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WAVE MOTION IN A MEDIUM WITH A CUBIC NONLINEARITY

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Abstract

An example of wave motion in a medium with a cubic nonlinearity is a transverse finite amplitude wave in an isotropic solid. The corresponding cubically nonlinear wave equation is derived with the nonlinearity expressed in terms of elastic constants. This nonlinear wave equation with dissipation is studied for standing and propagating waves. For standing waves in a resonator a simplified approach results in functional equations, from which frequency response curves are derived. These curves show the dependence of the amplitude on the difference between one of the resonator's ekgenfrequencies and the driving frequency. The frequency response curves are plotted for different values of the dissipation and are very different for quadratic and cubic nonlinearities. In the propagating wave case an N-wave evolution is studied, described by a modified Burgers' equation with a cubic nonlinearity. Approximate solutions to this equation are found for parts of the wave profile not studied in detail before.

1 Introduction

The literature on nonlinear acoustic waves in dissipative media is extensive. For a quadratic nonlinearity the waves fulfil Burgers' equation in a plane geometry and generalized Burgers' equations in other geometries [1,2,3,4]. For cubic nonlinearities few investigations are made. It is interesting to study if some of the methods used in refs. [1-4] can be generalized for application to cubic nonlinearities. In the present paper such studies are made, which show new features of cubically nonlinear waves in comparison with quadratically nonlinear waves.

Phywsical examples of quadratic nonlinearities are given by sound waves in fluids [3] and by longitudinal elastic waves in solids [5]. The nonlinear equation for transverse elastic waves in isotropic solids is derived by Zabolotskaya [6], who shows that the quadratic nonlinearity cancels. Thus the equation for transverse elastic waves in isotropic solids is cubically nonlinear. The derivation of this equation with a correction to Zabolotskaya's result [6] is the contents of section 2. Two examples of cubically nonlinear wave theory are studied. In section 3 standing waves in a cubically nonlinear resonator are studied and the frequency response function is calculated and compared with the corresponding function in the quadratically nonlinear case [7]. In section 4 cubically nonlinear propagation of N-waves is studied using methods developed for quadratically nonlinear waves [8].

2 Physical background

The equation of motion of a continuous medium is [5]

$$\rho_0 \frac{\partial^2 U_i(\vec{a},t)}{\partial t^2} = \frac{\partial}{\partial a_j}(P_{ij} + D_{ij}), \tag{1}$$

where
ρ_0 is the density in the undeformed state,
U_i is the displacement, $U_i = x_i - a_i$, of the particle originally at a_i (Lagrangian coord.),
P_{ij} is the stress tensor,
D_{ij} is the dissipative stress tensor.
The elastic energy W is expanded in the invariants of the strain tensor E_{ij},

$$E_{ij} = \frac{1}{2}(\frac{\partial U_i}{\partial a_j} + \frac{\partial U_j}{\partial a_i} + \frac{\partial U_k}{\partial a_i}\frac{\partial U_k}{\partial a_j}), \tag{2}$$

to the fourth order [6]:

$$\begin{aligned}\rho_0 W = (\frac{K}{2} - \frac{\mu}{3})(E_{kk})^2 + \mu E_{ij}E_{ji} + \frac{C}{3}(E_{kk})^3 + BE_{kk}E_{ij}E_{ji} \\ + \frac{A}{3}E_{ij}E_{jk}E_{ki} + H(E_{kk})^4 + G(E_{ij}E_{ji})^2 + F(E_{kk})^2E_{ij}E_{ji} \\ + EE_{kk}E_{ij}E_{jl}E_{li} + DE_{ij}E_{jk}E_{kl}E_{li}.\end{aligned} \tag{3}$$

Using the relation

$$P_{ij} = \rho_0 \frac{\partial W}{\partial(\frac{\partial U_i}{\partial a_j})} \tag{4}$$

we obtain from (1) and (3) the equations of motion, with the notation $U_{ij} = \frac{\partial U_i}{\partial a_j}$,

