac{B}{\sqrt{2}\,A} - \frac{A}{2\sqrt{2}}\,, \quad \frac{\partial \varrho}{\partial \tau} = C + \frac{A^2}{8} - \frac{B}{2} + \frac{B^2}{2A^2}\,\cdot$$

Integrating this system and ignoring the unessential constant of integration, we have:

$$\varrho = \left(\frac{B}{\sqrt{2}\,A} - \frac{A}{2\sqrt{2}}\right)y + \left(C + \frac{A^2}{8} - \frac{B}{2} + \frac{B^2}{2A^2}\right)\tau.$$

Hence, according to Eq. (3.13), the solution to Eq. (4.3) is given by

$$u(\tau, y) = \mathrm{e}^{\left(\frac{A}{2\sqrt{2}} - \frac{B}{\sqrt{2}\,A}\right)y + \left(\frac{B}{2} - \frac{A^2}{8} - \frac{B^2}{2A^2} - C\right)\tau}\, v(\tau, y), \tag{4.5}$$

where $v(\tau, y)$ is the solution of the heat equation (4.4).

4.2 Poisson's form of the solution to the Black-Scholes equation

According to Theorem 3.3 and Eq. (4.5), the solution to Eq. (4.3) is written

$$u(\tau, y) = \frac{1}{2\sqrt{\pi\tau}}\ \mathrm{e}^{\left(\frac{A}{2\sqrt{2}} - \frac{B}{\sqrt{2}\,A}\right)y + \left(\frac{B}{2} - \frac{A^2}{8} - \frac{B^2}{2A^2} - C\right)\tau} \int\limits_{-\infty}^{+\infty} f(z)\, \mathrm{e}^{-\frac{(y-z)^2}{4\tau}}\, dz, \quad \tau > 0.$$

Substituting the expressions (4.2) for τ and y, we obtain the integral representation of the general solution to the Black-Scholes equation (4.1) in the variables t, x :

$$u(t, x) = \frac{1}{2\sqrt{\pi(t_0 - t)}}\ \mathrm{e}^{\left(\frac{1}{2} - \frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right| + \left(\frac{B}{2} - \frac{A^2}{8} - \frac{B^2}{2A^2} - C\right)(t_0 - t)}$$

$$\times \int\limits_{-\infty}^{+\infty} f(z)\, \mathrm{e}^{-\frac{[\sqrt{2}(\ln|x| - \ln|x_o|) - Az]^2}{4A^2(t_o - t)}}\, dz, \quad t < t_0. \tag{4.6}$$

Black-Scholes [10] gave the solution of the Cauchy problem with a special initial data. The Cauchy problem can also be solved by using the fundamental solution for the Black-Scholes equation obtained in [11] by using the symmetries of the equation. Both ways are not simple. The integral representation (4.6) of the general solution allows one to solve the initial value problem

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0 \ \ (t < t_0), \quad u\big|_{t=t_0} = u_0(x) \tag{4.7}$$

with an arbitrary initial data $u_0(x)$. Indeed, letting in Eq. (4.6) $t \to t_0$ and using the initial condition we obtain:

$$u_0(x) = \mathrm{e}^{\left(\frac{1}{2}-\frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right|} \lim_{t\to t_0} \frac{1}{2\sqrt{\pi(t_0-t)}} \int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{1}{4(t_o-t)}\left[\frac{\sqrt{2}}{A}\ln\left|\frac{x}{x_o}\right|-z\right]^2} dz,$$

whence, according to Eq. (3.29),

$$u_0(x) = f\Big(\frac{1}{A}\sqrt{2}\,\ln\Big|\frac{x}{x_o}\Big|\Big)\,\mathrm{e}^{\left(\frac{1}{2}-\frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right|}.$$

Denoting $z = \frac{1}{A}\sqrt{2}\,\ln\left|\frac{x}{x_o}\right|$ we have $x = x_0\mathrm{e}^{\frac{Az}{\sqrt{2}}}$, and the above equation yields

$$f(z) = u_0\Big(x_0\mathrm{e}^{\frac{Az}{\sqrt{2}}}\Big)\,\mathrm{e}^{\left(\frac{B}{\sqrt{2}A}-\frac{A}{2\sqrt{2}}\right)z}.$$

Substituting this expression for f in Eq. (4.6), we obtain the following solution to the problem (4.7):

$$u(t,x) = \frac{1}{2\sqrt{\pi(t_0-t)}}\,\mathrm{e}^{\left(\frac{1}{2}-\frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right|+\left(\frac{B}{2}-\frac{A^2}{8}-\frac{B^2}{2A^2}-C\right)(t_0-t)}$$
$$\times\int\limits_{-\infty}^{+\infty} u_0\Big(x_0\mathrm{e}^{\frac{Az}{\sqrt{2}}}\Big)\,\mathrm{e}^{\left(\frac{B}{\sqrt{2}A}-\frac{A}{2\sqrt{2}}\right)z}\,\mathrm{e}^{-\frac{[\sqrt{2}(\ln|x|-\ln|x_o|)-Az]^2}{4A^2(t_o-t)}} dz, \quad t<t_0. \tag{4.8}$$

Remark 4.1. Along with the general solution (4.6), group invariant solutions can be useful as well, particularly those given explicitly by elementary functions. Numerous invariant solutions are obtained in [11]. It is not easy to recognize these invariant solutions from Poisson's form (4.6) of the solutions. Compare, e.g. one of the simplest invariant solutions from [11]:

$$u = x\,\mathrm{e}^{(B-C)(t_o-t)}$$

with its integral representation obtained from (4.8) by substituting $u_0(x) = x$:

$$u = \frac{1}{2\sqrt{\pi(t_0-t)}}\,\mathrm{e}^{\left(\frac{1}{2}-\frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right|+\left(\frac{B}{2}-\frac{A^2}{8}-\frac{B^2}{2A^2}-C\right)(t_0-t)}$$
$$\times\int\limits_{-\infty}^{+\infty} x_0\,\mathrm{e}^{\left(\frac{B}{\sqrt{2}A}+\frac{A}{2\sqrt{2}}\right)z}\,\mathrm{e}^{-\frac{[\sqrt{2}(\ln|x|-\ln|x_o|)-Az]^2}{4A^2(t_o-t)}} dz, \quad t<t_0. \tag{4.9}$$

4.3 Tikhonov's form of the solution to the Black-Scholes equation

Substituting in (4.5) the solution $v(\tau, y)$ of the heat equation (4.4) in Tikhonov's form (3.17), we obtain Tikhonov's representation of the solution to Eq. (4.3):

$$u(\tau, y) = \mathrm{e}^{\left(\frac{A}{2\sqrt{2}} - \frac{B}{\sqrt{2}\,A}\right)y + \left(\frac{B}{2} - \frac{A^2}{8} - \frac{B^2}{2A^2} - C\right)\tau} \Big[F(\tau) + yF_1(\tau) + \frac{y^2}{2!}\,F'(\tau)$$

$$+ \frac{y^3}{3!}\,F_1'(\tau) + \cdots + \frac{y^{2n}}{(2n)!}\,F^{(n)}(\tau) + \frac{y^{2n+1}}{(2n+1)!}\,F_1^{(n)}(\tau) + \cdots\Big]. \tag{4.10}$$

Now we replace τ and y by their expressions (4.2) and arrive at the following Tikhonov's form of the solution to the Black-Scholes equation (4.1) (cf. Eq. (4.6)):

$$u(t, x) = \mathrm{e}^{\left(\frac{1}{2} - \frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right| + \left(\frac{B}{2} - \frac{A^2}{8} - \frac{B^2}{2A^2} - C\right)(t_0 - t)} \Big[F(t_0 - t) + F_1(t_0 - t)\frac{\sqrt{2}}{A}\,\ln\Big|\frac{x}{x_0}\Big|$$

$$+ \frac{1}{2!}\,F'(t_0 - t)\Big[\frac{\sqrt{2}}{A}\,\ln\Big|\frac{x}{x_0}\Big|\Big]^2 + \frac{1}{3!}\,F_1'(t_0 - t)\Big[\frac{\sqrt{2}}{A}\,\ln\Big|\frac{x}{x_0}\Big|\Big]^3 + \cdots\Big]. \tag{4.11}$$

Example 4.1. Let us obtain Tikhonov's form of the invariant solution

$$u = x\,\mathrm{e}^{(B-C)(t_o - t)} \tag{4.12}$$

from Remark 4.1. The solution (4.12) is written in the variables τ and y given by (4.2) as follows:

$$u(\tau, y) = x_0\,\mathrm{e}^{\frac{A}{\sqrt{2}}y + (B-C)\tau}. \tag{4.13}$$

We note that if we substitute it in Eq. (4.5),

$$x_0\,\mathrm{e}^{\frac{A}{\sqrt{2}}y + (B_C)\tau} = \mathrm{e}^{\left(\frac{A}{2\sqrt{2}} - \frac{B}{\sqrt{2}\,A}\right)y + \left(\frac{B}{2} - \frac{A^2}{8} - \frac{B^2}{2A^2} - C\right)\tau}\,v(\tau, y),$$

and solve for v we have the following solution to the heat equation (4.4):

$$v(\tau, y) = x_0\,\mathrm{e}^{\alpha y + \alpha^2\tau}, \tag{4.14}$$

where

$$\alpha = \frac{A}{2\sqrt{2}} + \frac{B}{\sqrt{2}A}\,\cdot$$

Writing

$$x_0\,\mathrm{e}^{\alpha y}\,\mathrm{e}^{\alpha^2\tau} = x_0\,\mathrm{e}^{\alpha^2\tau}\Big(1 + \alpha y + \frac{y^2}{2!}\,\alpha^2 + \frac{y^3}{3!}\,\alpha^3 + \cdots\Big)$$

or

$$x_0\,\mathrm{e}^{\alpha y}\,\mathrm{e}^{\alpha^2\tau} = x_0\,\mathrm{e}^{\alpha^2\tau} + y\alpha\,x_0\,\mathrm{e}^{\alpha^2\tau} + \frac{y^2}{2!}\,\alpha^2\,x_0\,\mathrm{e}^{\alpha^2\tau} + \frac{y^3}{3!}\,\alpha^3\,x_0\,\mathrm{e}^{\alpha^2\tau} + \cdots$$

we see that Tikhonov's series for the solution (4.14):

$$x_0\,\mathrm{e}^{\alpha y}\,\mathrm{e}^{\alpha^2\tau} = F(\tau) + yF_1(\tau) + \frac{y^2}{2!}\,F'(\tau) + \frac{y^3}{3!}\,F_1'(\tau) + \cdots$$

is satisfied with

$$F(\tau) = x_0\,\mathrm{e}^{\alpha^2\tau}, \quad F_1(\tau) = \alpha F(\tau) \equiv \alpha x_0\,\mathrm{e}^{\alpha^2\tau}.$$

Hence, Tikhonov's form (4.11) of the solution (4.12) of the the Black-Scholes equation corresponds to

$$F(t_0 - t) = x_0\,\mathrm{e}^{\alpha^2(t_0-t)}, \quad F_1(t_0 - t) = \alpha x_0\,\mathrm{e}^{\alpha^2(t_0-t)}.$$

4.4 Fundamental and other particular solutions

Eqs. (4.5) and (4.2) provide the transition formula for mapping any exact solution $v(\tau, y)$ of the heat equation (4.4) to an exact solution $u(t, x)$ of the Black-Scholes equation (4.1). The transition formula can be used either in the form of the system (4.5), (4.2) or in the following explicit form:

$$u(t,x) = \mathrm{e}^{\left(\frac{1}{2}-\frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right| + \left(\frac{B}{2}-\frac{A^2}{8}-\frac{B^2}{2A^2}-C\right)(t_0-t)}\, v\Big(t_0 - t,\, \frac{\sqrt{2}}{A}\,\ln\Big|\frac{x}{x_0}\Big|\Big). \tag{4.15}$$

Let us find the Let us substitute in Eq. (4.15) the fundamental solution the fundamental solution $u = \mathcal{E}(t, x; t_0, x_0)$ of the Cauchy problem for the Black-Scholes equation defined by the equations

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0, \quad t < t_0,$$
$$u\big|_{t=t_0} \equiv u\big|_{t\to t_0} = \delta(x - x_0).$$

We will use the transition formula in the form of the system (4.5), (4.2). If we substitute in (4.5) the function

$$v(\tau, y) = \frac{K}{2\sqrt{\pi\tau}}\,\mathrm{e}^{-\frac{y^2}{4\tau}}, \quad \tau > 0,\ K = \text{const.},$$

which solves the heat equation (4.4) and is proportional to the fundamental solution of the Cauchy problem for the heat equation, we get:

$$u(\tau, y) = \frac{K}{2\sqrt{\pi\tau}}\,\mathrm{e}^{-\frac{y^2}{4\tau} + \left(\frac{A}{2\sqrt{2}} - \frac{B}{\sqrt{2}\,A}\right)y + \left(\frac{B}{2}-\frac{A^2}{8}-\frac{B^2}{2A^2}-C\right)\tau}, \quad \tau > 0. \tag{4.16}$$

Let us replace τ and y by their expressions (4.2), introduce the notation

$$z = \frac{\sqrt{2}}{A} \ln|x|$$

so that $y = z - z_0$, and rewrite (4.16) in the form

$$u = \frac{K}{2\sqrt{\pi(t_0 - t)}}\, \mathrm{e}^{-\frac{(z-z_o)^2}{4(t_o-t)} + \left(\frac{A}{2\sqrt{2}} - \frac{B}{\sqrt{2}\,A}\right)(z-z_o) + \left(\frac{B}{2} - \frac{A^2}{8} - \frac{B^2}{2A^2} - C\right)(t_o-t)}, \quad t < t_0.$$

Letting now $t \to t_0$ and proceeding as in Example 3.6 we obtain:

$$u\Big|_{t=t_0} = K\mathrm{e}^{\left(\frac{A}{2\sqrt{2}} - \frac{B}{\sqrt{2}\,A}\right)(z-z_o)}\, \delta(z - z_o) = K\delta(z - z_o).$$

Using the formula for the change of variables in the $delta$-function,

$$\delta(x - x_o) = \left|\frac{dz}{dx}\right|_{x=x_0} \delta(z - z_o) = \frac{\sqrt{2}}{A|x_0|}\, \delta(z - z_o)$$

we have:

$$u\Big|_{t=t_0} = K\, \frac{A|x_0|}{\sqrt{2}}\, \delta(x - x_o).$$

Therefore, we take

$$K = \frac{\sqrt{2}}{A|x_0|}\,,$$

substitute the expression for z and obtain the following fundamental solution of the Cauchy problem for the Black-Scholes equation:

$$\mathcal{E} = \frac{1}{A|x_0|\sqrt{2\pi(t_0 - t)}}\, \mathrm{e}^{-\frac{(\ln|x| - \ln|x_o|)^2}{2A^2(t_0-t)} + \left(\frac{1}{2} - \frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right| + \left(\frac{B}{2} - \frac{A^2}{8} - \frac{B^2}{2A^2} - C\right)(t_0-t)}, \tag{4.17}$$

where $t < t_0$. It was obtained in [11] by means of the invariance principle.

The transition formula (4.15) can be used for obtaining numerous particular solutions to the Black-Scholes equation in a closed form, in particular, those given by elementary functions. For example, one can take for v any invariant solution of the heat equation. In this way, one can obtain all invariant solutions of the Black-Scholes equation by enumerating independent invariant solutions of the heat equation provided by the optimal system of one-dimensional subalgebras of the symmetry algebra of the heat equation. Another way to find particular solutions of the Black-Scholes equation is to substitute in Eq. (4.11) any polynomials F and F_1.

Example 4.2. The infinitesimal symmetry

$$X = \frac{\partial}{\partial \tau} - \alpha^2 v \frac{\partial}{\partial v}\,, \quad \alpha = \text{const.},$$

of the heat equation $v_\tau - v_{yy} = 0$ provides the following invariant solution:

$$v = \big[C_1 \cos(\alpha y) + C_2 \sin(\alpha y)\big] \mathrm{e}^{-\alpha^2 \tau}, \quad C_1, C_2 = \text{const.}$$

Substituting it in (4.15) we obtain the following solution to the Black-Scholes equation:

$$u = \Big[C_1 \cos\Big(\beta \ln\Big|\frac{x}{x_0}\Big|\Big) + C_2 \sin\Big(\beta \ln\Big|\frac{x}{x_0}\Big|\Big)\Big] \mathrm{e}^{\left(\frac{1}{2} - \frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right| + \left(\frac{B}{2} - \frac{A^2}{8} - \frac{B^2}{2A^2} - C - \alpha^2\right)(t_0 - t)},$$

where $\beta = \frac{\alpha\sqrt{2}}{A}$.

4.5 Non-linearization of the Black-Scholes model

Recall that the heat equation (4.4) is connected with the Burgers equation

$$w_\tau - w w_y - w_{yy} = 0 \tag{4.18}$$

by the following differential substitution known as the Hopf-Cole transformation:

$$w = 2\frac{v_y}{v} \equiv \frac{\partial \ln v^2}{\partial y} \cdot \tag{4.19}$$

Eq. (4.18) has specific properties due to its nonlinearity and is used in turbulence theories, non-linear acoustics, etc. It seems reasonable to employ the remarkable connections of the heat equation with the Black-Scholes and the Burgers equations in order to describe nonlinear effects in finance ("financial turbulence"). Noting that the transformations (4.2) and (4.5) formally bring in the heat equation a "financial content", I rewrite the Hopf-Cole transformation (4.19) in the variables t, x, u given by (4.2), (4.5) and the Burgers equation (4.18) in the variables t, x given by (4.2) and obtain:

$$w = \sqrt{2}\Big(\frac{B}{A} - \frac{A}{2} + Ax\frac{u_x}{u}\Big), \tag{4.20}$$

$$w_t + \frac{A}{\sqrt{2}}\left(w + \frac{A}{\sqrt{2}}\right) x w_x + \frac{A^2}{2} x^2 w_{xx} = 0. \tag{4.21}$$

One can verify that the linear Black-Scholes equation is connected with the nonlinear equation (4.21) by the transformation (4.20).

4.6 Symmetries of the basic equations

For the convenience of the reader who might be interested in investigating invariant solutions of the Black-Scholes equation and/or of the nonlinear equation (4.21), I will list first of all the well-known symmetries of the heat equation (4.4) and of the Burgers equation (4.18). Then I will recall the symmetries of the Black-Scholes equation computed in [11] and will give the symmetries of the nonlinear equation (4.21). All these symmetries are mutually connected by the transformations (4.2), (4.5), (4.19) and (4.20). For the linear equations (4.4) and (4.1) I will omit the trivial infinite part of their symmetry algebras appearing due to the superposition principle. For example, for the heat equation this trivial part comprises the operator

$$X_* = v_*(\tau, y)\frac{\partial}{\partial v}\,,$$

where $v_*(\tau, y)$ is any particular solution of Eq. (4.4).

4.6.1 Symmetries of the heat and Burgers equations

The symmetries of the heat equation (4.4) are:

$$X_1 = \frac{\partial}{\partial \tau}\,, \quad X_2 = \frac{\partial}{\partial y}\,, \quad X_3 = 2\tau\frac{\partial}{\partial \tau} + y\frac{\partial}{\partial y}\,, \quad X_4 = v\frac{\partial}{\partial v}\,, \tag{4.22}$$

$$X_5 = 2\tau\frac{\partial}{\partial y} - yv\frac{\partial}{\partial v}\,, \quad X_6 = \tau^2\frac{\partial}{\partial \tau} + \tau y\frac{\partial}{\partial y} - \frac{1}{4}(2\tau + y^2)\,v\frac{\partial}{\partial v}\,\cdot$$

The symmetries of the Burgers equation (4.18) are:

$$X_1 = \frac{\partial}{\partial \tau}\,, \quad X_2 = \frac{\partial}{\partial y}\,, \quad X_3 = 2\tau\frac{\partial}{\partial \tau} + y\frac{\partial}{\partial y} - w\frac{\partial}{\partial w}\,, \tag{4.23}$$

$$X_4 = \tau\frac{\partial}{\partial y} - \frac{\partial}{\partial w}\,, \quad X_5 = \tau^2\frac{\partial}{\partial \tau} + \tau y\frac{\partial}{\partial y} - (y + \tau w)\frac{\partial}{\partial w}\,\cdot$$

To obtain the symmetries (4.23) of the Burgers equation there is no need to solve the determining equations. They can be obtained merely by rewriting the symmetries (4.22) of the heat equation in terms of the variable w given by Eq. (4.19). I will illustrate the procedure by the operators X_5 and X_6 from (4.22).

Let us begin with X_5. We extend it to v_y by the usual prolongation formula,

$$X_5 = 2\tau\frac{\partial}{\partial y} - yv\frac{\partial}{\partial v} - (v + yv_y)\frac{\partial}{\partial v_y}\,,$$

and act on the dependent variable w of the Burgers equation. Using the definition (4.19) of w we have:

$$X_5(w) = -\frac{2}{v}\left(v + yv_y\right) + 2(yv)\frac{v_y}{v^2} = -2.$$

Hence, X_5 is written in terms of the variables τ, y and w of the Burgers equations as follows:

$$X_5 = 2\Big[\tau\frac{\partial}{\partial y} - \frac{\partial}{\partial w}\Big].$$

Up to the unessential coefficient 2, this is the operator X_4 from (4.23).

Proceeding likewise with the operator X_6 we obtain:

$$X_6 = \tau^2\frac{\partial}{\partial \tau} + \tau y\frac{\partial}{\partial y} - \frac{1}{4}(2\tau + y^2)v\frac{\partial}{\partial v} - \Big[\frac{1}{4}(2\tau + y^2)v_y + \frac{1}{2}yv + \tau v_y\Big]\frac{\partial}{\partial v_y}\,,$$

$$X_6(w) = -y - 2\tau\frac{v_y}{v} = -(y + \tau w),$$

and hence we get

$$X_6 = \tau^2\frac{\partial}{\partial \tau} + \tau y\frac{\partial}{\partial y} - (y + \tau w)\frac{\partial}{\partial w}\,,$$

i.e. the operator X_5 from (4.23).

If we will perform the similar procedure with the operator X_4 from (4.22), we will see that its action on the variables of the Burgers equation is identically zero. Indeed, its prolongation to v_y is written

$$X_4 = v\frac{\partial}{\partial v} + v_y\frac{\partial}{\partial v_y}\,,$$

and hence $X_4(w) = 0$. Therefore X_4 does not provide any symmetry for the Burgers equation, and hence the Burgers equation possesses only five Lie point symmetries.

