Commensurable and Rational Triangles

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Abstract

One may ask which property the equilateral, the right isosceles, the half equilateral, and the two golden triangles, with angles $\frac{\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}$ and $\frac{\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{5}$, have in common. One answer is that their angles are commensurable with each other – such triangles are commensurable. We investigate properties of this class of triangles, which is a countable subset of the entire class of triangles – we do not distinguish between similar triangles. It can naturally be endowed with a family structure by integer triples. The equilateral is the only member of the first generation, and the other triangles mentioned above populate the first generations. A formula for the number of non-similar triangles that can be formed by triples of corners in a regular n-polygon is calculated, which gives the number of commensurable triangles at each generation.

Three "metatriangles" are described – so called because each possible triangle is represented as a point in each of them. The set of right triangles form a height in one of the metatriangles. The eye is the point of a metatriangle in the same metatriangle.

In the second part of this report, triangles are studied by side length. A rational triangle is a triangle where all sides and all heights are rational numbers. We show that the right rational triangles are the Pythagorean triangles, and each non-right rational triangle consists of two Pythagorean triangles. Almost all triangles are irrational.

It turns out that no Pythagorean triangle is commensurable. We prove that the only triangle with commensurable angles and also commensurable sides is the equilateral triangle.

1 Introduction

In this paper triangles are considered modulo similarity, i.e. we do not distinguish between two triangles that are similar. Also, we do not consider degenerate triangles – we suppose that all angles are strictly positive.

Triangles have two defining properties: angles and sides. This paper considers triangles from these two perspectives, and consists of three sections: *Triangles by angles*, *Triangles by sides*, and *Angles and sides*. Two subsets of trianges are defined and discussed. By the angle perspective we define *commensurable triangles* as triangles where the angles are commensurable with each other, i.e. integer multiples of a common quantity. In the side perspective we define *rational triangles* as triangles where all sides and also all heights are commensurable, so we can use integers for both side and height lengths. These two subsets contain most of the well known triangles, for example the equilateral, the right isosceles and the Pythagorean triangles. In both classes each triangle correspond to a certain triplet of integers, providing an interplay with number theory. Both sets are enumerable, in constrast to the entire set of triangles. In section *Angles and sides* we investigate the intersection of the two sets. One may ask the question if there exist triangles that are both commensurable and rational, and furthermore if there are any interesting or useful triangles that are neither commensurable nor rational.

It turns out that the set of commensurable triangles naturally define a tree structure as a family of triangles with distinct generations. A formula for the number of distinct (non-similar) triangles that can be constructed by three corners in a regular n-polygon is given and proved. By this formula the number of triangles at each generation of commensurable triangles can be deduced, which in turn reveals further surprising number theoretic properties.

Furthermore, geometric sets are described where each triangle, not only commensurable or rational ones, are represented uniquely as a point. By plotting all possible pairs of the two smallest angles against each other we obtain a triangle of triangles, which is one of the three *metatriangles*. By the side wiewpoint we fix the longest side and simiarly obtain a circle sector with area $\frac{\pi}{6} - \frac{\sqrt{3}}{8}$ $(= 0.307\,09)$, called the *triangle sector,* in which each point represents a unique triangle. In both these geometric sets the subset of right triangles appear as a certain line, and the isosceles triangles as two distinct boundary curves. The equilateral triangle is situated at the boundary – in the intersection of the two isosceles boundary curves.

We show that almost all triangles are irrational – the set of rational triangles as a set of points in the triangle sector is a zero set. This follows from the fact that each rational triangle consists of one or two Pythagorean triangles, so the set is enumerable. For example, the triangles with sides $5-5-6$ and $5-5-8$ are non-right rational triangles which both consist of two Pythagorean triangles with sides $3-4-5$.

The construction of commensurable triangles appear generally to contain more famous triangles and to be more far reaching than the construction of rational triangles, somewhat supporting the name "triangle" over the possible alternative "triside" for this extremely basic and well known geometrical figure. The constructions generates the following natural problem: Are there any triangles that are both commensurable and rational? The question is anwered negatively in the last section. Here we prove that the equilateral triangle is the only triangle where the angles are commensurable and the sides are also commensurable. These proofs are due to Anders Tengstrand, University of Växjö, Sweden, who has also kindly contributed with many other remarks and improvements in this paper. The proofs are done with elementary tools, but can be formulated shorter in terms of classic algebra using cyclotomic polynomials.

2 Triangles by angles

Apart from the Pythagorean triangles, which appear in the next section, five triangles may be regarded as particularly famous: the equilateral, the right isosceles, the half equilateral and the two golden triangles. The two golden triangles occur in a pentagram, and have sides $1, \varphi, \varphi$ and $1, 1, \varphi$, where $\varphi = \frac{\sqrt{5}+1}{2}$ is sometimes called the golden ratio. The angles of these triangles are $\frac{\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}$ and $\frac{\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{5}$, respectively, so there is one acute and one obtuse golden triangle. Which property do these five triangles have in common?

2.1 The family of commensurable triangles

The following answer is given and investigated in this paper: the angles of each of these five triangles are multiples of the smallest angle. In this spirit we define a commensurable triangle as a triangle where the three angles are integer multiples of a common quantity – not necessarily the smallest angle, but the so called root angle. In formula:

Definition 1 *A triangle with angles A, B and C is **commensurable** if there is a real number $q > 0$ and positive integers n, m, k so that $A = nq$, $B = mq$ and $C = kq$. The largest possible q is called the **root angle** of the commensurable triangle.*

One may define $q = 0$ for a triangle that is not commensurable.

The set of commensurable triangles is certainly a proper subclass of the set of triangles. An example of a non-commensurable triangle is the triangle with angles 1, 1, $\pi - 2$. The set of commensurable triangles is dense in the set of triangles in the following sense:

Theorem 2 *For any triangle T with angles A, B and C and for any $\varepsilon > 0$, there is a commensurable triangle T' with angles A', B' and C' so that $|\max(|A - A'|, |B - B'|, |C - C'|)| < \varepsilon$.*

Proof. For any triangle T with angles A, B and C we denote $a = A/\pi$ and $b = B/\pi$. Now take a' and b' with finite decimal expansions and so that $|A - a'\pi| < \varepsilon/2$ and $|B - b'\pi| < \varepsilon/2$. Define $A' = a'\pi$ and $B' = b'\pi$ and take $C' = \pi - A' - B'$. Then the triangle T' with angles A', B' and C' is commensurable. Furthermore, $|C - C'| \leq |A - A'| + |B - B'| < \varepsilon$, so the theorem follows. ∎

2.1.1 Commensurable triangles and integer triples

It is natural to represent a commensurable triangle by an integer triple (n, m, k), where the three integers denote the size relationship of the three angles. Thus, a triple (n, m, k) corresponds to a commensurable triangle with angles

$$\frac{n\pi}{n+m+k}, \frac{m\pi}{n+m+k}, \frac{k\pi}{n+m+k}.$$

This means that the root angle is $\frac{\pi}{n+m+k}$ if n, m, k has no common factor, and in general it is $\frac{\pi LCD(n,m,k)}{n+m+k}$, where $LCD(n, m, k)$ is the largest common divisor of n, m and k. The most famous five triangles correspond to the following integer triples: $(1, 1, 1)$ (equilateral), $(1, 1, 2)$ (right isosceles), $(1, 2, 3)$ (half equilateral), $(1, 2, 2)$ (acute golden triangle) and $(1, 1, 3)$ (obtuse golden triangle).

Note that the order between the integers n, m and k in (n, m, k) is irrelevant – the triples $(1, 1, 2)$, $(1, 2, 1)$ and $(2, 1, 1)$ all represent the right isosceles triangle. We may thus consider non-decreasing triples only – we usually use $(1, 1, 2)$ to represent the right isosceles triangle.

Furthermore, the triples (jn, jm, jk) represent the same triangle for all positive integers j, – the triples $(1, 1, 2)$, $(3, 3, 6)$ and $(10, 10, 20)$ all represent the right isosceles triangle. We may thus also require that n, m and k have no common factor – triples that are relatively prime. Hence:

Lemma 3 *Each commensurable triangle can uniquely be represented by a non-decreasing triple (n, m, k), i.e., $n \leq m \leq k$, where n, m and k are integers that are relatively prime.*

A triple that represents a commesurable triangle is usually assumed to fulfill the conditions of the lemma. This means that each corner in a commensurable triangle can be associated with an integer, denoting the multiple of the triangle's root angle. We may then use expressions such as that "the corner is divisible by 3", referring to the associated integer to that corner.

Commensurable triangles are thus connected to number theory by the representation by integer triples. One immediate number theoretical question is how many distinct such partitions in triples, g_n, that exist for a certain sum n.

The basic triangle properties of "isosceles" and "right" can easily be translated into the language of integer triples.

Remark 4 *For commensurable triangles, an isosceles triangle correspond to a triple where two entries are equal, either $n = m$ or $m = k$. A right triangle corresponds to a triple where $n + m = k$. A triangle is acute if $n + m > k$ and obtuse if $n + m > k$.*

We may say that triangles that have the same integer sum are related. For example, the two golden triangles $(1, 2, 2)$ and $(1, 1, 3)$ are related in that they are the only commensurable triangles (triples) that have the sum 5. We can impose a tree structure on the set of commensurable triangles. We use the sum to denote which generation a triangle belongs to, and we may generate children to a certain triple by adding 1 to one of the three entries – if such triples have no common factor and is non-decreasing. Each triangle then have at most three children and three parents.

Since the triple of positive integers with lowest sum is $(1, 1, 1)$, which represents the equilateral, the enumeration of generations of commensurable triangles starts with 3, perhaps fittingly for a structure of triangles. This generation obviously contains only the equilateral, and its only child is $(1, 1, 2)$, the right isosceles triangle. The generation number n of a certain commensurable triangle, sometimes called the order, is simply the sum of the entries in the triple. A triangle of order n has root angle $\frac{\pi}{n}$.

A commensurable triangle where the root angle q is the smallest angle may be called a **root triangle**. The triple of such a triangle starts with 1. We may note in the table below that all triangles in the first four generations are root triangles. In the same four generations, all triangles are either right or isosceles, or both. The first triangle that is not a root triangle, the $(2, 2, 3)$, is isosceles. The first that is neither right or isosceles, the $(1, 2, 4)$, is a root triangle. These two triangles belong to generation 7. In generation 9 we have the first commensurable triangle that is neither right, isosceles or root. It is the $(2, 3, 4)$ triangle.