$$\begin{aligned} &\rho_0\{\frac{\partial^2 U_i}{\partial t^2} - c_t^2 \frac{\partial^2 U_i}{\partial a_k^2} - (c_l^2 - c_t^2)\frac{\partial U_{kk}}{\partial a_i}\} \\ &= \frac{\partial D_i}{\partial t} + \frac{\partial}{\partial a_s}\{(\frac{K}{2} - \frac{\mu}{3})(U_{kl}U_{kl}\delta_{is} + 2U_{kk}U_{is} + U_{kl}U_{kl}U_{is}) \\ &\quad + \mu(U_{ik}U_{sk} + U_{ik}U_{ks} + U_{ki}U_{ks} + U_{ik}U_{rk}U_{rs}) + ...\}, \end{aligned} \tag{5}$$

where "..." stands for terms of second and third order in U_{ik} with coefficients A, B, C, D, E, F, G, H, and c_t and c_l are the transverse and longitudinal wave propagation velocities respectively,

$$c_t^2 = \frac{\mu}{\rho_0}, \;\; c_l^2 = \frac{K + \frac{4\mu}{3}}{\rho_0}. \tag{6}$$

The dissipative term $\frac{\partial D_i}{\partial t}$ in (5) is obtained from

$$D_i = \eta(\frac{\partial^2 U_i}{\partial a_k^2} + \frac{\partial^2 U_k}{\partial a_i \partial a_k}) + (\zeta - \frac{2}{3}\eta)\frac{\partial^2 U_k}{\partial a_i \partial a_k}, \tag{7}$$

where η, ζ are the shear and bulk viscosities respectively.

With the scalings

$$\begin{aligned} \overline{a}_1 = \epsilon^{\frac{1}{2}} a_1, \quad \overline{a}_2 = \epsilon^{\frac{1}{2}} a_2, \quad \overline{a}_3 = \epsilon a_3 \\ \overline{U}_1 = \epsilon^{-\frac{1}{2}} U_1, \quad \overline{U}_2 = \epsilon^{-\frac{1}{2}} U_2, \quad \overline{U}_3 = \epsilon^{-1} U_3 \\ \overline{\eta} = \epsilon^{-1}\eta, \quad \overline{\zeta} = \epsilon^{-1}\zeta \end{aligned} \tag{8}$$

a nonlinear beam equation can be derived [6] for transverse waves, vibrating in the 1-direction and propagating in the 3-direction. In this case the quadratic nonlinearity cancels and the equation for $\frac{\partial \overline{U}_1}{\partial t} = V$ becomes

$$\frac{\partial}{\partial \tau}\{\frac{\partial V}{\partial z} - \frac{\beta}{c_t^3} V^2 \frac{\partial V}{\partial \tau} - \delta \frac{\partial^2 V}{\partial \tau^2}\} = \frac{c_t}{2}\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\}V, \tag{9}$$

where (x, y, z, τ) is given as $(\overline{a}_1, \overline{a}_2, \overline{a}_3, t - \frac{\overline{a}_3}{c_3})$ and δ and β are given as

$$\delta = \frac{\overline{\eta}}{2\rho_0 c_t^3} \tag{10}$$

$$\beta = \frac{3}{2\rho_0 c_t^2}\{\frac{K}{2} + \frac{2\mu}{3} + B + \frac{A}{2} + G + \frac{D}{2}$$
$$-\frac{(\frac{K}{2} + \frac{2\mu}{3} + \frac{3B}{4} + \frac{A}{4})(K + \frac{4\mu}{3} + B + \frac{A}{2})}{K + \frac{\mu}{3}}\}. \tag{11}$$

The quotient term in (11), because of which β can be negative, is missing in the paper by Zabolotskaya [6]. The equation (9) is studied by Rudenko and Sapoznikov [9]. If K and μ are the only elastic constants different from zero in (11) we obtain using (6):

$$\beta = -\frac{3}{4}\frac{c_l^2}{c_l^2 - c_t^2}. \tag{12}$$

Thus β is negative for the simplest solid materials.

3 Standing waves in a cubically nonlinear resonator

Neglecting dissipation and transverse extension Eq. (9) becomes

$$\frac{\partial V}{\partial z} - \frac{\beta}{c^3}V^2\frac{\partial V}{\partial \tau} = 0, \tag{13}$$

where c_t is replaced by c in order that Eq. (13) be applied to other physical phenomena than transverse waves in solids. The choice of the negative sign in the definition of τ means that Eq. (13) describes rightgoing waves. Standing waves in a resonator are composed by waves propagating in both directions. A wave equation, which gives Eq. (13) when specialized to rightgoing waves, is (change z into x):

$$\frac{\partial^2 V}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{2\beta}{3c^4}\frac{\partial^2 V^3}{\partial t^2}. \tag{14}$$

