

# Standing and propagating waves in cubically nonlinear media

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**Abstract.** The paper has three parts. In the first part a cubically nonlinear equation is derived for a transverse finite-amplitude wave in an isotropic solid. The cubic nonlinearity is expressed in terms of elastic constants. In the second part a simplified approach for a resonator filled by a cubically nonlinear medium results in functional equations. The frequency response shows the dependence of the amplitude of the resonance on the difference between one of the resonator's eigenfrequencies and the driving frequency. The frequency response curves are plotted for different values of the dissipation and differ very much for quadratic and cubic nonlinearities. In the third part a propagating N-wave is studied, which fulfils a modified Burgers' equation with a cubic nonlinearity. Approximate solutions to this equation are found for new parts of the wave profile.

**Keywords:** Cubic nonlinear media, nonlinear acoustic resonator, cubic resonator, N-wave propagation

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## INTRODUCTION

Nonlinear acoustical waves in dissipative media have been extensively studied and model equations describing weakly nonlinear, weakly dissipative waves have been solved [1,2,3]. In most cases the nonlinearity is quadratic and the model equations are Burgers' or generalized Burgers' equations [4]. An interesting problem of principal interest is to investigate how the methods used for quadratic nonlinearities can be applied to cubic nonlinearities. Results of such investigations show new features in comparison with quadratic nonlinearities.

The quadratic nonlinearity is characteristic for waves in fluids [3] and also for longitudinal waves in solids [5]. However, transverse wave propagation is modelled by a cubically nonlinear wave equation, since the quadratic nonlinearity cancels [6]. The derivation of this equation is the contents of section 2. Two examples of cubically nonlinear wave theory are studied. In section 3 standing waves in a cubically nonlinear resonator are studied and the frequency response function is calculated and compared with the corresponding function in the quadratically nonlinear case [7]. In section 4 cubically nonlinear propagation of N-waves is studied using methods developed for quadratically nonlinear waves [8].

## PHYSICAL BACKGROUND

The equation of motion of a continuous medium is [5]

$$\rho_0 \frac{\partial^2 U_i(\vec{a}, t)}{\partial t^2} = \frac{\partial}{\partial a_j} (P_{ij} + D_{ij}), \quad (1)$$

where

$\rho_0$  is the density in the undeformed state,

$U_i$  is the displacement,  $U_i = x_i - a_i$ , of the particle originally at  $a_i$  (Lagrangian coord.),

$P_{ij}$  is the stress tensor,

$D_{ij}$  is the dissipative stress tensor.

The elastic energy  $W$  is expanded in the invariants of the strain tensor  $E_{ij}$ ,

$$E_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial a_j} + \frac{\partial U_j}{\partial a_i} + \frac{\partial U_k}{\partial a_i} \frac{\partial U_k}{\partial a_j} \right), \quad (2)$$

to the fourth order [6]:

$$\begin{aligned} \rho_0 W = & \left( \frac{K}{2} - \frac{\mu}{3} \right) (E_{kk})^2 + \mu E_{ij} E_{ji} + \frac{C}{3} (E_{kk})^3 + B E_{kk} E_{ij} E_{ji} + \frac{A}{3} E_{ij} E_{jk} E_{ki} \\ & + H (E_{kk})^4 + G (E_{ij} E_{ji})^2 + F (E_{kk})^2 E_{ij} E_{ji} + E E_{kk} E_{ij} E_{ji} + D E_{ij} E_{jk} E_{kl} E_{li}. \end{aligned} \quad (3)$$