4.6.2 Symmetries of the Black-Scholes equation

The Black-Scholes equation has the following symmetries obtained in [11] by solving the determining equations. They can also be obtained by subjecting the symmetries (4.22) of the heat equation to the transformations (4.2), (4.5).

$$\begin{gathered}
X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = x\frac{\partial}{\partial x}\,, \quad X_3 = 2t\frac{\partial}{\partial t} + \big(\ln x + Pt\big)x\frac{\partial}{\partial x} + 2Ctu\frac{\partial}{\partial u}\,,\\
X_4 = u\frac{\partial}{\partial u}\,, \quad X_5 = A^2tx\frac{\partial}{\partial x} + \big(\ln x - Pt\big)u\frac{\partial}{\partial u}\,, \qquad (4.24)\\
X_6 = 2A^2t^2\frac{\partial}{\partial t} + 2A^2tx\ln x\frac{\partial}{\partial x} + \Big[\big(\ln x - Pt\big)^2 + 2A^2Ct^2 - A^2t\Big]u\frac{\partial}{\partial u}\,,
\end{gathered}$$

where P is the constant defined by

$$P = B - \frac{1}{2}A^2.$$

4.6.3 Symmetries of the nonlinear equation (4.21)

Let us obtain the symmetries of the nonlinear equation (4.21) by subjecting the symmetries (4.23) of the Burgers equation to the transformation (4.2) written in the form

$$t = t_0 - \tau, \quad x = x_0 \mathrm{e}^{\frac{Ay}{\sqrt{2}}}. \tag{4.25}$$

It is manifest that the transformation (4.25) maps the first operator (4.23) into

$$X_1 = \frac{\partial}{\partial t}\,\cdot$$

For the second operator (4.23) we have

$$X_2(x) = \frac{A}{\sqrt{2}}\,x_0 \mathrm{e}^{\frac{Ay}{\sqrt{2}}} = \frac{A}{\sqrt{2}}\,x.$$

Hence, ignoring the unessential constant factor we obtain

$$X_2 = x\frac{\partial}{\partial x}\,\cdot$$

For the third operator (4.23) we have

$$X_3(t) = -2\tau = 2(t - t_0), \quad X_3(x) = \frac{A}{\sqrt{2}}\,yx = x(\ln|x| - \ln|x_0|).$$

Therefore the operator X_3 from (4.23) becomes

$$\begin{aligned} &2(t-t_0)\frac{\partial}{\partial t} + x(\ln|x| - \ln|x_0|)\frac{\partial}{\partial x} - w\frac{\partial}{\partial w} \\ &= 2t\frac{\partial}{\partial t} + x\ln|x|\frac{\partial}{\partial x} - w\frac{\partial}{\partial w} - 2t_0X_1 - \ln|x_0|X_2, \end{aligned}$$

where X_1 and X_2 are given above. Taking it modulo X_1, X_2, we obtain the following operator admitted by Eq. (4.21):

$$X_3 = 2t\frac{\partial}{\partial t} + x\ln|x|\frac{\partial}{\partial x} - w\frac{\partial}{\partial w}\,\cdot$$

Proceeding likewise with remaining two operators from (4.23) we arrive at the following symmetries of Eq. (4.21):

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = x\frac{\partial}{\partial x}\,, \quad X_3 = 2t\frac{\partial}{\partial t} + x\ln|x|\frac{\partial}{\partial x} - w\frac{\partial}{\partial w}\,, \\ &X_4 = Atx\frac{\partial}{\partial x} + \sqrt{2}\,\frac{\partial}{\partial w}\,, \\ &X_5 = t^2\frac{\partial}{\partial t} + tx\ln|x|\frac{\partial}{\partial x} + \left(\frac{\sqrt{2}}{A}\,\ln|x| - tw\right)\frac{\partial}{\partial w}\,\cdot \end{aligned} \tag{4.26}$$

4.7 Optimal system of one-dimensional subalgebras for Eq. (4.21)

The structure of the Lie algebra L_5 spanned by the operators (4.26) is described by the following commutator table.

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$2X_1$	AX_2	X_3
X_2	0	0	X_2	0	$\frac{1}{A}X_4$
X_3	$-2X_1$	$-X_2$	0	X_4	$2X_5$
X_4	$-AX_2$	0	$-X_4$	0	0
X_5	$-X_3$	$-\frac{1}{A}X_4$	$-2X_5$	0	0

(4.27)

The Lie algebra L_5 spanned the symmetries (4.26) provides a possibility to find invariant solutions of Eq. (4.21) based on any one-dimensional subalgebra of the algebra L_5, i.e. on any operator $X \in L_5$. However, there are infinite number of one-dimensional subalgebras of L_5 since an arbitrary operator from L_5 is written

$$X = l^1 X_1 + \ldots + l^5 X_5, \tag{4.28}$$

and hence depends on five arbitrary constants $l^1, \ldots, l^5$. In order to make this problem manageable, L.V. Ovsyannikov [12] (see also [13]) introduced the concept of *optimal systems of subalgebras*[2] by noting that if two subalgebras are *similar*, i.e. connected with each other by a transformation of the symmetry group, then their corresponding invariant solutions are connected with each other by the same transformation. Consequently, it is sufficient to deal with an optimal system of invariant solutions obtained, in our case, as follows. We put into one class all similar operators $X \in L_5$ and select a representative of each class. The set of the representatives of all these classes is an *optimal system of one-dimensional subalgebras*.

Now all invariant solutions can in principle be obtained by constructing the invariant solution for each member of the optimal system of subalgebras. The set of invariant solutions obtained in this way is an *optimal system of invariant solutions*. It is worth noting that the form of these invariant solutions depends on the choice of representatives. If one constructs an optimal system of invariant solutions and subjects these solutions to all transformations of the group admitted by the equation in question, one obtains *all invariant solutions* of this equation.

Let us construct an optimal system of one-dimensional subalgebras of the Lie algebra L_5 following the simple method used by Ovsyannikov [12].

[2] He considered subalgebras of any dimension. We need here only one-dimensional subalgebras.

The transformations of the symmetry group with the Lie algebra L_5 provide the 5-parameter group of linear transformations of the operators $X \in L_5$ (see examples in Section 4.8.12) or, equivalently, linear transformations of the vector

$$l = (l^1, \ldots, l^5), \tag{4.29}$$

where $l^1, \ldots, l^5$ are taken from (4.28). To find these linear transformations, we use their generators (see, e.g. [14], Section 1.4)

$$E_\mu = c^\lambda_{\mu\nu} l^\nu \frac{\partial}{\partial l^\lambda}\,, \quad \mu = 1, \ldots, 5, \tag{4.30}$$

where $c^\lambda_{\mu\nu}$ are the structure constants of the Lie algebra L_5 defined by

$$[X_\mu, X_\nu] = c^\lambda_{\mu\nu} X_\lambda.$$

Let us find, e.g. the operator E_1. According to (4.30), it is written

$$E_1 = c^\lambda_{1\nu} l^\nu \frac{\partial}{\partial l^\lambda}\,,$$

where $c^\lambda_{1\nu}$ are defined by the commutators $[X_1, X_\nu] = c^\lambda_{\mu\nu} X_\lambda$, i.e. by the first raw in Table (4.27). Namely, the non-vanishing $c^\lambda_{\mu\nu}$ are

$$c^1_{13} = 2, \quad c^2_{14} = A, \quad c^2_{15} = 1.$$

Therefore we have:

$$E_1 = 2l^3 \frac{\partial}{\partial l^1} + A l^4 \frac{\partial}{\partial l^2} + l^5 \frac{\partial}{\partial l^3}\,\cdot$$

Substituting in (4.30) all structure constants given by Table (4.27) we obtain:

$$E_1 = 2l^3 \frac{\partial}{\partial l^1} + A l^4 \frac{\partial}{\partial l^2} + l^5 \frac{\partial}{\partial l^3}\,, \qquad E_2 = l^3 \frac{\partial}{\partial l^2} + \frac{1}{A}\, l^5 \frac{\partial}{\partial l^4}\,,$$

$$E_3 = -2l^1 \frac{\partial}{\partial l^1} - l^2 \frac{\partial}{\partial l^2} + l^4 \frac{\partial}{\partial l^4} + 2l^5 \frac{\partial}{\partial l^5}\,, \quad E_4 = -A l^1 \frac{\partial}{\partial l^2} - l^3 \frac{\partial}{\partial l^4}\,,$$

$$E_5 = -l^1 \frac{\partial}{\partial l^3} - \frac{1}{A}\, l^2 \frac{\partial}{\partial l^4} - 2l^3 \frac{\partial}{\partial l^5}\,\cdot \tag{4.31}$$

Let us find the transformations provided by the generators (4.31). For the generator E_1, the Lie equations with the parameter a_1 are written

$$\frac{d\tilde{l}^1}{da_1} = 2\tilde{l}^3\,, \quad \frac{d\tilde{l}^2}{da_1} = A\tilde{l}^4\,, \quad \frac{d\tilde{l}^3}{da_1} = \tilde{l}^5\,, \quad \frac{d\tilde{l}^4}{da_1} = 0, \quad \frac{d\tilde{l}^5}{da_1} = 0.$$

Integrating these equations and using the initial condition $\tilde{l}\,\big|_{a_1=0} = l$, we obtain:

$$
\begin{aligned}
E_1: \qquad & \tilde{l}^{\,1} = l^1 + 2a_1 l^3 + a_1^2 l^5, \quad \tilde{l}^{\,2} = l^2 + a_1 A l^4, \\
& \tilde{l}^{\,3} = l^3 + a_1 l^5, \qquad \tilde{l}^{\,4} = l^4, \qquad \tilde{l}^{\,5} = l^5.
\end{aligned}
\tag{4.32}
$$

Taking the other operators (4.31) we obtain the following transformations:

$$
E_2: \quad \tilde{l}^{\,1} = l^1,\ \tilde{l}^{\,2} = l^2 + a_2 l^3,\ \tilde{l}^{\,3} = l^3,\ \tilde{l}^{\,4} = l^4 + \frac{a_2}{A}\,l^5, \quad \tilde{l}^{\,5} = l^5; \tag{4.33}
$$

$$
E_3: \quad \tilde{l}^{\,1} = a_3^{-2}\,l^1, \quad \tilde{l}^{\,2} = a_3^{-1}\,l^2, \quad \tilde{l}^{\,3} = l^3, \quad \tilde{l}^{\,4} = a_3 l^4, \quad \tilde{l}^{\,5} = a_3^2 l^5, \tag{4.34}
$$

where $a_3 > 0$ since the integration of the Lie equations yields, e.g. $l^4 = l^4 \mathrm{e}^{\tilde{a}_3} = a_3 l^4$;

$$
E_4: \quad \tilde{l}^{\,1} = l^1, \quad \tilde{l}^{\,2} = l^2 - a_4 A l^1, \quad \tilde{l}^{\,3} = l^3, \quad \tilde{l}^{\,4} = l^4 - a_4 l^3, \quad \tilde{l}^{\,5} = l^5; \tag{4.35}
$$

$$
\begin{aligned}
E_5: \qquad & \tilde{l}^{\,1} = l^1, \qquad \tilde{l}^{\,2} = l^2, \qquad \tilde{l}^{\,3} = l^3 - a_5 l^1, \\
& \tilde{l}^{\,4} = l^4 - \frac{a_5}{A}\,l^2, \quad \tilde{l}^{\,5} = l^5 - 2a_5 l^3 + a_5^2 l^1.
\end{aligned}
\tag{4.36}
$$

Note that the transformations (4.32)-(4.36) map the vector $X \in L_5$ given by (4.28) to the vector $\widetilde{X} \in L_5$ given by the following formula:

$$
\widetilde{X} = \tilde{l}^{\,1} X_1 + \ldots + \tilde{l}^{\,5} X_5. \tag{4.37}
$$

Now we can prove the following result on an optimal system of one-dimensional subalgebras of the five-dimensional Lie algebra of symmetries of Eq. (4.21).

Theorem 4.1. The following operators provide an optimal system of one-dimensional subalgebras of the Lie algebra L_5 with the basis (4.26):

$$
\begin{aligned}
& X_1, \quad X_2, \quad X_3 + kX_2, \quad X_4, \quad X_1 + X_4, \quad X_1 - X_4, \\
& X_5, \qquad X_1 + X_5, \qquad X_2 + X_5, \qquad X_2 - X_5,
\end{aligned}
\tag{4.38}
$$

where k is an arbitrary parameter.

Proof. We first clarify if the transformations (4.32)-(4.36) have invariants $J(l^1, \ldots, l^5)$. The reckoning shows that the 5×5 matrix $\|c^{\lambda}_{\mu\nu} l^{\nu}\|$ of the coefficients of the operators (4.31) has the rank four. It means that the transformations (4.32)-(4.36) have precisely one functionally independent invariant. The integration of the equations

$$
E_\mu(J) = 0, \quad \mu = 1, \ldots, 5,
$$

shows that the invariant is

$$J = (l^3)^2 - l^1 l^5. \tag{4.39}$$

Knowledge of the invariant (4.39) simplifies further calculations significantly.

Since X_5 is the most complicated operator among the symmetries (4.26), we will try to exclude it from the operators of the optimal system when it is possible. In other words, we have to annul $\bar{l}_5$ if possible. It is manifest from Eqs. (4.32)-(4.36) that we can annul $\bar{l}_5$ only by the transformation (4.36) provided that l^1 and l^3 do not vanish simultaneously.

The last equation in (4.36) shows that if $l^1 \neq 0$, we get $\bar{l}_5 = 0$ by solving the quadratic equation $l^5 - 2a_5 l^3 + a_5^2 l^1 = 0$ for a_5, i.e. by taking

$$a_5 = \frac{l^3 \pm \sqrt{J}}{l^1}, \tag{4.40}$$

where J is the invariant (4.39). We can use Eq. (4.40) only if $J \geq 0$.

Now we begin the construction of the optimal system. The method requires a simplification of the general vector (4.29) by means of the transformations (4.32)-(4.36). As a result, we will find the simplest representatives of each class of similar vectors (4.29). Substituting these representatives in (4.28), we will obtain the optimal system of one-dimensional subalgebras of L_5. We will divide the construction to several cases.

4.7.1 The case $l^1 = 0$

I will divide this case into the following two subcases.

1°. $l^3 \neq 0$. In other words, we consider the vectors (4.29) of the form

$$(0, l^2, l^3, l^4, l^5), \quad l^3 \neq 0.$$

First we take $a_5 = l^5/(2l^3)$ in (4.36) and reduce the above vector to the form

$$(0, l^2, l^3, l^4, 0).$$

Then we subject the latter vector to the transformation (4.35) with $a_4 = l^4/l^3$ and obtain the vector

$$(0, l^2, l^3, 0, 0).$$

Since the operator X is defined up to a constant factor and $l^3 \neq 0$, we divide the above vector by l^3 and transform it to the form

$$(0, k, 1, 0, 0).$$

Substituting it in (4.28), we obtain the operator

$$X_3 + kX_2 \tag{4.41}$$

with an arbitrary parameter k.

2°. $l^3 = 0$. Thus, we consider the vectors (4.29) of the form

$$(0, l^2, 0, l^4, l^5).$$

2°(1). If $l^2 \neq 0$, we can assume $l^2 = 1$ (see above), use the transformation (4.36) with $a_5 = Al^4$ and get the vector

$$(0, 1, 0, 0, l^5).$$

If $l^5 \neq 0$ we can make $l^5 = \pm 1$ by the transformation (4.34). Thus, taking into account the possibility $l^5 = 0$, we obtain the following representatives for the optimal system:

$$X_2, \quad X_2 + X_5, \quad X_2 - X_5. \tag{4.42}$$

2°(2). Let $l^2 = 0$. If $l^5 \neq 0$ we can set $l^5 = 1$. Now we apply the transformation (4.33) with $a_2 = -Al^4$ and obtain the vector $(0, 0, 0, 0, 1)$. If $l^5 = 0$ we get the vector $(0, 0, 0, 1, 0)$. Thus, the case $l^2 = 0$ provides the operators

$$X_4, \quad X_5. \tag{4.43}$$

4.7.2 The case $l^1 \neq 0, \ J > 0$

Now we can define a_5 by Eq. (4.40) and annul $\bar{l}^5$ by the transformation (4.36). Thus, we will deal with the vector

$$(l^1, l^2, l^3, l^4, 0), \quad l^1 \neq 0.$$

Since J is invariant under the transformations (4.32)-(4.36), the condition $J > 0$ yields that in the above vector we have $l^3 \neq 0$. Therefore we can use the transformation (4.35) with $a_4 = l^4/l^3$ and get $\bar{l}^4 = 0$. Then we apply the transformation (4.32) with $a_1 = -l^1/(2l^3)$ and obtain $\bar{l}^1 = 0$, thus arriving at the vector $(0, l^2, l^3, 0, 0)$, and hence at the previous operator (4.41). Hence, this case contributes no additional subgroups to the optimal system.

4.7.3 The case $l^1 \neq 0, \ J = 0$

In this case Eq. (4.40) reduces to $a_5 = l^3/l^1$.

If $l^3 \neq 0$, we use the transformation (4.36) with $a_5 = l^3/l^1$ and obtain $\bar{l}^5 = 0$. Due to the invariance of J we conclude that the equation $J = 0$ yields $(\bar{l}^3)^2 - \bar{l}^1\bar{l}^5 = 0$. Since $\bar{l}^5 = 0$, it follows that $\bar{l}^3 = 0$. Thus we can deal with the vectors of the form

$$(l^1, l^2, 0, l^4, 0), \quad l^1 \neq 0. \tag{4.44}$$

Furthermore, if $l^3 = 0$, we have $J = -l^1 l^5$, and the equation $J = 0$ yields $l^5 = 0$ since $l^1 \neq 0$. Therefore we again have the vectors of the form (4.44) where we can assume $l^1 = 1$. Subjecting the vector (4.44) with $l^1 = 1$ to the transformation (4.35) with $a_4 = l^2/A$ we obtain $\overline{l}^2 = 0$, and hence map the vector (4.44) to the form

$$(1, 0, 0, l^4, 0).$$

If $l^4 \neq 0$, we use the transformation (4.34) with an appropriately chosen a_3 and obtain $l^4 = \pm 1$. taking into account the possibility $l^4 = 0$, we see that this case contributes the following operators:

$$X_1, \quad X_1 + X_4, \quad X_1 - X_4. \tag{4.45}$$

4.7.4 The case $l^1 \neq 0,\ J < 0$

It is obvious from the condition $J = (l^3)^2 - l^1 l^5 < 0$ that $l^5 \neq 0$. Therefore we successively apply the transformations (4.36), (4.35) and (4.33) with $a_5 = l^3/l^1,\ a_4 = l^2/(Al^1)$ and $a_2 = -Al^4/l^5$, respectively and obtain $\overline{l}^3 = \overline{l}^2 = \overline{l}^4 = 0$. The components l^1 and l^5 of the resulting vector

$$(l^1, 0, 0, 0, l^5)$$

have the common sign since the condition $J < 0$ yields $l^1 l^5 > 0$. Therefore using the transformation (4.34) with an appropriate value of the parameter a_3 and invoking that we can multiply the vector l by any constant, we obtain $l^1 = l^5 = 1$, i.e. the operator

$$X_1 + X_5. \tag{4.46}$$

Finally, collecting the operators (4.41), (4.42), (4.43), (4.45) and (4.46), we arrive at the optimal system (4.38), thus completing the proof of the theorem.

Remark 4.2. The commutator table of the symmetries (4.23) of the Burgers equation (4.18) has the form (4.27) with $A = 1$:

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$2X_1$	X_2	X_3
X_2	0	0	X_2	0	X_4
X_3	$-2X_1$	$-X_2$	0	X_4	$2X_5$
X_4	$-X_2$	0	$-X_4$	0	0
X_5	$-X_3$	$-X_4$	$-2X_5$	0	0

(4.47)

Therefore an optimal system of one-dimensional subalgebras of the Lie algebra L_5 of the symmetries of the Burgers equation is given by (4.38),

$$\begin{aligned} &X_1, \quad X_2, \quad X_3 + kX_2, \quad X_4, \quad X_1 + X_4, \quad X_1 - X_4, \\ &X_5, \qquad X_1 + X_5, \qquad X_2 + X_5, \qquad X_2 - X_5, \end{aligned} \tag{4.48}$$

with the operators $X_1, \ldots, X_5$ taken from (4.23).

4.8 Invariant solutions of Eq. (4.21)

In order to construct an optimal system of invariant solutions, we have to find the invariant solution for each operator of the optimal system (4.38). For the sake of simplicity, we will make the calculations for positive values of the variables t and x.

4.8.1 Invariant solution for the operator X_1

Two functionally independent invariants for the operator X_1 are x and w. Consequently, the invariant solution is the stationary solution, $w = w(x)$. For this solution, Eq. (4.21) yields the following second-order ordinary differential equation:

$$\left(w + \frac{A}{\sqrt{2}}\right) x w' + \frac{A}{\sqrt{2}} x^2 w'' = 0. \tag{4.49}$$

By setting

$$\psi = \frac{\sqrt{2}}{A}\left(w + \frac{A}{\sqrt{2}}\right)$$

or

$$w = \frac{A}{\sqrt{2}}(\psi - 1),$$

and considering ψ as a function $\psi = \psi(z)$ of the new independent variable

$$z = \ln x,$$

we rewrite Eq. (4.49) in the form

$$\psi'' - \psi' + \psi\psi' = 0.$$

Integrating it once, we obtain

$$\psi' - \psi + \frac{1}{2}\psi^2 = \frac{1}{2}K_1. \tag{4.50}$$

The general solution to Eq. (4.50) is given by quadrature:

$$\int \frac{2d\psi}{\psi^2 - 2\psi - K_1} = -(z + K_2), \tag{4.51}$$

and can be written in terms of elementary fuinctions. Namely, consider the equation

$$\psi^2 - 2\psi - K_1 = 0.$$

According to the solutuion formula $\psi = 1 \pm \sqrt{1 + K_1}$ we deal with three cases:

$$\text{(i) } K_1 = -1, \quad \text{(ii) } 1 + K_1 = \alpha^2 > 0, \quad \text{(iii) } 1 + K_1 = -\alpha^2 < 0.$$

In the first case we have $\psi^2 - 2\psi - K_1 = (\psi - 1)^2$ and the integral in (4.51) is

$$\int \frac{2d\psi}{\psi^2 - 2\psi - K_1} = \int \frac{2d\psi}{(\psi - 1)^2} = -\frac{2}{\psi - 1}\,\cdot$$

Therefore Eq. (4.51) yields

$$\psi - 1 = \frac{2}{z + K_2}\,\cdot$$

Thus, the first case leads to the following solution of Eq. (4.50):

$$\text{(i)} \qquad w = \frac{A\sqrt{2}}{K_2 + \ln x}\,\cdot$$

In the second case we have $\psi^2 - 2\psi - K_1 = (\psi - 1 - \alpha)(\psi - 1 + \alpha)$, and hence

$$\int \frac{2d\psi}{\psi^2 - 2\psi - K_1} = \frac{1}{\alpha}\Big[\int \frac{d\psi}{\psi - 1 - \alpha} - \int \frac{d\psi}{\psi - 1 + \alpha}\Big] = -\frac{1}{\alpha}\,\ln \frac{\psi - 1 + \alpha}{\psi - 1 - \alpha}\,\cdot$$

Therefore Eq. (4.51) yields

$$\psi - 1 = \alpha\,\frac{\beta \mathrm{e}^{\alpha z} + 1}{\beta \mathrm{e}^{\alpha z} - 1}\,, \quad \beta = \text{const.}$$

Thus, the second case leads to the following solution of Eq. (4.50):

$$\text{(ii)} \qquad w = \frac{A\alpha}{\sqrt{2}}\,\frac{\beta x^\alpha + 1}{\beta x^\alpha - 1}\,, \quad \alpha, \beta = \text{const.}$$

In the third case the integral in (4.51) is written

$$\int \frac{2d\psi}{\psi^2 - 2\psi + 1 + \alpha^2} = \frac{2}{\alpha}\,\arctan\Big(\frac{\psi - 1}{\alpha}\Big).$$

Therefore Eq. (4.51) yields

$$\psi - 1 = \alpha \tan\Big(-\frac{\alpha}{2}\,(z + K_2)\Big).$$

Thus, the first case leads to the following solution of Eq. (4.50):

$$\text{(iii)} \qquad w = \frac{A\alpha}{\sqrt{2}} \tan\Big(\beta - \frac{\alpha}{2} \ln x\Big).$$

Summing up, we conclude that the operator X_1 provides the invariant solution given by the following formulae:

$$\begin{aligned}
&\text{(i)} \qquad w = \frac{A\sqrt{2}}{K + \ln x}\,;\\
&\text{(ii)} \qquad w = \frac{A\alpha}{\sqrt{2}}\,\frac{\beta x^\alpha + 1}{\beta x^\alpha - 1}\,;\\
&\text{(iii)} \qquad w = \frac{A\alpha}{\sqrt{2}} \tan\Big(\beta - \frac{\alpha}{2} \ln x\Big).
\end{aligned} \tag{4.52}$$

4.8.2 Invariant solution for the operator X_2

The operator X_2 from the optimal system (4.38) yields $w = w(t)$ and provides the trivial invariant solution

$$w = K, \quad K = \text{const}. \tag{4.53}$$

It can be obtained from the solution (4.52)(ii) by letting $\beta = 0$ and α be arbitrary.