We next present the first nine generations of commensurable triangles, which includes in total 36 triangles. Note that triangles in the same generation are often related in that they have angles of common size. Among commensurable triangles, the famous five turn out to populate the first four generations. In the fourth generation, number 6, also the triangle with angles $\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3}$ occurs.

order	*triple*	*name, if any*	g_n
3 :	$(1, 1, 1)$	equilateral	1
4 :	$(1, 1, 2)$	right isosceles	1
5 :	$(1, 2, 2), (1, 1, 3)$	golden triangles	2
6 :	$(1, 2, 3), (1, 1, 4)$	half equilateral	2
7 :	$(2, 2, 3), (1, 3, 3), (1, 2, 4), (1, 1, 5)$		4
8 :	$(2, 3, 3), (1, 3, 4), (1, 2, 5), (1, 1, 6)$		4
9 :	$(2, 3, 4), (1, 4, 4), (1, 3, 5),$ $(2, 2, 5), (1, 2, 6), (1, 1, 7)$		6
10 :	$(3, 3, 4), (2, 3, 5), (1, 4, 5),$ $(1, 3, 6), (1, 2, 7), (1, 1, 8)$		6
11 :	$(3, 4, 4), (3, 3, 5), (2, 4, 5), (2, 3, 6),$ $(1, 5, 5), (1, 4, 6).(1, 3, 7), (2, 2, 7),$ $(1, 2, 8), (1, 1, 9)$		10

Thus, two triangles that are identical but differ by one in one position are directly related, where the one with the smaller entry in the differing position is the parent. The equilateral triangle $(1, 1, 1)$ (with sum 3) is the very first anchestor, and the half equilateral triangle $(1, 2, 3)$ (with sum 6) is the first that actually has three distinct children, which $(2, 2, 3), (1, 3, 3)$ and $(1, 2, 4)$. Analogously, the $(2, 3, 5)$-triangle (with sum 10) is the first that actually has three distinct parents – $(1, 3, 5), (2, 2, 5)$ and $(2, 3, 4)$. We may talk about the family of commensurable triangles, consisting of generations.

2.1.2 Cutting commensurable triangles by angles

Definition 5 *An **angle sector** is a generalization of an angle bisector, and it is defined for a commensurable triangle. It is a line dividing an angle in two* *angles where both are multiples of the root angle.*

For the triangles $(1, 1, 2)$ and $(1, 2, 2)$, the only angle sectors are angle bisectors. We know that the bisectors of the three angles intersect in one point. Two angle trisectors divides an angle in three equal parts. By the famous Morley's theorem, sometimes called the Morley's miracle, the "neighbouring" angle trisectors of a triangle meet pairwise in three points that always form an equilateral triangle (see [1]). The angle sectors defined here generalizes bisectors and trisectors, but are required to divide the angles in multiples of the root angle.

A commensurable triangle (n, m, k) has $n - 1, m - 1$ and $k - 1$ possible angle sectors in each corner, respectively. If the triangle is divided along an angle sector, we get two new triangles – this is a **sector tiling**. In a sector tiling, each of the two new triangles obviously has one angle in common with the previous triangle. The part of the angle that is removed by the angle sector is added to the third triangle. It turns out that this division often produces triangles at the same generation. A sector tiling of the $(1, 1, 2)$ triangle gives two copies of the same triangle. All possible sector tilings of a golden triangle gives the two golden triangles. Sector tilings of all the triangles of the first four generations give triangles on the same generation with one exception. It is the sector tiling of the $(1, 2, 3)$ triangle, the half equilateral, that gives $(1, 1, 1)$ and $(1, 1, 4)$, where the equilateral $(1, 1, 1)$ of course does not belong to the same generation as the triangle being tiled.

Consider a triangle that is the result of a sector tiling of a triangle (n, m, k). Then one angle is tiled, one is unchanged, and the third is new. Suppose that the tiled angle is the first one and that it corresponds to the number n. It can be tiled in the two parts i and $n - i$, for any $i = 1, ..., n - 1$. In one of the new triangles m is unchanged, and what is removed in the first angle reappears in the third: this angle is $(i, m, k+n-i)$. The second is analogously $(n-i, m+i, k)$. Note that the sum is preserved in both triangles. However, a common multiple may occur, in which case we obtain a triangle at an earlier generation. We have shown the following lemma, in which we use the letter n for the generation number, not to be confused with n in (n, m, k). This is used in order to be consistent with the sequel, where n is an extremely natural choice of letter.

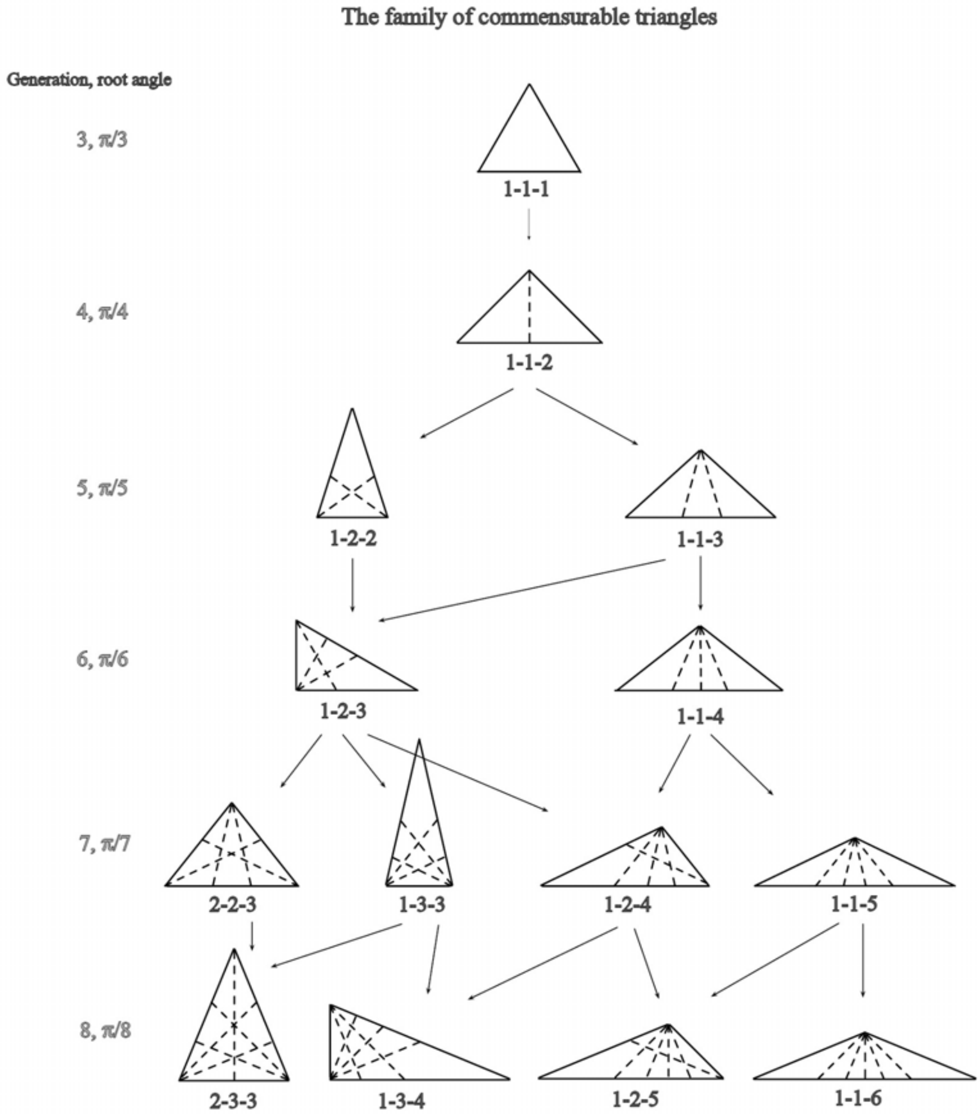
Lemma 6 *If a commensurable triangle belongs to generation , triangles produced by a sector tiling belongs to generation g/j for some positive integer j. In particular, the generation number of the appearing triangles cannot be larger than that of the original triangle. The number of possible sector tilings of a triangle at generation n is $n - 3$.*

A bisector tiling is not a sector tiling if we divide an odd angle, i.e. an angle which is an odd multiple of the root angle. Such tilings can be studied by replacing (n, m, k) by $(2n, 2m, 2k)$, which gives the triangles $(i, 2m, 2k + 2n - i)$ and $(2n + i, 2m + 1, 2k - i)$. The $(1, 2, 3)$ is such a bisector tiling of $(1, 1, 1)$, which doubles the generation number.

We end this subsection with graphics and a comparison between properties of commensurable triangles and integer triples.

commensurable triangle	*triple* (n, m, k)
right	$n + m = k$
acute	$n + m > k$
obtuse	$n + m < k$
isosceles	$n = m$ or $m = k$
root	$n = 1$
sector tiling of first angle	replace (n, m, k) by
	$(i, m, k + n - i)$ and $(n - i, m + i, k)$

In the graphics below, the first generations of commensurable triangles are shown with their sector tilings, and with the generation relations. Note that at generation n, the sector tilings and triangle sides of a triangle form in total n straight lines whose directions are distinct and evenly distributed in the available direction span of π – identifying opposite directions. Each "neighbouring" pair of straight lines is separated by the root angle $\frac{\pi}{n}$.



2.2 Regular polygons and commensurable triangles

Commensurable triangles can also be constructed by regular polygons. By regular polygons we will deduce a formula for the number of commensurable triangles in each generation.

Consider two corners in a regular n-polygon which are k corners apart, i.e., there are $k-1$ corners in between. Then the pheripherical angles based at the two corners are either $\frac{\pi k}{n}$ or $\frac{\pi(n-k)}{n}$. Hence, by joining three corners in an n-polygon, we obtain a triangle where all angles are multiples of $\frac{\pi}{n}$. This establishes the first part in the following theorem:

Theorem 7 *All triangles constructed by three corners in a regular polygon are commensurable. All commensurable triangles appear by joining three corners in a regular polygon.*

A commensurable triangle has angles $\frac{n\pi}{n+m+k}, \frac{m\pi}{n+m+k}, \frac{k\pi}{n+m+k}$, so by considering a $(n+m+k)$-polygon where two corners are n corners apart and two other are m corners apart, the second part follows.

The root angle for a commensurable triangle of order n is $\frac{\pi}{n}$. Some n-polygons contain also triangles of an earlier generation than n. For example, by taking every second corner in a hexagon (6-polygon), we get the equilateral triangle, with root angle $\frac{\pi}{3}$.

Theorem 8 *The root angle of a triangle constructed by an n-polygon is $\frac{\pi}{k}$ for some integer $k \geq 3$ that divides n.*

Our next question is the following: how many different triangles can be constructed by a regular n-polygon? The answer can be given as follows.

We here use the notation $\lfloor x \rfloor$, the so called floor function, for the closest integer that is not larger than x. The floor function is identical with the commonly used integer part, denoted by $[x]$. Similarly, the ceiling function $\lceil x \rceil$ is the integer that is closest to x that is not smaller. Thus, x is real, $\lfloor x \rfloor$ and $\lceil x \rceil$ are integers, and we have

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1,$$

with equalities if and only if x is an integer.

