Non-linear dynamics in granular materials

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ABSTRACT:

This short review is about non-linear behaviors in granular materials. It does not pretend to be exhaustive but instead, we have selected particular behaviors that seem to be relevant with the main topic of this book which is the nonlinearity relation between the microscale and macroscale properties of a material. At the mesoscale, granular material present intricate physical processes such as dynamical arches formations.

Vibrated granular materials are perfect example of such multi-scale nonlinearity. Density waves may have intricate patterns that oscillate between one symmetrical pattern to its inverse in resonance-like states.

Universal criteria in granular are found in many physical properties as for example in the infinite cluster of conductive grains. The conductivity in the proximity of conductivity threshold depends exponentially on a universal criteria that only depends on the dimensionality of the problem. Solidification and fluidization are two mechanical processes that have both tremendous industrial and scientific implications. We present a model based on 1D flow equations whose solution is a universal attractor representing an exponentially increasing stress with the solidity field at a critical value about 0.5.

Granular avalanches present segregation mechanisms that separate the larger grains from the smaller ones, also called Brazil nut effect. In consequence, the front of the avalanche is composed of large grains and the tail contains the smaller grains. One can also observed nonlinear upwards propagating dispersed shock waves.

We give also a stress-strain relationships typical for granular media with mesoscopic inhomogeneities. From the vibration of two grains the solution is

generalized to a distributed fluid-saturated grainy medium. The result show that when the velocity of wave propagation is equal to the speed of sound, a wave resonance phenomenon can be located in a layer at a particular depth of the granular medium.

KEYWORDS:: inertia nonlinearity, fluid-saturated granular media, grain vibration, universal criteria, grain fluidization, granular avalanches, multi-scale nonlinearity

1 Introduction

The global behavior of a large number of grains that only interact through collisions and friction is typically non-linear and different from any other standard and familiar forms of matter as solids, liquids and gases. Granular materials are often referred as a new form of matter on its own. Granular materials present different scales of interest. At the microscale, i.e. at the scale of a few grain, a granular material behaves as an inelastic solid.

At the macroscale, granular material behave very much like a liquid or a gas depending of the density of grains (e.g. [1]). But it is at the mesoscale of 5-7 grain sizes, that granular materials show intricate mechanisms. The existence of such mesoscale was estimated by P. Martinet in the context of internal erosion of Earth dams [2]. The same critical value of 5 grains was more precisely recalculated by Koenders et.al. [3]-[5] who also demonstrated the reality and existence of the mesoscale range in granular media.

Biot [6] in 1955 developed a theory of dynamic deformation of media with a solid and a fluid phase. He stated that there exist two pressure waves, and one displacement wave. In the laboratory Paterson in 1955 [7] measured two pressure waves in marine sands. He found a fluid wave, and one frame wave which he believed to be the pressure wave of second type of Biot, its velocity increases with pressure, and velocity decreases with fluid increase. The Biot based models corresponds relatively well with measurements [8].

Marine sands are homogeneous in grain sizes, sorted by natural processes. In experiments an ultrasound pulse separates in propagation into two parts: one high frequency with high velocity (fast wave), and one low frequency with low velocity (slow wave) [9]. The fast wave has the same frequency as the excitation, while the slow wave frequencies depend on the non-linear interaction in the sand internal structure. The slow longitudinal wave was confirmed in water saturated glass beads by Plona [10]. The slow wave exist also when the fluid is air [11]-[13]. A mathematical model of this phenomena was developed by Nikolaevskii [14], [15] which reduced the problem to Korteweg-de-Vries, Burgers, or a fourth order equation. In sands with low fluid saturation (less than approximately 15 %) only the slow wave with velocity 200 - 300 m/s was found. For fully fluid saturated sands only the fast wave with velocities of 1600 m/s was found. In sands with 17-10 % saturation, both kinds of waves appeared. As the velocity of the slow wave (the frame wave) increases with pressure, it might reach the same velocity as the fast wave at higher depths. A peculiar behavior is that the dissipative coefficient

of an acoustic wave decrease with increasing amplitude [16]. This dissipative coefficient is sensitively dependent on the moisture content. An increase in the static pressure results for high-frequency pulses in wave velocity increases, and amplitude increases because the attenuation decreases [17].