4.8.3 Invariant solution for the operator $X_3 + kX_2$

The operator

$$X_3 + kX_2 = 2t\frac{\partial}{\partial t} + (k + \ln x)x\frac{\partial}{\partial x} - w\frac{\partial}{\partial w}$$

has the invariants

$$\lambda = \frac{k + \ln x}{\sqrt{t}}\,, \quad \varphi = \sqrt{t}\, w.$$

Therefore the candidates for the invariant solutions are written

$$w = \frac{\varphi(\lambda)}{\sqrt{t}}\,, \quad \lambda = \frac{k + \ln x}{\sqrt{t}}\,\cdot \tag{4.54}$$

The reckoning shows that

$$w_t = -\frac{1}{2t\sqrt{2}}\Big(\varphi + \lambda\varphi'\Big), \quad w_x = \frac{\varphi'}{tx}\,, \quad w_{xx} = \frac{1}{x^2}\Big(\frac{\varphi''}{t\sqrt{2}} - \frac{\varphi'}{t}\Big).$$

Substituting (4.54) and the above expressions for the derivatives in Eq. (4.21) we obtain:

$$\frac{A^2}{2t\sqrt{t}}\Big[\varphi'' + \frac{\sqrt{2}}{A}\,\varphi\varphi' - \frac{1}{A^2}(\varphi + \lambda\varphi')\Big] = 0,$$

whence

$$\varphi'' + \frac{\sqrt{2}}{A}\,\varphi\varphi' - \frac{1}{A^2}\big(\varphi + \lambda\varphi'\big) = \Big(\varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{1}{A^2}\,\lambda\varphi\Big)' = 0.$$

Integrating the above equation once, we obtain the Riccati equation

$$\varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{1}{A^2}\,\lambda\varphi = K, \quad K = \text{const}. \tag{4.55}$$

Thus, the invariant solution for the operator $X_3 + kX_2$ has the form (4.54), where the function $\varphi(\lambda)$ is defined by Eq. (4.55).

4.8.4 Invariant solution for the operator X_4

The invariants for the operator X_4 are t and $\varphi = tw - \frac{\sqrt{2}}{A}\ln x$. Therefore the candidates for the invariant solutions in this case have the form

$$w = \frac{\sqrt{2}}{A}\,\frac{\ln x}{t} + \varphi(t).$$

One can easily verify that Eq. (4.21) yields

$$\varphi' + \frac{\varphi}{t} = 0,$$

and hence $\varphi(t) = K/t$. Thus, X_4 provides the following invariant solution

$$w = \frac{1}{t}\bigg(\frac{\sqrt{2}}{A}\,\ln x + K\bigg), \quad K = \text{const}. \tag{4.56}$$

4.8.5 Invariant solution for the operator $X_1 + X_4$

The operator

$$X_1 + X_4 = \frac{\partial}{\partial t} + Atx\frac{\partial}{\partial x} + \sqrt{2}\frac{\partial}{\partial w}$$

has the invariants

$$\lambda = \ln x - \frac{A}{2}\,t^2, \quad \varphi = w - \sqrt{2}\,t.$$

The candidates for the invariant solutions are obtained by letting $\varphi = \varphi(\lambda)$. Hence

$$w = \sqrt{2}\,t + \varphi(\lambda), \quad \lambda = \ln x - \frac{A}{2}\,t^2. \tag{4.57}$$

Calculating the derivatives:

$$w_t = \sqrt{2} - At\varphi', \quad w_x = \frac{1}{x}\,\varphi', \quad w_{xx} = \frac{1}{x^2}\,\big(\varphi'' - \varphi'\big),$$

and substituting in Eq. (4.21) we obtain for $\varphi(\lambda)$ the ordinary differential equation

$$\varphi'' + \frac{\sqrt{2}}{A}\,\varphi\varphi' + \frac{2\sqrt{2}}{A^2} = 0.$$

Integrating it once, we arrive at the following Riccati equation:

$$\varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{2\sqrt{2}}{A^2}\,\lambda = K, \quad K = \text{const.} \tag{4.58}$$

Thus, the invariant solution for the operator $X_1 + X_4$ has the form (4.57), where the function $\varphi(\lambda)$ is defined by Eq. (4.58).

4.8.6 Invariant solution for the operator $X_1 - X_4$

Proceedings as in the case of the operator $X_1 + X_4$, one can verify that the invariant solution for the operator $X_1 - X_4$ has the form

$$w = \varphi(\lambda) - \sqrt{2}\,t, \quad \lambda = \ln x + \frac{A}{2}\,t^2, \tag{4.59}$$

where the function $\varphi(\lambda)$ is defined by the following Riccati equation:

$$\varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{2\sqrt{2}}{A^2}\,\lambda = K, \quad K = \text{const.} \tag{4.60}$$

4.8.7 Invariant solution for the operator X_5

The reckoning shows (cf. Section 4.8.8) that the operator

$$X_5 = t^2\frac{\partial}{\partial t} + tx\ln x\frac{\partial}{\partial x} + \left(\frac{\sqrt{2}}{A}\,\ln x - tw\right)\frac{\partial}{\partial w}$$

has the invariants

$$\lambda = \frac{\ln x}{t}\,, \quad \varphi = tw - \frac{\sqrt{2}}{A}\,\ln x.$$

Letting $\varphi = \varphi(\lambda)$ and substituting the resulting expression

$$w = \frac{\sqrt{2}}{A}\,\frac{\ln x}{t} + \frac{\varphi(\lambda)}{t}$$

in Eq. (4.21) we obtain

$$\frac{A^2}{2t^3}\left[\varphi'' + \frac{\sqrt{2}}{A}\,\varphi\varphi'\right] = 0,$$

or

$$\varphi'' + \frac{\sqrt{2}}{A}\,\varphi\varphi' = \Big(\varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2\Big)' = 0,$$

whence

$$\varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 = \text{const}.$$

Integration of this equation yields

$$\text{(i)} \qquad \varphi(\lambda) = \frac{A\alpha}{\sqrt{2}}\,\frac{\beta \mathrm{e}^{\alpha\lambda} - 1}{\beta \mathrm{e}^{\alpha\lambda} + 1}\,,$$

$$\text{(ii)} \qquad \varphi(\lambda) = \frac{A\alpha}{\sqrt{2}}\,\frac{\beta \mathrm{e}^{\alpha\lambda} + 1}{\beta \mathrm{e}^{\alpha\lambda} - 1}\,,$$

$$\text{(iii)} \qquad \varphi(\lambda) = \frac{A\alpha}{\sqrt{2}}\,\tan\Big(\beta - \frac{\alpha}{2}\,\lambda\Big),$$

where $\alpha, \beta = \text{const}$. Hence, X_5 provides the following the invariant solutions:

$$\text{(i)} \qquad w = \frac{\sqrt{2}}{At}\,\Big[\ln x + \frac{A^2\alpha}{2}\,\frac{\beta \mathrm{e}^{\alpha\lambda} - 1}{\beta \mathrm{e}^{\alpha\lambda} + 1}\Big], \quad \lambda = \frac{\ln x}{t}\,, \tag{4.61}$$

$$\text{(ii)} \qquad w = \frac{\sqrt{2}}{At}\,\Big[\ln x + \frac{A^2\alpha}{2}\,\frac{\beta \mathrm{e}^{\alpha\lambda} + 1}{\beta \mathrm{e}^{\alpha\lambda} - 1}\Big],$$

$$\text{(iii)} \qquad w = \frac{\sqrt{2}}{At}\,\Big[\ln x + \frac{A^2\alpha}{2}\,\tan\Big(\beta - \frac{\alpha}{2}\,\lambda\Big)\Big].$$

4.8.8 Invariant solution for the operator $X_1 + X_5$

This case is useful for illustrating all steps in constructing invariant solutions based on a rather complicated symmetry. Therefore, I will give here detailed calculations.

In order to find the invariants for the operator

$$X_1 + X_5 = X_5 = (1+t^2)\frac{\partial}{\partial t} + tx\ln x\frac{\partial}{\partial x} + \Big(\frac{\sqrt{2}}{A}\,\ln x - tw\Big)\frac{\partial}{\partial w}\,,$$

we have to find two functionally independent first integrals of the characteristic system

$$\frac{dt}{1+t^2} = \frac{dx}{tx\ln x} = \frac{dw}{\frac{\sqrt{2}}{A}\,\ln x - tw} \tag{4.62}$$

of the equation

$$(X_1 + X_5)J(t,x,w) = 0$$

for the invariants. Rewriting the first equation of the characteristic system in the form

$$\frac{d\ln x}{\ln x} = \frac{tdt}{1+t^2} \equiv \frac{1}{2}\,\frac{d(1+t^2)}{1+t^2}$$

and integrating it we obtain the first integral

$$\ln(\ln x) = \ln(1+t^2) + \text{const.},$$

which is convenient to write in the form

$$\frac{\ln x}{\sqrt{1+t^2}} = \text{const.}$$

The left-hand side of this first integral provides one of the invariants:

$$\lambda = \frac{\ln x}{\sqrt{1+t^2}}\,\cdot$$

Let us integrate the second equation of the characteristic system (4.62),

$$\frac{dt}{1+t^2} = \frac{dw}{\frac{\sqrt{2}}{A}\,\ln x - tw}\,\cdot$$

We rewrite it in the form

$$\frac{dw}{dt} + \frac{tw}{1+t^2} = \frac{\sqrt{2}}{A}\,\frac{\ln x}{1+t^2}\,,$$

eliminate x by using the first integral found above, namely, replace $\ln x$ by $\lambda\sqrt{1+t^2}$, and obtain the following non-homogeneous linear first-order equation:

$$\frac{dw}{dt} + \frac{tw}{1+t^2} = \frac{\sqrt{2}}{A}\,\frac{\ln x}{1+t^2}\,\cdot$$

The method of variation of the parameter yields its general solution

$$w = \frac{\sqrt{2}}{A}\,\frac{t\ln x}{1+t^2} + \frac{\varphi}{\sqrt{1+t^2}}\,, \tag{4.63}$$

where φ is the constant of integration. Solving the above equation with respect to φ one obtains the second invariant

$$\varphi = \sqrt{1+t^2}\,w - \frac{\sqrt{2}}{A}\,\frac{t\ln x}{\sqrt{1+t^2}}\,\cdot$$

However, the last step is unnecessary. Indeed, we let $\varphi = \varphi(\lambda)$ directly in Eq. (4.63) and obtain the following candidates for the invariant solution:

$$w = \frac{\sqrt{2}}{A}\,\frac{t\ln x}{1+t^2} + \frac{\varphi(\lambda)}{\sqrt{1+t^2}}\,, \quad \lambda = \frac{\ln x}{\sqrt{1+t^2}}\,\cdot \tag{4.64}$$

According to the general theory of invariant solutions (Lie [15], Ovsyannikov [12]), the substitution of (4.64) in Eq. (4.21) will reduce Eq. (4.21) to an ordinary differential equation containing only $\lambda, \varphi(\lambda)$ and the derivatives φ', φ'' of $\varphi(\lambda)$. Let us proceed.

Eqs. (4.64) yield:

$$\lambda_t = -\frac{t\ln x}{(1+t^2)\sqrt{1+t^2}}\,, \quad \lambda_x = \frac{1}{x\sqrt{1+t^2}}$$

and

$$w_t = \frac{\sqrt{2}}{A}\,\frac{\ln x}{1+t^2} - \frac{2\sqrt{2}}{A}\,\frac{t^2\ln x}{(1+t^2)^2} - \frac{t\varphi}{(1+t^2)\sqrt{1+t^2}} - \frac{t\,\ln x\,\varphi'}{(1+t^2)^2}\,,$$

$$w_x = \frac{\sqrt{2}}{A}\,\frac{t}{x(1+t^2)} + \frac{\varphi'}{x(1+t^2)}\,,$$

$$w_{xx} = -\frac{\sqrt{2}}{A}\,\frac{t}{x^2(1+t^2)} - \frac{\varphi'}{x^2(1+t^2)} + \frac{\varphi''}{x^2(1+t^2)\sqrt{1+t^2}}\,\cdot$$

Accordingly, we have:

$$\begin{aligned}
& w_t + \frac{A}{\sqrt{2}}\left(w + \frac{A}{\sqrt{2}}\right) x w_x + \frac{A^2}{2}\,x^2 w_{xx} \\
&= \frac{\sqrt{2}}{A}\,\frac{\ln x}{1+t^2} - \frac{2\sqrt{2}}{A}\,\frac{t^2\ln x}{(1+t^2)^2} - \frac{t\varphi}{(1+t^2)\sqrt{1+t^2}} - \frac{t\,\ln x\,\varphi'}{(1+t^2)^2} \\
&+ \left(\frac{t\,\ln x}{1+t^2} + \frac{A}{\sqrt{2}}\,\frac{\varphi}{\sqrt{1+t^2}} + \frac{A^2}{2}\right)\left(\frac{\sqrt{2}}{A}\,\frac{t}{1+t^2} + \frac{\varphi'}{1+t^2}\right) \\
&- \frac{A}{\sqrt{2}}\,\frac{t}{1+t^2} - \frac{A^2}{2}\,\frac{\varphi'}{1+t^2} + \frac{A^2}{2}\,\frac{\varphi''}{(1+t^2)\sqrt{1+t^2}} \\
&= \frac{A^2}{2(1+t^2)\sqrt{1+t^2}}\left[\varphi'' + \frac{\sqrt{2}}{A}\,\varphi\varphi' + \frac{2\sqrt{2}}{A^3}\,\frac{\ln x}{\sqrt{1+t^2}}\right].
\end{aligned}$$

Replacing in the last line $\varphi\varphi'$ by $\frac{1}{2}(\varphi^2)'$ and noting that $\frac{\ln x}{\sqrt{1+t^2}} = \lambda$, we see that Eq. (4.21) reduces to the following ordinary differential equation for $\varphi(\lambda)$:

$$\varphi'' + \frac{1}{A\sqrt{2}}\,(\varphi^2)' + \frac{2\sqrt{2}}{A^3}\,\lambda = 0.$$

Integrating it once, we arrive at the following Riccati equation:

$$\varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{\sqrt{2}}{A^3}\,\lambda^2 = K, \quad K = \text{const.} \tag{4.65}$$

Thus, the invariant solution for the operator $X_1 + X_5$ has the form (4.64), where the function $\varphi(\lambda)$ is defined by the Riccati equation (4.65).

4.8.9 Invariant solution for the operator $X_2 + X_5$

For the operator

$$X_2 + X_5 = X_5 = t^2\frac{\partial}{\partial t} + x(1 + t\ln x)\frac{\partial}{\partial x} + \left(\frac{\sqrt{2}}{A}\ln x - tw\right)\frac{\partial}{\partial w}\,,$$

we have the following characteristic system for calculating the invariants:

$$\frac{dt}{t^2} = \frac{dx}{x(1+t\ln x)} = \frac{dw}{\frac{\sqrt{2}}{A}\ln x - tw}\,\cdot$$

Writing the first equation of this system in the form

$$\frac{d\ln x}{1 + t\ln x} = \frac{dt}{t^2}\,,$$

we obtain the non-homogeneous linear first-order equation

$$\frac{d\ln x}{dt} = \frac{1}{t}\,\ln x + \frac{1}{t^2}\,\cdot$$

Solving this equation, we have

$$\ln x = -\frac{1}{2t} + \lambda t, \tag{4.66}$$

where λ is the constant of integration. Hence, we have found the following invariant:

$$\lambda = \frac{\ln x}{t} + \frac{1}{2t^2}\,\cdot$$

Now we write the second equation of the characteristic system in the form

$$\frac{dw}{dt} + \frac{w}{t} = \frac{\sqrt{2}}{A}\,\frac{\ln x}{t^2}$$

which upon replacing $\ln x$ by its expression (4.66) becomes:

$$\frac{dw}{dt} + \frac{w}{t} = \frac{\sqrt{2}}{A}\left(\frac{\lambda}{t} - \frac{1}{2t^3}\right).$$

Solving this linear first-order equation we have:

$$w = \frac{\sqrt{2}}{A}\Big(\lambda + \frac{1}{2t^2}\Big) + \frac{\varphi}{t},$$

where φ is the constant of integration. Replacing here λ by its expression given above and letting $\varphi = \varphi(\lambda)$, we obtain the following candidates for the invariant solution:

$$w = \frac{\sqrt{2}}{A}\left(\frac{\ln x}{t} + \frac{1}{t^2}\right) + \frac{\varphi(\lambda)}{t}, \quad \lambda = \frac{\ln x}{t} + \frac{1}{2t^2}\,. \tag{4.67}$$

Substituting (4.67) in Eq. (4.21) we obtain the following equation:

$$\varphi'' + \frac{1}{A\sqrt{2}}\,(\varphi^2)' - \frac{2\sqrt{2}}{A^3} = 0$$

which, upon integrating, yields the Riccati equation (4.60),

$$\varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{2\sqrt{2}}{A^3}\,\lambda = K, \quad K = \text{const}.$$

Thus, the invariant solution for the operator $X_2 + X_5$ has the form (4.67), where the function $\varphi(\lambda)$ is defined by the Riccati equation (4.60).

4.8.10 Invariant solution for the operator $X_2 - X_5$

The invariant solution for the operator $X_2 - X_5$ has the form

$$w = \frac{\sqrt{2}}{A}\left(\frac{\ln x}{t} - \frac{1}{t^2}\right) + \frac{\varphi(\lambda)}{t}, \quad \lambda = \frac{\ln x}{t} - \frac{1}{2t^2}\,, \tag{4.68}$$

where the function $\varphi(\lambda)$ is defined by the Riccati equation (4.58),

$$\varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{2\sqrt{2}}{A^2}\,\lambda = K, \quad K = \text{const}.$$

4.8.11 Optimal system of invariant solutions

Summing up the results of Section 4.8 and noting that invariant solutions based on two-dimensional subalgebras are particular cases of those based on one-dimensional subalgebras, we can formulate the following theorem.

Theorem 4.2. Every invariant solution of Eq. (4.21) is given either by elementary functions or by solving a Riccati equation. The following solutions provide an optimal system of invariant solutions so that any invariant solution can be obtained from them by transformations of the 5-parameter group admitted by Eq. (4.21).

$$\text{(i)}\quad w = \frac{A\sqrt{2}}{K + \ln x}\,; \tag{4.69}$$

$$\text{(ii)}\quad w = \frac{A\alpha}{\sqrt{2}}\,\frac{\beta x^{\alpha} + 1}{\beta x^{\alpha} - 1}\,;$$

$$\text{(iii)}\quad w = \frac{A\alpha}{\sqrt{2}}\,\tan\Big(\beta - \frac{\alpha}{2}\,\ln x\Big).$$

$$w = \frac{\varphi(\lambda)}{\sqrt{t}}\,, \quad \lambda = \frac{k + \ln x}{\sqrt{t}}\,, \tag{4.70}$$

$$\text{where}\quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{1}{A^2}\,\lambda\varphi = K.$$

$$w = \frac{1}{t}\bigg(\frac{\sqrt{2}}{A}\,\ln x + K\bigg). \tag{4.71}$$

$$w = \sqrt{2}\,t + \varphi(\lambda), \quad \lambda = \ln x - \frac{A}{2}\,t^2, \tag{4.72}$$

$$\text{where}\quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{2\sqrt{2}}{A^2}\,\lambda = K.$$

$$w = \varphi(\lambda) - \sqrt{2}\,t, \quad \lambda = \ln x + \frac{A}{2}\,t^2, \tag{4.73}$$

$$\text{where}\quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{2\sqrt{2}}{A^2}\,\lambda = K.$$

$$\text{(i)}\quad w = \frac{\sqrt{2}}{At}\left[\ln x + \frac{A^2\alpha}{2}\,\frac{\beta \mathrm{e}^{\alpha\lambda} - 1}{\beta \mathrm{e}^{\alpha\lambda} + 1}\right], \quad \lambda = \frac{\ln x}{t}\,, \tag{4.74}$$

$$\text{(ii)}\quad w = \frac{\sqrt{2}}{At}\left[\ln x + \frac{A^2\alpha}{2}\,\frac{\beta \mathrm{e}^{\alpha\lambda} + 1}{\beta \mathrm{e}^{\alpha\lambda} - 1}\right],$$

$$\text{(iii)}\quad w = \frac{\sqrt{2}}{At}\left[\ln x + \frac{A^2\alpha}{2}\,\tan\Big(\beta - \frac{\alpha}{2}\,\lambda\Big)\right].$$

$$w = \frac{\sqrt{2}}{A}\,\frac{t\ln x}{1+t^2} + \frac{\varphi(\lambda)}{\sqrt{1+t^2}}\,, \quad \lambda = \frac{\ln x}{\sqrt{1+t^2}}\,, \tag{4.75}$$

$$\text{where} \quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{\sqrt{2}}{A^3}\,\lambda^2 = K.$$

$$w = \frac{\sqrt{2}}{A}\left(\frac{\ln x}{t} + \frac{1}{t^2}\right) + \frac{\varphi(\lambda)}{t}\,, \quad \lambda = \frac{\ln x}{t} + \frac{1}{2t^2}\,, \tag{4.76}$$

$$\text{where} \quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{2\sqrt{2}}{A^2}\,\lambda = K.$$

$$w = \frac{\sqrt{2}}{A}\left(\frac{\ln x}{t} - \frac{1}{t^2}\right) + \frac{\varphi(\lambda)}{t}\,, \quad \lambda = \frac{\ln x}{t} - \frac{1}{2t^2}\,, \tag{4.77}$$

$$\text{where} \quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{2\sqrt{2}}{A^2}\,\lambda = K.$$

In these solutions, α, β, k and K are arbitrary constants.