Theorem 9 *By joining triples of corners in a regular n-polygon it is possible to construct*

$$p_n = \left\lfloor \frac{n-3}{6} \right\rfloor (n - 3 \left\lfloor \frac{n-3}{6} \right\rfloor) + \tau_{n \bmod 6}$$

distinct non-similar triangles. Here $\tau_3 = 1$, $\tau_4 = 1$, $\tau_5 = 2$, $\tau_6 = 3 (= \tau_0)$, $\tau_7 = 4 (= \tau_1)$, $\tau_8 = 5 (= \tau_2)$ are the six first values, i.e. $p_n = \tau_n$ if $n \leq 8$.

Proof. Given a regular n-polygon, we count triangles in disjoint subsets C_i, $i = 1, 2, \ldots$. Here $c_i = |C_i|$ is the number of triangles where the two closest corners are exactly i corners away. For example, C_1 consists of all triangles where two corners are adjacent in the regular polygon. Let us fix one corner in the polygon, called corner number 1, as the first triangle corner. To calculate the number of triangles in C_1 we fix an adjacent corner as the second corner. Then the following corners numbered $3, 4, \ldots, \lceil \frac{n}{2} \rceil + 1$, which requires $n \geq 3$ (of course, the regular 3-polygon is the equilateral triangle) give distinct non-similar triangles as third corners in a triangle. The third corner essentially moves to the opposite of the polygon relatively to two first corners, since the the third corner in the other half of the polygon corresponds to similar triangles by reflection. This counting argument is fundamental in this proof. So $c_1 = |C_1| = \lceil \frac{n}{2} \rceil - 1$.

Similarly, to calculate c_2, we start with three corners separated by two single corners, and count by moving the third corner similarly throughout essentially the half of the polygon up to a symmetry point defined by the two first corners of the n-polygon. In general we calculate c_i by starting with three corners separated by two sets of $i-1$ corners, and move the third corner to a point to the opposite of the two first corners. This means that for c_1 we need at least 3 corners, for c_2 at least 6, and so on. Since $n = 3$ represents the first entry in each sequence c_i – the 3-polygon, the equilateral triangle, is the first regular polygon, we have in formula that $c_i = 0$ if $n < 3i$ and $c_i > 0$ if $n \geq 3i$. We now know how all sequences c_i start, i.e. according to the pattern

$$\begin{aligned} c_1 &= (*, *, *, *, *, *, *, *, *, *, ...) \\ c_2 &= (0, 0, 0, *, *, *, *, *, *, *, ...) \\ c_3 &= (0, 0, 0, 0, 0, 0, *, *, *, *, *, ...) \\ c_4 &= (0, 0, 0, 0, 0, 0, 0, 0, 0, *, *, *, *, ...), \end{aligned}$$

where $*$ is a non-zero entry. This information can alternatively be described as that the sequence c_i has exactly $3(i-1)$ zeros in the beginning.

To calculate c_2, the corners number $5, 6, ..., \lfloor \frac{n}{2} \rfloor + 2$ give distinct triangles in C_2 as third corners, requiring $n \geq 6$. Hence, $c_2 = \lfloor \frac{n}{2} \rfloor - 2$ and

$$\begin{aligned} c_1 &= (1, 1, 2, 2, 3, 3, 4, 4, 5, 5, ...) \\ c_2 &= (0, 0, 0, 1, 1, 2, 2, 3, 3, 4, 4, ...). \end{aligned}$$

Since these two sequences increse by one at every second entry, except in the very beginning, and they increase always on different entries, we may add them to get a linear sequence, again except in the beginning. This gives

$$c_1 + c_2 = (1, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...).$$

Thus, $c_1 + c_2 = n - 3$ for all $n \geq 6$, and something slightly different for $n < 6$. This is consistent to adding $c_1 = \lceil \frac{n}{2} \rceil - 1$ and $c_2 = \lfloor \frac{n}{2} \rfloor - 2$, since $\lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor = n$ for all integers n. We next claim that also the sums $c_3 + c_4$ and $c_5 + c_6$ and so increase linearly except in the beginning. For this we only need to show that all sequences consist of $..., 0, 0, 0, 1, 1, 2, 2, 3, 3, 4, 4, ...$ and the only difference between the sequences is the number of zeros before the first 1. This number we already know – it is $3(i-1)$.

To prove that all sequences consist of $1, 1, 2, 2, 3, 3, 4, 4, ...$ after some initial zeros, we only need to consider the triangles that can be formed by the possible positions of the third corner in an n-polygon. We now let the third corner be at any of the two halfs of the polygon as described in the beginning of the proof, and not only in one of them. Then there are two possibilities. If the number of possible positions for the third corner is even, each single triangle is similar to another on the other side. If the number of possible positions is odd, each triangle is similar to another on the other side exept one, the one corresponding to the corner in the middle. If we in the even case add one corner, turning the n-polygon into an $(n+1)$-polygon, c_i will not increase, since this single triangle will be replaced by two triangles that are similar to each other. If we in the odd case increment n by one, also c_i will increase by one, since a triangle will appear that is not similar to another. Hence, Incrementing n will every second time increment c_i $(= c_{i-1} + 1)$ and every second time will leave c_i unchanged $(= c_{i-1})$.

We can thus deduce that

$$\begin{aligned}
c_1 &= (1,1,2,2,3,3,4,\ 4,\ 5,\ 5,...) \\
c_2 &= (0,0,0,1,1,2,2,\ 3,\ 3,\ 4,\ 4,...) \\
c_3 &= (0,0,0,0,0,0,1,\ 1,\ 2,\ 2,\ 3,\ 3,\ 4,...) \\
c_4 &= (0,0,0,0,0,0,0,\ 0,\ 0,\ 1,\ 1,\ 2,\ 2,...) \\
(\text{index; } n &= (3,4,5,6,7,8,9,10,11,12,...))
\end{aligned}$$

so that

$$\begin{aligned}
c_1 + c_2 &= n - 3 \text{ for all } n \geq 6, \\
c_3 + c_4 &= n - 9 \text{ for all } n \geq 12, \\
c_5 + c_6 &= n - 15 \text{ for all } n \geq 18,
\end{aligned}$$

and so on. This means that

$$c_{2j-1} + c_{2j} = n + 3 - 6j = m - 3(2j-1)$$

for any positive integer j. Adding k pairs of sequences $c_{2j-1} + c_{2j}$ gives

$$\begin{aligned}
\sum_{j=1}^{k} (c_{2j-1} + c_{2j}) &= kn - 3 \sum_{j=1}^{k} (2j-1) = \\
&= kn - 3k^2 = k(n-3k).
\end{aligned}$$

If we choose $k = \left\lfloor \frac{n-3}{6} \right\rfloor$ we get the sum $\left\lfloor \frac{n-3}{6} \right\rfloor (n - 3 \left\lfloor \frac{n-3}{6} \right\rfloor)$. We then start the summation of pairs of sequences at $n = 9, 15, 21, ...,$ and not at $6, 12, 18, ...$ as described above. At $n = 9$, for example, the first two sequences are added, but $n = 8$ gives zero. However, these two sequences started already at $n = 3$ and $n = 6$, respectively. We have in $\left\lfloor \frac{n-3}{6} \right\rfloor (n - 3 \left\lfloor \frac{n-3}{6} \right\rfloor)$ added only the main part of the sequences, namely

$$\begin{aligned}
c_1 &= (0,0,0,0,0,0,4,\ 4,\ 5,\ 5,...) \\
c_2 &= (0,0,0,0,0,0,2,\ 3,\ 3,\ 4,\ 4,...) \\
c_3 &= (0,0,0,0,0,0,0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 4,...) \\
c_4 &= (0,0,0,0,0,0,0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 2,...) \\
(\text{index; } n &= (3,4,5,6,7,8,9,10,11,12,...))
\end{aligned}$$

All that remains now is what happens for values between $n = 9, 15, 21, ..$, i.e. in the intervals each consisting of six numbers. We then need to add $1, 1, 2, 2, 3, 3$ and $0, 0, 0, 1, 1, 2$, which is $1, 1, 2, 3, 4, 5$. This sequence is denoted as the sequence τ, and gives the first six numbers, $p_n = \tau_{n \bmod 6}$ if $n \leq 9$. These numbers then repeates during each following interval of six numbers, added with the sum of the sequences. The term $\tau_{n \bmod 6}$ takes care of this finer structure if we define

$$\tau_j = \begin{cases} 3 \text{ if } j = 0 \bmod 6 \\ 4 \text{ if } j = 1 \bmod 6 \\ 5 \text{ if } j = 2 \bmod 6 \\ 1 \text{ if } j = 3 \bmod 6 \\ 1 \text{ if } j = 4 \bmod 6 \\ 2 \text{ if } j = 5 \bmod 6 \end{cases}$$

for all integers j. Then, $\tau_{n \bmod 6} = \tau_n$ for all n, but we for clarity choose to write $\tau_{n \bmod 6}$. The formula $p_n = \left\lfloor \frac{n-3}{6} \right\rfloor \left(n - 3 \left\lfloor \frac{n-3}{6} \right\rfloor\right) + \tau_{n \bmod 6}$ is proved. ∎

One interpretation of the formula in terms of the proof is the following way to calculate values of p_n by summing columns. The first column repeats $1, 1, 2, 3, 4, 5$ endlessly, corresponding to the term $\tau_{n \bmod 6}$. All other columns contains the sequence $6, 7, 8, 9, ...$, and each copy of this sequence starts in a position where $1, 1, 2, 3, 4, 5$ restarts. I.e., the first copy of $6, 7, 8, ...$ starts in 9, the second in 15, the third in 21, and so on. These columns correspond to $c_1 + c_2$, $c_3 + c_4$, $c_5 + c_6$ and so on. The enumeration of the rows starts with $n = 3$. Now, the sum of row n is p_n. Here is the start of such a table.

n	1:st	2:nd	3:rd	4:th	p_n
3	1				1
4	1				1
5	2				2
6	3				3
7	4				4
8	5				5
9	1	6			7
10	1	7			8
11	2	8			10
12	3	9			12
13	4	10			14
14	5	11			16
15	1	12	6		19
16	1	13	7		21
17	2	14	8		24
18	3	15	9		27
19	4	16	10		30
20	5	17	11		33
21	1	18	12	6	37
22	1	19	13	7	40

Note that the first and second difference sequences of p_n have the following behaviour:

$$\begin{aligned} Dp_n &= p_{n+1} - p_n = (0,1,1,1,1,2,1,2,2,2,2,3,2,3,3,3,4,3,4,4...) \\ D^2 p_n &= p_{n+2} - 2p_{n+1} + p_n = (1,0,0,0,1,-1,1,0,0,0,1,-1,1,0,0,...). \end{aligned}$$

Thus, the second difference $D^2 p_n$ is periodic, and the period $(1,0,0,0,-1,1)$ of $D^2 p_n$ has sum 1. To estimate p_n for large n, we may in the formula $p_n = \left\lfloor \frac{n-3}{6} \right\rfloor (n - 3 \left\lfloor \frac{n-3}{6} \right\rfloor) + \tau_{n \bmod 6}$ estimate $\left\lfloor \frac{n-3}{6} \right\rfloor$ by the inequalities $\frac{n-2}{6} - 1 \leq \left\lfloor \frac{n-3}{6} \right\rfloor \leq \frac{n-3}{6}$, which are sharp, and estimate $1 \leq \tau_{n \bmod 6} \leq 5$.

