NONLINEAR STANDING WAVES IN A CLOSED TUBE

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Simplified nonlinear evolution equations describing nonsteady-state forced vibrations in an acoustic resonator having one closed end and the other end periodically oscillating are derived. An approach is used based on a nonlinear functional equation. This approach is shown to be equivalent to the version of the successive approximation method developed in 1964 by Chester. It is explained how the acoustic field in the cavity is described as a sum of counterpropagating waves with no cross-interaction. The nonlinear Q-factor and the nonlinear frequency response of the resonator are calculated for steady-state oscillations of both inviscid and dissipative media. The general expression for the mean intensity of the acoustic wave in terms of the characteristic value of a Mathieu function is derived. Some results from a perturbation calculation of the wave profile are given.

1 Introduction and basic nonlinear equations

The resonance is known as one of the most interesting phenomena in the physics of vibrations and waves. It manifests itself markedly, if the dependence of the amplitude of a forced oscillation on frequency (i.e. the frequency response) has a sharp maximum. The ratio of the central frequency ω_0 of the spectral line imaging the response to the characteristic width of this line is called the Q-factor and is used as a measure of the "quality" of a resonant system. This work is devoted to the analysis of the frequency response and Q-factor of a nonlinear acoustic reaonator. The treatment of the problem is based on Kuznetsov's equation [1]

$$\frac{\partial^2\Phi}{\partial t^2} - c^2\Delta\Phi = \frac{\partial}{\partial t}[(\mathrm{grad}\,\Phi)^2 + \frac{1}{c^2}(\epsilon-1)\Big(\frac{\partial\Phi}{\partial t}\Big)^2 + \frac{b}{\rho}\Delta\Phi] \ , \qquad (1)$$

written for the potential Φ of the particle velocity $\vec{u} = -\nabla\Phi$. Here c is the sound velocity, ρ is the density of the medium, b is the effective viscosity and ϵ is the nonlinearity parameter, defined, for example, in reference 2.

The equation (1) is applied to a one-dimensional system of length L with the boundary conditions formulated for the particle velocity $u = -\frac{\partial \Phi}{\partial x}$:

$$u(x=0,t) = 0 \tag{2}$$

$$u(x=L,t) = Af(\omega t), \tag{3}$$

where A is an amplitude constant and ω is an imposed frequency.

For the case $b = 0$ the solution of Eq. (1) can be written as a sum of two travelling Riemann waves:

$$u = u_1 + u_2 = F_1(\omega t - \kappa x + \frac{\epsilon}{c^2}\omega x F_1) + F_2(\omega t + \kappa x + \frac{\epsilon}{c^2}\omega x F_2), \tag{4}$$

On the other hand, for the first-approximation solution of Eq. (1), the velocity is written as the sum of two waves travelling in opposite directions [3]

$$u^{(1)}(x,t) = F_1(\eta_1 = t - \frac{x}{c}) + F_2(\eta_2 = t + \frac{x}{c}), \tag{5}$$

where from (2) we obtain
$F_1 = F_2 = F$.
Two assumptions are now made in order to make appropriate assumptions.

First, the length of the resonator must be small in comparison to the shock formation length [2]:

$$L << \frac{c^2}{\epsilon\omega |F|_{max}}, \tag{6}$$

where $|F|_{max}$ is the maximum amplitude of the function F.

Second, the frequency ω of the vibration of the righthand boundary must differ slightly from a resonant frequency $n\omega_0$:

$$\omega - n\omega_0 = \frac{\Delta}{\pi}\omega_0, \ \Delta << 1, \tag{7}$$

where Δ is the discrepancy and $\omega_0 = \frac{\pi c}{L}$ is the frequency corresponding to the fundamental eigenmode $n = 1$.

Now the boundary condition (3), applied both to a form of (4), generalized to the case $b \neq 0$, and to (5), after some manipulations gives the "inhomogeneous Burgers equation" with "slow time" T and "fast time" ξ:

$$\frac{\partial U}{\partial T} + \Delta\frac{\partial U}{\partial \xi} - \pi\epsilon U\frac{\partial U}{\partial \xi} - D\frac{\partial^2 U}{\partial \xi^2} = -\frac{M}{2}f(\xi - \pi) \tag{8}$$

with the notation

$$U = \frac{F}{c},\ M = \frac{A}{c},\ \xi = \omega t + \pi,\ T = \frac{\omega t}{\pi},\ D = \frac{b\omega^2 L}{2c^3\rho} << 1. \tag{9}$$

Thus it is shown that Chester's approach is equivalent to an approach based on a nonlinear functional equation approach using (4).