4.8.12 How to use the optimal system of invariant solutions

Subjecting the optimal system of invariants solutions to the group transformations generated by the operators (4.26), one obtains all (regular) invariant solutions of Eq. (4.21). One can easily verify that the optimal system of invariant solutions (4.69)-(4.77) contains 25 parameters including k in (4.70). Since each of the thirteen types of of solutions (4.69)-(4.77) will gain four group parameters after subjecting it to the transformations generated by the operators (4.26) (not five due to the invariance of the solution under consideration with respect to one operator), we see that the invariant solutions provide a wide class of exact solutions containing 77 parameters.

I will illustrate the method by means of the group transformations generated by the last operator from (4.26):

$$X_5 = t^2\frac{\partial}{\partial t} + tx\ln x\frac{\partial}{\partial x} + \left(\frac{\sqrt{2}}{A}\,\ln x - tw\right)\frac{\partial}{\partial w}\cdot$$

The Lie equations

$$\frac{d\bar t}{da} = \bar t^2, \qquad \bar t\big|_{a=0} = t,$$
$$\frac{d\bar x}{da} = \bar t\bar x\ln\bar x, \qquad \bar x\big|_{a=0} = x,$$
$$\frac{d\bar w}{da} = \frac{\sqrt2}{A}\ln\bar x - \bar t\bar w, \qquad \bar w\big|_{a=0} = w,$$

provide the following group transformations:

$$\bar t = \frac{t}{1-at}, \quad \bar x = \exp\Big(\frac{\ln x}{1-at}\Big),$$
$$\bar w = (1-at)w + \frac{\sqrt2}{A}\,a\ln x. \tag{4.78}$$

Note that the inverse transformation is obtained from (4.78) by exchanging the variables $\bar t, \bar x, \bar w$ with t, x, w and replacing the group parameter a by $-a$. Hence:

$$t = \frac{\bar t}{1+a\bar t}, \quad x = \exp\Big(\frac{\ln\bar x}{1+a\bar t}\Big),$$
$$w = (1+a\bar t\,)\bar w - \frac{\sqrt2}{A}\,a\ln\bar x. \tag{4.79}$$

Let us apply the transformation (4.78) to the trivial invariant solution (4.53) obtained by using the operator X_2. Since Eq. (4.21) is invariant under transformation (4.78), let us write the solution (4.53) in the form $\bar w = K$. Using (4.78), we obtain

$$(1-at)w + \frac{\sqrt2}{A}\,a\ln x = K.$$

Upon solving this equation for w, we obtain the following new solution to Eq. (4.21):

$$w = \frac{1}{1-at}\,\Big(K - \frac{\sqrt2}{A}\,a\ln x\Big). \tag{4.80}$$

According to our construction, the solution (4.80) is invariant under the operator $\widetilde X_2$ obtained from X_2 by the transformation (4.78). Let us find $\widetilde X_2$.

Since the solution $\bar w = K$ is written in the variables $\bar t, \bar x, \bar w$, the operator X_2 leaving it invariant should also be written in these variables. Therefore we take it in the form

$$\overline X_2 = \bar x\frac{\partial}{\partial\bar x}$$

and denote by $\widetilde{X}_2$ its expression

$$\widetilde{X}_2 = \overline{X}_2(t)\frac{\partial}{\partial t} + \overline{X}_2(x)\frac{\partial}{\partial x} + \overline{X}_2(w)\frac{\partial}{\partial w} \tag{4.81}$$

written in terms of the variables t, x, w defined by Eqs. (4.79). We have $\overline{X}_2(t) = 0$ and

$$\overline{X}_2(x) = \bar{x}\frac{\partial x}{\partial \bar{x}} = \frac{x}{1 + a\bar{t}} = (1 - at)x\,, \quad \overline{X}_2(w) = -\frac{\sqrt{2}}{A}\,a$$

because $1 + a\bar{t} = (1 - at)^{-1}$. Therefore

$$\widetilde{X}_2 = (1 - at)x\frac{\partial}{\partial x} - \frac{\sqrt{2}}{A}\,a\frac{\partial}{\partial w} = x\frac{\partial}{\partial x} - \frac{a}{A}\Big(Atx\frac{\partial}{\partial x} + \sqrt{2}\,\frac{\partial}{\partial w}\Big).$$

Comparing the result with the operators (4.26) we conclude that the transformation generated by X_5 maps X_2 to the operator

$$\widetilde{X}_2 = X_2 - \frac{a}{A}\,X_4. \tag{4.82}$$

One can easily check that the solution (4.80) is invariant under the operator (4.82).

Applying the above procedure to the invariant solution (4.52) one can verify that the transformation (4.78) maps the solution (4.52) to the new solution

$$w = \frac{A\alpha}{\sqrt{2}\,(1 - at)}\,\frac{\beta\exp\big(\frac{\alpha \ln x}{1 - at}\big) + 1}{\beta\exp\big(\frac{\alpha \ln x}{1 - at}\big) - 1} - \frac{\sqrt{2}\,a}{A}\,\frac{\ln x}{1 - at}\,. \tag{4.83}$$

The corresponding operator X_1 is transformed by Eq. (4.81) where X_2 is replaced by X_1, namely:

$$\begin{aligned}\widetilde{X}_1 &= \frac{\partial t}{\partial \bar{t}}\,\frac{\partial}{\partial t} + \frac{\partial x}{\partial \bar{t}}\,\frac{\partial}{\partial x} + \frac{\partial w}{\partial \bar{t}}\,\frac{\partial}{\partial w}\\ &= \frac{1}{(1 + a\bar{t})^2}\,\frac{\partial}{\partial t} - \frac{a \ln \bar{x}}{(1 + a\bar{t})^2}\,x\,\frac{\partial}{\partial x} + a\bar{w}\,\frac{\partial}{\partial w}\,.\end{aligned}$$

Replacing here $\bar{t}, \bar{x}, \bar{w}$ by their expressions (4.78), we obtain:

$$\widetilde{X}_1 = (1 - at)^2\frac{\partial}{\partial t} - a(1 - at)x\ln x\frac{\partial}{\partial x} + a\Big[(1 - at)w + \frac{\sqrt{2}}{A}\,a\ln x\Big]\frac{\partial}{\partial w} = \frac{\partial}{\partial t}$$

$$- a\Big[2t\frac{\partial}{\partial t} + x\ln x\frac{\partial}{\partial x} - w\frac{\partial}{\partial w}\Big] + a^2\Big[t^2\frac{\partial}{\partial t} + tx\ln x\frac{\partial}{\partial x} + \Big(\frac{\sqrt{2}}{A}\,\ln x - tw\Big)\frac{\partial}{\partial w}\Big].$$

Hence, the image $\widetilde{X}_1$ of the operator X_1 is presented by the following linear combination of X_1, X_3 and X_5 :

$$\widetilde{X}_1 = X_1 - a\,X_3 + a^2X_5. \tag{4.84}$$

One can verify that (4.83) is the invariant solution under the operator $\widetilde{X}_1$.

The same procedure is applicable to the invariant solutions defined by solving Riccati equations. Starting, e.g. with the invariant solution (4.57) written in the form

$$\bar{w} = \sqrt{2}\,\bar{t} + \varphi(\bar{\lambda}), \quad \bar{\lambda} = \ln \bar{x} - \frac{A}{2}\,\bar{t}^2,$$

we arrive to the new solution

$$w = \sqrt{2}\Big[\frac{t}{(1-at)^2} - \frac{a}{A}\,\frac{\ln x}{1-at}\Big] + \frac{\varphi(\mu)}{1-at}\,, \quad \mu = \frac{\ln x}{1-at} - \frac{At^2}{2(1-at)^2}\,, \tag{4.85}$$

where $\varphi(\mu)$ is defined by the Riccati equation (4.55):

$$\frac{d\varphi}{d\mu} + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{2\sqrt{2}}{A^2}\,\mu = K, \quad K = \text{const.} \tag{4.86}$$

Applying the previous procedure to the operator X_3+kX_2 giving the invariant solution (4.57) we obtain that it is mapped to the operator

$$\widetilde{X}_3 + k\widetilde{X}_2 = X_3 + kX_2 - a\left(2X_5 + \frac{k}{A}X_4\right) \tag{4.87}$$

and that (4.85) is the invariant solution with respect to the operator (4.87).

It is easier to find the transformations (4.82), etc. of operators by using Eqs. (4.37) and (4.36) instead of the more complicated transformation rule (4.81). Let us find, e.g. the transformations of the operators X_1 and X_2 given by Eqs. (4.37) and (4.36). For the operator X_1 the vector l has the form

$$l = (1,0,0,0,0).$$

It is mapped by (4.36) with $a_5 = a$ to the vector

$$\tilde{l} = (1,0,-a,0,a^2).$$

Substituting the coordinates of this vector in Eq. (4.37) we obtain

$$\widetilde{X}_1 = X_1 - a\,X_3 + a^2 X_5,$$

i.e. the operator (4.84). Likewise, the coordinate vector

$$l = (0,1,0,0,0)$$

for the operator X_2 is mapped by (4.36) to the coordinate vector

$$\tilde{l} = (0,1,0,-\frac{a}{A}\,,0)$$

of the operator (4.82):

$$\widetilde{X}_2 = X_2 - \frac{a}{A}\,X_4.$$

Furthermore, the coordinate vector

$$l = (0, k, 1, 0, 0)$$

for the operator $X_3 + kX_2$ is mapped by (4.36) with $a_5 = a$ to the coordinate vector

$$\tilde{l} = (0, k, 1, -\frac{k\,a}{A}\,, -2a)$$

of the operator (4.87):

$$\widetilde{X}_3 + k\widetilde{X}_2 = X_3 + kX_2 - \frac{k\,a}{A}\,X_4 - 2aX_5.$$

4.9 Optimal system of invariant solutions for the Burgers equation

An optimal system of invariant solutions for the Burgers equation (4.18) can be obtained by using the optimal system (4.48) of one-dimensional subalgebras for the Burgers equation and proceeding as in Section 4.8.

An alternative way is to rewrite the optimal system of invariant solutions (4.69)-(4.77) for Eq. (4.21) in the variables τ, y given by (4.2). Let us apply this method to the solution (4.69). Substituting the expressions for t, x given by Eqs. (4.25):

$$t = t_0 - \tau, \quad x = x_0 \mathrm{e}^{\frac{Ay}{\sqrt{2}}}$$

we have:

$$\text{(i)}\ \ w = \frac{A\sqrt{2}}{K + \ln x} = \frac{A\sqrt{2}}{K + \ln x_0 + \frac{Ay}{\sqrt{2}}} = \frac{2}{y + \gamma}\,;$$

$$\text{(ii)}\ \ w = \frac{A\alpha}{\sqrt{2}}\,\frac{\beta x^\alpha + 1}{\beta x^\alpha - 1} = \frac{A\alpha}{\sqrt{2}}\,\frac{\beta x_0 \mathrm{e}^{\frac{A\alpha}{\sqrt{2}}\,y} + 1}{\beta x_0 \mathrm{e}^{\frac{A\alpha}{\sqrt{2}}\,y} - 1} = \sigma\,\frac{\gamma \mathrm{e}^{\sigma y} + 1}{\gamma \mathrm{e}^{\sigma y} - 1}\,;$$

$$\text{(iii)}\ \ w = \frac{A\alpha}{\sqrt{2}}\,\tan\Big(\beta - \frac{\alpha}{2}\,\ln x\Big)$$

$$= \frac{A\alpha}{\sqrt{2}}\,\tan\Big[\beta - \frac{\alpha}{2}\Big(\ln x_0 + \frac{Ay}{\sqrt{2}}\Big)\Big] = \sigma \tan\Big(\gamma - \frac{\sigma}{2}\,y\Big).$$

Note that the solution (ii) can also be written in the following form:

$$w = \tilde{\sigma} \tanh\Big(\tilde{\gamma} + \frac{\tilde{\sigma}}{2}\,y\Big).$$

Taking the solution (4.71) and proceeding likewise, we obtain the following solution to the Burgers equation:

$$w = \frac{y+K}{t_0-\tau}\,, \quad K = \text{const.}$$

The similar approach is applicable to the solutions involving the Riccati equation. For example, the solution (4.72) leads to the following solution to the Burgers equation:

$$w = \sqrt{2}\,(t_0-\tau) + \varphi(\lambda), \quad \lambda = \ln|x_0| + \frac{Ay}{\sqrt{2}} - \frac{A^2}{2}(t_0-\tau)^2,$$

where $\varphi(\lambda)$ is defined by the same Riccati equation as in (4.72).

Dealing likewise with all solutions (4.69)-(4.77) for Eq. (4.21), we arrive at the following statement similar to Theorem 4.2.

Theorem 4.3. Every invariant solution of the Burgers equation (4.18) is given either by elementary functions or by solving a Riccati equation. The following solutions provide an optimal system of invariant solutions so that any invariant solution can be obtained from them by transformations of the group admitted by the Burgers equation.

$$\begin{aligned} &\text{(i)} \quad w = \frac{2}{y+\gamma}\,;\\ &\text{(ii)} \quad w = \sigma\,\frac{\gamma e^{\sigma y}+1}{\gamma e^{\sigma y}-1} \equiv \tilde{\sigma}\tanh\Big(\tilde{\gamma} + \frac{\tilde{\sigma}}{2}\,y\Big)\,; \\ &\text{(iii)} \quad w = \sigma\,\tan\Big(\gamma - \frac{\sigma}{2}\,y\Big). \end{aligned} \tag{4.88}$$

$$w = \frac{\varphi(\lambda)}{\sqrt{t_0-\tau}}\,, \quad \lambda = \frac{1}{\sqrt{t_0-\tau}}\Big(k + \ln|x_0| + \frac{Ay}{\sqrt{2}}\Big), \tag{4.89}$$

$$\text{where} \quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{1}{A^2}\,\lambda\varphi = K.$$

$$w = \frac{y+K}{t_0-\tau}\,\cdot \tag{4.90}$$

$$w = \sqrt{2}\,(t_0-\tau) + \varphi(\lambda), \quad \lambda = \ln|x_0| + \frac{Ay}{\sqrt{2}} - \frac{A^2}{2}(t_0-\tau)^2, \tag{4.91}$$

$$\text{where} \quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{2\sqrt{2}}{A^2}\,\lambda = K.$$

$$w = \varphi(\lambda) - \sqrt{2}\,(t_0 - \tau), \quad \lambda = \ln|x_0| + \frac{Ay}{\sqrt{2}} + \frac{A^2}{2}(t_0 - \tau)^2, \tag{4.92}$$

$$\text{where} \quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{2\sqrt{2}}{A^2}\,\lambda = K.$$

$$\text{(i)} \quad w = \frac{1}{t_0 - \tau}\left[K + y + \frac{A\sigma}{\sqrt{2}}\,\frac{\gamma e^{\sigma\lambda} - 1}{\gamma e^{\sigma\lambda} + 1}\right], \quad \lambda = \frac{1}{t_0 - \tau}\Big(\ln|x_0| + \frac{Ay}{\sqrt{2}}\Big);$$

$$\text{(ii)} \quad w = \frac{1}{t_0 - \tau}\left[K + y + \frac{A\sigma}{\sqrt{2}}\,\frac{\gamma e^{\sigma\lambda} + 1}{\gamma e^{\sigma\lambda} - 1}\right]; \tag{4.93}$$

$$\text{(iii)} \quad w = \frac{1}{t_0 - \tau}\left[K + y + \frac{A\sigma}{\sqrt{2}}\,\tan\Big(\beta - \frac{\alpha}{2}\,\lambda\Big)\right].$$

$$w = \frac{(t_0 - \tau)(y + K)}{1 + (t_0 - \tau)^2} + \frac{\varphi(\lambda)}{\sqrt{1 + (t_0 - \tau)^2}}\,, \quad \lambda = \frac{\ln|x_0| + \frac{Ay}{\sqrt{2}}}{\sqrt{1 + (t_0 - \tau)^2}}\,, \tag{4.94}$$

$$\text{where} \quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{\sqrt{2}}{A^3}\,\lambda^2 = K.$$

$$w = \frac{Ay + \sqrt{2}\,\ln|x_0|}{A(t_0 - \tau)} + \frac{\sqrt{2}}{A(t_0 - \tau)^2} + \frac{\varphi(\lambda)}{t_0 - \tau}\,, \tag{4.95}$$

$$\lambda = \frac{Ay + \sqrt{2}\,\ln|x_0|}{\sqrt{2}(t_0 - \tau)} + \frac{1}{2(t_0 - \tau)^2}\,,$$

$$\text{where} \quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{2\sqrt{2}}{A^2}\,\lambda = K.$$

$$w = \frac{Ay + \sqrt{2}\,\ln|x_0|}{A(t_0 - \tau)} - \frac{\sqrt{2}}{A(t_0 - \tau)^2} + \frac{\varphi(\lambda)}{t_0 - \tau}\,, \tag{4.96}$$

$$\lambda = \frac{Ay + \sqrt{2}\,\ln|x_0|}{\sqrt{2}(t_0 - \tau)} - \frac{1}{2(t_0 - \tau)^2}\,,$$

$$\text{where} \quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{2\sqrt{2}}{A^2}\,\lambda = K.$$

In these solutions, σ, γ, k and K are arbitrary constants.

Bibliography

[1] L. Euler, *Integral Calculus*. Vol. III, Part 1, Chapter II, 1769/1770.

[2] L. Petrén, *Extension de la méthode de Laplace aux équations* $\sum_{i=0}^{n-1} A_{1i}(x,y)\frac{\partial^{i+1}z}{\partial x \partial y^i} + \sum_{i=0}^{n} A_{0i}(x,y)\frac{\partial^i z}{\partial y^i} = 0$. Lund: PhD Thesis. Lunds Universitets Årsskrift. N. F. Afd.2. Bd. 7. Nr 3, Kongl. Fysiografiska Sällskapets Handlingar. N.F. Bd. 22. Nr 3, 1911. English translation of "Introduction" to L. Petrén's Thesis is available in "Archives of ALGA", vol. 3, 2006, pp. 13-31.

[3] P. S. Laplace, "Recherches sur le calcul intégral aux différences partielles," *Mémoires de l'Académie royale des Sciences de Paris*, vol. 23, No. 24, pp. 341–402, 1773. Reprinted in: P. S. Laplace, *Oevres complètes,* t. IX, Gauthier-Villars, Paris, 1893, pp. 5-68; English Translation, New York, 1966.

[4] N. H. Ibragimov, "Laplace type invariants for parabolic equations," *Nonlinear Dynamics*, vol. 28 (2), pp. 125–133, 2002.

[5] N. H. Ibragimov, "Extension of Euler's method to parabolic equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. doi: 10.1016/j.cnsns.2008.04.010, pp. 1–19, 2008.

[6] N. H. Ibragimov, *A practical course in differential equations and mathemtical modelling*. Karlskrona: ALGA Publications, 3rd ed., 2006.

[7] S. Lie, "Über die Integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen," *Archiv for Matematik og Naturvidenskab (*Abbr. *Arch. for Math.)*, vol. 6, Heft 3, pp. 328–368, 1881. English transl., *CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 2: Applications in Engineering and Physical Sciences,* ed. N.H. Ibragimov, CRC Press, Boca Roton, 1995. Reprinted also in the book *Lie group analysis: Classical heritage,* ed. N.H. Ibragimov, ALGA Publications, Karlskrona, 2004, pp. 1-64.

[8] I. G. Petrovsky, *Lectures on partial differential equations*. Moscow: Fizmatgiz, 3 ed., 1961. English transl., Interscience, New York, 1964. Republished by Dover, 1991. Translated from Russian by A. Shenitzer.

[9] A. N. Tikhonov, “Uniqueness theorems for the heat equation,” *Dokl. Akad. Nauk SSSR*, vol. 1, No. 6, pp. 294–300, 1935. In Russian and French.

[10] F. Black and M. Scholes, “The pricing of options and corporate liabilities,” *J. Political Economy*, vol. 81, pp. 637–654, 1973.

[11] R. K. Gazizov and N. H. Ibragimov, “Lie symmetry analysis of differential equations in finance,” *Nonlinear Dynamics*, vol. 17, No. 4, pp. 387–407, 1998.

[12] L. V. Ovsyannikov, *Group properties of differential equations*. Novosibirsk: Siberian Branch, USSR Academy of Sciences, 1962. (Russian).

[13] L. V. Ovsyannikov, *Group analysis of differential equations*. Moscow: Nauka, 1978. English transl., ed. W.F. Ames, Academic Press, New York, 1982.

[14] N. H. Ibragimov, ed., *CRC Handbook of Lie group analysis of differential equations.* Vol. 2: *Applications in engineering and physical sciences*. Boca Raton: CRC Press Inc., 1995.

[15] S. Lie, “Zur allgemeine Theorie der partiellen Differentialgleichungen beliebiger Ordnung,” *Leipzig. Ber.*, vol. 1, pp. 53–128, 1895. Reprinted in Ges. Abhandl., Bd. 4, pp. 320–384. English translation “General theory of partial differential equations of an arbitrary order” is available in the book *Lie group analysis: Classical heritage,* ed. N.H. Ibragimov, ALGA Publications, Karlskrona, Sweden, 2004, pp. 1-63.

Archives of ALGA
Volume 5, 2008

CLASSICAL AND NEW RESULTS ON INTEGRATING FACTORS

NAIL H. IBRAGIMOV
Department of Mathematics and Science,
Research Centre ALGA: Advances in Lie Group Analysis,
Blekinge Institute of Technology,
SE-371 79 Karlskrona, Sweden

Abstract. Current developments in the theory of integrating factors for higher-order ordinary differential equations are compared with classical results on integrating factors for arbitrary first-order equations and multipliers for higher-order linear differential operators. It is shown how to employ the potential of integrating factors for reducing the non-homogeneous second-order linear equation to the homogeneous equation. This approach, unlike Lagrange's method of variation of parameters, requires only one solution of the homogeneous equation.

Keywords: Integrating factor for first-order equations, Multipliers for linear operators, Integrating factors for nonlinear higher-order equations, Existence of integrating factors.

1 Introduction

An arbitrary non-homogeneous linear first-order ordinary differential equation can be solved by quadrature using either the method of variation of parameters suggested by Jean Bernoulli in 1697 or the method of integrating factors suggested by A.C. Clairaut for arbitrary first-order equations in 1739. In 1774, the method of variation of parameters was extended by Lagrange to higher-order linear equations. Lagrange's method is commonly used for solving by quadrature non-homogeneous linear equations of the second and higher order. An alternative method, namely the method of multipliers for linear differential operators of an arbitrary order, is less known in modern textbooks. In this paper I treat the multipliers in the general framework of our recent work on integrating factors for nonlinear higher-order equations (see [1, 2, 3]). For linear equations the concepts of integrating factor and solutions of adjoint equations coincide. These concepts are different for nonlinear equations. The classical and new approaches are illustrated by various second-order equations.