Using the relation

$$P_{ij} = \rho_0 \frac{\partial W}{\partial \left( \frac{\partial U_i}{\partial a_j} \right)} \quad (4)$$

we obtain from (1) and (3) the equations of motion, with the notation  $U_{ij} = \frac{\partial U_i}{\partial a_j}$ ,

$$\begin{aligned} & \rho_0 \left\{ \frac{\partial^2 U_i}{\partial t^2} - c_t^2 \frac{\partial^2 U_i}{\partial a_k^2} - (c_l^2 - c_t^2) \frac{\partial U_{kk}}{\partial a_i} \right\} \\ & = \frac{\partial D_i}{\partial t} + \frac{\partial}{\partial a_s} \left\{ \left( \frac{K}{2} - \frac{\mu}{3} \right) (U_{kl} U_{kl} \delta_{is} + 2U_{kk} U_{is} + U_{kl} U_{kl} U_{is}) \right. \\ & \quad \left. + \mu (U_{ik} U_{sk} + U_{ik} U_{ks} + U_{ki} U_{ks} + U_{ik} U_{rk} U_{rs}) + \dots \right\}, \end{aligned} \quad (5)$$

where "..." stands for terms of second and third order in  $U_{ik}$  with coefficients  $A, B, C, D, E, F, G, H$ , and  $c_t$  and  $c_l$  are the transverse and longitudinal wave propagation velocities respectively,

$$c_t^2 = \frac{\mu}{\rho_0}, \quad c_l^2 = \frac{K + \frac{4\mu}{3}}{\rho_0}. \quad (6)$$

The dissipative term  $\frac{\partial D_i}{\partial t}$  in (5) is obtained from

$$D_i = \eta \left( \frac{\partial^2 U_i}{\partial a_k^2} + \frac{\partial^2 U_k}{\partial a_i \partial a_k} \right) + \left( \zeta - \frac{2}{3} \eta \right) \frac{\partial^2 U_k}{\partial a_i \partial a_k}, \quad (7)$$

where  $\eta$ ,  $\zeta$  are the shear and bulk viscosities respectively.

With the scalings

$$\begin{aligned}\bar{a}_1 &= \varepsilon^{\frac{1}{2}} a_1, & \bar{a}_2 &= \varepsilon^{\frac{1}{2}} a_2, & \bar{a}_3 &= \varepsilon a_3 \\ \bar{U}_1 &= \varepsilon^{-\frac{1}{2}} U_1, & \bar{U}_2 &= \varepsilon^{-\frac{1}{2}} U_2, & \bar{U}_3 &= \varepsilon^{-1} U_3 \\ \bar{\eta} &= \varepsilon^{-1} \eta, & \bar{\zeta} &= \varepsilon^{-1} \zeta\end{aligned}\quad (8)$$

a nonlinear beam equation can be derived [6] for transverse waves, vibrating in the 1-direction and propagating in the 3-direction. In this case the quadratic nonlinearity cancels and the equation for  $\frac{\partial \bar{U}_1}{\partial t} = V$  becomes

$$\frac{\partial}{\partial \tau} \left\{ \frac{\partial V}{\partial z} - \frac{\beta}{c_t^3} V^2 \frac{\partial V}{\partial \tau} - \delta \frac{\partial^2 V}{\partial \tau^2} \right\} = \frac{c_t}{2} \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\} V, \quad (9)$$

where  $(x, y, z, \tau)$  is given as  $(\bar{a}_1, \bar{a}_2, \bar{a}_3, t - \frac{\bar{a}_3}{c_3})$  and  $\delta$  and  $\beta$  are given as

$$\delta = \frac{\bar{\eta}}{2\rho_0 c_t^3} \quad (10)$$

$$\beta = \frac{3}{2\rho_0 c_t^2} \left\{ \frac{K}{2} + \frac{2\mu}{3} + B + \frac{A}{2} + G + \frac{D}{2} - \frac{(\frac{K}{2} + \frac{2\mu}{3} + \frac{3B}{4} + \frac{A}{4})(K + \frac{4\mu}{3} + B + \frac{A}{2})}{K + \frac{\mu}{3}} \right\}. \quad (11)$$

The quotient term in (11), because of which  $\beta$  can be negative, is missing in the paper by Zabolotskaya [6]. The equation (9) is studied by Rudenko and Sapoznikov [9]. If  $K$  and  $\mu$  are the only elastic constants different from zero in (11) we obtain using (6):

$$\beta = -\frac{3}{4} \frac{c_t^2}{c_l^2 - c_t^2}. \quad (12)$$

Thus  $\beta$  is negative for the simplest solid materials.