The equation with a quadratic nonlinearity corresponding to the cubic nonlinear equation (14) is studied for a resonator by the present authors [7]. The resonator boundary conditions are

$$V(x = 0, t) = A \sin \omega t, \qquad V(x = L, t) = 0. \tag{15}$$

In analogy with the solution attempted by the present authors for a quadratic nonlinearity [7] we attempt a solution to Eq. (14) in the form

$$V = V_+ + V_-,\ V_\pm = \pm F(\omega t \mp \frac{\omega}{c}(x-L) \pm \frac{\beta\omega}{c^3}(I+F^2)(x-L)),$$
$$I = \langle V_\pm^2 \rangle. \quad (16)$$

Inserting (16) into the boundary conditions (15) gives a functional equation

$$F(\omega t + kL - \frac{\beta\omega}{c^3}(I+F^2)(x-L)) - F(\omega t - kL + \frac{\beta\omega}{c^3}(I+F^2)(x-L))$$
$$= A\sin\omega t, \qquad k = \frac{\omega}{c}. \quad (17)$$

The functional equation (17) can be reduced to a differential equation with dimensionless variables:

$$\frac{\partial W}{\partial t} + (\Delta - \pi\beta J - \pi\beta W^2)\frac{\partial W}{\partial \xi} - D\frac{\partial^2 W}{\partial \xi^2} = \frac{M}{2}\sin\xi, \quad (18)$$

with

$$W = \frac{F}{c},\ M = \frac{A}{c},\ \xi = \omega t + \pi,\ T = \frac{\omega t}{\pi}, \qquad J = \frac{I}{c^2}. \quad (19)$$

The discrepancy Δ in Eq. (18) is defined as

$$\Delta = \frac{(\omega - \omega_0)}{\omega_0}, \quad (20)$$

with $\omega = \frac{\pi L}{c}$ (lowest resonance frequency of the resonator). The dissipation coefficient D in Eq. (18) is defined as

$$D = \frac{b\omega L}{2c^3\rho_0} << 1, \quad (21)$$

where the absorption coefficient b ($\sim \eta$ and ζ) can be introduced in Eq. (18) in analogy with its occurrence in Burgers' equation [1].

Frequency response functions for a quadratic and a cubic nonlinear resonator are plotted in Fig. 1 and Fig. 2 respectively. In Fig. 1 we plot the rms normalized particle velocity $\sqrt{\overline{W^2}}$ as function of the normalized discrepancy $\frac{\Delta}{\pi\varepsilon}$, where ε is the quadratic nonlinearity parameter. Curves 1-5 are constructed for different values of boundary vibration $[\frac{M}{\pi\varepsilon}\cdot 10^2 = 1, 4, 9, 16$ and $25]$. In Fig. 2 a frequency response function $y(\delta)$ with

$$y = \frac{I}{c^2}\left(\frac{3\sqrt{2}\pi\beta}{M}\right)^{\frac{2}{3}}, \qquad \delta = \Delta\left(\frac{16}{3\pi\beta M^2}\right)^{\frac{1}{3}} \tag{22}$$

is constructed for different values of the normalized absorption coefficient $[d = D(\frac{3\sqrt{2}\pi\beta}{M})^{\frac{1}{3}} = 2, 1.25, 0.75, 0.5$ and $0.4]$.

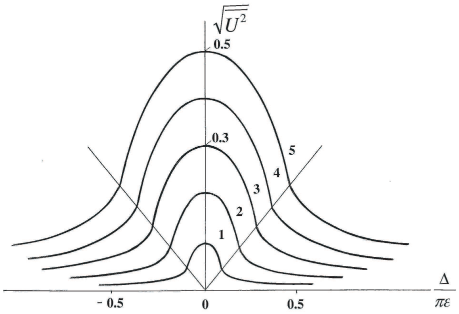

Fig. 1. Quadratic resonator root-mean-square velocity as function of discrepancy parameter $\Delta/(\pi\epsilon)$. The curves are for different boundary vibration amplitudes.