1.1 Integrating factors for first-order equations

For the first-order equation written in the differential form

$$a(x,y)dy + b(x,y)dx = 0 \tag{1.1}$$

the integrating factor is a function $\mu(x,y)$ such that

$$\mu(x,y)\,[a(x,y)dy + b(x,y)dx] = d\phi(x,y) \tag{1.2}$$

with a certain function $\phi(x,y)$. Existence of a function $\mu(x,y)$ satisfying Eq. (1.2) is equivalent to the following equation:

$$(\mu b)_y - (\mu a)_x = 0, \tag{1.3}$$

where the subscripts y and x denote the partial derivatives with respect to y and x, respectively. Writing the differential equation (1.1) in the form

$$a(x,y)y' + b(x,y) = 0 \tag{1.4}$$

and dividing both sides of Eq. (1.2) by dx, we rewrite the definition of an integrating factor μ in the following form:

$$\mu a\, y' + \mu b = D_x(\phi), \tag{1.5}$$

where instead of $\frac{d}{dx}$ the notation

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots + y^{(s+1)}\frac{\partial}{\partial y^{(s)}} + \cdots \tag{1.6}$$

is used for the total differentiation with respect to x. This reformulation does not change the determining equation (1.3) for the integrating factor (see Example 2.1).

Eqs. (1.4), (1.5) yield the general solution to Eq. (1.4) in the implicit form

$$\phi(x,y) = C, \quad C = \text{const.}, \tag{1.7}$$

provided that the function $\phi(x,y)$ has been found from Eq. (1.5).

An integrating factor $\mu(x,y)$ exists for any first-order equation (1.4) since it is determined by *one equation* (1.3) for one unknown function μ.

1.2 Multipliers for linear differential operators

Consider the general nth-order linear equation

$$L_n[y] = f(x), \tag{1.8}$$

where L_n is following nth-order linear differential operator:

$$L_n = a_0(x)D_x^n + a_1(x)D_x^{n-1} + \ldots + a_{n-1}(x)D_x + a_n(x), \quad a_0(x) \neq 0, \tag{1.9}$$

where D_x is the operator (1.6). Thus,

$$L_n[y] = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y.$$

Definition 1.1. (Cf. [4], Chap. 5, §4: "Adjoint equation"). A function $z = z(x) \neq 0$ is called a *multiplier* for the differential operator L_n if the equation

$$zL_n[y] = D_x(\phi) \tag{1.10}$$

with $\phi = \phi\big(x, y, y', \ldots, y^{(n-1)}\big)$ is satisfied identically in the variables $x, y, y', \ldots, y^{(n)}$.

Remark 1.1. Eq. (1.10) can be written in the form

$$z(x)L_n = D_x \circ \phi,$$

where $\circ$ indicates the composition of the mappings D_x and ϕ.

Since the variational derivative of $D_x(\phi)$ vanishes (see, e.g. Lemma 6.6.2 in [2]), Eq. (1.10) yields:

$$\frac{\delta\,(zL_n[y])}{\delta y} = 0, \tag{1.11}$$

where

$$\frac{\delta}{\delta y} = \frac{\partial}{\partial y} - D_x\frac{\partial}{\partial y'} + D_x^2\frac{\partial}{\partial y''} - D_x^3\frac{\partial}{\partial y'''} + \cdots \tag{1.12}$$

is the variational derivative. It follows from Eq. (1.11) that $z = z(x)$ is a solution of the *adjoint equation* (see [1])

$$L_n^*[z] \equiv (-D_x)^n \left(a_0\, z\right) + (-D_x)^{n-1} \left(a_1\, z\right) + \ldots - D_x\big(a_{n-1}\, z\big) + a_n z = 0. \tag{1.13}$$

Thus, $L_n[y]$ becomes a total derivative after multiplying it by any solution $z(x)$ of the adjoint equation (1.13). Furthermore, Eq. (1.10) yields that

$$z(x)[L_n[y] - f(x)] = D_x(\psi), \tag{1.14}$$

where $\psi = \psi\big(x, y, y', \ldots, y^{(n-1)}\big)$ is given by

$$\psi = \phi - \int f(x)z(x)dx. \tag{1.15}$$

The result is formulated as follows.

Theorem 1.1. Any solution $z(x)$ of the adjoint equation (1.13) is an integrating factor for Eq. (1.8) and furnishes us with the following first integral for Eq. (1.8):

$$\psi\big(x, y, y', \ldots, y^{(n-1)}\big) = C. \tag{1.16}$$

Example 1.1. Let us integrate the general non-homogeneous first-order equation

$$y' + P(x)y = Q(x). \tag{1.17}$$

Here the operator (1.9) is $L_1 = D_x + P(x)$ and its adjoint L_1^* is defined by

$$L_1^*[z] = -z' + P(x)z.$$

Hence the adjoint equation (1.13) yields

$$z' - P(x)z = 0.$$

The solution of this is given by

$$z = \mathrm{e}^{\int P(x)dx}$$

and provides an integrating factor for Eq. (1.17). We use it to integrate Eq. (1.17). We find $\phi(x, y)$ from Eq. (1.10) which is written [since $D_x\left(\phi(x, y)\right) = y'\phi_y + \phi_x$]

$$y'\mathrm{e}^{\int P(x)dx} + P(x)y\mathrm{e}^{\int P(x)dx} = y'\phi_y + \phi_x. \tag{1.18}$$

Comparing the coefficients for y' we get

$$\phi_y = \mathrm{e}^{\int P(x)dx},$$

whence

$$\phi = y\mathrm{e}^{\int P(x)dx} + g(x).$$

Substitution in (1.18) yields $g'(x) = 0$, i.e.

$$\phi = y\mathrm{e}^{\int P(x)dx} + K, \quad K = \mathrm{const}. \tag{1.19}$$

When we substitute (1.19) in the first integral (1.16) we can ignore the constant K due to the presence of an arbitrary constants C in (1.16). Thus, Eqs. (1.15)-(1.16) yield:

$$y\mathrm{e}^{\int P(x)dx} - \int Q(x)\mathrm{e}^{\int P(x)dx}dx = C.$$

Solving for y we obtain the general solution to Eq. (1.17):

$$y = \left[C + \int Q(x)\mathrm{e}^{\int P(x)dx}dx\right]\mathrm{e}^{-\int P(x)dx}. \tag{1.20}$$

Example 1.2. Let us solve the following third-order equation:

$$y''' - y = f(x). \tag{1.21}$$

The adjoint equation $z''' + z = 0$ has the following linearly independent solutions:

$$z_1 = \mathrm{e}^{-x}, \quad z_2 = \mathrm{e}^{x/2}\cos\theta, \quad z_3 = \mathrm{e}^{x/2}\sin\theta, \quad \text{where} \quad \theta = \frac{\sqrt{3}}{2}x\cdot$$

For $z = z_1$ and $\psi = \psi(x, y, y', y'')$ Eq. (1.14) is written:

$$\mathrm{e}^{-x}\left(y''' - y - f(x)\right) = y'''\psi_{y''} + y''\psi_{y'} + y'\psi_y + \psi_x. \tag{1.22}$$

Comparing the coefficients for y''' we see that $\psi_{y''} = \mathrm{e}^{-x}$. Hence

$$\psi = \mathrm{e}^{-x}\,y'' + \alpha(x, y, y').$$

Substituting this in (1.22) and noting that the left-hand side of Eq. (1.22) does not contain y'', we obtain $\alpha_{y'} = \mathrm{e}^{-x}$, i.e. $\alpha = y'\,\mathrm{e}^{-x} + \beta(x, y)$. Hence

$$\psi = \mathrm{e}^{-x}\left(y'' + y'\right) + \beta(x, y).$$

Substituting this in (1.22) and annulling the coefficient for y' we obtain

$$\beta = y\,\mathrm{e}^{-x} + \gamma(x, y),$$

and hence

$$\psi = \mathrm{e}^{-x}\left(y'' + y' + y\right) + \gamma(x).$$

The final substitution in Eq. (1.22) yields

$$\gamma'(x) = -\mathrm{e}^{-x} f(x),$$

so that we have:

$$\psi = \mathrm{e}^{-x}\left(y'' + y' + y\right) - \int \mathrm{e}^{-x} f(x)dx. \tag{1.23}$$

Therefore the first integral (1.16) can be written in the following form:

$$y'' + y' + y = \left(C_1 + \int \mathrm{e}^{-x} f(x)dx\right)\mathrm{e}^{x}. \tag{1.24}$$

Repeating this procedure for the solutions $z = z_2$ and $z = z_3$ we obtain two more first integrals, namely:

$$y''\cos\theta + \frac{1}{2}\left(\sqrt{3}\sin\theta - \cos\theta\right)y' - \frac{1}{2}\left(\sqrt{3}\sin\theta + \cos\theta\right)y = \left(C_2 + \int f(x)\mathrm{e}^{x/2}\cos\theta dx\right)\mathrm{e}^{-x/2} \tag{1.25}$$

and

$$y'' \sin\theta - \frac{1}{2}\left(\sqrt{3}\cos\theta + \sin\theta\right) y' + \frac{1}{2}\left(\sqrt{3}\cos\theta - \sin\theta\right) y$$
$$= \left(C_3 + \int f(x)\mathrm{e}^{x/2}\sin\theta dx\right)\mathrm{e}^{-x/2}. \tag{1.26}$$

Eliminating y'' and y' from three equations, (1.24), (1.25) and (1.26) we will obtain the solution $y(x)$ of Eq. (1.21). To this end, we first multiply Eq. (1.25) by $\cos\theta$ and add Eq. (1.26) multiplied by $\sin\theta$, then multiply Eq. (1.25) by $\sin\theta$ and sybtract Eq. (1.26) multiplied by $\cos\theta$. In this way, we replace Eqs. (1.25), (1.26) by the equations

$$y'' - \frac{1}{2}(y' + y) = \bigg(C_2\cos\theta + C_3\sin\theta + \cos\theta\int f(x)\mathrm{e}^{x/2}\cos\theta dx$$
$$+ \sin\theta\int f(x)\mathrm{e}^{x/2}\sin\theta dx\bigg)\mathrm{e}^{-x/2}, \tag{1.27}$$

$$y' - y = \frac{2}{\sqrt{3}}\bigg(C_2\sin\theta - C_3\cos\theta + \sin\theta\int f(x)\mathrm{e}^{x/2}\cos\theta dx$$
$$- \cos\theta\int f(x)\mathrm{e}^{x/2}\sin\theta dx\bigg)\mathrm{e}^{-x/2}. \tag{1.28}$$

Eliminating y'' and y' from Eqs. (1.24), (1.27) and (1.28) we obtain the following general solution to Eq. (1.21):

$$y = \left[\tilde{C}_1 + \frac{1}{3}\int \mathrm{e}^{-x} f(x)dx\right]\mathrm{e}^{x} + \bigg[\tilde{C}_2\cos\theta + \tilde{C}_3\sin\theta$$
$$- \left(\frac{1}{3}\cos\theta + \frac{1}{\sqrt{3}}\sin\theta\right)\int f(x)\mathrm{e}^{x/2}\cos\theta)dx \tag{1.29}$$
$$- \left(\frac{1}{3}\sin\theta - \frac{1}{\sqrt{3}}\cos\theta\right)\int f(x)\mathrm{e}^{x/2}\sin\theta)dx\bigg]\mathrm{e}^{-x/2},$$

where

$$\theta = \frac{\sqrt{3}}{2}x\,, \quad \tilde{C}_1 = \frac{1}{3}C_1, \quad \tilde{C}_2 = -\frac{1}{3}C_2 + \frac{1}{\sqrt{3}}C_3, \quad \tilde{C}_3 = -\frac{1}{3}C_3 - \frac{1}{\sqrt{3}}C_2.$$

Note that in the particular case $f(x) = x,$ Eq. (1.29) provides the solution

$$y = \tilde{C}_1\mathrm{e}^{x} + \left(\tilde{C}_2\cos\theta + \tilde{C}_3\sin\theta\right)\mathrm{e}^{-x/2} - x$$

which can also be obtained simply by reducing the equation

$$y''' - y = x$$

to the homogeneous equation

$$v''' - v = 0$$

by the substitution $y = v - x.$

2 Integrating factors for nonlinear higher-order ODEs

Integrating factors for arbitrary higher-order ordinary differential equations are defined in [1] as follows.

Definition 2.1. An integrating factor μ for an nth-order ordinary differential equation

$$a\big(x, y, y', \ldots, y^{(n-1)}\big)\, y^{(n)} + b\big(x, y, y', \ldots, y^{(n-1)}\big) = 0 \tag{2.1}$$

is a function $\mu\left(x, y, y', \ldots, y^{(n-1)}\right)$ such that

$$\mu a\, y^{(n)} + \mu b = D_x(\phi), \tag{2.2}$$

where D_x is the operator of total differentiation (1.6) and

$$\phi = \phi\big(x, y, y', \ldots, y^{(n-1)}\big)$$

is a certain a function.

Provided that an integrating factor is known, Eqs. (2.1) and (2.2) provide the following first integral for Eq. (2.1):

$$\phi\big(x, y, y', \ldots, y^{(n-1)}\big) = \text{const.}$$

A practical tool for finding integrating factors is given by the following theorem (for more details, see [1]).

Theorem 2.1. A function

$$\mu = \mu\left(x, y, y', \ldots, y^{(n-1)}\right)$$

is an integrating factor for Eq. (2.1) if and only if the equation (termed the *determining equation for integrating factors*)

$$\frac{\delta}{\delta y}\big(\mu a\, y^{(n)} + \mu b\big) = 0 \tag{2.3}$$

is satisfied *identically in all the variables*[1] $x, y, y', \ldots, y^{(2n-2)}$, where $\delta/\delta y$ is the variational derivative (1.12).

[1]If we release this condition, we obtain the definition of an adjoint equation for nonlinear equations.

Example 2.1. In the case of first-order equations (1.4) the determining equation (2.3) coincides with Eq. (1.3).

For the higher-order equations (2.1) the determining equation (2.3) provides an overdetermined system for one function μ.

Example 2.2. Investigation of the determining equation (2.3) in the case of the second-order equations

$$a(x, y, y')y'' + b(x, y, y') = 0 \tag{2.4}$$

shows that the integrating factors $\mu(x, y, y')$ are determined by the following system of two equations (see [1] or [2], Section 6.6.2):

$$y'\,(\mu a)_{yy'} + (\mu a)_{xy'} + 2(\mu a)_y - (\mu b)_{y'y'} = 0, \tag{2.5}$$

$$y'^2\,(\mu a)_{yy} + 2y'(\mu a)_{xy} + (\mu a)_{xx} - y'(\mu b)_{yy'} - (\mu b)_{xy'} + (\mu b)_y = 0. \tag{2.6}$$

It is not clear *a priory* if the system (2.5)-(2.6) is always solvable, in other words, if an integrating factor exists for any second-order equation (2.4). For higher-order equations the situation is similar. Namely, the integrating factors for third-order equations are determined by three equations for one function $\mu = \mu(x, y, y', y'')$. In general, the determining equation Eq. (2.3) imposes the system of n equations on the integrating factors μ for the nth-order equation (2.1).

The existence of integrating factors for equations of any order n has been proved recently in [3]. It is shown there that upon introducing the *potential*

$$\psi = \psi\left(x, y, y', \ldots, y^{(n-1)}\right),$$

connected with an integrating factor

$$\mu = \mu\big(x, y, y', \ldots, y^{(n-1)}\big)$$

by the equation

$$\mu = \frac{\partial\psi\left(x, y, y', \ldots, y^{(n-1)}\right)}{\partial y^{(n-1)}},$$

the overdetermined system provided by the determining equation (2.3) reduces to *one equation* for the potential. In consequence, the integrating factor exists for any higher-order equation.

Another advantage of introducing the potential ψ is that upon finding an integrating factor there is no need to find the function ϕ by solving Eq. (2.2) in order to have a first integral

$$\phi\big(x, y, y', \ldots, y^{(n-1)}\big) = \text{const}.$$

Instead, a first integral is given by

$$\psi\left(x, y, y', \ldots, y^{(n-1)}\right) = C = \text{const}. \tag{2.7}$$

The case of second order equations is discussed further in Section 4.

Definition 2.1 applies to systems of linear and nonlinear ordinary differential equations of an arbitrary order.

Example 2.3. Consider the following system of first-order nonlinear equations with two dependent variables y and z :

$$F_1 \equiv 2xyz' - xzy' - yz = 0, \tag{2.8}$$

$$F_2 \equiv 2xzz' + y' + 2z^2 + \frac{y}{x} = 0. \tag{2.9}$$

The integrating factors $\mu^1(x,y,z)$ and $\mu^2(x,y,z)$ are determined by the following equations (cf. Eq. (2.3))

$$\begin{aligned} &\frac{\delta}{\delta y}\big(\mu^1 F_1\big) = 0, \quad \frac{\delta}{\delta z}\big(\mu^1 F_1\big) = 0,\\ &\frac{\delta}{\delta y}\big(\mu^2 F_2\big) = 0, \quad \frac{\delta}{\delta z}\big(\mu^2 F_2\big) = 0. \end{aligned} \tag{2.10}$$

Letting, e.g. $\mu^1 = \mu^1(z)$ and $\mu^2 = \mu^2(x)$, we easily solve the determining equations (2.10) and find

$$\mu^1 = z^3, \quad \mu^2 = x.$$

With these integrating factors, we have

$$\mu^1 F_1 = D_x\left(\frac{xy}{z^2}\right), \quad \mu^2 F_2 = D_x\left(xy + x^2 z^2\right).$$

Hence,

$$xy = C_1 z^2, \quad xy + x^2 z^2 = C_2.$$

These equations can be readily solved for y, z. Indeed, using the first of the above equations we rewrite the second equation in the form

$$(x^2 + C_1) z^2 = C_2$$

and find z. Then we substitute it in the first equation and find y. Thus we obtain the following general solution of the system under consideration:

$$z = \pm\sqrt{\frac{C_2}{x^2 + C_1}}, \quad y = \frac{C_1 C_2}{x(x^2 + C_1)}.$$

Remark 2.1. Integrating factors of the form

$$\mu = \mu(x, y, \ldots, y^{(n-1)})$$

for differential equations

$$F(x, y, \dots, y^{(n)}) = 0$$

may not exist if the differential equation in questions is not solved for the highest derivative $y^{(n)}$, i.e. not written in the form (2.1). For example, let us consider Clairaut's equation

$$\varphi(y') + xy' = y$$

which is not of the form (1.4). In this case, the determining equation (2.3),

$$\frac{\delta}{\delta y}\big[(\varphi(y') + xy' - y)\mu\big] = 0,$$

for the integrating factors $\mu = \mu(x, y)$ yields

$$\mu\varphi'' y'' + (y + y'\varphi' - \varphi)\mu_y + (\varphi' + x)\mu_x + 2\mu = 0,$$

whence $\varphi'' = 0$. It means that Clairaut's equation does not have the integrating factors of the form $\mu = \mu(x, y)$ unless the function $\varphi(y')$ is linear.

Remark 2.2. It is interesting to compare the theory of integrating factors for *higher-order* nonlinear equations with Jacobi's theory of multipliers for systems of nonlinear *first-order* ordinary differential equations (see, e.g. [5], Section 32: "Multipliers").

3 First-order linear equations

Let us begin with an illustration of the new method by first-order linear equations. Consider the first-order equations

$$y' + h(x, y) = 0. \tag{3.1}$$

The potential ψ of the integrating factor $\mu(x, y)$ for Eq. (3.1) is defined by $\mu = \psi_y$. According to [3], the determining equation (1.3) for the integrating factor is written as the following equation for the potential:

$$\psi_x - h(x, y)\psi_y = 0. \tag{3.2}$$

If $\psi(x, y)$ solves Eq. (3.2), then the equation

$$\psi(x, y) = C \tag{3.3}$$

with an arbitrary constant C provides the general solution to Eq. (3.1).

3.1 Potential of the integrating factor for linear equations

Let us apply the method of the potential of integrating factors to the non-homogeneous linear equation (1.17). Thus, we will apply Eqs. (3.2) and (3.3) to the equation

$$y' + P(x)y = Q(x).$$

In this case we have

$$h(x, y) = P(x)y - Q(x), \tag{3.4}$$

and Eq. (3.2) is written

$$\psi_x - h\psi_y = 0. \tag{3.5}$$

Eq. (3.5) shows that one can look for the potentials that are linear in y, i.e.

$$\psi = \alpha(x)y + \beta(x). \tag{3.6}$$

This is equivalent to the assumption that the integrating factor $\mu(x, y)$ depends on x only. Substituting (3.4) and (3.6) in Eq. (3.2) we obtain

$$\alpha'(x)y + \beta'(x) - [P(x)y - Q(x)]\alpha(x) = 0,$$

or

$$[\alpha'(x) - \alpha(x)P(x)]y + [\beta'(x) + Q(x)\alpha(x)] = 0,$$

whence

$$\alpha'(x) - \alpha(x)P(x) = 0, \quad \beta'(x) + Q(x)\alpha(x) = 0. \tag{3.7}$$

Integrating the first equation (3.7) and substituting the result in the second equation (3.7) we get

$$\alpha = K\mathrm{e}^{\int P(x)dx}, \quad \beta' = -KQ(x)\mathrm{e}^{\int P(x)dx}, \quad K = \text{const}.$$

Integrating the resulting differential equation for β we have:

$$\alpha(x) = K\mathrm{e}^{\int P(x)dx}, \quad \beta(x) = -K\int Q(x)\mathrm{e}^{\int P(x)dx}dx + K_1.$$

Substituting them in Eq. (3.6) we finally obtain

$$\psi(x, y) = K\Big[y\mathrm{e}^{\int P(x)dx} - \int Q(x)\mathrm{e}^{\int P(x)dx}dx\Big] + K_1.$$

Now Eq. (3.3) yields

$$y\mathrm{e}^{\int P(x)dx} - \int Q(x)\mathrm{e}^{\int P(x)dx}dx = C,$$

i.e. the general solution (1.20):

$$y = \Big[C + \int Q(x)\mathrm{e}^{\int P(x)dx}dx\Big]\mathrm{e}^{-\int P(x)dx}. \tag{1.20}$$

4 Second-order linear equations

4.1 Determining equation for the potential

For the second-order equation

$$y'' + h(x, y, y') = 0 \tag{4.1}$$

the integrating factors

$$\mu = \mu(x, y, p), \quad \text{where} \quad p = y',$$

are determined by the following two equations:

$$\begin{aligned} &p\mu_{yp} + \mu_{xp} + 2\mu_y - (\mu h)_{pp} = 0, \\ &p^2\mu_{yy} + 2p\mu_{xy} + \mu_{xx} - p(\mu h)_{yp} - (\mu h)_{xp} + (\mu h)_y = 0. \end{aligned} \tag{4.2}$$

They are obtained from Eqs. (2.5)-(2.6) by setting $a = 1, b = h$. Upon introducing the potential $\psi(x, y, p)$ defined by $\mu = \psi_p$, the system of determining equations (4.2) reduces to one equation, namely (see [3]):

$$\psi_x + p\psi_y - h(x, y, p)\psi_p = 0. \tag{4.3}$$

If $\psi(x, y, p)$ solves Eq. (4.3), we have the following first integral for Eq. (4.1):

$$\psi(x, y, y') = C. \tag{4.4}$$

Example 4.1. Consider the equation

$$y'' + \left(2y + \frac{1}{y}\right) y'^2 - \frac{2}{x}\, y' = 0.$$

It has the form (4.1) with

$$h = \left(2y + \frac{1}{y}\right) y'^2 - \frac{2}{x}\, y'.$$

Looking for the integrating factors of the particular form $\mu = \mu(p)$ and solving the determining equations (4.2) and ignoring the inessential constant factor, we obtain

$$\mu(p) = p^{-1}.$$

The potential of this integrating factor is

$$\psi = \ln p + g(x, y).$$

The unknown function $g(x,y)$ is determined by Eq. (4.3) which in our case is written:

$$g_x + pg_y - \left(2y + \frac{1}{y}\right)p + \frac{2}{x} = 0.$$

Solving it, we arrive at the following potential:

$$\psi = y^2 + \ln\left|\frac{yp}{x^2}\right|.$$

The first integral (4.4) is written:

$$y\mathrm{e}^{y^2}y' = Cx^2.$$

It can be written

$$D_x\left(\frac{1}{2}\,\mathrm{e}^{y^2}\right) = D_x\left(\frac{1}{3}\,Cx^3\right).$$

The integration yields the following general solution to our equation:

$$y = \pm\sqrt{\ln(\alpha x^3 + \beta)}, \quad \alpha, \beta = \mathrm{const}.$$

4.2 Non-homogeneous linear equations

Let us apply Eqs. (4.3) and (4.4) to the non-homogeneous linear equation

$$y'' + a(x)y' + b(x)y = f(x). \tag{4.5}$$

Here we have

$$h(x,y,p) = a(x)p + b(x)y - f(x). \tag{4.6}$$

Reasoning as in the case of first-order linear equations, we look for the potentials having the same structure as the function h, i.e.

$$\psi(x,y,p) = \alpha(x)p + \beta(x)y + \gamma(x). \tag{4.7}$$

Substituting (4.6) and (4.7) in Eq. (4.3) we obtain

$$[\alpha'(x) - a(x)\alpha(x) + \beta(x)]p + [\beta'(x) - b(x)\alpha(x)]y + [\gamma'(x) + f(x)\alpha(x)] = 0.$$

Since the expressions in the square brackets depend only on the variable x, the above equation yields the following system of first-order linear differential equations for the unknown functions α, β, γ :

$$\begin{aligned} &\frac{d\alpha}{dx} - a(x)\,\alpha + \beta = 0,\\ &\frac{d\beta}{dx} - b(x)\,\alpha = 0,\\ &\frac{d\gamma}{dx} + f(x)\,\alpha = 0. \end{aligned} \tag{4.8}$$

4.3 Examples

In order to understand how to tackle the system (4.8), let us consider an example.