## STANDING WAVES IN A CUBICALLY NONLINEAR RESONATOR

Neglecting dissipation and transverse extension Eq. (9) becomes

$$\frac{\partial V}{\partial z} - \frac{\beta}{c^3} V^2 \frac{\partial V}{\partial \tau} = 0, \quad (13)$$

where  $c_t$  is replaced by  $c$  in order that Eq. (13) be applied to other physical phenomena than transverse waves in solids. The choice of the negative sign in the definition of  $\tau$  means that Eq. (13) describes rightgoing waves. Standing waves in a resonator are

composed by waves propagating in both directions. A wave equation, which gives Eq. (13) when specialized to rightgoing waves, is (change  $z$  into  $x$ ):

$$\frac{\partial^2 V}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = -\frac{2\beta}{3c^4} \frac{\partial^2 V^3}{\partial t^2}. \quad (14)$$

The equation with a quadratic nonlinearity corresponding to the cubic nonlinear equation (14) is studied for a resonator by the present authors [7]. The resonator boundary conditions are

$$V(x=0, t) = A \sin \omega t, \quad V(x=L, t) = 0. \quad (15)$$

In analogy with the solution attempted by the present authors for a quadratic nonlinearity [7] we attempt a solution to Eq. (14) in the form

$$V = V_+ + V_-, \quad V_{\pm} = \pm F\left(\omega t \mp \frac{\omega}{c}(x-L) \pm \frac{\beta\omega}{c^3}(I+F^2)(x-L)\right), \\ I = \langle V_{\pm}^2 \rangle. \quad (16)$$

Inserting (16) into the boundary conditions (15) gives a functional equation

$$F\left(\omega t + kL - \frac{\beta\omega}{c^3}(I+F^2)(x-L)\right) - F\left(\omega t - kL + \frac{\beta\omega}{c^3}(I+F^2)(x-L)\right) \\ = A \sin \omega t, \quad k = \frac{\omega}{c}. \quad (17)$$

The functional equation (17) can be reduced to a differential equation with dimensionless variables:

$$\frac{\partial W}{\partial t} + (\Delta - \pi\beta J - \pi\beta W^2) \frac{\partial W}{\partial \xi} - D \frac{\partial^2 W}{\partial \xi^2} = \frac{M}{2} \sin \xi, \quad (18)$$

with

$$W = \frac{F}{c}, \quad M = \frac{A}{c}, \quad \xi = \omega t + \pi, \quad T = \frac{\omega t}{\pi}, \quad J = \frac{I}{c^2}. \quad (19)$$

The discrepancy  $\Delta$  in Eq. (18) is defined as

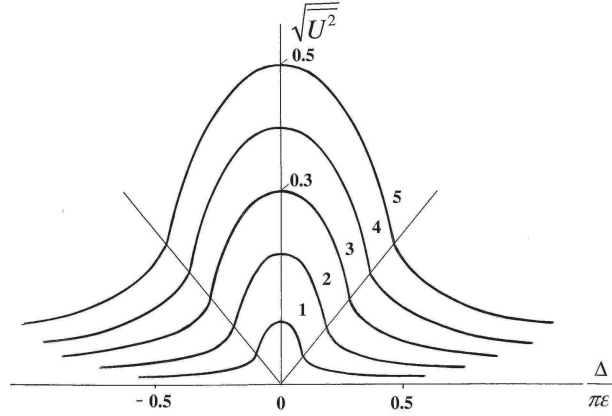
$$\Delta = \frac{(\omega - \omega_0)}{\omega_0}, \quad (20)$$

with  $\omega = \frac{\pi L}{c}$  (lowest resonance frequency of the resonator).