2 Steady-state vibrations

The equilibrium state reached at $T \to \infty$ can be described by the ordinary differential equation, obtained by integration of Eq. (8) at $\frac{\partial U}{\partial T} = 0$, and with $f = \sin\xi$, i.e. harmonic vibration of the boundary:

$$D\frac{dU}{d\xi} + \frac{\pi\epsilon}{2}(U^2 - C^2) - \Delta U = \frac{M}{2}\cos\xi. \tag{10}$$

From (10) follows that the constant C can be interpreted as the normalized intensity of one of two counterpropagating waves:

$$\overline{U^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} U^2 d\xi = C^2. \tag{11}$$

The mean value of U is assumed to be zero:

$$\overline{U} = \frac{1}{2\pi}\int_{-\pi}^{\pi} U d\xi = 0. \tag{12}$$

For negligible weak linear absorption, $D \to 0$, the solution of the quadratic equation corresponding to Eq. (10) is

$$U = \frac{\Delta}{\pi\epsilon} \pm \sqrt{(\frac{\Delta}{\pi\epsilon})^2 + C^2 + \frac{M}{\pi\epsilon}\cos\xi}. \tag{13}$$

Because of the condition (12) the "-" solution of (13) is valid for $\Delta > 0$ and the "+" solution for $\Delta < 0$.

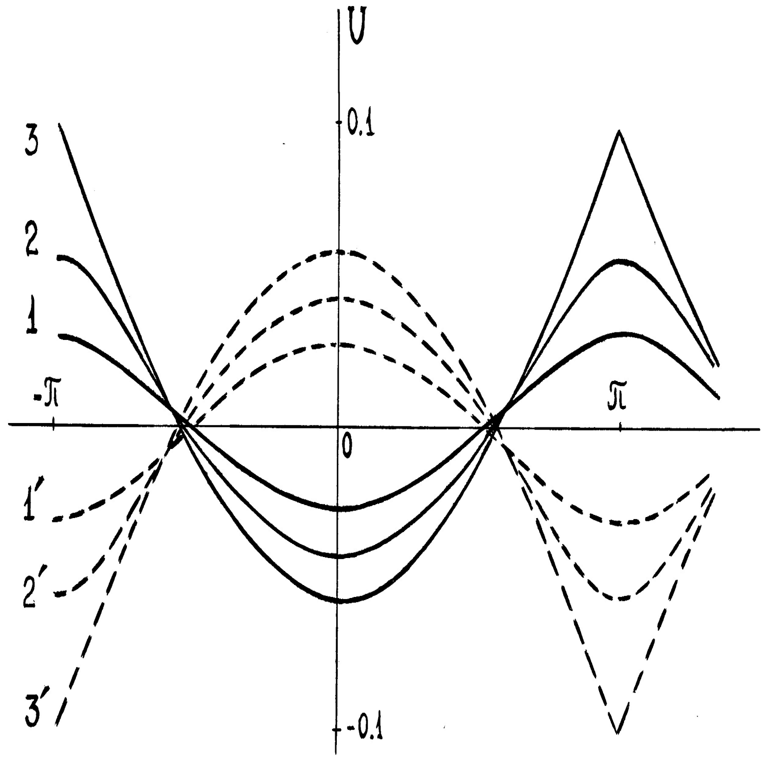

Figure 1: Temporal profile U of one of two counter-propagating waves forming the steady-state vibration in a non-dissipative layer. The dimensionless amplitude $M = \frac{A}{c}$ of the vibration of the boundary is small enough and U does not contain shocks. $(\frac{M}{\pi\epsilon})\cdot 10^3$ equals to 5.6, 9.1 and 12.3 for the curves 1, 2 and 3 correspondingly.

We will here give an example of the jump between the branches in (13), giving a shock. At $M = M_* \equiv \frac{\pi}{8\epsilon}\Delta^2$ the bifurcation happens and the stady-state waveform becomes discontinuous. The shock front appears at each period of the wave, connecting the two branches of the solution (13). At the moment $\xi = \xi_{SH}$, found by applying the condition $\overline{U} = 0$, the solution (13) jumps from the "-" branch to the "+" branch. This jump corresponds to a shock of compression. Because a shock of rarefaction is prohibited, both branches of the solution (13) must have one common point in each period. If and only if the common point exists, the transition can go on in the opposite direction, from the "+" to the "-" branch, without jump.