Example 4.2. Let us find an integrating factor for the following equation:

$$y'' - \omega^2 y = f(x), \quad \omega = \text{const.} \tag{4.9}$$

The homogeneous equation

$$y'' - \omega^2 y = 0 \tag{4.10}$$

has the following fundamental set of solutions

$$y_1 = \mathrm{e}^{-\omega x}, \quad y_2 = \mathrm{e}^{\omega x}. \tag{4.11}$$

Let us calculate the potential (4.7) of the integrating factor. We have

$$a(x) = 0, \quad b(x) = -\omega^2,$$

and hence Eqs. (4.8) are written:

$$\frac{d\alpha}{dx} + \beta = 0, \quad \frac{d\beta}{dx} + \omega^2\,\alpha = 0, \quad \frac{d\gamma}{dx} + f(x)\,\alpha = 0. \tag{4.12}$$

The first two equations of the system (4.12) can be easily solved, e.g. by reducing them via differentiation to the second-order homogeneous equation (4.10):

$$\frac{d^2\alpha}{dx^2} - \omega^2\,\alpha = 0. \tag{4.13}$$

Integrating Eq. (4.13) and using the first and third Eqs. (4.14) we obtain:

$$\begin{aligned}
&\alpha = C_1\mathrm{e}^{\omega x} + C_2\mathrm{e}^{-\omega x},\\
&\beta = -C_1\,\omega\,\mathrm{e}^{\omega x} + C_2\,\omega\,\mathrm{e}^{-\omega x},\\
&\gamma = -C_1\int f(x)\mathrm{e}^{\omega x}dx - C_2\int f(x)\mathrm{e}^{-\omega x}dx + C_3.
\end{aligned}$$

Hence, Eq. (4.7) yields:

$$\begin{aligned}
\psi(x,y,p) = &\Big[C_1\mathrm{e}^{\omega x} + C_2\mathrm{e}^{-\omega x}\Big]p + \Big[C_2\mathrm{e}^{-\omega x} - C_1\mathrm{e}^{\omega x}\Big]\,\omega\,y\\
&- C_1\int f(x)\mathrm{e}^{\omega x}dx - C_2\int f(x)\mathrm{e}^{-\omega x}dx + C_3.
\end{aligned} \tag{4.14}$$

Now Eq. (4.4) provides the following non-homogeneous first-order linear equation:

$$\Big[C_1 e^{\omega x} + C_2 e^{-\omega x}\Big]\frac{dy}{dx} + \Big[C_2 e^{-\omega x} - C_1 e^{\omega x}\Big]\,\omega\, y$$
$$= C_1 \int f(x) e^{\omega x} dx + C_2 \int f(x) e^{-\omega x} dx + C. \tag{4.15}$$

Any solution of Eq. (4.15) will solve Eq. (4.9). It suffices to find any particular solution to Eq. (4.9) due to its linearity. For example, one can find a particular solution to Eq. (4.15) by using only one of solutions (4.11) of the homogeneous equation. Taking, e.g. the first solution from (4.11), i.e. setting $C_1 = 1,\ C_2 = 0$ and in addition letting $C = 0$ in (4.15) we rewrite Eq. (4.15) in the form

$$y' - \omega y = e^{-\omega x} \int f(x) e^{\omega x} dx. \tag{4.16}$$

Solving this simple first-order linear equation, e.g. by using Eq. (1.20) with $C = 0$, we obtain the following particular solution:

$$y_* = e^{\omega x} \int e^{-2\omega x} \Big(\int f(x)\, e^{\omega x} dx \Big) dx. \tag{4.17}$$

Thus, using *one particular solution* $y_1(x)$ of the homogeneous equation we can reduce the non-homogeneous equation (4.9) to the homogeneous equation (4.10) for $y - y_*$. Hence

$$y - y_* = C_1 e^{\omega x} + C_2 e^{-\omega x}$$

and therefore the general solution to Eq. (4.9) is given by

$$y = e^{\omega x} \int e^{-2\omega x} \Big(\int f(x)\, e^{\omega x} dx \Big) dx + C_1 e^{\omega x} + C_2 e^{-\omega x}. \tag{4.18}$$

Note that application of the method of variation of parameters leads to the following particular solution of (4.9):

$$y_* = \frac{1}{2\omega}\Big[e^{\omega x} \int f(x)\, e^{-\omega x} dx - e^{-\omega x} \int f(x)\, e^{\omega x} dx \Big]. \tag{4.19}$$

It is obtained, unlike (4.17), by using both solutions (4.11) of the homogeneous equation. Let us compare the solutions (4.17) and (4.19) for some functions $f(x)$.

Example 4.3. Let us take $f(x) = x$ and express the particular solutions (4.17) and (4.19) in terms of elementary functions. The formula (4.19) yields:

$$
\begin{aligned}
y_* &= \frac{1}{2\omega}\Big[\mathrm{e}^{\omega x}\int x\mathrm{e}^{-\omega x}dx - \mathrm{e}^{-\omega x}\int x\mathrm{e}^{\omega x}dx\Big]\\
&= \frac{1}{2\omega^3}\Big[-\mathrm{e}^{\omega x}\big(1+\omega x\big)\mathrm{e}^{-\omega x} + \mathrm{e}^{-\omega x}\big(1-\omega x\big)\mathrm{e}^{\omega x}\Big]\\
&= \frac{1}{2\omega^3}\Big[-\big(1+\omega x\big) + \big(1-\omega x\big)\Big] = -\frac{x}{\omega^2}\,\cdot
\end{aligned}
$$

One can readily verify that the formula (4.17) yields the same result. Indeed:

$$
\begin{aligned}
y_* &= \mathrm{e}^{\omega x}\int \mathrm{e}^{-2\omega x}\Big(\int x\mathrm{e}^{\omega x}dx\Big)dx\\
&= \frac{1}{\omega^2}\,\mathrm{e}^{\omega x}\int\big(\omega x - 1\big)\mathrm{e}^{-\omega x}\,dx\\
&= \frac{1}{\omega^2}\,\mathrm{e}^{\omega x}\Big[\big(-\omega x - 1\big)\mathrm{e}^{-\omega x} + \mathrm{e}^{-\omega x}\Big] = -\frac{x}{\omega^2}\,\cdot
\end{aligned}
$$

Substituting the above y_* in (4.18) we obtain the following general solution to Eq. (4.9) with $f(x) = x$:

$$
y = -\frac{x}{\omega^2} + C_1\mathrm{e}^{\omega x} + C_2\mathrm{e}^{-\omega x}.
$$

Example 4.4. Consider one more example by letting

$$
f(x) = \mathrm{e}^{\frac{1}{2}\omega x}.
$$

The formula (4.19) yields:

$$
\begin{aligned}
y_* &= \frac{1}{2\omega}\Big[\mathrm{e}^{\omega x}\int \mathrm{e}^{-\frac{1}{2}\omega x}dx - \mathrm{e}^{-\omega x}\int \mathrm{e}^{\frac{3}{2}\omega x}dx\Big]\\
&= -\frac{1}{\omega^2}\mathrm{e}^{\omega x}\,\mathrm{e}^{-\frac{1}{2}\omega x} - \frac{1}{3\omega^2}\mathrm{e}^{-\omega x}\,\mathrm{e}^{\frac{3}{2}\omega x} = -\frac{4}{3\omega^2}\,\mathrm{e}^{\frac{1}{2}\omega x}.
\end{aligned}
$$

The formula (4.17) yields the same result. Indeed:

$$
y_* = \mathrm{e}^{\omega x}\int \mathrm{e}^{-2\omega x}\Big(\int \mathrm{e}^{\frac{3}{2}\omega x}dx\Big)dx = \frac{2}{3\omega}\,\mathrm{e}^{\omega x}\int \mathrm{e}^{-\frac{1}{2}\omega x}\,dx = -\frac{x}{\omega^2}\,\cdot
$$

4.4 Integrating factor for a special non-homogeneous equation

Let us consider the equations (4.5) not containing the first derivative, i.e. having the form

$$y'' + b(x)y = f(x). \tag{4.20}$$

The corresponding homogeneous equations is

$$y'' + b(x)y = 0. \tag{4.21}$$

For the equation (4.20) the system (4.8) is written

$$\frac{d\alpha}{dx} + \beta = 0,$$

$$\frac{d\beta}{dx} - b(x)\,\alpha = 0,$$

$$\frac{d\gamma}{dx} + f(x)\,\alpha = 0$$

and can be easily solved. Indeed, differentiating the first equation of this system and using the second equation, we obtain:

$$\alpha'' + b(x)\alpha = 0. \tag{4.22}$$

Thus, the coefficient $\alpha(x)$ of the potential (4.7) solves the homogeneous equation (4.21). If we know any nontrivial solution $y_1(x) \neq 0$ of the homogeneous equation (4.21), we take

$$\alpha(x) = y_1(x), \tag{4.23}$$

then obtain the coefficients $\beta(x)$ and $\gamma(x)$ of the potential (4.7) by differentiation and a quadrature, respectively. Namely:

$$\beta(x) = -\alpha'(x), \quad \gamma(x) = -\int f(x)\alpha(x)dx. \tag{4.24}$$

Hence, the potential (4.7) is given by

$$\psi(x, y, p) = \alpha(x)p - \alpha'(x)y - \int f(x)\alpha(x)dx, \tag{4.25}$$

where $\alpha(x)$ is any solution of Eq. (4.22).

Eqs. (4.4) and (4.25) yield:

$$\alpha(x)y' - \alpha'(x)y - \int f(x)\alpha(x)dx = C.$$

Taking here $C = 0$ for the sake of simplicity and dividing by $\alpha(x)$ we obtain:

$$y' = \frac{\alpha'(x)}{\alpha(x)}\, y + \frac{1}{\alpha(x)} \int f(x)\alpha(x)dx.$$

It is a non-homogeneous linear first-order equation (1.17) with the coefficients

$$P(x) = -\frac{\alpha'(x)}{\alpha(x)} = -\frac{d\ln\alpha(x)}{dx}\,, \quad Q(x) = \frac{1}{\alpha(x)} \int f(x)\alpha(x)dx.$$

Substituting these expressions for $P(x)$ and $Q(x)$ in Eq. (1.20) were we can let $C = 0$, and noting that

$$\mathrm{e}^{-\int P(x)dx} = \mathrm{e}^{\int d\ln\alpha(x)} = \alpha(x), \quad \mathrm{e}^{\int P(x)dx} = \frac{1}{\alpha(x)}\,,$$

we obtain the following particular solution for the non-homogeneous equation (4.20):

$$y_*(x) = \alpha(x) \int \left[\frac{1}{\alpha^2(x)} \int \alpha(x) f(x)dx\right] dx. \tag{4.26}$$

Here $\alpha(x) \neq 0$ is any particular solution of the homogeneous equation (4.21).

Remark 4.1. Our method of computation of the particular solution $y_*(x)$ requires knowledge of only one particular solution $\alpha(x) = y_1(x)$ of the homogeneous equation (4.21), whereas Lagrange's method of variation of parameters needs two linearly independent solutions of Eq. (4.21).

4.5 Integrating factor for the general non-homogeneous equation

We can extend the method of Section 4.4 to all linear second-order equations (4.5) due to the following well-known result (the proof can be found, e.g. in [2], Section 3.3.2).

Lemma 4.1. The general linear equation (4.5) is mapped by the transformation

$$\tilde{y} = y\,\mathrm{e}^{\frac{1}{2}\int a(x)dx} \tag{4.27}$$

to an equation of the form (4.20), namely:

$$\tilde{y}'' + \tilde{b}(x)\,\tilde{y} = \tilde{f}(x), \tag{4.28}$$

where

$$\tilde{b}(x) = b(x) - \frac{1}{4}\,a^2(x) - \frac{1}{2}\,a'(x), \quad \tilde{f}(x) = f(x)\,\mathrm{e}^{\frac{1}{2}\int a(x)dx}. \tag{4.29}$$

The following example clarifies how to use Lemma 4.1 together with the method from Section 4.4.

Example 4.5. Let us find a particular solution to the non-homogeneous equation

$$y'' - xy' + y = f(x) \tag{4.30}$$

using the obvious particular solution

$$y_1 = x \tag{4.31}$$

of the homogeneous equation

$$y'' - xy' + y = 0. \tag{4.32}$$

Substituting the coefficients $a(x) = -x,\ b(x) = 1$ in Eqs. (4.27) and (4.29) we have

$$\tilde{y} = y\,\mathrm{e}^{-\frac{1}{4}x^2} \tag{4.33}$$

and

$$\tilde{b}(x) = \frac{3}{2} - \frac{x^2}{4}, \quad \tilde{f}(x) = f(x)\,\mathrm{e}^{-\frac{1}{4}x^2},$$

respectively.

Hence Eq. (4.30) has been transformed, in accordance with (4.28), to the following special form (4.20):

$$\tilde{y}\,'' + \left(\frac{3}{2} - \frac{x^2}{4}\right)\tilde{y} = f(x)\,\mathrm{e}^{-\frac{1}{4}x^2}. \tag{4.34}$$

The transformation (4.33) maps the particular solution (4.31) of Eq. (4.32) to the particular solution

$$\tilde{y}_1 = x\,\mathrm{e}^{-\frac{1}{4}x^2} \tag{4.35}$$

of the homogeneous equation

$$\tilde{y}\,'' + \left(\frac{3}{2} - \frac{x^2}{4}\right)\tilde{y} = 0. \tag{4.36}$$

Applying Eqs. (4.23), (4.24) to the solution (4.35) we obtain:

$$\tilde{\alpha}(x) = x\,\mathrm{e}^{-\frac{1}{4}x^2}, \quad \tilde{\beta}(x) = -\alpha'(x) = \left(\frac{x^2}{2} - 1\right)\mathrm{e}^{-\frac{1}{4}x^2} \tag{4.37}$$

and

$$\tilde{\gamma}(x) = -\int \tilde{f}(x)\alpha(x)dx = -\int f(x)\,\mathrm{e}^{-\frac{1}{4}x^2}\,x\,\mathrm{e}^{-\frac{1}{4}x^2}dx$$
$$= -\int xf(x)\mathrm{e}^{-\frac{1}{2}x^2}dx. \tag{4.38}$$

Substituting these expressions in the potential (4.7):

$$\psi(x,\tilde{y},\tilde{p}) = \tilde{\alpha}(x)\tilde{p} + \tilde{\beta}(x)\tilde{y} + \tilde{\gamma}(x)$$

and using Eq. (4.4) we obtain the following first integral for Eq. (4.34):

$$x\,\mathrm{e}^{-\frac{1}{4}x^2}\,\tilde{y}\,' + \Big(\frac{x^2}{2} - 1\Big)\mathrm{e}^{-\frac{1}{4}x^2}\,\tilde{y} - \int x f(x)\mathrm{e}^{-\frac{1}{2}x^2}dx = C.$$

Taking $C = 0$ we write it in the form

$$\frac{d\tilde{y}}{dx} = \Big(\frac{1}{x} - \frac{x}{2}\Big)\tilde{y} + \frac{1}{x}\,\mathrm{e}^{\frac{1}{4}x^2}\int x f(x)\mathrm{e}^{-\frac{1}{2}x^2}dx.$$

Integrating this first-order linear equation and ignoring the constant integration we obtain the particular solution to Eq. (4.34):

$$\tilde{y}_* = x\,\mathrm{e}^{-\frac{1}{4}x^2}\int \frac{1}{x^2}\,\mathrm{e}^{\frac{1}{2}x^2}\Big(\int x f(x)\mathrm{e}^{-\frac{1}{2}x^2}dx\Big)dx.$$

Substituting it in (4.33) we arrive at the following particular solution of Eq. (4.30):

$$y_* = x\int \frac{1}{x^2}\,\mathrm{e}^{\frac{1}{2}x^2}\Big(\int x f(x)\mathrm{e}^{-\frac{1}{2}x^2}dx\Big)dx. \tag{4.39}$$

Finally, recall that a knowledge of a particular solution of a homogeneous linear second-order equation allows one to find a second solution linearly independent of the first solution (see, e.g. [2], Section 6.5.5). Applying this method to the solution (4.31) we obtain the following second solution of Eq. (4.32):

$$y_2 = x\int \frac{1}{x^2}\,\mathrm{e}^{\frac{1}{2}x^2}dx. \tag{4.40}$$

Thus, the general solution of Eq. (4.30) has the form

$$y = x\bigg[\int \frac{1}{x^2}\,\mathrm{e}^{\frac{1}{2}x^2}\Big(\int x f(x)\mathrm{e}^{-\frac{1}{2}x^2}dx\Big)dx + C_1 + C_2\int \frac{1}{x^2}\,\mathrm{e}^{\frac{1}{2}x^2}dx\bigg]. \tag{4.41}$$

4.6 Alternative method for the general non-homogeneous equation

Let us extend the transformation (4.27) to the derivative $p = y'$ and write it in the form

$$y = \tilde{y}\,\mathrm{e}^{-\frac{1}{2}\int a(x)dx}, \quad p = \Big(\tilde{p} - \frac{a}{2}\,\tilde{y}\Big)\mathrm{e}^{-\frac{1}{2}\int a(x)dx}. \tag{4.42}$$

Substituting (4.42) in (4.7) yields:

$$\psi(x,y,p) = \alpha\,\tilde{p}\,\mathrm{e}^{-\frac{1}{2}\int a(x)dx} + \Big(\beta - \frac{a}{2}\,\alpha\Big)\tilde{y}\,\mathrm{e}^{-\frac{1}{2}\int a(x)dx} + \gamma. \tag{4.43}$$

The potential of the integrating factor for Eq. (4.28) is written

$$\psi(x, \tilde{y}, \tilde{p}) = \tilde{\alpha}\,\tilde{p} + \tilde{\beta}\,\tilde{y} + \tilde{\gamma}. \tag{4.44}$$

Its coefficients are defined by the following equations (see Section 4.4):

$$\begin{aligned}
&\frac{d\tilde{\alpha}}{dx} + \tilde{\beta} = 0,\\
&\frac{d\tilde{\beta}}{dx} - \tilde{b}(x)\,\tilde{\alpha} = 0,\\
&\frac{d\tilde{\gamma}}{dx} + \tilde{f}(x)\,\tilde{\alpha} = 0.
\end{aligned} \tag{4.45}$$

Comparing (4.43) and (4.43) we conclude that

$$\tilde{\alpha} = \alpha\,\mathrm{e}^{-\frac{1}{2}\int a(x)dx}, \quad \tilde{\beta} = \Big(\beta - \frac{a}{2}\,\alpha\Big)\,\mathrm{e}^{-\frac{1}{2}\int a(x)dx}, \quad \tilde{\gamma} = \gamma,$$

whence, integrating the third equation of the system (4.45) and invoking the second equation (4.29) we obtain:

$$\begin{aligned}
&\alpha = \tilde{\alpha}\,\mathrm{e}^{\frac{1}{2}\int a(x)dx},\\
&\beta = \Big(\tilde{\beta} + \frac{a}{2}\,\tilde{\alpha}\Big)\,\mathrm{e}^{\frac{1}{2}\int a(x)dx},\\
&\gamma = -\int \tilde{\alpha}\, f(x)\,\mathrm{e}^{\frac{1}{2}\int a(x)dx}\,dx.
\end{aligned} \tag{4.46}$$

One can verify that the functions $\alpha(x), \beta(x), \gamma(x)$ given by (4.46) furnish the solution to Eqs. (4.8) provided that $\tilde{\alpha}, \tilde{\beta}), \tilde{\gamma}$ solve Eqs. (4.45). Hence, the equation

$$\alpha(x)\,y' + \beta(x)\,y + \gamma(x) = C \tag{4.47}$$

with the coefficients given by Eqs. (4.46) provides a first integral for Eq. (4.5).

Example 4.6. Consider again Equation (4.30),

$$y'' - xy' + y = f(x).$$

Substituting in (4.46) the expressions for $\tilde{\alpha},\ \tilde{\beta}$ and $\tilde{\gamma}$ given by (4.37) and (4.38), respectively, we have

$$\alpha = x\,\mathrm{e}^{-\frac{1}{2}x^2}, \quad \beta = -\mathrm{e}^{-\frac{1}{2}x^2}, \quad \gamma = -\int x\,\mathrm{e}^{-\frac{1}{2}x^2} f(x)dx.$$

Inserting them in the first integral (4.47) and letting $C = 0$, we obtain the following simple non-homogeneous first-order linear equation:

$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{x}\,\mathrm{e}^{\frac{1}{2}x^2}\int x\,\mathrm{e}^{-\frac{1}{2}x^2} f(x)dx.$$

Its integration provides the particular solution (4.39).

Bibliography

[1] N.H. Ibragimov, "Integrating factors, adjoint equations and Lagrangians," *Journal of Mathematical Analysis and Applications*, vol. 318, No. 2, pp. 742–757, 2006.

[2] N.H. Ibragimov, *A practical course in differential equations and mathemtical modelling*, Karlskrona, ALGA Publications, 3rd ed., 2006.

[3] N.H. Ibragimov and S.V. Khabirov, "Existence of integrating factors for higher-order ordinary differential equations," *Archives of ALGA*, vol. 4, pp. 61–70, 2007.