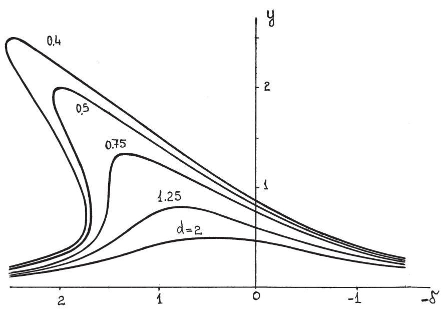

Fig. 2. The cubic resonator frequency response as a function of discrepancy parameter δ. The curves are (from above) for different absorptions: d=0.4, 0.5, 0.75, 1.25, 2.

4 Propagating N-waves in a cubically nonlinear medium

The modified Burgers' equation for plane waves

$$\frac{\partial V}{\partial z} - \frac{\beta}{c_t^3} V^2 \frac{\partial V}{\partial \tau} = \delta \frac{\partial^2 V}{\partial \tau^2}, \tag{23}$$

following from (9), has to be made dimensionless. To this end we introduce a fundamental period $\nu = t_0^{-1}$, where t_0 is the duration of the N-wave at the boundary, and the velocity amplitude V_0 at the boundary. The dimensionless equation derived from (23) is

$$\frac{\partial W}{\partial X} + W^2 \frac{\partial W}{\partial \theta} = \epsilon \frac{\partial^2 W}{\partial \theta^2} \tag{24}$$

with

$$W = \frac{V}{V_0}, \qquad X = \frac{2|\beta| V_0^2 \nu}{c_t^3} z, \qquad \theta = 2\nu\tau,\ \epsilon = \frac{\overline{\eta}\nu}{|\beta|\rho_0 V_0^2}. \tag{25}$$

We assume $\beta < 0$ and $\epsilon << 1$.

The equation (24) has been studied by Lee-Bapty and Crighton [10]. The qualitative deformation of an original N-wave for increasing X-values according to Eq. (24) with $\epsilon \to 0$ is shown in Fig. 3.

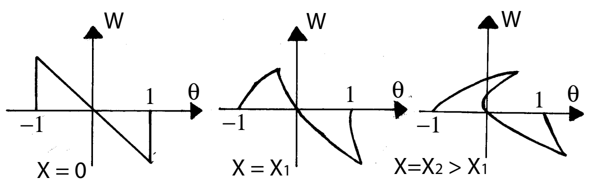

Fig. 3. Schematic evolution of an original N-wave in a cubically nonlinear medium.

The multivalued parts of the profile in the last picture of Fig. 3 are replaced [10] by a tail shock at $\theta = C_t(X) = -1 + d^2 X^{\frac{1}{4}}$ and a head shock at $\theta = C_h(X) = -1 + 3d^2 X^{\frac{1}{4}}$ with $d \approx 0.95$. The structures of the shocks and of the tail at $\theta \approx -1$ are found by rescaling Eq. (24), so that the righthand side is no longer small. By this procedure seven parts of the profile are discerned, satisfying different scaled versions of Eq. (24) with different dominant terms (Fig. 4).

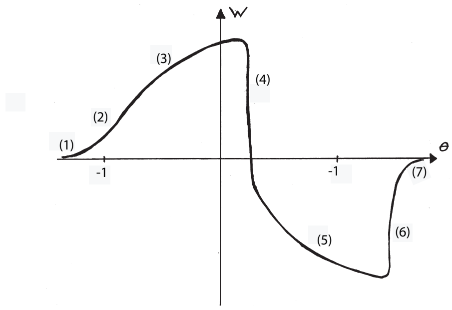

Fig. 4. The seven parts of the deformed N-wave in a cubically nonlinear medium.

The new results in the present paper concern part 1 (left tail) and part 2 (connection at $\theta \approx -1$ between left tail and left curve). For parts 3 (left curve), 4 (tail shock), 5 (right curve) and 6 (head shock) analytic lowest approximation solutions are given by Lee-Bapty and Crighton [10]. For part 7 (right tail) an analytic solution is still not yet found.