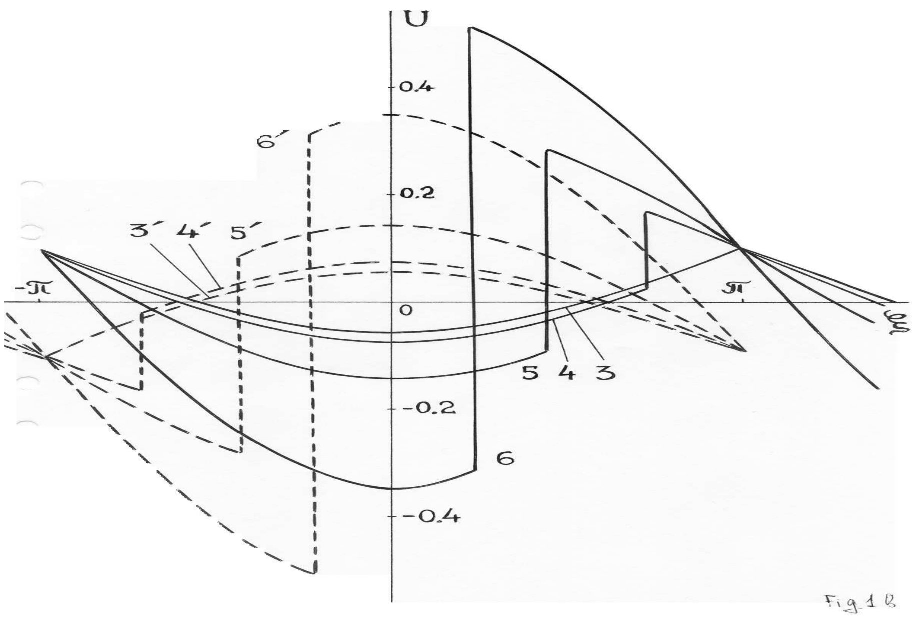

Figure 2: The same profiles as in Figure 1 are constructed for greater magnitudes of M. The profiles 4, 5 and 6 corresponding to $(\frac{M}{\pi\epsilon})\cdot 10^2 = 1.5$, 3 and 10 contain shocks. The profile 3 is identical with profile 3 in Figure 1.

The common point exists if the expression under the square root in (13) is equal to zero, or

$$C^2 = \frac{M}{\pi\epsilon} - (\frac{\Delta}{\pi\epsilon})^2. \tag{14}$$

For given eigenvalue (14) the solution (13) reduces to

$$U = \frac{\Delta}{\pi\epsilon} \pm \sqrt{\frac{2M}{\pi\epsilon}}|\cos\frac{\xi}{2}|. \tag{15}$$

By use of (12) the position of the jump is calculated:

$$\sin(\frac{\xi_{SH}}{2}) = \frac{\Delta}{2}\sqrt{\frac{\pi}{2\epsilon M}}. \tag{16}$$

The frequency response is analysed not for the frequency dependence of the amplitude, as is customary for linear vibration, but for the frequency dependence of the root-mean-square (rms) particle velocity $\sqrt{\overline{U^2}}$. However, by analogy with the use of the frequency response in the evaluation of the Q-factor of a linear resonator, it is possible to evaluate the Q-factor of a nonlinear resonator. Two definitions, analogous to the definitions in the case of a linear oscillator, are used. The first definition of the Q-factor gives

$$Q_{NL} = \frac{c(\sqrt{\overline{U^2}})_{\Delta=0}}{A} = \frac{c}{A}\sqrt{\frac{M}{\pi\epsilon}} = \frac{1}{\sqrt{M\pi\epsilon}}. \tag{17}$$

The second definition gives, using (12)

$$Q_{NL} = \frac{1}{\Delta} = \frac{\pi}{2\sqrt{2}}\frac{1}{\sqrt{M\pi\epsilon}}, \tag{18}$$

which differs only slightly from (17).

