On Nonlinear Parameter Estimation with Random Noise Signals

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NOMENCLATURE

H	Frequency Response Function
G_{xx}	Single sided, auto-spectrum of signal x
G_{xy}	Single sided, cross-spectrum of x with y
$\mathcal{F}(.)$	Fourier Transform
F , X	Force and displacement, frequency domain
f , x	Force and displacement, time domain
γ_m^2	Multiple Coherence function

ABSTRACT

In the field of nonlinear dynamics it is essential to have well tested and reliable tools for estimating the nonlinear parameters from measurement data. This paper presents an identification technique based on using random noise signals, as initially developed by Julius S. Bendat. With this method the nonlinearity is treated as a feedback forcing term acting on an underlying linear system. The parameter estimation is then performed in the frequency domain by using conventional MISO/MIMO techniques.

To apply this method successfully it is necessary to have some pre-information about the model structure and thus methods for nonlinear characterization and localization are studied. The paper also demonstrates the various ways the method can be formulated for multiple-degree-of-freedoms.

The implementation of the method is illustrated with simulated data as well as a practical application, where the method is used to create a dynamic model of a test-rig with a significant nonlinearity.

1. INTRODUCTION

The overall aim with parameter estimation is to find suitable parameters to a mathematical model, based on measurements of the inputs and outputs of a system. The mathematical model can, for instance, be based on a beforehand known model structure. This is known as parametric modeling, which will be studied in this paper. Furthermore, the systems studied contain significant nonlinearities that must be included when creating the model, hence nonlinear parameter estimation.

Systems used in engineering can generally be separated into two distinct groups; linear or nonlinear. A system, *H*, which does not satisfy the following properties, is defined as nonlinear:

$$H\{c \cdot x\} = c \cdot H\{x\} \tag{1.1}$$

$$H\{x_1 + x_2\} = H\{x_1\} + H\{x_2\} \tag{1.2}$$

A useful tool for detecting nonlinearities is the frequency response function. For a significant nonlinearity the amplitude and/or the resonance frequencies will be dependent on the excitation amplitude. Thus, the FRF's gives a linear approximation at the specific excitation amplitude and fails to fully describe the dynamics.

This work will study a method where random noise signals are used and the parameter estimation is done in the frequency domain. The methods are described in Chapter 2 and Chapter 3. This will be followed by a brief analysis of the errors introduced when doing the parameter estimation. Finally an experimental test is performed, in Chapter 5, where a real nonlinear structure is studied.

2. USING RANDOM NOISE SIGNALS FOR NONLINEAR SYSTEMS

One of the most promising identification techniques for nonlinear parameter estimation was initially developed by Bendat [1]. The method estimates the parameters in the frequency domain with conventional Multiple-Input-Multiple/Single-Output (MIMO/MISO) procedures. The latter is often used in linear theory and is demonstrated below.

For a general linear system, with N inputs and M outputs, and noise on the measured outputs, as shown in Figure 2.1, a least-square solution that minimize the noise on the outputs (*H1*-estimator) can be determined at each frequency as

$$\mathbf{H}_1 = \mathbf{G}_{YX} \cdot \mathbf{G}_{XX}^{-1} \tag{2.1}$$

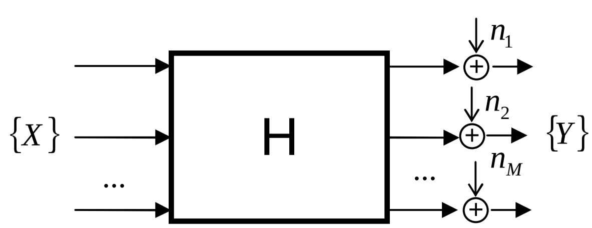

Figure 2.1: A Multiple-Input-Multiple-Output Model (MIMO) with noise on the outputs.

In Equation (2.1), $\mathbf{G}_{YX}$ is the cross-spectral density matrix with size $(M \times N)$, $\mathbf{G}_{XX}$ is the auto-spectral density matrix with size $(N \times N)$ and $\mathbf{H}$ is the estimated transfer matrix with size $(M \times N)$. Furthermore, a multiple coherence function can be calculated for each output as

$$\gamma_m^2 = \frac{\mathbf{G}_{YX} \cdot \mathbf{G}_{XX}^{-1} \cdot \mathbf{G}_{YX}^H}{\mathbf{G}_{YY}} \tag{2.2}$$

The size of $\mathbf{G}_{YX}$ will be $(1 \times N)$ and $\mathbf{G}_{YY}$ will be a scalar since only one output is selected. Thus, M number of multiple coherence functions can be calculated for the system shown in Figure 2.1. A multiple coherence equal to one for all frequencies indicates that the measured response can be explained totally by the measured inputs, without the effect of extraneous noise.