This gives
$$\frac{1}{12} n^2 - \frac{13}{3} \leq p_n \leq \frac{1}{12} n^2 + \frac{17}{4}.$$
So p_n increases roughly as $\frac{1}{12} n^2$. In particular, $p_{6k+3} = k(6k+3-3k)+1 = 3k^2 + 3k + 1$, or $p_n = 3(\frac{n-3}{6})^2 + 3(\frac{n-3}{6}) + 1 = \frac{1}{12} n^2 + \frac{1}{4}$ when $n = 6k+3$.

We now return from the calculation of p_n to the definition of p_n, i.e. to the triangles. For example, $p_6 = 3$, represents the triangles with angles $\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3}$, with $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ (the half equlateral) and with $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$ (the equilateral). The first two belong to G_6, while the equilateral is the only member of G_3. Certainly, in any n-polygon we can form triangles from the $n/2$-polygon if $n/2$ is an integer, by picking corners with one polyon corner between the first and the second corner and also between the second and third corner, for example. We can do similarly in the $n/3$-polygon, if 3 divides n and $n/3 \geq 3$, and in general for any n/k-polygon if n/k is an integer and $n/k \geq 3$ We can thus iteratively calculate g_n from p_n as follows:

Theorem 10 *For all $n \geq 3$,*
$$g_n = p_n - \sum_{\{k \geq 2, : n/k \text{ integer and } n/k \geq 3\}} g_{n/k}.$$

This gives the following sequence of the first 20 values of g_n.

n	p_n	g_n
3	1	1
4	1	1
5	2	2
6	3	2 $(= p_6 - g_3)$
7	4	4
8	5	4 $(= p_8 - g_4)$
9	7	6 $(= p_9 - g_3)$
10	8	6 $(= p_{10} - g_5)$
11	10	10
12	12	8 $(= p_{12} - g_6 - g_4 - g_3)$
13	14	14
14	16	12 $(= p_{14} - g_7)$
15	19	16 $(= p_{15} - g_5 - g_3)$
16	21	16 $(= p_{16} - g_8 - g_4)$
17	24	24
18	27	18 $(= p_{18} - g_9 - g_6 - g_3)$
19	30	30
20	33	24 $(= p_{20} - g_{10} - g_5 - g_4)$
21	37	32 $(= p_{21} - g_7 - g_3)$
22	40	30 $(= p_{22} - g_{11})$

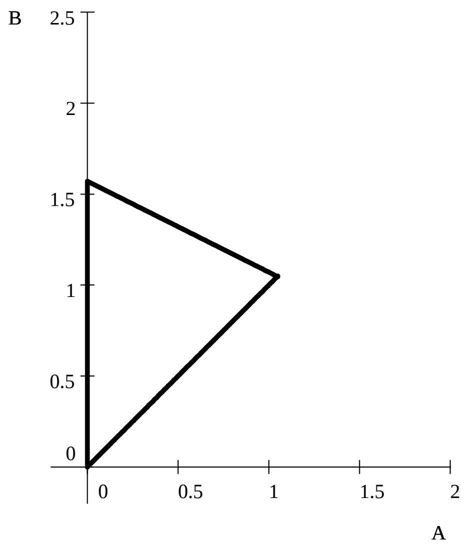
Note that the sequence g_n is not increasing. For example, $g_{11} = 10 > g_{12} = 8$. It is an open question whether all g_n for $n \geq 5$ are even numbers. It is also remarkable that in the prime factorizations of the first 20 g_n the smallest primes are strongly overrepresented. Adding the number of prime factors among these gives in total 35 2:s, 9 3:s, 3 5:s, one 7, and no other primes. This suggests more symmetries. Only $g_3 = g_4 = 1$ are odd among the first 20 g_n:s, and only $g_5 = g_6 = 2$ are primes.

2.3 Parametrizing the set of triangles - the meta triangles

We next consider the entire set of triangles and not only commensurable triangles. We parametrize the set of triangles by two of the angles. It is practical to denote an isosceles triangle where the third side is larger than the equal sides as a *wide isosceles triangle*. If the third side is smaller we talk about a *narrow isosceles triangle*.

For triangles with angles A, B and C, we assume that A is the smallest and C is the largest, i.e.: $A \leq B \leq C = \pi - A - B$. The set of triangles modulo similarity can be described in the parameters A and B, so it is a two parameter family. Which are the restrictions of A and B? Certainly, both are positive and $A \leq B$. Furthermore, using $B \leq C$ and $C = \pi - A - B$, we get $B \leq \pi - A - B$, i.e. $A + 2B \leq \pi$. In the A, B-plane these limitations gives a triangle, which we call a meta triangle since each point (A, B) fulfilling $A > 0$, $A \leq B$ and $A + 2B \leq \pi$ represents a unique triangle.

At the side $A = 0$ we have degenerate triangles, so this value is not allowed. Along the lower side $A = B$ we have the wide isosceles triangles. At the upper side we have $A + 2B = \pi$, meaning that $B = C$, so here we have all narrow isosceles triangles. In the intersection of $A = B$ and $B = C$, in the corner, we of course have the eqilateral triangle.



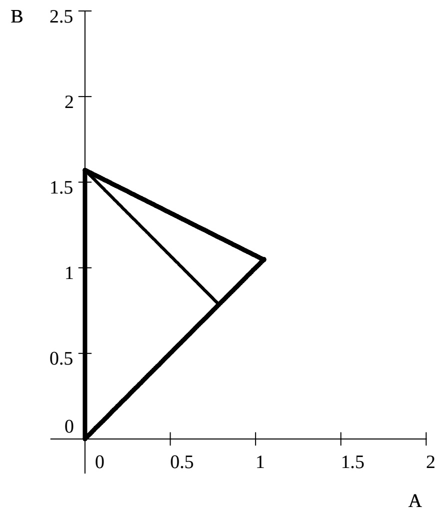
The first metatriangle.

Triangles of different kinds of course appears in different parts of this meta-triangle. In the right corner are the "round" or equilateral-like triangles, where all three angles are approximately equal. In the upper corner we have the "narrow" triangles, where one angle is much smaller than the other two, so the other two are both not far from $\pi/2$. In the lower corner we may talk about the "wide" triangles, where two angles are much smaller than the third, so the third angle is not far from π.

In Section 3 a slightly similar division of the set of triangles is done. Here distinct borders are drawn in terms of the golden ratio as a bound for the ratio of side lengths, in which terms "round", "narrow" and "wide" may be defined distinctly.

The first meta triangle is acute and rather close to the equilateral and even closer being isosceles. Its sides are $\frac{\pi}{2}$, $\frac{\pi\sqrt{2}}{3}$ and $\frac{\pi\sqrt{5}}{6}$, which are approximately 1.

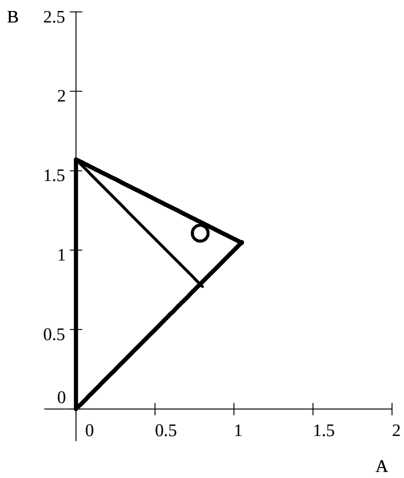
570 8, 1. 481 0 and 1. 170 8. Where do right triangles occur in the meta triangle? Well, since C is the largest angle, we have $A + B = \frac{\pi}{2}$ for right triangles. This means that the right triangles form a straight line, which in fact is a height in the meta triangle to the second largest side. The base point certainly represents the right isosceles triangle, since it is right (on $A + B = \frac{\pi}{2}$) and isosceles (on $A = B$).



The first metatriangle with height of right triangles.

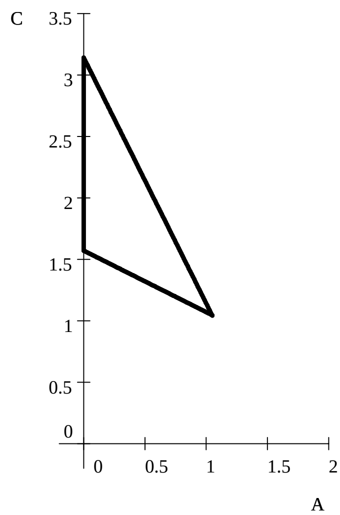
Below the height of right triangles we have the obtuse triangles, and above are all acute triangles.

Since the meta triangle is a triangle, it is also represented by a point in the meta triangle. This is a point that is very different in nature than triangular points that are constructed geometrically, such as the intersection of the three heights, so a different kind of name is appropriate. We call this point the **eye** of the meta triangle.



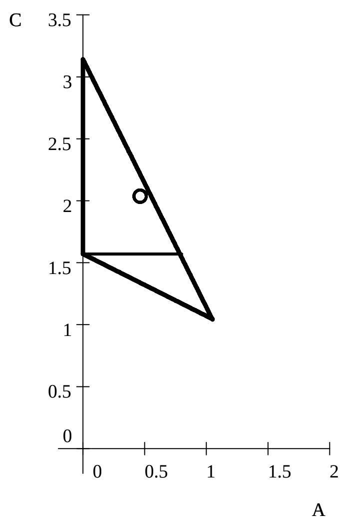
The eye of the first meta triangle.

If we plot the largest angle C against the smallest A we get the second metatriangle. We eliminate B in the equalities $A \leq B$ and $B \leq C$ by $B = \pi - A - C$ which gives $2C \geq \pi - A$ and $C \leq \pi - 2A$. So the boundaries are given by $C = (\pi - A)/2$ and $C = \pi - 2A$. Again every point in the triangle including the boundaries with $A > 0$ uniquely represent the set of triangles modulo similarity. At the lower side we have $A = B \leq C$, which are the wide isosceles triangles, and at the upper side we have the narrow isosceles triangle. The equilateral triangle is of course the same point $(\frac{\pi}{3}, \frac{\pi}{3})$ as before.



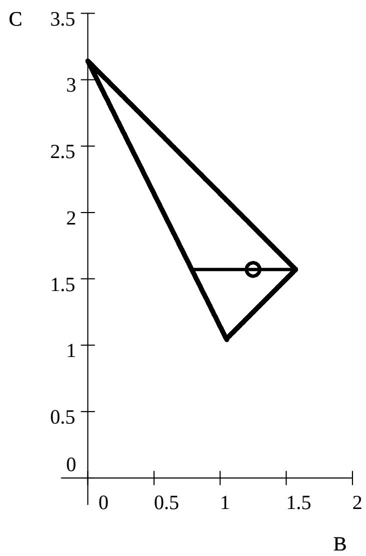
The second metatriangle.