For $D \neq 0$ we have to study the differential equation (10). Using the transformation

$$U = \frac{2D}{\pi\epsilon}\frac{d}{d\xi}\ln W, \tag{19}$$

the nonlinear first order differential equation (10) is reduced to the linear second order differential equation

$$\frac{d^2W}{d\xi^2} - \frac{\Delta}{D}\frac{dW}{d\xi} = (\frac{\pi\epsilon}{2D})^2[C^2 + \frac{M}{\pi\epsilon}\cos\xi]W. \tag{20}$$

For zero discrepancy ($\Delta = 0$) the equation (20) can be transformed into the canoncal form of the Mathieu equation

$$\frac{d^2W}{dz^2} + [-(\frac{\pi\epsilon}{D})^2C^2 - \frac{\pi\epsilon M}{D^2}\cos 2z]W = 0,\ z = \frac{\xi}{2}. \tag{21}$$

Because of (12) the function W must be periodic and thus W can be written in terms of Mathieu functions[4]

$$W = ce_0(z, q = \frac{\pi\epsilon M}{2D^2}). \tag{22}$$

The intensity (11) of the wave (19), (22) is determined by the characteristic value $a_0(q)$ of the Mathieu function ce_0 [4]:

$$\overline{U^2} = C^2 = -(\frac{D}{\pi\epsilon})^2 a_0(q). \tag{23}$$

Approximate values of $a_0(q)$ for $q << 1$ and $q >> 1$ are respectively

$$a_0 \approx -\frac{q^2}{2} + \frac{7q^4}{128},\ q << 1 \tag{24}$$

$$a_0 \approx -2q + 2\sqrt{q} - \frac{1}{4} - \frac{1}{32\sqrt{q}},\ q >> 1. \tag{25}$$

For arbitrary values of dissipation D the Q-factor equals to

$$Q = Q_{NL}\sqrt{-d^2 a_0(q = \frac{1}{2d^2})},\ d = \frac{D}{\sqrt{\pi\epsilon M}}. \tag{26}$$

For the case $\Delta \neq 0$ the solution to (10) cannot be expressed through Mathieu functions. Approximate solutions to (10) can then be sought for by perturbation theory in the cases $q << 1$ and $q >> 1$.

3 Perturbation solution for $q >> 1$

Since the solution of the nonlinear equation (10) cannot be expressed through Mathieu functions, another approach will be chosen to the problem of finding the mean intensity depending on M, Δ and D in the opposite limiting case, i.e. $q >> 1$, for standing waves containing shocks. Combining (14), (23) and (25) we write the two first terms in a series expansion C^2 in the small parameter $q^{-\frac{1}{2}}$:

$$C^2 = \frac{M}{\pi\epsilon} - \frac{\Delta^2}{\pi^2\epsilon^2} - k\frac{M}{\pi\epsilon}q^{-\frac{1}{2}}, \tag{27}$$

where k is a constant to be determined. Inserting (27) into (26) and making the substitution

$$U - \frac{\Delta}{\pi\epsilon} = \frac{\sqrt{2}M^{\frac{1}{2}}}{\sqrt{\pi\epsilon}}V \tag{28}$$

we obtain

$$V^2 + \frac{1}{2q^{\frac{1}{2}}}\frac{dV}{d\tau} = \frac{1}{2}(1 + \cos 2\tau) - \frac{k}{2q^{\frac{1}{2}}}. \tag{29}$$

Introducing the small parameter $\nu \equiv 2q^{-\frac{1}{2}}$ we obtain from (71):

$$V^2 + \nu\frac{dV}{d\tau} = \cos^2\tau - \nu k. \tag{30}$$

The solution to Eq. (30) is assumed to have a jump at $\tau = \tau_0, \ -\pi < \tau_0 < \pi$. The matched asymptotic expansion method [5] on Eq. (30) has been used and an "outer" solution, i.e. outside the jump at $\tau = \tau_0$, and a corresponding "inner" solution are expanded in powers of ν [6]. The result is a uniform solution for $-\frac{\pi}{2} < \tau < \frac{\pi}{2}$. Curves of $u(x,t)$ with x-dependence at given t and t-dependence at given x are given for $\Delta = 0$. The curves are compared with the corresponding curves obtained with neglect of nonlinearity.
The following parameter values are used:
$L = 0.5$ m
$c = 330$ m/s
$\omega_0 = \frac{\pi c}{L} \approx 2000$ s^{-1}
$\epsilon = 1.4$
$\frac{b}{\rho_0} \approx 10^{-2}$ m^2/s
$D = \frac{b\omega^2 L}{2c^3\rho_0} \approx 0.3 \cdot 10^{-3}$
$q \approx 120$.