The MIMO/MISO technique can also be suitable to model nonlinear mechanical systems. In many cases, a nonlinear problem can be viewed as several nonlinear feedback forces acting on an underlying linear system. Thus, artificial inputs are created and, together with the measured quantities, used as input to the MIMO/MISO model when doing the analysis. This can best be illustrated with a simple example.

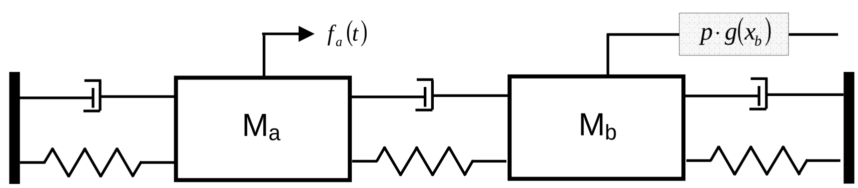

Figure 2.2: A nonlinear system with two degrees of freedom and a nonlinear element connected to the second dof. The nonlinearity is described with a nonlinear operator *g(.)* with a coefficient *p*.

The system shown in Figure 2.2 can be modeled in frequency domain as

$$\begin{pmatrix} H_{aa} & H_{ab} \\ H_{ba} & H_{bb} \end{pmatrix} \begin{pmatrix} F_a \\ -P \cdot \mathcal{F}(g(x_b)) \end{pmatrix} = \begin{pmatrix} X_a \\ X_b \end{pmatrix} \tag{2.3}$$

Alternatively, equation (2.3) can be rewritten, for each row, as

$$X_a \cdot H_{aa}^{-1} + \frac{H_{ab}}{H_{aa}} \cdot P \cdot \mathcal{F}(g(x_b)) = F_a \tag{2.4}$$

$$X_b \cdot H_{ba}^{-1} + \frac{H_{bb}}{H_{ba}} \cdot P \cdot \mathcal{F}(g(x_b)) = F_a \tag{2.5}$$

There are several possible ways to formulate this problem with MIMO/MISO. With the original *Reverse-Path technique* (RP) [1], Equation (2.4)-(2.5) can be used and the analysis is accomplished as illustrated in Figure 2.3. With *Nonlinear Identification Through Feedback Of Outputs* (NIFO) [2], Equation (2.3) is used directly and the linear system and the nonlinear coefficients are estimated as shown in Figure 2.4.

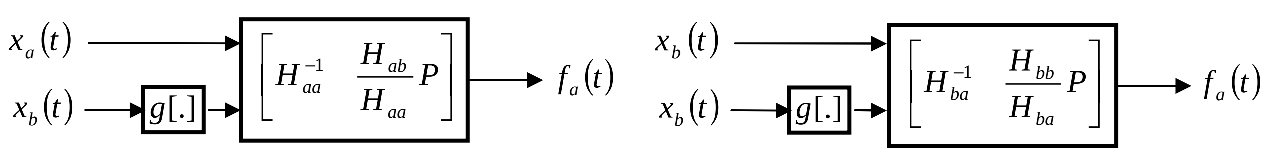

Figure 2.3: *Reverse-Path* (RP): Two MISO analyses are performed and using the reciprocity of the underlying linear system H_{aa}^{-1}, H_{ba}^{-1} and the nonlinear coefficient *P* can be identified. Here, *g(.)* is a nonlinear operator.

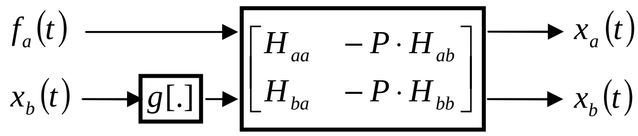

Figure 2.4: *Nonlinear Identification Through Feedback of Outputs* (NIFO): A single MIMO analysis is performed and using the reciprocity of the underlying linear system H_{aa}, H_{ba} and the nonlinear coefficient *P* can be identified. As in Figure 2.3, *g(.)* is a nonlinear operator.

In the above figures, all inputs and outputs quantities are given in the time domain, to emphasize that the input signal $x_b(t)$ must pass through the nonlinear function, for instance $g(x) = x^3$, before the spectral densities can be calculated. The transfer functions are then given by Equation (2.1) and, in both cases, two multiple coherence functions can be calculated with Equation (2.2).

Notice that the transfer function *P(f)* can be both frequency dependent and complex. However, in this example *P(f)* will be, after a sufficient number of averages, real-valued and constant throughout the frequency range. It is also possible to use several nonlinear inputs and obtain a polynomial fit to the true nonlinearity, as may be the case with a gap- or clearance-nonlinearity.

As shown in the above example, the NIFO-technique is more compact and easier to work with - especially for larger systems with several nonlinear elements - since MIMO is used as shown in [2,3]. With the reverse-path technique the system must always be rewritten with force as output as in Equation (2.4)-(2.5).