In the second metatriangle the right triangles correspond to $C = \frac{\pi}{2}$ which in this case form a part of a height in the triangle. This meta triangle has sides $\frac{\pi}{2}, \frac{\pi\sqrt{5}}{6}, \frac{\pi\sqrt{5}}{3}$, which is approximately $1.570\,8, 1.\,170\,8$ and $2.\,341\,6$ 1. This gives the eye at $A = 2.\,036\,9$ and $C = 0.463\,74$ radians, so the second meta triangle is obtuse.



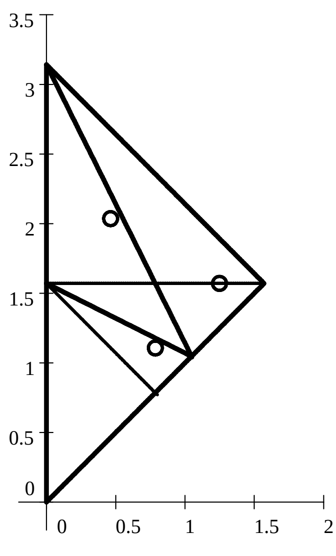
The second meta triangle with right triangle line and eye.

Analoguously, the third metatriangle contains the locus of all the possible angle pairs (B, C) for a triangle – the two largest angles. This gives the limitations $\max(\pi - 2B, B) \leq C \leq \pi - B$ and the side lengths $\frac{\pi}{\sqrt{2}}, \frac{\pi}{3\sqrt{2}}$ and $\frac{\pi\sqrt{5}}{3}$, which are approximately $2.\,221\,4$, $0.740\,48$, $2.\,341\,6$. In the third meta triangle, the right triangle line do not form a height. On the other hand, this meta triangle is in itself a right triangle. This means that the eye is on the right triangle line. The eye has coordinates $(1.\,249\,2, 1.570\,8)$.



The third meta triangle with right triangle line and eye.

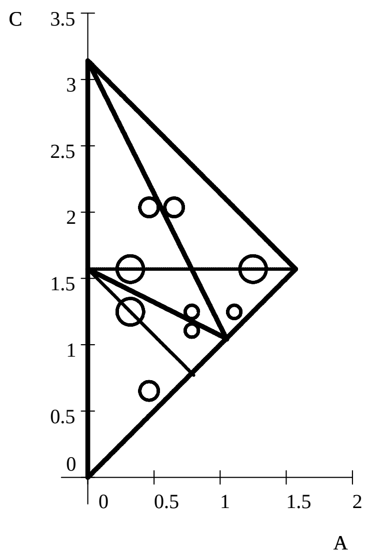
The three meta triangles can be depicted in the same graph, although they correspond to different coordinates. Because of $A \leq B \leq C$, they do not overlap except on the boundaries, which correspond to isosceles triangles. They meet in one point only – the point of the equilateral triangle.



The three meta triangles with right triangle lines and eyes.

The three meta triangles have the same area. This can be seen as the first two has the same base length on the y-axis, and the same height, while the first and the second have base on the line $y = x$ which relate $1 : 2$ and heights which by similarity relate $2 : 1$. The area of a meta triangle is $\pi^2/12$.

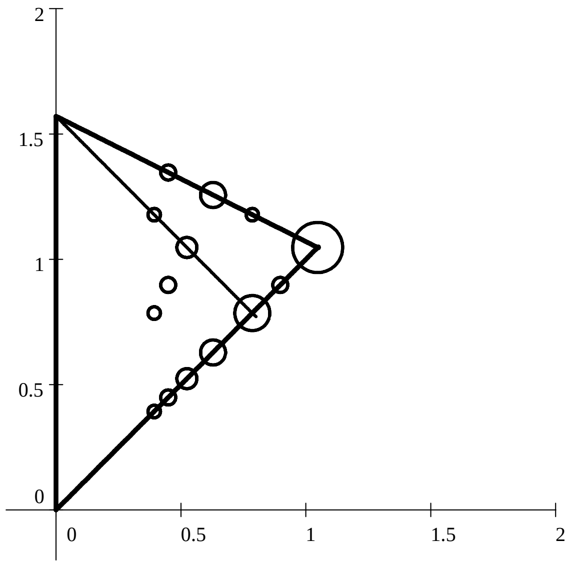
In each meta triangle, we can certainly represent the other two meta trianges a points. In the following figure, the eye of the first meta triangle is in all meta triangles a small circle, the second is larger and the third is largest. This graph also gives an idea of how points/triangles correspond in the three triangles. Points corresponding to the same triangle are non-orthogonal reflections in the boundary lines. Projections where the second triangle is involved are parallel to the x-axis or the y-axis, and preserves the distance to the dividing line between the triangles.



The three meta triangles with nine eyes.

2.4 Commensurable triangles and the meta triangles

In a meta triangle we may certainly depict the family of commensurable triangles. The following graph contains the first six generations of commensurable triangles, denoted with circles of decreasing size.



The first six generations of commensurable trianges in the first metatriangle.

One can note that in the first four generations, all are either isosceles or right. In the first six generations, three are right, ten are isosceles (one is both), and only two are neither isosceles or right. In the iteration, a triangle (n, m, k) gives the three triangles $(n+1, m, k)$, $(n, m+1, k)$ and $(n, m, k+1)$, where those of the three triples which are non-decreasing or relatively prime are discarded since the same triangle otherwise would be represented twice. This means that from (n, m, k), the first goes back towards the equilateral corner, the second towards the corner of the sticklike triangles in the lower left corner, and the third towards the corner of the flat triangles in the upper left. This does not usually take place along Euclidean straight lines. The generations of triangles start at the equilateral triangle, and are spreading to the left and throughout the metatriangle as the root angle $\frac{\pi GLCD(n,m,k)}{n+m+k}$ grows smaller. In each generation $n+m+k$ there is at least one triangle where the root angle is the smallest angle, which in this meta triangle is on the horizontal axis. One example of such triangles is the set of wide isosceles commensurable triangles $(1, 1, n+m+k-2)$ at the lower edge. Rememember that the commensurable triangles are dense among the set of triangles, similarly to the rational numbers among the real. There are of course other types of wide isosceles commensurable triangles, such as $(2, 2, 2j+1)$, $(3, 3, 3j+1)$ and $(3, 3, 3j+2)$, etc., for positive integers j.

3 Triangles by sides

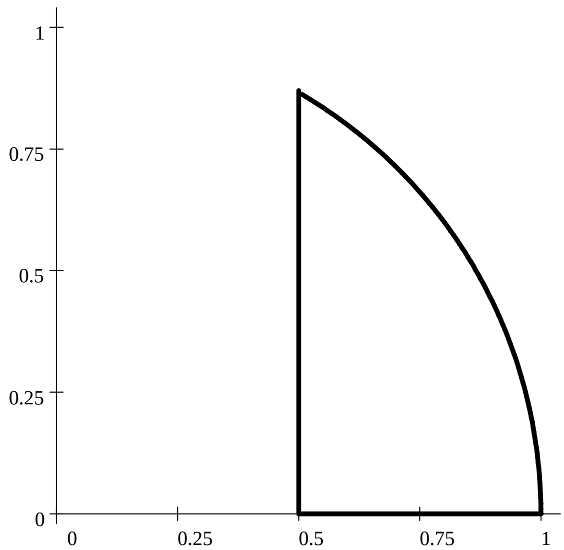
We in this section construct a side analogue to the metatriangles. We consider all triangles modulo similarity in terms of their sides, and we only consider non-trivial triangles, i.e. both angles and sides are strictly positive. We normalize by choosing all triangles to have largest side one, and we locate the corners of this side in the points $(0,0)$ and $(1,0)$. The third corner is in the point (x,y) where $y>0$. Concerning the size order of the sides, we assume that

$$|(0,0)-(1,0)| \geq |(0,0)-(x,y)| \geq |(x,y)-(1,0)|.$$

I.e., the shortest side is to the right – the side with end points (x,y) and $(1,0)$. The second equality is equivalent to $x \geq \frac{1}{2}$. We also have $|(x,y)| \leq 1$, otherwise the second side would be larger than the first, so we are inside the unit circle. The **triangle sector** is the set

$$T = \{(x,y) : \frac{1}{2} \leq x, x^2+y^2 \leq 1, y>0\},$$

where each triangles modulo similarity is represented uniquely. The side length relations implies that the smallest angle is in the $(0,0)$ corner, the middle in $(1,0)$ and the largest in (x,y).

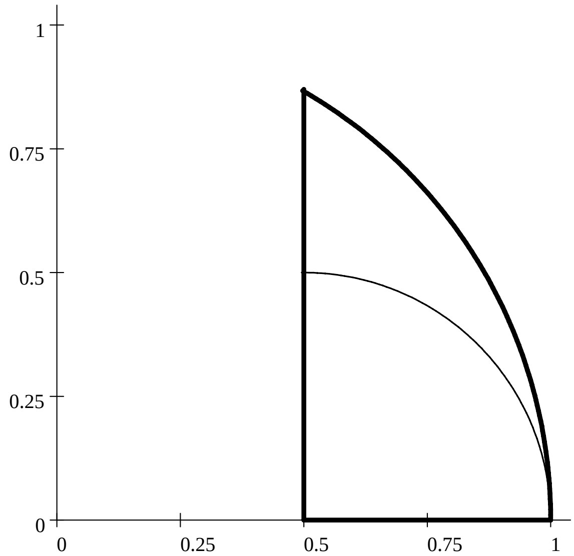


The set T, containing each triangle uniquely when disregarding congruences.

Note that T does not contain the lower boundary $y=0$, this side correspond to degenerated triangles. The set T has area $\frac{1}{6}\pi - \frac{1}{8}\sqrt{3} \approx 0.307\,09$, which is the value of $\int_{\frac{1}{2}}^{1} \sqrt{1-x^2}dx$.

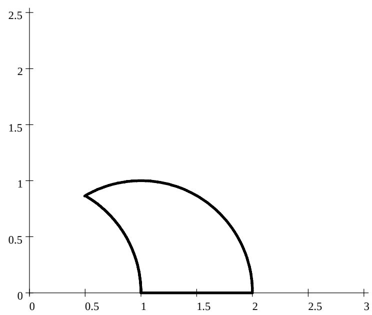
In the previous section we introduced the distinction of isosceles triangles in two types: narrow isosceles triangles, where the third side is smaller than the equal sides, and the wide isosceles triangles, where the third side is larger. By the construction, the wide isosceles triangles are here at the straight line $x = \frac{1}{2}$ to the left. The narrow isosceles triangles are at the right side, at the circular boundary $x^2 + y^2 = 1$. In the top point, in the intersection point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, we find the equilateral triangle.

The right triangles are in T represented with the hypothenuse as the side on the x-axis. Since the peripheral angle of a diameter is a right angle, we can conclude that all right triangles occur on the circular arc $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$. We call the circle arc of right triangles the **Pythagoras line**.