[4] V. V. Stepanov, *Course of differential equations*, Moscow, State Publisher of Phys.-Math. Literature, 7th ed., 1958. In Russian.

[5] E. Goursat, *Differential equations*, Boston, Ginn and Co., 1917. (English transl. by E.R. Hedrick and O. Dunkel of Goursat's classical *Cours d'analyse mathématique*, vol. 2, Part II).

Archives of ALGA
Volume 5, 2008

SELF-ADJOINTNESS AND QUASI-SELF-ADJOINTNESS OF AN EQUATION MODELLING MELT MIGRATION THROUGH THE EARTH'S MANTLE. NONLOCAL CONSERVATION LAWS

RAISA KHAMITOVA
Department of Mathematics and Science,
Blekinge Institute of Technology,
SE-371 79 Karlskrona, Sweden

Abstract. The recent theorem on nonlocal conservation laws [1] is applied to a magma equation modelling a melt migration through the Earth's mantle. It is shown that the equation in question is quasi-self-adjoint in the terminology of [2]. The self-adjoint equations are singled out. Nonlocal and local conservation densities are obtained using the symmetries of the magma equation.

Keywords: Magma equation, Self-adjointness, Quasi-self-adjointness, Conservation laws.

1 Introduction

The equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z}\Big\{u^n\Big[1 - \frac{\partial}{\partial z}\Big(\frac{1}{u^m}\frac{\partial u}{\partial t}\Big)\Big]\Big\} = 0 \tag{1.1}$$

models the migration of melt through the Earth's mantle. It follows from the equations suggested by Scott and Stevenson [3] for $2 \leq n \leq 5$ and $0 \leq m \leq 1$. However other authors discussed Eq. (1.1) for any values of n and m.

Using differential variables we can rewrite the equation as follows:

$$F \equiv u_t + D_z\Big\{u^n\Big[1 - D_z\Big(u^{-m}u_t\Big)\Big]\Big\} = 0 \tag{1.2}$$

where

$$D_z = \frac{\partial}{\partial z} + u_z\frac{\partial}{\partial u} + u_{zj}\frac{\partial}{\partial u_j} + u_{zjk}\frac{\partial}{\partial u_{jk}} + \ldots$$

is the operator of total differentiation with respect to z. Since

$$D_z\big(u^{-m}u_t\big) = -mu^{-m-1}u_zu_t + u^{-m}u_{tz}$$

we obtain that

$$D_z\big\{u^n\big[1-D_z\big(u^{-m}u_t\big)\big]\big\}=D_z\big\{u^n+mu^{n-m-1}u_zu_t-u^{n-m}u_{tz}\big\}$$
$$=nu^{n-1}u_z+m(n-m-1)u^{n-m-2}u_z^2u_t+mu^{n-m-1}u_{zz}u_t$$
$$+mu^{n-m-1}u_zu_{tz}-(n-m)u^{n-m-1}u_zu_{tz}-u^{n-m}u_{tzz}\,.$$

Thus Eq. (1.2) transforms to

$$\begin{aligned}F\equiv{}&u_t-u^{n-m}u_{tzz}+(2m-n)u^{n-m-1}u_zu_{tz}+mu^{n-m-1}u_tu_{zz}\\&+m(n-m-1)u^{n-m-2}u_tu_z^2+nu^{n-1}u_z=0.\end{aligned}\tag{1.3}$$

The adjoint equation $F^*=0$ is defined according to [1]:

$$F^*\equiv\frac{\delta}{\delta u}(vF)=0$$

where $v=v(t,z)$ is a new dependent variable and

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_i\frac{\partial}{\partial u_i}+D_iD_j\frac{\partial}{\partial u_{ij}}-D_iD_jD_k\frac{\partial}{\partial u_{ijk}}+...$$

is the variational derivative. We have

$$v\frac{\partial F}{\partial u}-D_t\Big[v\frac{\partial F}{\partial u_t}\Big]-D_z\Big[v\frac{\partial F}{\partial u_z}\Big]+D_tD_z\Big[v\frac{\partial F}{\partial u_{tz}}\Big]+D_z^2\Big[v\frac{\partial F}{\partial u_{zz}}\Big]-D_tD_z^2\Big[v\frac{\partial F}{\partial u_{tzz}}\Big]=0.$$

Separate calculation of each term gives the following results:

$$v\frac{\partial F}{\partial u}=v\big[(m-n)u^{n-m-1}u_{tzz}+(2m-n)(n-m-1)u^{n-m-2}u_zu_{tz}$$
$$+m(n-m-1)u^{n-m-2}u_tu_{zz}+m(n-m-1)(n-m-2)u^{n-m-3}u_tu_z^2$$
$$+n(n-1)u^{n-1}u_z\big],$$

$$-D_t\Big(v\frac{\partial F}{\partial u_t}\Big)=-v_t\big[1+mu^{n-m-1}u_{zz}+m(n-m-1)u^{n-m-2}u_z^2\big]$$
$$-v\big[m(n-m-1)u^{n-m-2}u_tu_{zz}+mu^{n-m-1}u_{tzz}+m(n-m-1)$$
$$\times(n-m-2)u^{n-m-3}u_tu_z^2+2m(n-m-1)u^{n-m-2}u_zu_{tz}\big],$$

$$-D_z\Big(v\frac{\partial F}{\partial u_z}\Big) = -v_z\big[(2m-n)u^{n-m-1}u_{tz} + 2m(n-m-1)u^{n-m-2}u_tu_z + nu^{n-1}\big]$$
$$- v\big[(4m-n)(n-m-1)u^{n-m-2}u_zu_{tz} + (2m-n)u^{n-m-1}u_{tzz}$$
$$+ 2m(n-m-1)(n-m-2)u^{n-m-3}u_tu_z^2 + 2m(n-m-1)u^{n-m-2}u_tu_{zz}$$
$$+ n(n-1)u^{n-2}u_z\big],$$

$$D_tD_z\Big(v\frac{\partial F}{\partial u_{tz}}\Big) = D_tD_z\big\{v\big[(2m-n)u^{n-m-1}u_z\big]\big\}$$
$$= (2m-n)D_t\big\{v_zu^{n-m-1}u_z + v\big[(n-m-1)u^{n-m-2}u_z^2 + u^{n-m-1}u_{zz}\big]\big\}$$
$$= (2m-n)\big\{v_{tz}u^{n-m-1}u_z + v_t\big[(n-m-1)u^{n-m-2}u_z^2 + u^{n-m-1}u_{zz}\big]$$
$$+ v_z\big[(n-m-1)u^{n-m-2}u_tu_z + u^{n-m-1}u_{tz}\big] + v(n-m-1)$$
$$\times \big[(n-m-2)u^{n-m-3}u_tu_z^2 + 2u^{n-m-2}u_zu_{tz} + u^{n-m-2}u_tu_{zz})\big] + vu^{n-m-1}u_{tzz}\big\},$$

$$D_z^2\Big(v\frac{\partial F}{\partial u_{zz}}\Big) = D_z^2\big(mvu^{n-m-1}u_t\big)$$
$$= mv_{zz}u^{n-m-1}u_t + 2mv_z\big[(n-m-1)u^{n-m-2}u_tu_z + u^{n-m-1}u_{tz}\big]$$
$$+ mv(n-m-1)\big[(n-m-2)u^{n-m-3}u_tu_z^2 + u^{n-m-2}u_tu_{zz} + 2u^{n-m-2}u_zu_{tz}\big]$$
$$+ mvu^{n-m-1}u_{tzz}\,,$$

$$-D_tD_z^2\Big(v\frac{\partial F}{\partial u_{tzz}}\Big) = D_tD_z^2\big(vu^{n-m}\big) = D_t\big\{v_{zz}u^{n-m} + 2v_z(n-m)u^{n-m-1}u_z$$
$$+ v(n-m)\big[(n-m-1)u^{n-m-2}u_z^2 + u^{n-m-1}u_{zz}\big]\big\}$$
$$= v_{tzz}u^{n-m} + v_{zz}(n-m)u^{n-m-1}u_t + 2v_{tz}(n-m)u^{n-m-1}u_z$$
$$+ 2v_z(n-m)\big[(n-m-1)u^{n-m-2}u_tu_z + u^{n-m-1}u_{tz}\big]$$
$$+ v_t(n-m)\big[(n-m-1)u^{n-m-2}u_z^2 + u^{n-m-1}u_{zz}\big]$$
$$+ v(n-m)(n-m-1)\big[(n-m-2)u^{n-m-3}u_tu_z^2 + 2u^{n-m-2}u_zu_{tz}$$
$$+ u^{n-m-2}u_tu_{zz}\big] + v(n-m)u^{n-m-1}u_{tzz}\,.$$

Summing up the above results we obtain:

$$\begin{aligned} F^* \equiv & - v_t + v_{tzz}u^{n-m} + nv_{tz}u^{n-m-1}u_z + nv_{zz}u^{n-m-1}u_t \\ & + nv_z\big[2u^{n-m-1}u_{tz} + (n-m-1)u^{n-m-2}u_tu_z - u^{n-1}\big] = 0. \end{aligned} \tag{1.4}$$

2 Self-adjointness and quasi-self-adjointness

In order to check whether Eq. (1.3) is self-adjoint, we let $v = u$, then

$$\begin{aligned} F^* = & - u_t + u^{n-m}u_{tzz} + 3nu^{n-m-1}u_zu_{tz} + nu^{n-m-1}u_tu_{zz} \\ & + n(n-m-1)u^{n-m-2}u_tu_z^2 - nu^{n-1}u_z\,. \end{aligned}$$

Comparison with

$$\begin{aligned} -F = & - u_t + u^{n-m}u_{tzz} - (2m-n)u^{n-m-1}u_zu_{tz} - mu^{n-m-1}u_tu_{zz} \\ & - m(n-m-1)u^{n-m-2}u_tu_z^2 - nu^{n-1}u_z = 0 \end{aligned}$$

gives that $F^* = -F = 0$ if $m = -n$.
Thus, *Eq. (1.3) is self-adjoint if* $m = -n$, i.e. when

$$\begin{aligned} F = & u_t - u^{2n}u_{tzz} - 3nu^{2n-1}u_zu_{tz} - nu^{2n-1}u_tu_{zz} \\ & - n(2n-1)u^{2n-2}u_tu_z^2 + nu^{n-1}u_z\,. \end{aligned} \tag{2.1}$$

According to [2] Eq. (1.3) is quasi-self-adjoint, if the substitution $v = h(u)$ gives $F^* = \lambda(u)F$. Since

$$v_t = h'u_t, \quad v_z = h'u_z, \quad v_{tz} = h''u_tu_z + h'u_{tz}, \quad v_{zz} = h''u_z^2 + h'u_{zz},$$
$$v_{tzz} = h'''u_tu_z^2 + 2h''u_zu_{tz} + h''u_tu_{zz} + h'u_{tzz},$$

F^* obtains the following form:

$$\begin{aligned} F^* = & - h'u_t + \big(h'''u_tu_z^2 + 2h''u_zu_{tz} + h''u_tu_{zz} + h'u_{tzz}\big)u^{n-m} \\ & + n\big(h''u_tu_z + h'u_{tz}\big)u^{n-m-1}u_z + n\big(h''u_z^2 + h'u_{zz}\big)u^{n-m-1}u_t \\ & + nh'u_z\big[2u^{n-m-1}u_{tz} + (n-m-1)u^{n-m-2}u_tu_z - u^{n-1}\big], \end{aligned}$$

whence

$$\begin{aligned} F^* = & -h'(u_t - u_{tzz}u^{n-m}) + (2uh'' + 3nh')u^{n-m-1}u_zu_{tz} - h'nuu_z \\ & + (uh'' + nh')u^{n-m-1}u_tu_{zz} + [u^2h''' + 2nuh'' + n(n-m-1)h']u^{n-m-2}u_tu_z^2. \end{aligned} \tag{2.2}$$

Comparing with Eq. (1.3),

$$\begin{aligned} F \equiv & u_t - u^{n-m}u_{tzz} + (2m-n)u^{n-m-1}u_z u_{tz} + mu^{n-m-1}u_t u_{zz} \\ & + m(n-m-1)u^{n-m-2}u_t u_z^2 + nu^{n-1}u_z, \end{aligned}$$

we obtain $\lambda(u) = -h'$ and

$$2uh^{''} + 3nh' = -(2m-n)h', \quad uh^{''} + nh' = -mh',$$

$$u^2 h^{'''} + 2nuh^{''} + n(n-m-1)h' = -m(n-m-1)h'.$$

Two first equations yield the same result:

$$uh'' + (n+m)h' = 0.$$

Rewriting the equation in the form

$$\frac{h^{''}}{h'} = -\frac{n+m}{u}$$

and integrating once we obtain

$$h' = \frac{C_1}{u^{n+m}}, \quad C_1 = \text{const.},$$

whence for $n + m \neq 1$

$$h = -\frac{C_1}{(n+m-1)u^{n+m-1}} + C_2, \quad C_2 = \text{const.}$$

and for $n + m = 1$

$$h = C_1 \ln|u| + C_2.$$

The third equation is also satisfied; for $n + m \neq 1$

$$u^2 h^{'''} + 2nuh^{''} + (n+m)(n-m-1)h' = C_1(n+m)u^{-(n+m)}[n+m+1-2n+n-m-1] = 0$$

and for $n + m = 1$

$$u^2 h^{'''} + 2nuh^{''} + (n-m-1)h' = C_1 u^{-1}[2 - 2n + n - m - 1] = 0.$$

Thus *the equation $F = 0$ is quasi-self-adjoint for arbitrary m and n.*

We can choose $h(u) = u^{1-n-m}$ and $h(u) = \ln|u|$ for $n + m \neq 1$ and $n + m = 1$ correspondingly.

3 Conservation laws: General form

According to [4] and [1] it is possible to introduce a formal Lagrangian $\mathcal{L} \equiv vF$. Since the mixed derivatives u_{tx} and u_{txx} are present in F, we have to choose Lagrangian in the symmetrized form:

$$\begin{aligned}\mathcal{L} =& v\big\{u_t - \frac{1}{3}u^{n-m}(u_{tzz} + u_{ztz} + u_{zzt}) + \frac{1}{2}(2m-n)u^{n-m-1}u_z(u_{tz} + u_{zt}) \\ &+ mu^{n-m-1}u_t u_{zz} + m(n-m-1)u^{n-m-2}u_t u_z^2 + nu^{n-1}u_z\big\}.\end{aligned} \tag{3.1}$$

A conservation law corresponding to an operator

$$X = \xi^1(t,z,u)\frac{\partial}{\partial t} + \xi^2(t,z,u)\frac{\partial}{\partial x} + \eta(t,z,u)\frac{\partial}{\partial u}$$

has the form

$$D_t(C^1) + D_z(C^2) = 0$$

where

$$\begin{aligned}C^i =& \xi^i\mathcal{L} + W\Big[\frac{\partial \mathcal{L}}{\partial u_i} - D_j\Big(\frac{\partial \mathcal{L}}{\partial u_{ij}}\Big) + D_j D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}}\Big)\Big] \\ &+ D_j(W)\Big[\frac{\partial \mathcal{L}}{\partial u_{ij}} - D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}}\Big)\Big] + D_j D_k(W)\frac{\partial \mathcal{L}}{\partial u_{ijk}}\,,\end{aligned} \tag{3.2}$$

$W = \eta - \xi^i u_i$.

Since the Lagrangian $\mathcal{L}$ is equal to zero on solutions of the equation $F = 0$ we can calculate C^i without the term $\xi^i\mathcal{L}$. Furthermore, we can rewrite the density

$$\begin{aligned}C^1 =& W\Big[\frac{\partial \mathcal{L}}{\partial u_t} - D_z\Big(\frac{\partial \mathcal{L}}{\partial u_{tz}}\Big) + D_z^2\Big(\frac{\partial \mathcal{L}}{\partial u_{tzz}}\Big)\Big] \\ &+ D_z(W)\Big[\frac{\partial \mathcal{L}}{\partial u_{tz}} - D_z\Big(\frac{\partial \mathcal{L}}{\partial u_{tzz}}\Big)\Big] + D_z^2(W)\frac{\partial \mathcal{L}}{\partial u_{tzz}}\end{aligned}$$

using the following:

$$\begin{aligned}-WD_z\Big(\frac{\partial \mathcal{L}}{\partial u_{tz}}\Big) &= -D_z\Big(W\frac{\partial \mathcal{L}}{\partial u_{tz}}\Big) + D_z(W)\,\frac{\partial \mathcal{L}}{\partial u_{tz}}\,, \\ WD_z^2\Big(\frac{\partial \mathcal{L}}{\partial u_{tzz}}\Big) &= D_z^2\Big(W\frac{\partial \mathcal{L}}{\partial u_{tzz}}\Big) - 2D_z(W)D_z\Big(\frac{\partial \mathcal{L}}{\partial u_{tzz}}\Big) - D_z^2(W)\frac{\partial \mathcal{L}}{\partial u_{tzz}}\,.\end{aligned}$$

Hence we obtain

$$C^1 = W\frac{\partial \mathcal{L}}{\partial u_t} + D_z(W)\Big[2\frac{\partial \mathcal{L}}{\partial u_{tz}} - 3D_z\Big(\frac{\partial \mathcal{L}}{\partial u_{tzz}}\Big)\Big]. \tag{3.3}$$

The remaining term

$$-D_z\Big(W\frac{\partial\mathcal{L}}{\partial u_{tz}}\Big)+D_z^2\Big(W\frac{\partial\mathcal{L}}{\partial u_{tzz}}\Big)$$

will be included in C^2, where it will have the form

$$-D_t\Big(W\frac{\partial\mathcal{L}}{\partial u_{tz}}\Big)+D_tD_z\Big(W\frac{\partial\mathcal{L}}{\partial u_{tzz}}\Big)=-D_t(W)\frac{\partial\mathcal{L}}{\partial u_{tz}}-WD_t\Big(\frac{\partial\mathcal{L}}{\partial u_{tz}}\Big) \tag{3.4}$$

$$+D_tD_z(W)\frac{\partial\mathcal{L}}{\partial u_{tzz}}+D_t(W)D_z\Big(\frac{\partial\mathcal{L}}{\partial u_{tzz}}\Big)+D_z(W)D_t\Big(\frac{\partial\mathcal{L}}{\partial u_{tzz}}\Big)+WD_tD_z\Big(\frac{\partial\mathcal{L}}{\partial u_{tzz}}\Big)\cdot$$

We have from the formula (3.2)

$$\begin{aligned}C^2=&W\Big[\frac{\partial\mathcal{L}}{\partial u_z}-D_t\Big(\frac{\partial\mathcal{L}}{\partial u_{zt}}\Big)-D_z\Big(\frac{\partial\mathcal{L}}{\partial u_{zz}}\Big)+D_tD_z\Big(\frac{\partial\mathcal{L}}{\partial u_{ztz}}\Big)+D_zD_t\frac{\partial\mathcal{L}}{\partial u_{zzt}}\Big)\Big]\\&+D_t(W)\frac{\partial\mathcal{L}}{\partial u_{zt}}+D_z(W)\frac{\partial\mathcal{L}}{\partial u_{zz}}-D_t(W)D_z\Big(\frac{\partial\mathcal{L}}{\partial u_{ztz}}\Big)-D_z(W)D_t\Big(\frac{\partial\mathcal{L}}{\partial u_{zzt}}\Big)\\&+D_tD_z(W)\frac{\partial\mathcal{L}}{\partial u_{ztz}}+D_zD_t(W)\frac{\partial\mathcal{L}}{\partial u_{zzt}}\cdot\end{aligned}$$

Using the fact that the Lagrangian is symmetrical with respect to the mixed derivatives we can simplify the formula for C^2:

$$C^2=W\Big[\frac{\partial\mathcal{L}}{\partial u_z}-D_t\Big(\frac{\partial\mathcal{L}}{\partial u_{tz}}\Big)-D_z\Big(\frac{\partial\mathcal{L}}{\partial u_{zz}}\Big)+2D_tD_z\Big(\frac{\partial\mathcal{L}}{\partial u_{tzz}}\Big)\Big]+D_t(W)\frac{\partial\mathcal{L}}{\partial u_{tz}}$$

$$+D_z(W)\frac{\partial\mathcal{L}}{\partial u_{zz}}-D_t(W)D_z\Big(\frac{\partial\mathcal{L}}{\partial u_{tzz}}\Big)-D_z(W)D_t\Big(\frac{\partial\mathcal{L}}{\partial u_{tzz}}\Big)+2D_tD_z(W)\frac{\partial\mathcal{L}}{\partial u_{tzz}}\cdot$$

Adding the expression (3.4) we obtain:

$$\begin{aligned}C^2=&W\Big[\frac{\partial\mathcal{L}}{\partial u_z}-2D_t\Big(\frac{\partial\mathcal{L}}{\partial u_{tz}}\Big)-D_z\Big(\frac{\partial\mathcal{L}}{\partial u_{zz}}\Big)+3D_tD_z\Big(\frac{\partial\mathcal{L}}{\partial u_{tzz}}\Big)\Big]\\&+D_z(W)\frac{\partial\mathcal{L}}{\partial u_{zz}}+3D_tD_z(W)\frac{\partial\mathcal{L}}{\partial u_{tzz}}\cdot\end{aligned} \tag{3.5}$$

Invoking (3.1) we get from (3.3) and (3.5) the final expressions for nonlocal conservation laws:

$$C^1=v\Big\{W\big[1+mu^{n-m-1}u_{zz}+m(n-m-1)u^{n-m-2}u_z^2\big]+(2m-n)u^{n-m-1}u_zD_z(W)\Big\}$$

$$+D_z(W)D_z(vu^{n-m}) \tag{3.6}$$

and

$$C^2 = W\big\{v\big[(2m-n)u^{n-m-1}u_{tz} + 2m(n-m-1)u^{n-m-2}u_t u_z + nu^{n-1}\big] \\ -(2m-n)D_t\big(vu^{n-m-1}u_z\big) - mD_z\big(vu^{n-m-1}u_t\big) - D_tD_z\big(vu^{n-m}\big)\big\} \\ + mvu^{n-m-1}u_tD_z(W) - vu^{n-m}D_tD_z(W). \tag{3.7}$$

4 Computation of conservation laws

4.1 Translation of time

It is obvious that Eq. (1.3) with arbitrary m and n is invariant under the translation of time t. The operator for the time translation, $X_1 = \frac{\partial}{\partial t}$, has $\xi^1 = 1,\ \xi^2 = 0,\ \eta = 0$, therefore $W = -u_t$. Hence the density C^1 has the following form:

$$C^1 = -v\big\{u_t\big[1 + mu^{n-m-1}u_{zz} + m(n-m-1)u^{n-m-2}u_z^2\big] \\ + (2m-n)u^{n-m-1}u_zu_{tz}\big\} - u_{tz}D_z(vu^{n-m}).$$

It follows from Eq. (1.3), $F = 0$, that

$$[u_t + (2m-n)u^{n-m-1}u_zu_{tz} + mu^{n-m-1}u_tu_{zz} + m(n-m-1)u^{n-m-2}u_tu_z^2] \\ = u^{n-m}u_{tzz} - nu^{n-1}u_z \tag{4.1}$$

which gives

$$C^1 = v[nu^{n-1}u_z - u^{n-m}u_{tzz}] - u_{tz}D_z(vu^{n-m}) = D_z[-vu^{n-m}u_{tz} + u^n] - u^nv_z.$$

Hence

$$C^1 = -u^nv_z.$$

We see that the nonlocal conservation law provided by the operator X_1 is nontrivial, but the corresponding local conservation law is trivial.