In an ideal case, without any noise, both methods will give the same estimates since they are mathematically identical. However, in the presence of bias errors and - for a real measurement - external noise, the estimates are not identical as will be discussed in chapter 4.

3. CLASSIFY NONLINEARITIES AND FINDING NONLINEAR NODES

With the method studied in previous chapter the type of nonlinearity and its location in the system must be known for the analysis to work which may seem like a major drawback. However, as will be shown in this chapter, the MIMO/MISO analysis can actually be a good tool to determine the model structure.

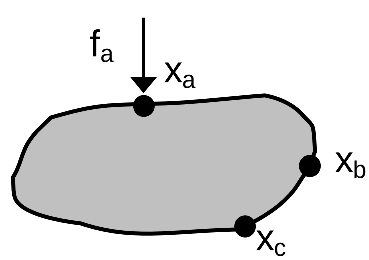

Figure 3.1: A blackbox system used in simulations with three degree of freedoms. A nonlinear element is connected either to a single node, or is dependent on the relative motion between two nodes.

A simulated blackbox system with three degrees-of-freedom is studied, as shown in Figure 3.1. The system does not fulfill the properties given by Equation (1.1)-(1.2) since a nonlinear element is connected somewhere. By studying the FRF data it was possible to see the presence of a hardening spring nonlinearity.

Random data was simulated at all three dofs and, for the analysis, a reverse-path model was created for the system as illustrated in Figure 3.2. Here, three polynoms is used to describe the nonlinearity but the input signal *q(t)* is unknown. The nonlinear element may be connected to ground from any dof or may depend on the relative motion between two dofs.

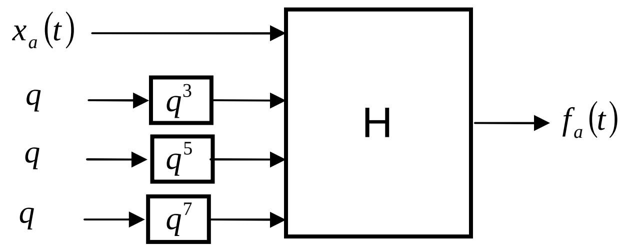

Figure 3.2: A reverse-path model created for the unknown system shown in Figure 3.1. The input signal *q(t)* is unknown and depends on the nonlinearity.

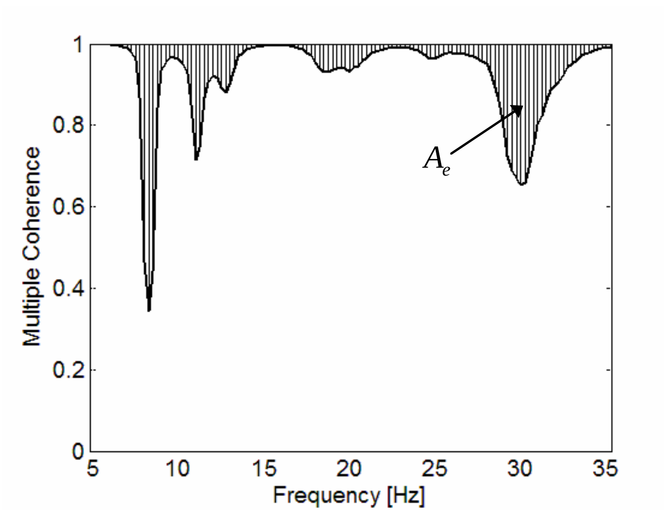

Figure 3.3: Multiple Coherence Function. The area shown in the figure was calculated for each combination of inputs and an error function was defined as 1/ A_e.

To solve this problem all possible combinations of inputs was tested and the multiple coherence function was used to evaluate each combination. If the model is correct the multiple coherence function should be close to one for all frequency. The area, A_e, shown in Figure 3.3, was therefore calculated for each combination of inputs. The error function is then defined as 1/ A_e, i.e the highest value for the overall best multiple coherence.

As shown in Figure 3.4, this method correctly detects that the nonlinear element is connected between x_b and x_c. With the nonlinear nodes identified the test was repeated but with different nonlinear functions as shown in Figure 3.5. Nonlinear functions that depend on the velocity, as well as the unphysical x^2, were included for comparison. The true nonlinearity in the black-box system was a cubic function that depends on the relative motion between x_b and x_c. As illustrated with this example, random data can in many cases be used to classify nonlinearities and finding nonlinear nodes.