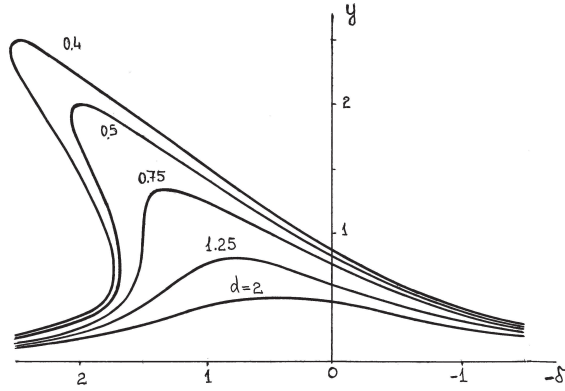
The dissipation coefficient  $D$  in Eq. (18) is defined as

$$D = \frac{b\omega L}{2c^3\rho_0} \ll 1, \quad (21)$$

where the absorption coefficient  $b$  ( $\sim \eta$  and  $\zeta$ ) can be introduced in Eq. (18) in analogy with its occurrence in Burgers' equation [1].



**FIGURE 1.** Quadratic resonator root-mean-square velocity as function of discrepancy parameter  $\Delta/(\pi\varepsilon)$ . The curves are for different boundary vibration amplitudes.

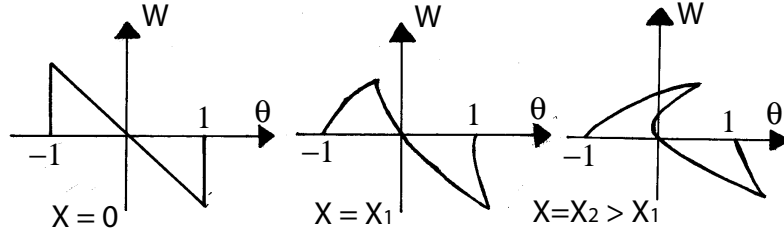


**FIGURE 2.** The cubic resonator frequency response as a function of discrepancy parameter  $\delta$ . The curves are (from above) for different absorptions:  $d=0.4, 0.5, 0.75, 1.25, 2$ .

Frequency response functions for a quadratic and a cubic nonlinear resonator are plotted in Fig. 1 and Fig. 2 respectively. In Fig. 1 we plot the rms normalized particle velocity  $\sqrt{W^2}$  as function of the normalized discrepancy  $\frac{\Delta}{\pi\varepsilon}$ , where  $\varepsilon$  is the quadratic nonlinearity parameter. Curves 1-5 are constructed for different values of boundary vibration [ $\frac{M}{\pi\varepsilon} \cdot 10^2 = 1, 4, 9, 16$  and  $25$ ]. In Fig. 2 a frequency response function  $y(\delta)$  with

$$y = \frac{I}{c^2} \left( \frac{3\sqrt{2}\pi\beta}{M} \right)^{\frac{2}{3}}, \quad \delta = \Delta \left( \frac{16}{3\pi\beta M^2} \right)^{\frac{1}{3}} \quad (22)$$

is constructed for different values of the normalized absorption coefficient [ $d = D \left( \frac{3\sqrt{2}\pi\beta}{M} \right)^{\frac{1}{3}} = 2, 1.25, 0.75, 0.5$  and  $0.4$ ].



**FIGURE 3.** Schematic evolution of an original N-wave in a cubically nonlinear medium.

## PROPAGATING N-WAVES IN A CUBICALLY NONLINEAR MEDIUM

The modified Burgers' equation for plane waves

$$\frac{\partial V}{\partial z} - \frac{\beta}{c_t^3} V^2 \frac{\partial V}{\partial \tau} = \delta \frac{\partial^2 V}{\partial \tau^2}, \quad (23)$$

following from (9), has to be made dimensionless. To this end we introduce a fundamental period  $\nu = t_0^{-1}$ , where  $t_0$  is the duration of the N-wave at the boundary, and the velocity amplitude  $V_0$  at the boundary. The dimensionless equation derived from (23) is

$$\frac{\partial W}{\partial X} + W^2 \frac{\partial W}{\partial \theta} = \varepsilon \frac{\partial^2 W}{\partial \theta^2} \quad (24)$$