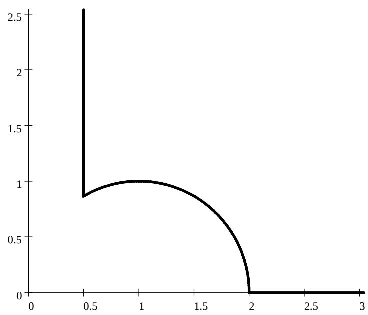


Triangle sector with the Pythagoras line.

We obtain different triangle sets if we fix the middle or the smallest side in the interval $(0, 0)$ and $(0, 1)$.



The middle side fixed.



The smallest side fixed.

As in the first section, there is a bijection B between the points of T and each equivalence class of similar triangles. For a point $(x, y) \in T$ we sometimes denote the corresponding similarity class of triangles by $t(x, y)$, although we will usually discuss representatives of the classes in T. Thus, $B : t(x, y) \mapsto (x, y)$ and $B^{-1} : (x, y) \mapsto t(x, y)$.

3.1 Rational triangles

Among other things, Pythagoras is known for his theorem about thriangles (which was known long before him, at least by for example the babylonians and the chinese) and for his dislike of irrational numbers. The Pythagorean triangles are right triangles with integer sides. We next define a slightly larger class of Pythagoras-friendly triangles, where the triangles are not necessarily right.

Definition 11 *A triangle is* ***rational*** *if all sides and all height are rational numbers when one side is normalized to one. All other triangles are* ***irrational triangles****.*

We define

$$R = \{(x, y) \in T : t(x, y) \text{ is rational}\}.$$

Can rational triangles be described in terms of other known types of triangles? For an answer, we need a simple lemma.

Lemma 12 *Suppose all sides of a triangle are rational. Then either all heights are rational, or all heights are irrational.*

With sides a, b and c and corresponding heights h_a, h_b and h_c, the area of the triangle can be written in three ways:

$$\frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c.$$

Since a, b and c are rational, one of these equations would be violated if one height is rational and another is irrational. The lemma is proved.

As an example, the Pythagorean triangle with sides $3, 4, 5$ is right, so two sides are heights. Hence, the third height is also rational, and it is 2.4.

So Pythagorean triangles are rational triangles, and form a discrete set on the circle arc $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$. Are there any more rational triangles? Yes, but not so many more.

Theorem 13 *The right rational triangles are the Pythagorean triangles. The non-right rational triangles consist of two Pythagorean triangles, separated by a height in the triangle.*

Proof. Consider a rational triangle, and draw a height in it. Then the height is rational, and so the two right subtriangles are Pythagorean.

Consider a triangle that consists of two Pythagorean triangles. Then all sides and one height of the triangle is rational. By Lemma 12, all heights are rational, so the triangle is rational. The theorem is proved. ■

A point $(x, y) \in T$ corresponds to a triangle with side lengths $\sqrt{(x-1)^2 + y^2}$, $\sqrt{x^2 + y^2}$ and 1. Some work with the Pythagorean theorem also yields the following lemma in the opposite direction:

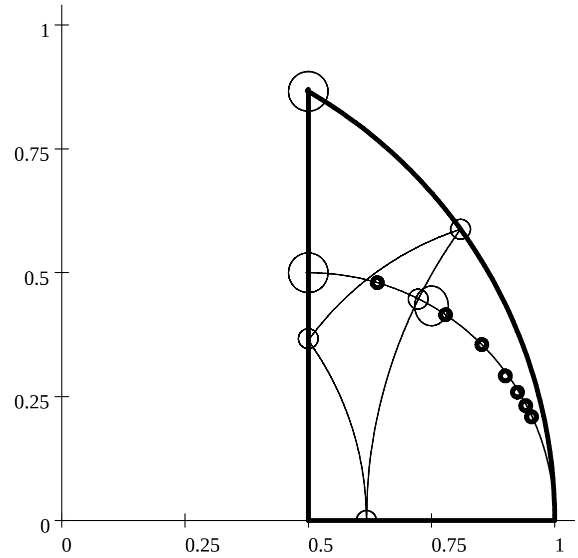
Lemma 14 *A triangle with sides a, b, c, where $a \le b \le c$, corresponds to the point $(\frac{b^2-a^2+1}{2c}, \frac{\sqrt{4b^2c^2-(b^2-a^2+1)^2}}{2c^2})$ in T.*

Denote two-dimensional Lebesgue measure of a set S by $m(S)$. For sets bounded by continuous and differentiable curves, this measure coincides with the usual area. Then we have

Theorem 15 $m(R) = 0$, *so almost all triangles are irrational.*

Proof. By Theorem 13, R is a countable subset of T. All countable sets have Lebesgue measure zero. Note that $m(T) = \frac{\pi}{6} - \frac{\sqrt{3}}{8} > 0$. In the terminology of integration theory, we say that a statement is valid for almost all $x \in S$ if the exception set has measure zero. Thus, in terms of the bijection from the set of triangles to T, we have motivated the second statement in the theorem. ■

The world view of the pythagoreans was disturbed by the discovery of that the diagonal and the side of a square are incommensurable. However, one may regard irrational numbers as inevitable and common as irrational triangles. In the following graphic, some Pythagorean thriangles are represented by small bold circles. The equilateral, the half equilateral and the right isosceles are marked with large circles and the two golden triangles by smaller circles. On the lines that occur in the graph, certain side lengths ratios are constant. They are described in Section 3.3.



Triangle sector with some well known triangles.

3.2 Non-right rational triangles

By Theorem 13 we know that the Pythagorean triangles are the only right rational triangles, and the only non-right rational triangles consists of two Pythagorean triangles. In this section we construct a few of the first non-right rational triangles.

It is well known that all Pythagorean triangles can be described as a triple $(m^2-n^2, 2mn, m^2+n^2)$, for some positive integers m and n. Such a triple fulfills the Pythagorean relation $a^2+b^2=c^2$, since

$$(m^2-n^2)^2+(2mn)^2=(m^2+n^2)^2.$$

The description in m and n is not unique. To begin with we can clearly limit ourselves to consider pairs (n, m) that are relatively prime (with no common factor), since otherwise the common factor can be cancelled in the equation. We also may assume for convenience that $n > m$. For a given sum $n+m$ this leaves $\phi(n+m)/2$ distinct alternatives. Here $\phi(k)$ is the Euler totient function, which denotes the number of integers smaller than k that are relatively prime with k. If k is a prime we have $\phi(k)=k-1$, and $\phi(k)$ is even if $k > 2$. Still under these restrictions, some of the pairs (n, m) give the same Pythagoran triangle. In the following table all distinct Pythagorean triangles are shown up to $n+m \leq 10$. They are in the following table ordered by the ratio a/b i.e. by their closeness to the right isosceles triangle $(1, 1, 2)$. Certainly, $a/b = \tan A$, where A is the smallest angle. We assume that the sides a, b and c fulfill $a < b < c$.

$a-b-c$	a/b	$m-n$
$20-21-29$	0.952	$5-2, 7-3$
$3-4-5$	0.750	$2-1, 3-1$
$8-15-17$	0.533	$4-1, 5-3$
$28-45-53$	0.538	$7-2$
$5-12-13$	0.417	$5-1, 3-2$
$12-35-37$	0.343	$6-1$
$7-24-25$	0.292	$7-1, 4-3$
$16-63-65$	0.254	$8-1$
$9-40-41$	0.225	$9-1, 5-4$

This list is not exhaustive. There are surely Pythagorean triangles that are closer to the right isosceles triangle than the $20-21-29$ triangle. However, we can enumerate the triangles by the integer pairs (m, n), firstly with increasing $m+n$, and then secondly with decreasing m, while skipping triangles that have occured before. This gives the following table, containing the same triangles but enumerated in this order.

no.	triangle sides	$m-n$
1	$3-4-5$	$2-1, 3-1$
2	$8-15-17$	$4-1, 5-3$
3	$5-12-13$	$5-1, 3-2$
4	$12-35-37$	$6-1$
5	$20-21-29$	$5-2, 7-3$
6	$7-24-25$	$7-1, 4-3$
7	$16-63-65$	$8-1$
8	$28-45-53$	$7-2$
9	$9-40-41$	$9-1, 5-4$

To produce non-right rational triangles, we need to glue two cathetii together. For two Pythagorean triangles, this can be done in at most four ways. Thus, each pair of entries in the list need to be considered, and each such pair may result in at most four entries.

In gluing we can work with integer sides only. We then before gluing expand each of the two triangles with the side length of the other triangle to be glued, after having divided the side lengths with any common factor. For example, let us consider gluing 3-4-5 and 3-4-5. When gluing 3 and 3 we get the rational triangle 8-5-5. When gluing 3 and 4 the 3-4-5 triangle reappear. Finally, gluing 4 and 4 gives 6-5-5.

Similarly, gluing 5-12-13 with another copy of itself gives three distinct rational triangles: 13-13-24, 65-156-169 and 10-13-13. Gluing 3-4-5 and 5-12-13 gives four distinct rational triangles: 50-39-25, 21–20-13, 15-14-13, and 63-52-25.

Pairing	side lengths of non-right rational triangles
$3-4-5$ and $3-4-5$	$5-5-6, 5-5-8$
$3-4-5$ and $8-15-17$	$10-17-21, 17-25-28, 40-51-77,$
	$72-75-77.$
$8-15-17$ and $8-15-17$	$17-17-30, 16-17-17, 136-275-289$
$3-4-5$ and $5-12-13$	$50-39-25, 21-20-13, 15-14-13,$
	$63-52-25$
$5-12-13$ and $5-12-13$	$3-13-24, 65-156-169, 10-13-13$
$8-15-17$ and $5-12-13$	$17-39-44$

Each non-right rational triangle has of course two more heights than the one that was part of its construction. By dividing by one of these heights we get two Pythagoran triangles. For example, dividing the $5-5-6$ triangle in this way gives the Pythagorean triangles $7-24-25$ and $3-4-5$.

3.3 Lines and regions of T

In this subsection we quantify and name different parts of T. Inspired by the golden triangles of the third generation of the family of commensurable triangles, we partly use the golden ratio for this.

The golden ratio is often denoted $\varphi = \frac{\sqrt{5}+1}{2}$. It is one of the solutions of the equation $x^2 = x + 1$, the other is φ^{-1}. Approximately we have $\varphi \approx 1.618$, and the equation immediately gives that $\varphi^2 = \varphi + 1 \approx 2.618$ and $\varphi^{-1} = \varphi - 1 \approx 0.618$. Certain triangles have this ratio between one of the three possible pairs of sides. The golden triangles are special in that they have this ratio between two pairs of sides. Such triangles appear at three distinct curves, one for each ratio: largest/smallest, largest/middle and middle/smallest that has the golden ratio. Sides of a triangle represented by a point $(x, y) \in T$ has lengths $\sqrt{(x-1)^2 + y^2}$, $\sqrt{x^2 + y^2}$ and 1. So we get the following three curves of triangles where one pair of sides has the golden ratio. The middle/smallest ratio gives the equation

$$\frac{\sqrt{x^2 + y^2}}{\sqrt{(x-1)^2 + y^2}} = \varphi,$$

which can be written as

$$\varphi^2(x-1)^2 - x^2 + (\varphi^2 - 1)y^2 = 0.$$

By using $\varphi^2 = \varphi + 1$ and $\varphi^{-1} = \varphi - 1$, from which $(\varphi - 1)^{-1} = \varphi$ and $\varphi^2 - 1 = \varphi$ also follows, we get eventually

$$(x - \varphi)^2 + y^2 = 1.$$

So this curve is a circle with center in $(\varphi, 0)$ and radius 1.