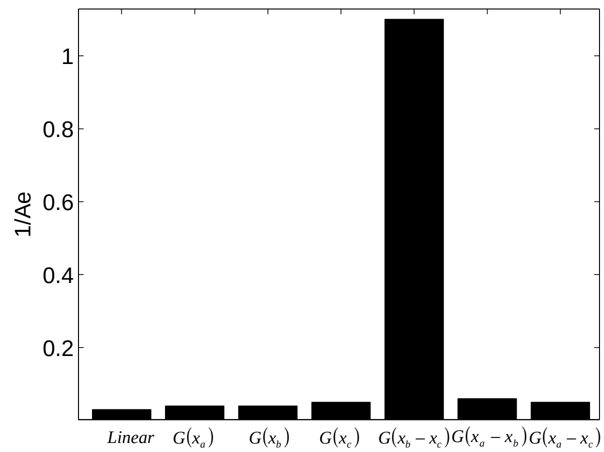

Figure 3.4: The method correctly detects that detects that the nonlinear element is connected between x_b and x_c. The true nonlinearity in the black-box system was a cubic function that depends on the relative motion between x_b and x_c.

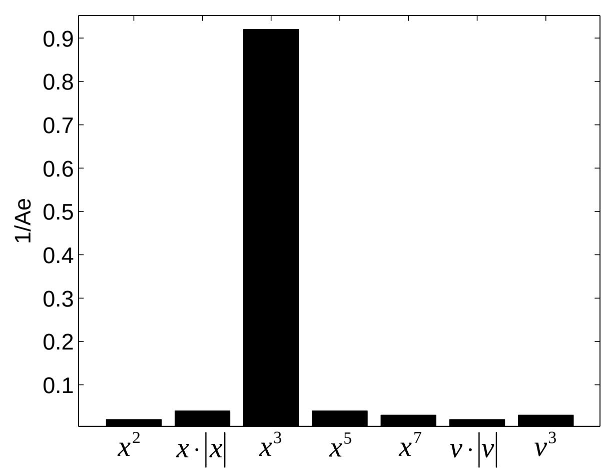

Figure 3.5: The presence of a cubic stiffness could clearly be seen. Nonlinear functions that depend on the velocity, as well as the unphysical x^2, were included for comparison.

4. ERRORS IN THE ANALYSIS

When using random noise signals the estimated spectral densities will contain *random* and *bias* (systematic) errors. The effect from random errors can be controlled with a sufficient number of averages and normally, overlap processing is applied for more effective use of data. The bias error, on the other hand, does not necessarily decrease with averaging since this error comes from consistently estimating an incorrect spectrum, for instance due to aliasing or leakage.

The time-leakage effect can give a considerable random error. This error occurs when the signal is truncated and an output signal is measured which is not totally correlated with the input signal. In the beginning of the output signal, for a certain time block, there is data which depends on the input signal from the previous time block. This effect will be even more significant for lightly damped structure. The leakage error can be reduced by increasing the block length. The bias error is a function of the truncation and the window used and may also be reduced by increasing the block length.

When using MIMO/MISO to identify nonlinear systems it is essential to use a sufficient amount of data due to a relatively high sensitivity to these errors, as can clearly be seen in simulations.

Bias errors can also be introduced by contaminating noise in the measurement. The *H1-estimator*, Equation (2.1), assumes that the noise on the input channels in the MIMO/MISO model is small and can be neglected, while the *H2-estimator*, Equation (4.1), neglects the noise on the output channels.

$$\mathbf{H}_2 = \mathbf{G}_{YY} \cdot \mathbf{G}_{XY}^{-1} \tag{4.1}$$

However, the *H2-algorithm* is less useful for multiple-input models since it is necessary to calculate an inverse to a non-square matrix if the number of inputs and outputs are not equal. For this reason, the *H1-estimator* must be used instead. The effect this will have on the parameter estimation can be demonstrated with a simulated example.

A duffing system is used with the following parameters: $M = 150\ kg$, $C = 800\ N/ms$, $K = 4e6\ N/m$ and a cubic term $P = 2e12\ N/m^3$. The excitation force was set to an amplitude were the resonance would increase approximately 15%-20% relative to the underlying linear system.

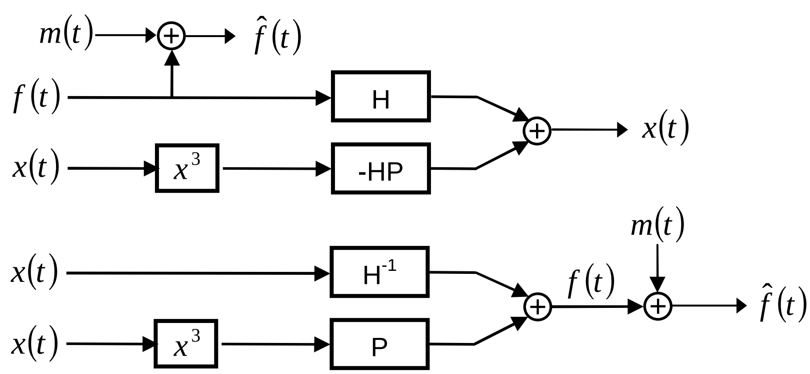

Figure 4.1: Analysis of a duffing system with a NIFO model (above) and the reverse-path technique (below). A small amount of noise is added to the force channel as shown in the figure.