with

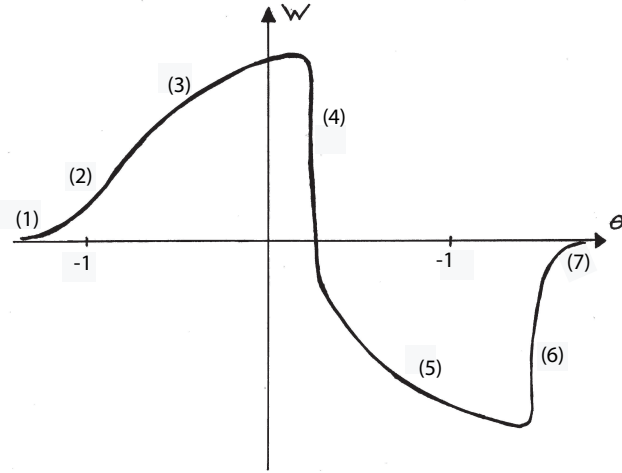
$$W = \frac{V}{V_0}, \quad X = \frac{2|\beta|V_0^2\nu}{c_t^3}z, \quad \theta = 2\nu\tau, \quad \varepsilon = \frac{\bar{\eta}\nu}{|\beta|\rho_0V_0^2}. \quad (25)$$

We assume  $\beta < 0$  and  $\varepsilon \ll 1$ .

The equation (24) has been studied by Lee-Bapty and Crighton [10]. The qualitative deformation of an original N-wave for increasing  $X$ -values according to Eq. (24) with  $\varepsilon \rightarrow 0$  is shown in Fig. 3.

The multivalued parts of the profile in the last picture of Fig. 3 are replaced [10] by a tail shock at  $\theta = C_t(X) = -1 + d^2X^{\frac{1}{4}}$  and a head shock at  $\theta = C_h(X) = -1 + 3d^2X^{\frac{1}{4}}$  with  $d \approx 0.95$ . The structures of the shocks and of the tail at  $\theta \approx -1$  are found by rescaling Eq. (24), so that the righthand side is no longer small. By this procedure seven parts of the profile are discerned, satisfying different scaled versions of Eq. (24) with different dominant terms (Fig. 4).

The new results in the present paper concern part 1 (left tail) and part 2 (connection at  $\theta \approx -1$  between left tail and left curve). For parts 3 (left curve), 4 (tail shock), 5 (right curve) and 6 (head shock) analytic lowest approximation solutions are given by Lee-Bapty and Crighton [10]. For part 7 (right tail) an analytic solution is still not yet found.



**FIGURE 4.** The seven parts of the deformed N-wave in a cubically nonlinear medium.

We first attempt to find a solution for part 2, because this solution has to be consistent with the part 1 and part 3 solutions. The scalings

$$X^* = \varepsilon^{\frac{1}{2}}X, \quad \theta^* = \frac{\theta + 1}{\varepsilon^{\frac{1}{2}}}, \quad W^* = \varepsilon^{\frac{1}{2}}W = W_0^* + O(\varepsilon^{\frac{1}{2}}) \quad (26)$$

are inserted into (24) and give in lowest order

$$W_0^{*2} \frac{\partial W_0^*}{\partial \theta^*} = \frac{\partial^2 W_0^*}{\partial \theta^{*2}}. \quad (27)$$

The solution of (27) with the necessary consistency properties is

$$W_0 = \varepsilon^{-\frac{1}{2}}W_0^* = \sqrt{\frac{3}{2}} \frac{\varepsilon^{\frac{1}{4}}}{\sqrt{C - \varepsilon^{\frac{1}{2}}(\theta + 1)}}, \quad (28)$$

where the constant  $C$  can be determined numerically. The solution (28) is valid for