We first attempt to find a solution for part 2, because this solution has to be consistent with the part 1 and part 3 solutions. The scalings

$$X^* = \epsilon^{\frac{1}{2}} X, \qquad \theta^* = \frac{\theta + 1}{\epsilon^{\frac{1}{2}}}, \qquad W^* = \epsilon^{\frac{1}{2}} W = W_0^* + O(\epsilon^{\frac{1}{2}}) \tag{26}$$

are inserted into (24) and give in lowest order

$$W_0^{*2} \frac{\partial W_0^*}{\partial \theta^*} = \frac{\partial^2 W_0^*}{\partial \theta^{*2}}. \tag{27}$$

The solution of (27) with the necessary consistency properties is

$$W_0 = \epsilon^{-\frac{1}{2}} W_0^* = \sqrt{\frac{3}{2}} \frac{\epsilon^{\frac{1}{4}}}{\sqrt{C - \epsilon^{\frac{1}{2}}(\theta + 1)}}, \tag{28}$$

where the constant C can be determined numerically. The solution (28) is valid for

$$|\theta + 1| << 2\epsilon^{\frac{1}{2}} X^{\frac{1}{2}}, \qquad \theta < -1 \tag{29}$$

$$\theta + 1 < \frac{3}{2} \epsilon^{\frac{1}{2}} \frac{X}{C}, \qquad \theta > -1. \tag{30}$$

The independence of the solution (28) on X is seen in the numerically calculated solution in Fig. 5, where all profiles for different X have approximately the same value for $\theta = -1$.

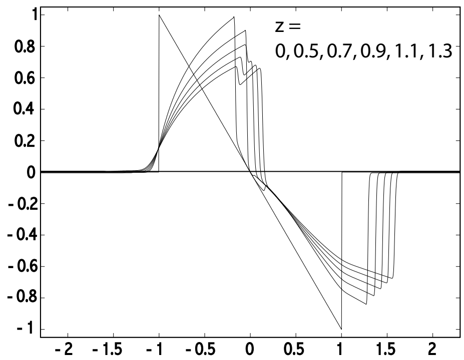

Fig. 5. Numerical solutions of Eq. (24), evaluated with $\epsilon = 0.005$ for $X = 0, 0.5, 0.7, 0.9, 1.1, 1.3$.

For part 1 (left tail) we make the scalings

$$\xi = \epsilon^{\frac{3}{2}} X, \quad \zeta = -\epsilon^{\frac{1}{4}}(\theta + 1), \quad U = \epsilon^{-\frac{7}{8}} W = U_0 + O(\epsilon^{\frac{1}{2}}) \,, \tag{31}$$

which are inserted into (24) and give a lowest order linear equation

$$\frac{\partial U_0}{\partial \xi} = \frac{\partial^2 U_0}{\partial \zeta^2}. \tag{32}$$

The equation (32) is identically solved by the integral representation

$$U_0 = K \int_0^{\infty} h(\lambda) \exp(\mathrm{i}\lambda\zeta - \lambda^2 \xi) d\lambda + \text{c.c.}$$

$$K = \sqrt{\frac{3}{8\pi}} \epsilon^{-\frac{1}{2}}, \; h(\lambda) = \lambda^{-\frac{1}{2}}, \tag{33}$$

where c.c. stands for complex conjugated and K and $h(\lambda)$ are chosen so that the solution (33) matches the solution (28). This is seen by evaluating the integral in (33) by the steepest descent method with X and θ inserted from (31) with the result

$$W_0 = \epsilon^{\frac{7}{8}} U_0 = \sqrt{\frac{3}{2}} |\theta + 1|^{-\frac{1}{2}} \exp(-\frac{(\theta + 1)^2}{4\epsilon X}). \tag{34}$$

Solutions (28) and (34) match each other in the region where (29) is fulfilled together with

$$|\theta + 1| >> C\epsilon^{-\frac{1}{2}}. \tag{35}$$

For part 3 (left curve) the expansion $W = W_0 + O(\epsilon)$ gives using (24) [10]

$$\frac{\partial W_0}{\partial X} + W_0^2 \frac{\partial W_0}{\partial \theta} = 0 \implies W_0 = \sqrt{\frac{\theta + 1}{X}}. \tag{36}$$

For the solution (36) to be valid we must require (cf. Eq. 24)

$$\epsilon \left| \frac{\partial^2 W_0}{\partial \theta^2} \right| << \left| \frac{\partial W_0}{\partial X} \right| \implies \frac{(\theta + 1)^2}{\epsilon X} >> 1. \tag{37}$$

Thus for $X = O(1)$ the transition from the solution (37) to the solution (28) is completed at $\theta + 1 = O(\epsilon^{\frac{1}{2}})$. A representation similar to (34) for the right tail remains to be found, as well as an exponentially decreasing profile at both ends for $X >> \epsilon^{-2}$ (cf. the scaling (31)).

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