Two possible MISO-models (NIFO and Reverse-Path) are shown in Figure 4.1. When doing the analysis a small amount of noise was added to the force channel as shown in the figure. The noise was designed with an rms-value equal to approximately 5%-10% of the rms-value of the force signal.

The simulation was done with a sampling frequency of 512 Hz and enough data was simulated so that the analysis could be done with 2^{16} in blocksize (fft) and 200 averages with 50% overlap.

As shown in Figure 4.2, the NIFO formulation gives relatively large bias error while the reverse-path formulation performs better since the *H1-estimator* minimizes the effect from noise at the output. This is an undesirable effect since in reality the dominating noise may be on the force signal due to force-dropouts and a very weak force signal at the resonance.

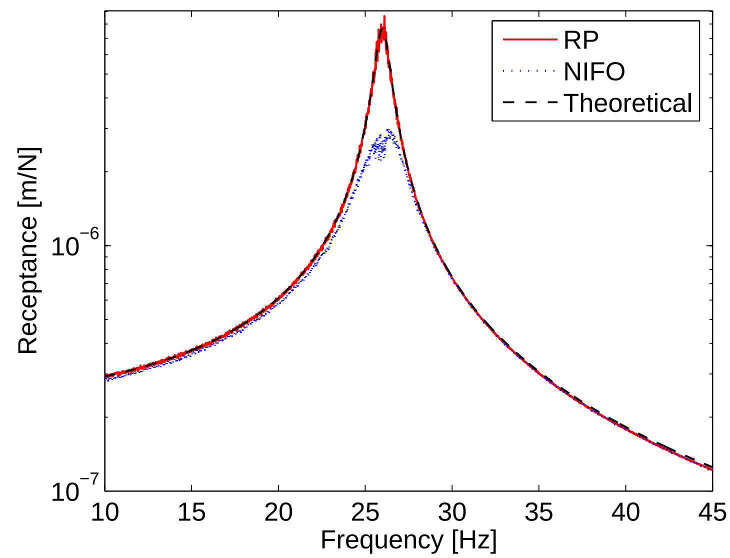

Figure 4.2: Estimated linear system - comparison between the two models shown in Figure 4.1. After averaging a large bias error is left with NIFO while the reverse-path formulation performs better since the *H1-estimator* minimizes the effect from noise at the output.

5. EXPERIMENTAL TEST

In this chapter the methods described previously is used to model a real structure with a small but significant nonlinearity. The test structure is described in section 5.1 and the experimental setup is explained in Section 5.2 followed by results from the nonlinear analysis in Section 5.3.

5.1 THE NONLINEAR TESTRIG

The structure under test consists of a clamped beam supported at point *A* with two clamped thin beams as shown in Figure 5.1. The design comes from a similar test-rig that was developed by J. Ferreira [4] and has since then been used to verify various identification methods [4, 5].

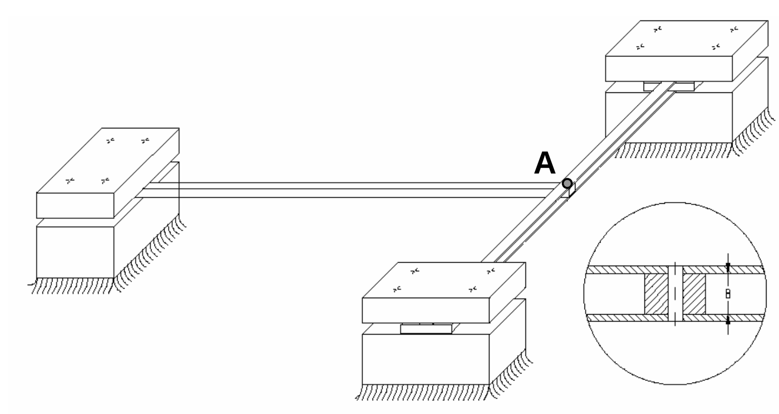

Figure 5.1: The structure used in the experiment consists of a clamped beam with two fixed transversal beams connected at point B. The clamped beam has a cross section of (15x8) mm with a free length of 420 mm and the two transversal beams has a cross section of (13x1.4) mm and free length 242 mm.

The properties of the beam are interesting since it can be described as a linear system (the clamped beam) with a local nonlinearity at the free end. Due to the thin geometry of the transversal beams the linear assumption is no longer valid. As illustrated in Figure 5.2, the geometry is changed as the beams deflect and the length along the center line of the beams (L') becomes longer than the original length L. This will introduce axial forces (N) which will increase the stiffness of the beam [6]. However, the material is still elastic and when the force is removed, the system will return to its initial position without any change in material properties. Hence, this effect is known as *geometric nonlinearity* since the nonlinearity is only related to changes in geometry.