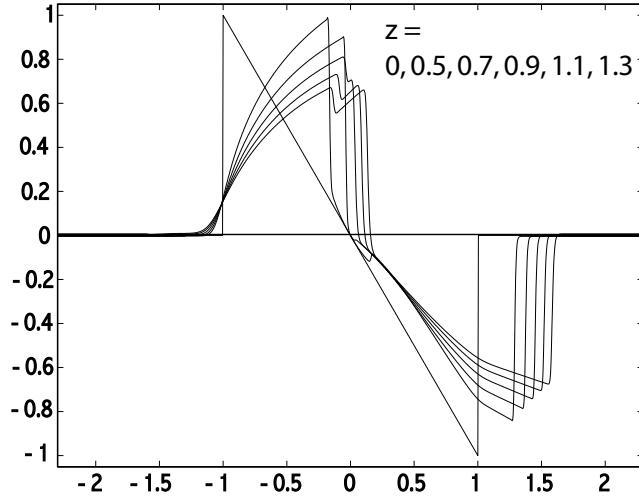
$$|\theta + 1| \ll 2\varepsilon^{\frac{1}{2}}X^{\frac{1}{2}}, \quad \theta < -1 \quad (29)$$

$$\theta + 1 < \frac{3}{2}\varepsilon^{\frac{1}{2}}\frac{X}{C}, \quad \theta > -1. \quad (30)$$

The independence of the solution (28) on  $X$  is seen in the numerically calculated solution in Fig. 5, where all profiles for different  $X$  have approximately the same value for  $\theta = -1$ .

For part 1 (left tail) we make the scalings

$$\xi = \varepsilon^{\frac{3}{2}}X, \quad \zeta = -\varepsilon^{\frac{1}{4}}(\theta + 1), \quad U = \varepsilon^{-\frac{7}{8}}W = U_0 + O(\varepsilon^{\frac{1}{2}}), \quad (31)$$



**FIGURE 5.** Numerical solutions of Eq. (24), evaluated with  $\varepsilon = 0.005$  for  $X = 0, 0.5, 0.7, 0.9, 1.1, 1.3$ .

which are inserted into (24) and give a lowest order linear equation

$$\frac{\partial U_0}{\partial \xi} = \frac{\partial^2 U_0}{\partial \zeta^2}. \quad (32)$$

The equation (32) is identically solved by the integral representation

$$U_0 = K \int_0^\infty h(\lambda) \exp(i\lambda \zeta - \lambda^2 \xi) d\lambda + \text{c.c.}$$

$$K = \sqrt{\frac{3}{8\pi}} \varepsilon^{-\frac{1}{2}}, \quad h(\lambda) = \lambda^{-\frac{1}{2}}, \quad (33)$$

where c.c. stands for complex conjugated and  $K$  and  $h(\lambda)$  are chosen so that the solution (33) matches the solution (28). This is seen by evaluating the integral in (33) by the steepest descent method with  $X$  and  $\theta$  inserted from (31) with the result

$$W_0 = \varepsilon^{\frac{7}{8}} U_0 = \sqrt{\frac{3}{2}} |\theta + 1|^{-\frac{1}{2}} \exp\left(-\frac{(\theta + 1)^2}{4\varepsilon X}\right). \quad (34)$$

Solutions (28) and (34) match each other in the region where (29) is fulfilled together with

$$|\theta + 1| \gg C\varepsilon^{-\frac{1}{2}}. \quad (35)$$

For part 3 (left curve) the expansion  $W = W_0 + O(\varepsilon)$  gives using (24) [10]

$$\frac{\partial W_0}{\partial X} + W_0^2 \frac{\partial W_0}{\partial \theta} = 0 \implies W_0 = \sqrt{\frac{\theta + 1}{X}}. \quad (36)$$

For the solution (36) to be valid we must require (cf. Eq. 24)

$$\varepsilon \left| \frac{\partial^2 W_0}{\partial \theta^2} \right| \ll \left| \frac{\partial W_0}{\partial X} \right| \implies \frac{(\theta + 1)^2}{\varepsilon X} \gg 1. \quad (37)$$



Thus for  $X = O(1)$  the transition from the solution (37) to the solution (28) is completed at  $\theta + 1 = O(\varepsilon^{\frac{1}{2}})$ . A representation similar to (34) for the right tail remains to be found, as well as an exponentially decreasing profile at both ends for  $X \gg \varepsilon^{-2}$  (cf. the scaling (31)).

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