If we fix the relation largest/smallest to φ we have the equation

$$\frac{1}{\sqrt{(x-1)^2+y^2}} = \varphi$$

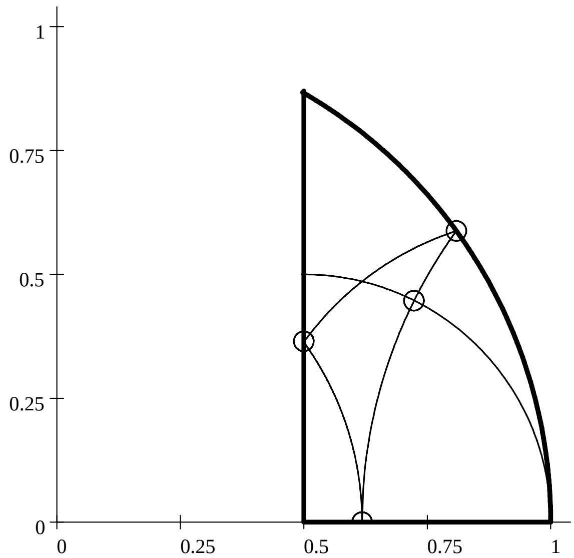
for all points fulfilling this condition. This is a circle

$$(x-1)^2 + y^2 = \varphi^{-1}$$

with center in $(1, 0)$ and radius φ^{-1}. Finally, if we the relation largest/middle is fixed to the golden ratio we get

$$\frac{1}{\sqrt{x^2+y^2}} = \varphi,$$

i.e. a circle $x^2 + y^2 = \varphi^{-1}$with center in $(0, 0)$ and radius φ^{-1}.



Borders of the regions – Theano's, euqilateral, wide and narrow – and the Pythagoras line.

Theano is supposed to be the wife of Pythagoras and credited to the first text on the golden ratio, although not much is certain from this time. In her honor we may call the three lines of triangles where two sides relate as the golden ratio **the Theano lines**, the first ($(x-\varphi)^2+y^2 = 1$), second ($(x-1)^2+y^2 = \varphi^{-1}$) and third ($x^2+y^2 = \varphi^{-1}$). The first Theano line is perpendicular to the Pythagoras line, see Section 4.1. We may similarly call the area bounded by the Theano lines the **Theano region**. In this region, the ratio of the largest to the smallest side is at least φ, and the other two ratios are at most φ. Note that at the intersection of the first Theano line and the Pythagoras line we have half of a golden rectangle, having side proportion φ. This triangle is derived from the rectangle by cutting along a diagonal, and has sides $1, \varphi, \sqrt{\varphi^2+1}$. Close to it on the Pythagoras line is the half equilateral, which is commensurable and corresponds to the triple $(1, 2, 3)$. At the other intersection with the Pythagoras line is the triangle with sides $\sqrt{\varphi^2-1}, 1, \varphi$, which is close to the Pythagorean triangle with sides $3, 4$ and 5.

In two of the corners of the Theano region we have the two golden triangles – the acute (sides: $1, \varphi, \varphi$, angles: $\frac{\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}$) and the obtuse (sides: $1, 1, \varphi$, angles: $\frac{\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{5}$). Only in these two intersections two pairs of sides fulfill the golden ratio in the same triangle. We remark that φ and $\frac{\pi}{5}$ are related by the formula $\varphi = 2\cos\frac{\pi}{5}$.

The third corner of the Theano region lies at the side of degenerate triangles. This corner divides the line from 0 to 1 in the golden ratio - in "a mean and extreme ratio", as Euclid formulated it. The Theano region also separates three other subregions. This is the **equilateral region** of triangles close to the top point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ where the ratio of the largest to the smallest is at most φ and all three angles have comparable size, the **narrow region** close to $(1, 0)$ where the middle to the smallest is at least φ and one angle is considerably smaller than the other two, and the **wide region** close to $(\frac{1}{2}, 0)$ where the largest to the middle side is at least φ and two angles are considerably smaller than the third. Note that the equilateral region is bounded by one of the Theano lines and by the wide and narrow isosceles boundaries. Only the golden triangles are at the boundary of three of the four regions.

An isosceles triangle can be divided in two identical right triangles by a height. This is a well defined mapping from isosceles triangles to the right triangles. How does this mapping work in T? We can describe this as follows. If we start at the boundary in $(\frac{1}{2}, 0)$ and move along the front through the top (equilateral) point and along the back to $(1, 0)$ we go through all isosceles triangles. The corresponding half triangle point will then start in $(1, 0)$, move along the Pythagoras line, hit $(\frac{1}{2}, \frac{1}{2})$ simultaneuosly as the whole triangle point, since it is both isosceles and right. Then it goes back again, passes $(\frac{3}{4}, \frac{\sqrt{3}}{4})$ which is the point of the half equilateral (also called $(1, 2, 3)$), when the whole triangle point is in the top point. Then the half triangle point returns to $(1, 0)$, where the two points meet a second time.

4 Angles and sides

In this section we prove the following:

Theorem 16 *There are no triangles that are both commensurable and rational.*

This theorem follows immediately by the following, since the equilateral is not a rational triangle.

Theorem 17 *The only triangle where the angles are commensurable with each other and the sides are commensurable with each other is the equilateral triangle.*

Proof. Consider a triangle with angles A, B and C and sides a, b and c, where $A \leq B \leq C$ and $a \leq b \leq c$. Suppose that both angles and sides are commensurable. We may then assume that a, b and c are rational numbers. From the cosine theorem $a^2 = b^2 + c^2 - 2bc\cos A$ it follows that also $\cos A$ is rational, so let $\cos A = \frac{p}{q}$, where $LCD(p,q) = 1$. We then also have $\sin A = \frac{\sqrt{p^2-q^2}}{q}$, and we may define a complex number z having $\cos A$ as its real part:

$$z = \cos A + i\sin A = e^{iA} = \frac{p}{q} + i\frac{\sqrt{q^2-p^2}}{q}.$$

Since a commensurable triangle has angles $\frac{n\pi}{n+m+k}, \frac{m\pi}{n+m+k}, \frac{k\pi}{n+m+k}$ (see page 3), they necessarily have the form $\frac{k}{n}\pi$ for some positive integers n and k. So;

$$z = \cos A + i\sin A = e^{iA} = e^{i\frac{k}{n}\pi}.$$

Furthermore, a number $e^{i\frac{k}{n}\pi}$ where k and n are positive integers is precisely the type of number e^{ix}, x real, that fulfills an equation

$$z^n = 1$$

for some positive integer n. This is a useful observation, since a formulation in terms of an equation often is efficient in a proof.

We summarize. Because of the side commensurability, we know concerning the smallest angle A that

$$z = e^{iA} = \frac{p}{q} + i\frac{\sqrt{q^2-p^2}}{q}.$$

The angle commensurability provides the equation $z^n = 1$, i.e.

$$(\frac{p}{q} + i\frac{\sqrt{q^2-p^2}}{q})^n = 1.$$

We next show that $p = \pm 1$ is the only possibility in this equation. Since A is the smallest angle we know that $A \leq \frac{\pi}{3}$ and $\frac{1}{2} \leq \cos A \leq 1$. Then from $p = \pm 1$ and $\frac{1}{2} \leq \cos A$ it follows that $q = \pm 2,$, so $\frac{p}{q} = \frac{1}{2}$ which forces $A = B = C = \frac{\pi}{3}$, and the proof is complete.

So let us prove that $p = \pm 1$. This is easy if n is odd. Let $r = q^2 - p^2$. Then the binomial expansion of $(\frac{p}{q} + i\frac{\sqrt{r}}{q})^n$ gives

$$p^n + i\binom{n}{1}p^{n-1}\sqrt{r} - \binom{n}{2}p^{n-2}r - ... + (-1)^n\binom{n}{1}p(\sqrt{r})^{n-1} + (-1)^{\frac{n-1}{2}}i(\sqrt{r})^n = q^n$$

when multiplied with q^n. Now the real part of this equality provides

$$p(p^{n-1} - \binom{n}{2}p^{n-3}r + ... + (-1)^n\binom{n}{1}(\sqrt{r})^{n-1}) = q^n.$$

Since r is an integer and the exponents of $\sqrt{r}$ are even integers, the number in the large parenthesis is an integer. It follows that p divides q. But since $LCD(p,q)=1$, the only possibility is that $p=\pm 1$. So the theorem follows if n is odd.

If n is even we define $n=2^k u$, where u is odd, and reduce the problem to the equation $z^u=1$. Then the result follows as before, since u is odd.

If $n=2^k u$ we have $z^n=z^{2^k u}=(z^u)^{2^k}=w^{2^k}$, so by denoting $w=z^u$, the equation $z^n=1$ transforms into $z^u=w$ and $w^{2^k}=1$.

Now, if $\sqrt{r}$ is irrational, we denote $Q[i\sqrt{r}]=\{a+bi\sqrt{r}:a,b\in\mathbf{Q}\}$. Then, of course, $z\in Q[i\sqrt{r}]$. Furthermore, since $w=z^u$, we have $w\in Q[i\sqrt{r}]$. Now, the only solutions in $Q[i\sqrt{r}]$ to $w^{2^k}=1$ are $w=\pm 1$ – see Lemma 18. Then we obtain the equations $z^u=\pm 1$, which has odd exponent. So this part of the proof is complete by the same argument as above once we have established Lemma 18.

If $\sqrt{r}=p'$ is rational, we have by $r=q^2-p^2$ the relation $q^2=p^2+p'^2$, so by $LCD(p,q)$, also $LCD(p',q)$ follows. Lemma 19 tells us that $w=\pm 1$ and $w=\pm i$ are the only possible solutions to $w^{2^k}=1$ in $Q[i]$, so we get either $z^u=\pm 1$ or $z^u=\pm i$. $z^u=\pm 1$ we have $(p+ip')^u=\pm q^u$, and from the real part we again can see that $p=1$.

$z^u=\pm 1$ we have

$$(p+ip')^u=\pm i q^u,$$

which is equivalent to

$$(p'-ip)^u=\pm q^u.$$

Hence, $p'=1$, and by $q^2=p^2+p'^2$ it follows that $p=0$ and $q=1$, hence $\cos A=0$ and $A=\pi/2$, which contradicts $A\leq\pi/3$. So this case cannot happen, and the proof is complete when Lemma 18 and 19 are proved. ■

Lemma 18 *Suppose that* $r\in\mathbf{N}, r>1$. *Then the only possible solutions to* $z^{2^k}=1$ *in* $Q[i\sqrt{r}]=\{a+bi\sqrt{r}:a,b\in\mathbf{Q}\}$ *for any positive integer* k *are* $z=\pm 1$.