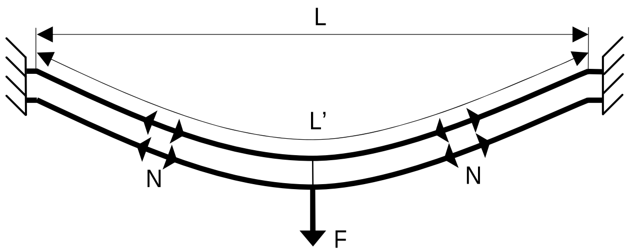

Figure 5.2: The transversal beams will behave nonlinearly at higher displacements. As the force is increased the beams becomes longer than the original length L, and an axial force is introduced along the beams. The stiffness will therefore increase. This effect is known as geometric nonlinearity.

Using ABAQUS® [7], a nonlinear static model could be created. ABAQUS® uses an iterative scheme where the force is applied incrementally and a Newton-Raphson solver is used to balance the nonlinear equations. Using this solver, the theoretical nonlinear force-displacement relationship at point *A* could be calculated as

$$F(x) = 1.76 \cdot 10^4 \cdot x + 2.97 \cdot 10^9 \cdot x^3 \quad (5.1)$$

5.2 EXPERIMENTAL SETUP

The complete setup for the experimental test is shown in Figure 5.3. The structure is firmly fixed to a rigid table and a rigid steel module is used to fix the shaker. The free end of the cantilever beam was selected as a response point since the nonlinear element is connected to this point. With the identification method described previously it is necessary to measure the response at the nonlinear degree of freedoms. However, the reference point (force) can be selected at any point along the cantilever. For practical reasons the shaker is fixed to the same point and thus the linear point-receptance and the nonlinear coefficients will be estimated.

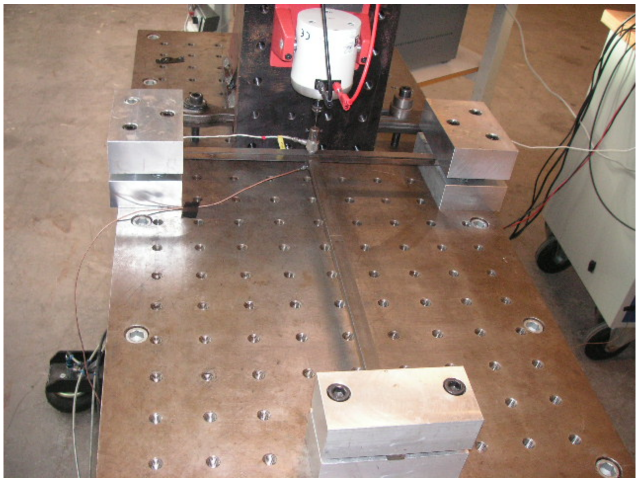

Figure 5.3: The complete setup for the experimental test. The structure is fixed to a rigid table and a shaker is used to excite the structure.

Figure 5.4: A detail view of the experimental setup. A force transducer and an accelerometer are used at the selected measurement point.

A force transducer was mounted on the top of the beam and the accelerometer was mounted under the bottom beam as shown in Figure 5.4. The force signals were created in MATLAB [8] and Signal Calc Mobilyzer [9] was used to acquire the raw time data. All data analysis was done in MATLAB.

Initially, no evident nonlinear phenomena could be seen in the measurements due to force dropouts which only increased with higher excitation amplitude and thus avoiding the high displacement at the resonance. It was therefore necessary to create voltage signals which added more power at the resonances and carefully make sure that the force spectrum obtained was constant over the desired frequency range.

5.3 ANALYSIS OF EXPERIMENTAL DATA

The structure behaves linearly at low excitations and it was therefore necessary to a find level of the excitation where the nonlinear effects are clear. A force amplitude which made the resonance move approximately 15% was selected for nonlinear analysis.

In the following two figures, two nonlinear analyses are compared with the raw FRF. The latter is the FRF obtained with conventional linear theory (SISO). Thus, the raw FRF is calculated without any compensation for nonlinear effects.

The nonlinear effects can clearly be seen in the raw measured FRF. A disturbance occurs at approximately $2 \cdot f_0$ and $3 \cdot f_0$. This disturbance can also be seen in the coherence function. A part of the response that is obtained at the resonance frequency will, due to the nonlinearity, create higher frequency response which the linear model fails to describe.

Two reverse-path models were created. First, a cubic function was used to describe the nonlinearity and the result can be studied in Figure 5.5. In the second model, shown in Figure 5.6, a cubic- and a square-polynom are used to describe the nonlinearity. It could be seen that the true nonlinearity was a non-symmetric function and thus a cubic function was not enough to describe this behavior.