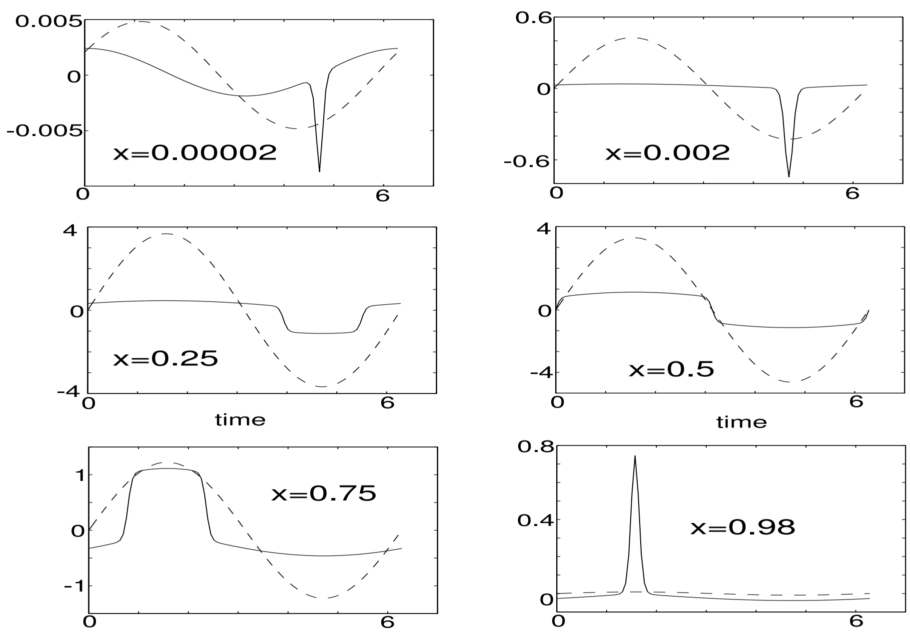

Figure 3: One time period of the wave at different distances x from the moving boundary in a resonator of length 1. The solid line is the nonlinear solution, which is compared to the solution with neglect of nonlinearity (dashed line).

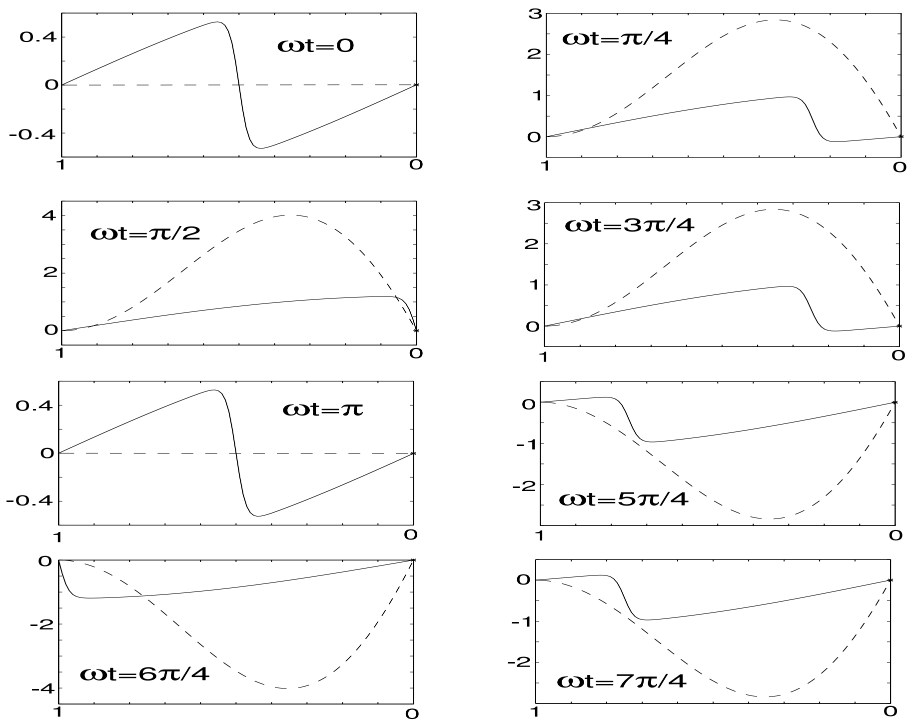

Figure 4: The wave amplitude along the length of a resonator for different ωt showing the wave form, where x is the distance from the moving boundary. The solid line is the nonlinear solution, which is compared to the solution with neglect of nonlinearity (dashed line).

4 References

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