4.2 Translation of the space coordinate z

4.2.1 Nonlocal conservation law

Eq. (1.3) with arbitrary m and n is also invariant under the translation of the coordinate z with the operator $X_2 = \frac{\partial}{\partial z}$. In this case $\xi^1 = 0, \xi^2 = 1,\ \eta = 0$, therefore $W = -u_z$. From (3.6) the corresponding density C^1 has the form:

$$C^1 = -v\Big\{u_z\big[1 + mu^{n-m-1}u_{zz} + m(n-m-1)u^{n-m-2}u_z^2\big] \\ + (2m-n)u^{n-m-1}u_zu_{zz}\Big\} - u_{zz}D_z(vu^{n-m})$$

or

$$C^1 = -v\Big\{u_z + m(n-m-1)u^{n-m-2}u_z^3 + 2mu^{n-m-1}u_z u_{zz}\Big\} - v_z u^{n-m}u_{zz}$$
$$= [u + mu^{n-m-1}u_z^2 - u^{n-m}u_{zz}]v_z + D_z\big[-v(u + mu^{n-m-1}u_z^2)\big].$$

Hence we can choose

$$C^1 = [u + mu^{n-m-1}u_z^2 - u^{n-m}u_{zz}]v_z \tag{4.2}$$

as the density of a nonlocal conservation law corresponding to the translation of z. Adding to C^2 in (3.7) the corresponding to $D_z\big[-v(u + mu^{n-m-1}u_z^2)\big]$ term $D_t\big[-v(u + mu^{n-m-1}u_z^2)\big]$ we obtain:

$$C^2 = -u_z\big\{v\big[(2m-n)u^{n-m-1}u_{tz} + 2m(n-m-1)u^{n-m-2}u_t u_z + nu^{n-1}\big]$$
$$-(2m-n)D_t\big(vu^{n-m-1}u_z\big) - mD_z\big(vu^{n-m-1}u_t\big) - D_t D_z\big(vu^{n-m}\big)\big\}$$
$$-mvu^{n-m-1}u_t u_{zz} + vu^{n-m}u_{tzz} + D_t\big[-v(u + mu^{n-m-1}u_z^2)\big]\cdot$$

Separate calculation of each term gives the following:

$$-u_z\big\{v\big[(2m-n)u^{n-m-1}u_{tz} + 2m(n-m-1)u^{n-m-2}u_t u_z + nu^{n-1}\big]$$
$$-mvu^{n-m-1}u_t u_{zz} + vu^{n-m}u_{tzz} = -v\big[(2m-n)u^{n-m-1}u_z u_{tz}$$
$$+2m(n-m-1)u^{n-m-2}u_t u_z^2 + nu^{n-1}u_z + mu^{n-m-1}u_t u_{zz} - u^{n-m}u_{tzz}\big]$$
$$= -vF + v[u_t - m(n-m-1)u^{n-m-2}u_t u_z^2],$$

$$(2m-n)u_z D_t\big(vu^{n-m-1}u_z\big) = (2m-n)v_t u^{n-m-1}u_z^2$$
$$+(2m-n)v\big[(n-m-1)u^{n-m-2}u_t u_z^2 + u^{n-m-1}u_z u_{tz}\big],$$

$$mu_z D_z\big(vu^{n-m-1}u_t\big) = mv_z u^{n-m-1}u_t u_z + mv\big[(n-m-1)u^{n-m-2}u_t u_z^2$$
$$+u^{n-m-1}u_z u_{tz}\big],$$

$$u_z D_t D_z\big(vu^{n-m}\big) = u_z D_t\big(v_z u^{n-m} + (n-m)vu^{n-m-1}u_z\big)$$
$$= v_{tz}u^{n-m}u_z + (n-m)v_z u^{n-m-1}u_t u_z + (n-m)v_t u^{n-m-1}u_z^2$$
$$+(n-m)v\big[(n-m-1)u^{n-m-2}u_t u_z^2 + u^{n-m-1}u_z u_{tz}\big],$$

$$D_t\big[-v(u+mu^{n-m-1}u_z^2)\big]=-v_t(u+mu^{n-m-1}u_z^2)$$
$$-\,v\big[u_t+m(n-m-1)u^{n-m-2}u_tu_z^2+2mu^{n-m-1}u_zu_{tz}\big].$$

Hence

$$C^2=v_{tz}u^{n-m}u_z-v_tu+nv_zu^{n-m-1}u_tu_z-vF.$$

Excluding the trivial part vF we have finally:

$$C^2=v_{tz}u^{n-m}u_z-v_tu+nv_z^{n-m-1}u_tu_z. \tag{4.3}$$

Thus Eqs. (4.2) and (4.3) define a nonlocal conservation law corresponding to the translation of the coordinate z. Indeed, calculations give that

$$D_tC^1+D_zC^2=v_zF+u_zF^*$$

which equals to zero on solutions of Eqs. (1.3) and (1.4).

4.2.2 Local conservation laws

Eq. (1.3) is self-adjoint when $n=-m$. Substitution of $v=u$ in (4.2) yields

$$C^1=[u+mu^{-2m-1}u_z^2-u^{-2m}u_{zz}]u_z=D_z\Big[\frac{1}{2}(u^2-u^{-2m}u_z^2)\Big].$$

In the case of quasi-self-adjointness of the equation $F=0$, when $n+m\neq 1$ and $v=u^{1-n-m}$ the density has the following form:

$$\begin{aligned}C^1&=[u+mu^{n-m-1}u_z^2-u^{n-m}u_{zz}](1-n-m)u^{-n-m}u_z\\&=(n+m-1)[-u^{1-n-m}u_z-mu^{-2m-1}u_z^3+u^{-2m}u_zu_{zz}]\\&=D_z\Big\{(n+m-1)\Big[\frac{u^{2-n-m}}{n+m-2}+\frac{u^{-2m}u_z^2}{2}\Big]\Big\},\quad\text{if}\quad n+m\neq 2\end{aligned}$$

or

$$C^1=D_z\Big(-\ln u+\frac{u^{-2m}u_z^2}{2}\Big),\quad\text{if}\quad n+m=2.$$

In the second case of quasi-self-adjointness, when $n+m=1$ and $v=\ln u$, we obtain

$$C^1=[u+mu^{-2m}u_z^2-u^{1-2m}u_{zz}]u^{-1}u_z=D_z(u-u^{-2m}u_z^2).$$

Thus for arbitrary m and n only trivial local conservation laws correspond to the operator X_2.

4.3 Dilations

4.3.1 Nonlocal conservation law

When the parameter $n \neq 0$, Eq.(1.3) admits the following operator of dilation [5]:

$$X_3 = (2-n-m)t\frac{\partial}{\partial t} + (n-m)z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u}\cdot$$

For this operator

$$\xi^1 = (2-n-m)t, \quad \xi^2 = (n-m)z \quad \text{and} \quad \eta = 2u,$$

therefore

$$W = 2u - (2-n-m)tu_t - (n-m)zu_z.$$

From (3.6),

$$C^1 = v\Big\{W\big[1 + mu^{n-m-1}u_{zz} + m(n-m-1)u^{n-m-2}u_z^2\big]$$
$$+(2m-n)u^{n-m-1}u_zD_z(W)\Big\} + D_z(W)D_z(vu^{n-m}),$$

we have:

$$\begin{aligned}C^1 =& v\big\{[2u-(2-n-m)tu_t-(n-m)zu_z]\\ &\times\big[1+mu^{n-m-1}u_{zz}+m(n-m-1)u^{n-m-2}u_z^2\big]\\ &+(2m-n)u^{n-m-1}u_zD_z[2u-(2-n-m)tu_t-(n-m)zu_z]\big\}\\ &+\big[v_zu^{n-m}+(n-m)vu^{n-m-1}u_z)\big]D_z[2u-(2-n-m)tu_t-(n-m)zu_z].\end{aligned}$$

Let's start with the terms containing tu_t:

$$\begin{aligned}C_1^1 =& v\Big\{-(2-n-m)tu_t\big[1+mu^{n-m-1}u_{zz}+m(n-m-1)u^{n-m-2}u_z^2\big]\\ &+(2m-n)u^{n-m-1}u_zD_z[-(2-n-m)tu_t]\Big\}\\ &+\big[v_zu^{n-m}+(n-m)vu^{n-m-1}u_z\big]D_z[-(2-n-m)tu_t]\\ =& -(2-n-m)t\Big\{v\big[u_t+mu^{n-m-1}u_tu_{zz}+m(n-m-1)u^{n-m-2}u_tu_z^2\\ &+(2m-n)u^{n-m-1}u_zu_{tz}\big]+(n-m)vu^{n-m-1}u_zu_{tz}+v_zu^{n-m}u_{tz}\Big\}.\end{aligned}$$

Invoking (4.1) we obtain:

$$C_1^1 = (2-n-m)t\big\{v\big[nu^{n-1}u_z - u^{n-m}u_{tzz} - (n-m)u^{n-m-1}u_zu_{tz}\big] - v_zu^{n-m}u_{tz}\big\}$$

$$= -(2-n-m)tu^nv_z + D_z\big\{(2-n-m)tv\big(u^n - u^{n-m}u_{tz}\big)\big\}.$$

For the terms containing $2u-(n-m)zu_z$ we have:

$$\begin{aligned}
C_2^1 =& v\Big\{\big[2u-(n-m)zu_z\big]\big[1+mu^{n-m-1}u_{zz}+m(n-m-1)u^{n-m-2}u_z^2\big]\\
&+mu^{n-m-1}u_zD_z\big[2u-(n-m)zu_z\big]\Big\}+v_zu^{n-m}D_z\big[2u-(n-m)zu_z\big]\\
=&v\Big\{2\big[u+mu^{n-m}u_{zz}+m(n-m-1)u^{n-m-1}u_z^2\big]\\
&-(n-m)z\big[u_z+mu^{n-m-1}u_zu_{zz}+m(n-m-1)u^{n-m-2}u_z^3\big]\\
&+m(2-n+m)u^{n-m-1}u_z^2-m(n-m)zu^{n-m-1}u_zu_{zz}\Big\}\\
&+v_zu^{n-m}\big[(2-n+m)u_z-(n-m)zu_{zz}\big]\\
=&v\Big\{2\big[u+mu^{n-m}u_{zz}+m(n-m)u^{n-m-1}u_z^2\big]\\
&-(n-m)z\big[u_z+2mu^{n-m-1}u_zu_{zz}+m(n-m-1)u^{n-m-2}u_z^3\big]\\
&-m(n-m)u^{n-m-1}u_z^2\Big\}+v_zu^{n-m}\big[(2-n+m)u_z-(n-m)zu_{zz}\big]\\
=&2vu+2mv\big[u^{n-m}u_{zz}+(n-m)u^{n-m-1}u_z^2\big]-(n-m)zvu_z\\
&-m(n-m)v\big\{z\big[2u^{n-m-1}u_zu_{zz}+(n-m-1)u^{n-m-2}u_z^3\big]+u^{n-m-1}u_z^2\big\}\\
&+v_z\big[(2-n+m)u^{n-m}u_z-(n-m)zu^{n-m}u_{zz}\big]\\
=&2vu+D_z\big(2mvu^{n-m}u_z\big)-2mv_zu^{n-m}u_z-D_z\big[(n-m)zvu\big]+(n-m)vu\\
&+(n-m)zv_zu-D_z\big[m(n-m)zvu^{n-m-1}u_z^2\big]+m(n-m)zv_zu^{n-m-1}u_z^2\\
&+v_z\big[(2-n+m)u^{n-m}u_z-(n-m)zu^{n-m}u_{zz}\big].
\end{aligned}$$

whence

$$\begin{aligned}
C_2^1 =& v_z\big[(n-m)z(u+mu^{n-m-1}u_z^2-u^{n-m}u_{zz})+(2-n-m)u^{n-m}u_z\big]\\
&(2+n-m)vu+D_z\big\{v\big[2mu^{n-m}u_z-(n-m)z\big(u+mu^{n-m-1}u_z^2\big)\big]\big\}.
\end{aligned}$$

Combining all terms in C_1^1 and C_1^2 we obtain the density of a nonlocal conservation law:

$$C^1 = v_z\big[(n-m)z(u + mu^{n-m-1}u_z^2 - u^{n-m}u_{zz}) + (2-n-m)(u^{n-m}u_z - tu^n)\big]$$
$$+(2+n-m)vu. \tag{4.4}$$

The term corresponding to

$$D_z\{v[2mu^{n-m}u_z - (n-m)z(u + mu^{n-m-1}u_z^2) + (2-n-m)t(u^n - u^{n-m}u_{tz})]\}$$

we add to C^2.

4.3.2 Local conservation laws

Case 1. Eq. (1.3) is self-adjoint: $n = -m$.

Substitution of $v = u$ in (4.4) yields:

$$C^1 = u_z\big[-2mz(u + mu^{-2m-1}u_z^2 - u^{-2m}u_{zz}) + 2(u^{-2m}u_z - tu^{-m})\big] + (2-2m)u^2$$
$$= mz(-2uu_z - 2mu^{-2m-1}u_z^2 + 2u^{-2m}u_{zz}) + 2u^{-2m}u_z^2 - 2tu^{-m}u_z + (2-2m)u^2$$
$$= -D_z\big[mz(-u^2 + u^{-2m}u_z^2)\big] + mu^2 - mu^{-2m}u_z^2 + 2u^{-2m}u_z^2 - 2tu^{-m}u_z + (2-2m)u^2.$$

Since

$$-2tu^{-m}u_z = \begin{cases} D_z\Big(\frac{2t}{m-1}\,u^{1-m}\Big) & \text{if} \quad m \neq 1 \\ D_z(-2t\ln u) & \text{if} \quad m = 1 \end{cases}$$

the density has the form:

$$C^1 = (2-m)u^{-2m}u_z^2 + (2-m)u^2 = (2-m)(u^{-2m}u_z^2 + u^2).$$

Therefore if $m \neq 2$, $m \neq 1$ and $n + m = 0$

$$C^1 = u^{-2m}u_z^2 + u^2. \tag{4.5}$$

The case $m = 2$ and $n + m = 0$ gives a trivial conservation law. The density (4.5) corresponds up to a constant factor $1/2$ to the density of the conservation law discussed in [5], Case (B.1) (see Table 2).

For $m = 1$, $n = -1$ the density

$$C^1 = u^{-2}u_z^2 + u^2 \tag{4.6}$$

corresponds up to a constant factor $1/2$ to Case (B.2) in the same table.

Case 2. Eq. (1.3) is quasi-self-adjoint.

a) $n+m\neq 1$ and $v=u^{1-n-m}$.
According to (4.4) the density has the following form:

$$\begin{aligned} C^1 =&(1-n-m)u^{-n-m}u_z\big[(n-m)z(u+mu^{n-m-1}u_z^2-u^{n-m}u_{zz})\\ &+(2-n-m)(u^{n-m}u_z-tu^n)\big]+(2+n-m)u^{2-n-m}\\ =&(2+n-m)u^{2-n-m}+(n-m)(1-n-m)zu^{1-n-m}u_z\\ &+(n-m)(1-n-m)z(mu^{-2m-1}u_z^3-u^{-2m}u_zu_{zz})\\ &+(1-n-m)(2-n-m)u^{-2m}u_z^2-(1-n-m)(2-n-m)tu^{-m}u_z. \end{aligned}$$

After transforming the terms:

$$zu^{1-n-m}u_z=\begin{cases} D_z\Big(\frac{1}{2-n-m}\,zu^{2-n-m}\Big)-\frac{1}{2-n-m}\,u^{2-n-m} & \text{if}\quad n+m\neq 2\\ D_z(z\ln u)-\ln u & \text{if}\quad n+m=2; \end{cases}$$

$$\begin{aligned} &(n-m)(1-n-m)[z(mu^{-2m-1}u_z^3-u^{-2m}u_zu_{zz})]\\ &=-D_z\Big[\frac{1}{2}(n-m)(1-n-m)zu^{-2m}u_z^2\Big]+\frac{1}{2}(n-m)(1-n-m)u^{-2m}u_z^2\,; \end{aligned}$$

$$-(1-n-m)(2-n-m)tu^{-m}u_z=\begin{cases} -D_z\Big[\frac{(1-n-m)(2-n-m)}{1-m}\,tu^{1-m}\Big] & \text{if } m\neq 1\\ D_z[n(n-1)t\ln u] & \text{if } m=1, n\neq 0 \end{cases}$$

and choosing $n+m\neq 2$ we have:

$$\begin{aligned} C^1 =&(2+n-m)u^{2-n-m}-\frac{(n-m)(1-n-m)}{2-n-m}\,u^{2-n-m}\\ &+\frac{1}{2}(n-m)(1-n-m)u^{-2m}u_z^2+(1-n-m)(2-n-m)u^{-2m}u_z^2 \end{aligned}$$

or

$$C^1=(4-n-3m)\big[\frac{1}{2}(1-n-m)u^{-2m}u_z^2+\frac{1}{2-n-m}\,u^{2-n-m}\big].$$

Thus if $n+3m\neq 4$, $n+m\neq 1$, $n+m\neq 2$ and $m\neq 1$ we obtain the density of a nontrivial conservation law (see [5], Case (B.1) in Table 2):

$$C^1=\frac{1}{2}(1-n-m)u^{-2m}u_z^2+\frac{1}{2-n-m}\,u^{2-n-m}. \tag{4.7}$$

When $n+m=2$ and $m \neq 1$

$$(2+n-m)u^{2-n-m} + \frac{1}{2}(4-n-3m)(1-n-m)u^{-2m}u_z^2 = 2n - (1-m)u^{-2m}u_z^2$$

therefore (ignoring the constant $2n$) we have:

$$C^1 = -(1-m)u^{-2m}u_z^2 + 2(1-m)\ln u.$$

Since $m \neq 1$ we can choose

$$C^1 = -\frac{1}{2}\,u^{-2m}u_z^2 + \ln u \tag{4.8}$$

as the density of the conservation law in this case (it corresponds to Case (B.6) in Table 2).

For $n+m=2$ and $m=1$, we have $n=1$ and $C^1=2$, the conservation law is trivial.

b) $n+m=1$ and $v=\ln u$.

For the second case of quasi-self-adjointness of Eq. (1.3) we have from (4.4) the following density:

$$\begin{aligned}
C^1 =& u^{-1}u_z\big[(1-2m)z(u+mu^{-2m}u_z^2 - u^{1-2m}u_{zz}) + u^{1-2m}u_z - tu^{1-m}\big] \\
&+ (3-2m)u\ln u \\
=&(1-2m)zu_z + (1-2m)z(mu^{-2m-1}u_z^3 - u^{-2m}u_zu_{zz}) + u^{-2m}u_z^2 - tu^{-m}u_z \\
&+ (3-2m)u\ln u \\
=&(3-2m)u\ln u + D_z[(1-2m)zu] - (1-2m)u - D_z\big[\frac{1}{2}(1-2m)zu^{-2m}u_z^2\big] \\
&+ \frac{1}{2}(1-2m)u^{-2m}u_z^2 + u^{-2m}u_z^2 - tu^{-m}u_z
\end{aligned}$$

where

$$-tu^{-m}u_z = \begin{cases} D_z\big(\frac{1}{m-1}\,tu^{1-m}\big) & \text{if } m \neq 1 \\ -D_z(t\ln u) & \text{if } m=1,\ n=0. \end{cases}$$

Thus for $n+m=1$ and $m \neq 1$ the density has the form:

$$C^1 = \frac{1}{2}(3-2m)u^{-2m}u_z^2 + (3-2m)u\ln u - (1-2m)u. \tag{4.9}$$

Since a linear combination of two conservation laws is again a conservation law we can come to the result presented in Case (B.5) by combining the latter density with the density $\widetilde{C}^1 = u$ (see Case A in Table 2):

$$C^1 - 2\widetilde{C}^1 = \frac{1}{2}(3-2m)u^{-2m}u_z^2 + (3-2m)u\ln u - (3-2m)u$$

Hence for $m \neq \frac{3}{2}$

$$C^1 = \frac{1}{2}\,u^{-2m}u_z^2 + u\ln u - u$$

coincides with the density in Case (B.5).

Other conservation laws can be calculated in a similar way.

5 Conclusion

The local conservation laws presented in [5], Table 2, were obtained by Barcilon and Ritcher [6] and Harris [7] using the direct method of calculating coordinates of conserved vectors. The conservation densities presented in my article are obtained by applying the recent theorem on nonlocal conservation laws [1] to the infinitesimal symmetries of the magma equation (1.3). The obtained conserved vectors have the coordinates of the form

$$C^1 = C^1(u, u_z) \quad \text{and} \quad C^2 = A(u, u_z)u_t + B(u, u_z)u_{tz} + S(u, u_z),$$

which are presented in [5], Case B.

Bibliography

[1] IBRAGIMOV, N.H. A new conservation theorem, *J. Math. Anal. Appl.*, **333**, No. 1, 2007, 311–328.

[2] IBRAGIMOV, N.H. Quasi-self-adjoint differential equations, *Archives of ALGA*, **4**, 2007, 55–60.

[3] SCOTT, D. R. AND STEVENSON, D. J. Magma solutions,*Geophys. Res. Lett.* **11**, 1984, 1161–1164.

[4] ATHERTON, R. W. AND HOMSY, G. M. On the existance and formulation of variational principles for nonlinear differential equations, *Studies Apll. Math.*, **LIV**, No. 1, 1975, 31.

[5] MALULEKE, G. H. AND MASON, D. P. Derivation of conservation laws for a nonlinear wave equation modelling melt migration using Lie point symmetry generators, *Communications in Nonlinear Science and Numerical Simulation,* **12**, 4, 2007, 423–433.

[6] BARCILON, V. AND RITCHER F. M. Nonlinear waves in compacting media, *J. Fluid. Mech.*, **164**, 1986, 429–448.

[7] HARRIS, S. E. Conservation laws for a nonlinear wave equation, *Nonlinearity*, **9**, 1996, 187–208.

Archives of ALGA

Brief facts about the centre ALGA: Advances in Lie Group Analysis

ALGA at Blekinge Institute of Technology, Sweden, is an international research and educational centre aimed at producing new knowledge of Lie group analysis of differential equations and enhancing the understanding of the classical results and modern developments.

The main objectives of ALGA are:

- To make available to a wide audience the classical heritage in group analysis and to teach courses in Lie group analysis as well as new mathematical programs based on the philosophy of group analysis.
- To advance studies in modern group analysis, differential equations and nonlinear mathematical modelling and to implement a database containing all the latest information in this field.

For more information, contact the director of ALGA, Professor N.H. Ibragimov.
E-mail: nib@bth.se
Homepage: www.bth.se/alga
Address: ALGA, Blekinge Institute of Technology, S-371 79 Karlskrona, Sweden.

Aims and Scope of *Archives*

Aim: The aim of the Archives is to provide an international forum for classical and modern group analysis by means of rapid communication of new results, review articles and publications of historical heritage. The publications are related to the activities of ALGA.

Scope: The scope of Archives encompasses Lie group analysis with a focus on nonlinear differential equations and mathematical models in science and engineering.