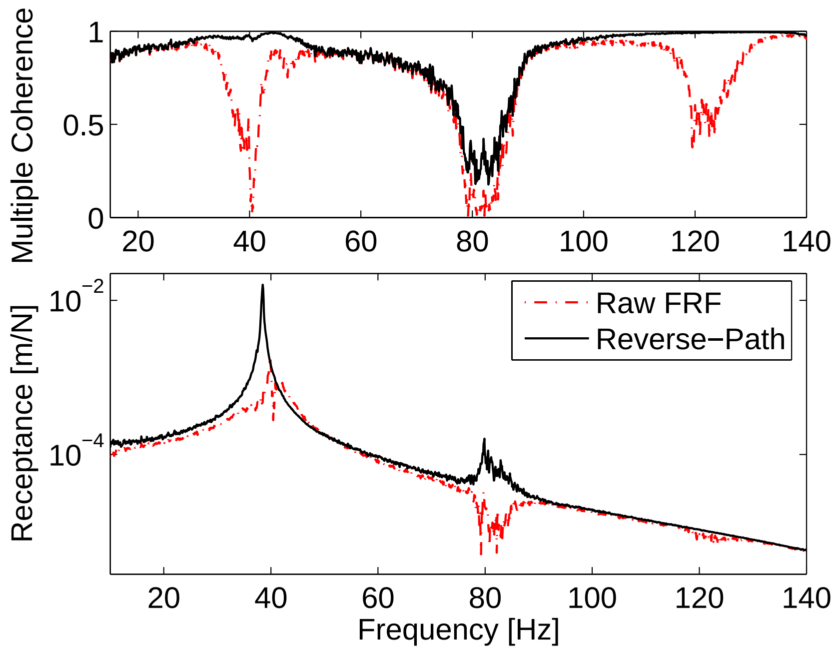

Figure 5.5: The Raw FRF (linear theory) compared with the linear system estimated with reverse-path technique. A cubic function is used to describe the nonlinearity. As shown, the FRF and the coherence are improved at the resonance and at three times the resonance frequency.

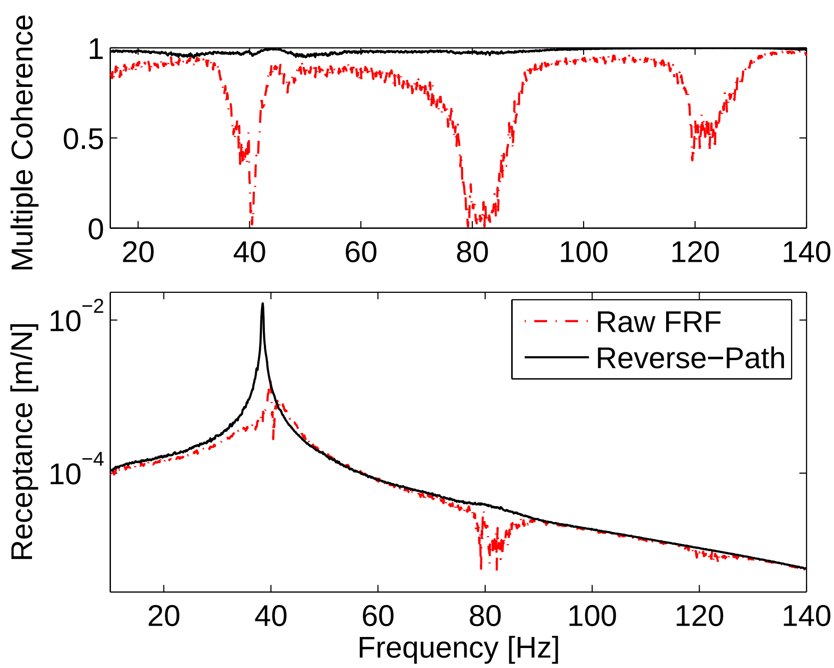

Figure 5.6: The Raw FRF (linear theory) compared with the linear system estimated with reverse-path technique. A cubic- and a square- function are used to describe the nonlinearity. As shown, the FRF and the coherence are considerably improved over the frequency range.

Figure 5.7 shows the estimated linear system with different methods. The true linear FRF was measured at very low excitation where all nonlinearities are unaffected. As shown in the figure, the estimated linear system with NIFO techniques gives larger amplitude errors. One possible explanation for this was given in chapter 4; if the force signal is contaminated with noise, the reverse-path technique will perform better.