Proof. In order to achieve a contradiction we suppose that there exist values of k where there is a solution to $z^{2^k}=1$ other than $z=\pm 1$. Suppose that k is the lowest such number, and that z is such a solution. But then we know that $w=z^2\in Q[i\sqrt{r}]$ and that w is a solution to $w^{2^{k-1}}=1$, which has lower degree. So by the assumption $w=\pm 1$ are the only solutions to $w^{2^{k-1}}=1$. This means that $z^2=\pm 1$. The equation $z^2=1$ has the only solutions $z=\pm 1$ in $Q[i\sqrt{r}]$ if $r\in\mathbf{N}$,$r>1$. This contradicts the assumption on z. It follows that there cannot be any other solutions to $z^{2^k}=1$ in $Q[i\sqrt{r}]$ than $z=\pm 1$. ■

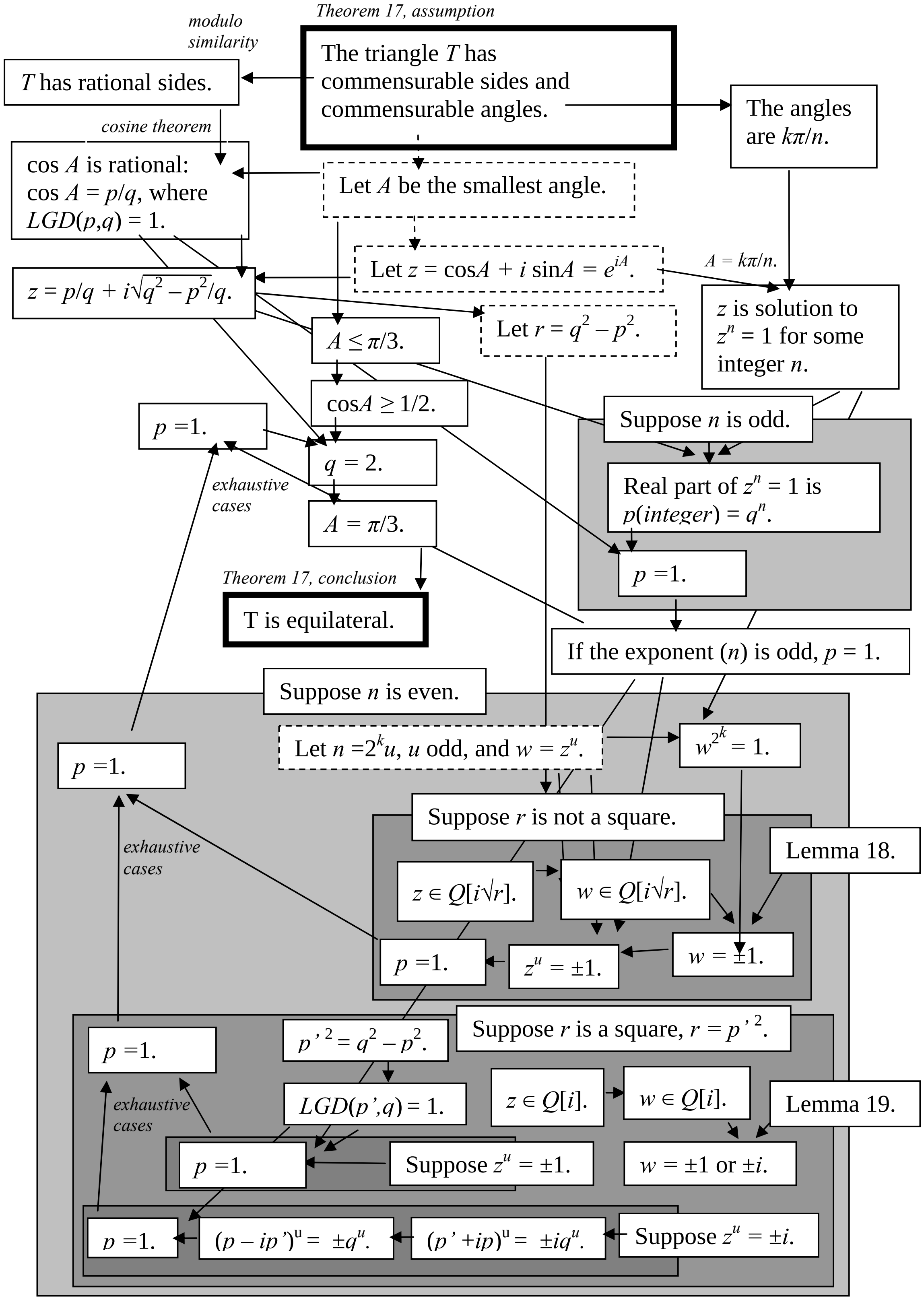
Lemma 19 *The only possible solutions to* $z^{2^k}=1$ *in* $Q[i]=\{a+bi:a,b\in\mathbf{Q}\}$ *for any positive integer* k *are* $z=\pm 1$ *and* $z=\pm i$.

Proof. In order to achieve a contradiction we suppose that there exist values of k where there is a solution to $z^{2^k} = 1$ in $Q[i]$ other than $z = \pm 1$ or $z = \pm i$. Suppose that k is the lowest such number, and that z is a solution which is neither $z = \pm 1$ nor $z = \pm i$. But then we know that $w = z^2 \in Q[i]$ and that w is a solution to $w^{2^{k-1}} = 1$, which has lower degree. So by the assumption, $w = \pm 1$ or $w = \pm i$ are the only solutions to $w^{2^{k-1}} = 1$. This means that $z^2 = \pm 1$ or $z^2 = \pm i$, which has the only solutions $z = \pm 1$ or $z = \pm i$ in $Q[i]$, contradicting the assumption on k and z. It follows that there cannot be any other solutions to $z^{2^k} = 1$ in $Q[i\sqrt{r}]$ than $z = \pm 1$ or $z = \pm i$. ■

Per-Anders Svensson, Växjö University, Sweden, has kindly remarked that the proofs in this section can be made shorter using cyklotomic polynomials (see [5]).

At the next page the proof of Theorem 17 is also given as a so called logical graph, a concept that was introduced in [6]. Here an implication $A \wedge B \Rightarrow C$ is denoted in three boxes, A, B and C, with arrows from A and B, both pointing to C. Thus, the arrows are implications. Note that, logically, the "AND" operation is used much more often in mathematical arguments than "OR". Boxes contain statements, except boxes with dotted borders, which only contains new notation. Dotted arrows are not implications, but show how notations are connected in terms of each other. When an assumption is made, the area where this assumption is valid is marked by a grey color. Implications may cross the border of such an assumption domain only from outside to the inside, except at the conclusion of the assumption domain or by the effect of several assumptions that represent exhaustive cases.

The logical structure of a mathematical proof is usually not linear, although the narrative in an article, a book or during a lecture is linear, since time itself is linear. A logical graph representation is consistent with the logical structure of the proof, rather than with the linear nature of time. It allows the reader to study a proof in any order - from assumption or from conclusion. Statements are not restated when they are needed again, instead an arrow is drawn from the single representation of it.

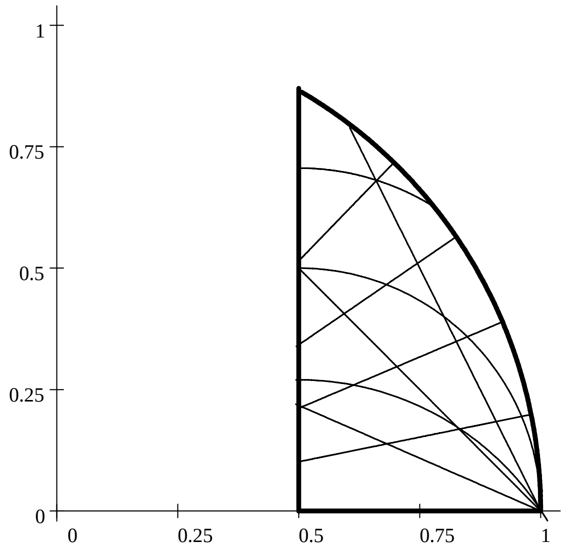


Proof of Theorem 17 in logical graph form.

4.1 Isoangle curves and isoside curves

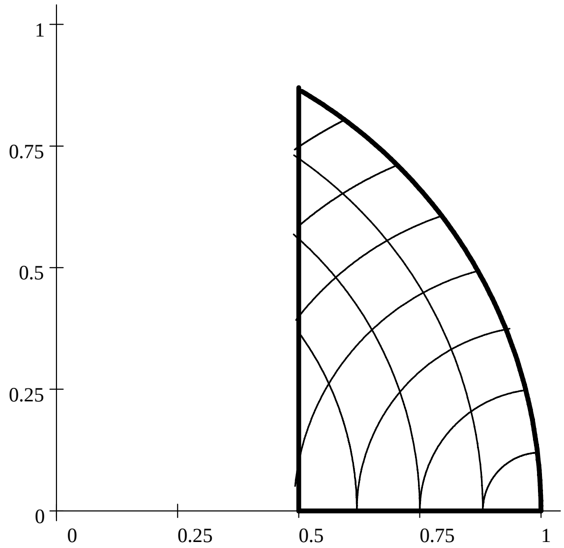
We conclude the paper with some remarks on the correspondence of lines where an angle or a side is constant.

On the Theano lines a certain ratio of sides is fixed. Here we consider lines where one angle or one side has fixed size. On which curves in the triangle sector is one angle constant? The angles at $(0,0)$ and $(1,0)$ are of course constant at straight lines through these points. Since the peripheral corresponding to a certain corda in a circle are the same, the points with the same third angle lies on a circular arc passing $(0,0)$ and $(1,0)$. The center of such a circular arc is at $(\frac{1}{2}, a)$ for some a. If we in $(\frac{1}{2} + r\cos t, a + r\sin t)$ eliminate t, we obtain the equation $r^2 = \frac{1}{4} + a^2$, relating the center and the radius of these circles. The radius is smallest for the Pythagoras line, which is an isoangle line with $a = 0$ and $r = \frac{1}{2}$. The Pythagoras line is also the only one in this family that has a common tangent to the narrow isosceles boundary.



Isoangle lines - keeping one angle constant.

There are only two families of curves at which the sidelengths are constant, since the length of one side is fixed to 1.



Isoside lines - keeping one side length constant.

Both families consist of concentric curves, while the narrow isosceles boundary is part of the family where the middle side is fixed, in this case being 1.

The isoside lines of the middle side are perpendicular to the isoangle lines of the smallest angle. The isoside lines of the smallest side are perpendicular to the isoangle linesof the middle angle. The lines perpendicular to the isoangle lines of the largest angle have no side counterpart. This family consists also of circles, of which the first Theano line is one. This family is related to both the other sides, since they are perpendicular to the curves on which the ratio of the two smallest sides is constant.

Certainly, most of the constructions done in the triangle sector can also be done in the metatriangles, and vice versa.

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