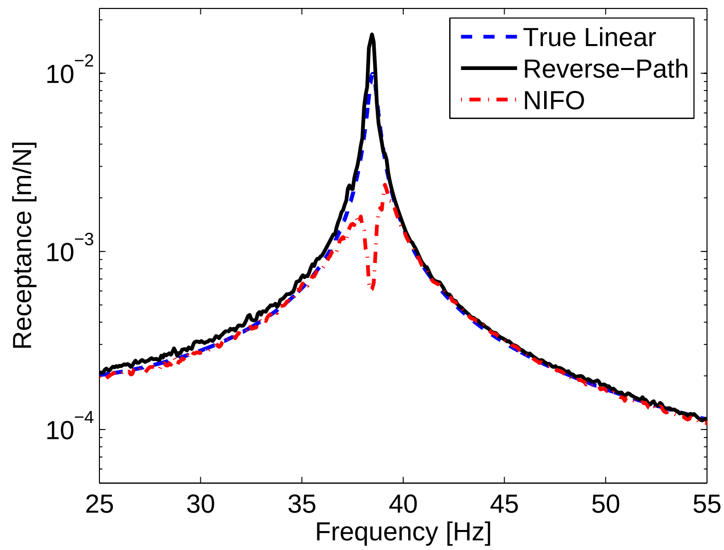

Figure 5.7: The estimated linear system with different methods. The true linear FRF, used as reference, was measured at very low excitation. As shown, the estimated linear system with NIFO techniques gives larger amplitude errors.

The real parts of the nonlinear coefficients are shown in Figure 5.8. By definition, for a zero-memory nonlinear system, the imaginary parts of the nonlinear coefficients should be relatively small after a sufficient number of averages. For this experiment, the imaginary part was approximately ten times smaller than the real part after averaging. This may indicate a small amount of nonlinear damping which was not considered in the modeling.

The estimated cubic coefficient was $2.02 \cdot 10^9$ N/m^3, which was considered to be reasonable in comparison with theoretical value from section 5.1. Due to the non-symmetric force, a square coefficient was identified as $4 \cdot 10^6$ N/m^2. When studying the test rig it could be seen that the slender beams were not perfectly aligned with the cantilever beam, and thus creating a non-symmetric force. When this effect was included in the nonlinear static model, a better correlation between the experimental results and the simulated results could be obtained.

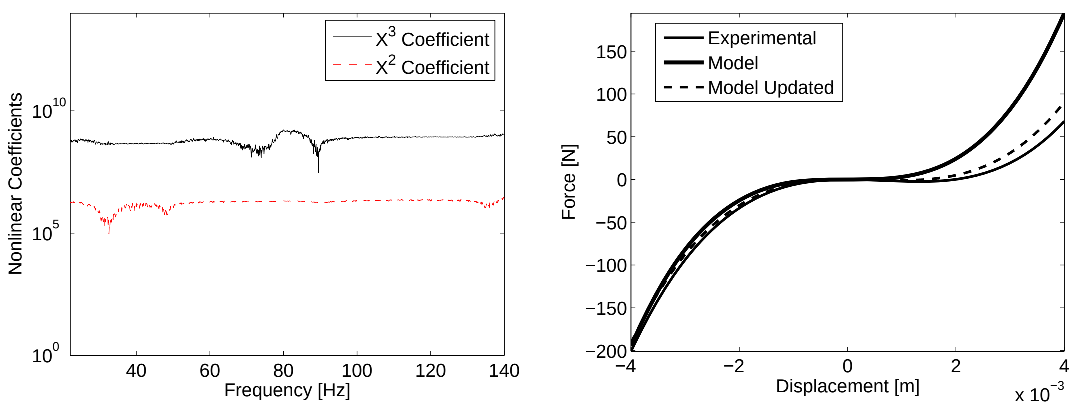

Figure 5.8: The nonlinear coefficients (left) and the estimated nonlinearity compared with the model (right). The nonlinear static model could be updated with the experimental parameters.

6. CONCLUSION

Nonlinear parameter estimation can be performed with random noise signals by using conventional MIMO/MISO techniques. The methods described in this work treat the nonlinearity as a feedback forcing term acting on an underlying linear system. Thus, the true linear system and the nonlinear coefficients can be estimated by measuring the input and output quantities.

The approach that is used in this work requires some pre-information about the type of nonlinearity present, and its location in the system. If this information cannot be obtained before the analysis it may be possible to try a set of different combinations and then utilize the multiple coherence function as shown in Chapter 3.

When using MIMO/MISO there are several possible ways to formulate a nonlinear problem. Two methods have been studied in simulations and experiments - referred to as NIFO and Reverse-Path. As shown in this work, it is essential to create nonlinear models which best compensate for the errors introduced with contaminating noise.

An experimental test was done on a structure with a significant nonlinearity and the results, presented in section 5.3, shows that the underlying linear system and the coefficients describing the nonlinearity could be identified. When comparing the results with linear theory a clear improvement could be seen: The estimated linear system is free from disturbance and with a correct resonance frequency. Thus, a simple but very useful application of the methods may be to make a better estimate of the linear system, for instance when comparison are made to a numerical linear model.

7. ACKNOWLEDGE

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