A perfect saturation is difficult to attain as even a small few tenths of a percent of air change the sound velocity from 1800 m/s for fully saturated, to 200 m/s [18]. The cause is the same which stands as a fundament in the phenomena in this book - very small soft partitions are introduced between the harder bulk part of the material leading to major change in the macroscopic behavior.

The elastic waves that propagate thorough the frame part of a fluid-grain material are well described by the grain contact mechanics [18], like Hertz contact description. The idea to replace the real sand-like medium by a model of packed spherical grains was used many years ago to describe the properties of seismic waves propagating in geological structures. An exhaustive review of these works is given by White [19]. More recent studies devoted to nonlinear properties of such models are reviewed in reference [20].

2 Vibrated granular materials

Under vertical vibrations granular materials behave following two different processes: A fast motion due to the collision of the grains with the vibrated plane and slow convective motion located in the bulk of the material.

For shallow granular beds that are confined inside a regular shape, surface waves may appear and typically, produce square patterns. These waves correspond to variations in the local density and are also called density waves. In general, the waves oscillate between two different patterns at a frequency $f/2$ where, f is the frequency of the vertical vibration, [21].

FIGURE 1. Snapshots of five-fold symmetry in resonance-like states of shallow granular beds vibrating vertically (a) and the inverse symmetry after one period (b). (a): Conductivity σ as a function of the probability p to have a grain that is conductor. (b): Infinite conductive cluster in a granular material with a proportion of p conductive grains and $(1-p)$ isolated grains.

3 Solidification and Fluidization

The flow equations for highly compacted granular materials that are externally stressed present strong non-linearities. One example is non-linear consolidation. A first order quantitative estimate was done by Koenders (1997), [4]. The result gives a clue on how semi-stable filtration takes place. The model is based on 1D flow equations that are: continuity equations for the solid and liquid phases and the force equilibrium equations. Two critical values for the velocity are obtained and, very reasonably, agree with measured values. The presence of waves in the medium is required by the presences of exponentially increasing solutions of the characteristic equation for the first order term of the linearized system of equations. A universal attractor is found and the solutions represents an exponentially increasing stress with the solidity field at a critical value about 0.5. If the granular flow appears in narrow pipes periodical clogging and declogging appears spontaneously [22]. In the case of a sand-clock this effect give rise to a "tic tac" like sound that is a combination of fluidization and solidification of the thin colon of sand located along the narrow canal.

4 Granular avalanches

If the slope of a sandpile is less than the angle of repose, the friction between the grains is larger than the gravitational forces and the granular material behaves as a solid. If the slope is higher than the angle of repose, a fluid-like motion appears that we called granular flow. In contrary with a classical fluid, the motion is limited in the top layer. For simple binary granular materials that contain only two sorts of grains with two different sizes, observations show granular avalanches with two different waves; typical roll waves and upwards propagating dispersed shock waves [23]. Segregation, i.e. size separation, is one of the first observed behavior of granular materials and is often referred to as the Brazil Nut effect. It is a simple mechanical process that inhabit the large grains that are obliged to move upward. In avalanches, segregation results in a front of large grains moving downward with the front of the avalanche.

FIGURE 2. (a): Typical roll wave in granular avalanche. The large grains (white) overlie the small grains (dark). (b): Upwards propagating shock wave. The grains below the shock are almost at rest whilst the grains above are flowing rapidly downslope.

5 Universal Critability

If granular material is compressed, a non-uniform stress field composed of "backbones" appears [24],[25]. The density threshold between the rigid and non-rigid phase is a first-order transition because there is discontinuity in the backbone density. With $\epsilon = p - p_c$ being the distance to the critical point representing the density threshold p_c , the correlation length ξ diverges as ϵ goes to zero as

$$\xi \approx \epsilon^{-\nu}$$

where, ν is a universal correlation length exponent. Inside regions that have a size smaller than the correlation length ξ , mean values depend of the size of the system and homogeneous approaches can not be applied.

An example of such critical behavior is given by the classical experiment of a mixture of isolated and conductive mono-size grains. At a critical density p_c of conductive grains the entire medium becomes conductive ($\sigma = 1$) because an infinite conductive cluster appears in the medium. The value of the critical density is an universal parameter that only depends on the dimensionality of the problem.

FIGURE 3. (a): Conductivity σ as a function of the probability p to have a grain that is conductor. (b): Infinite conductive cluster in a granular material with a proportion of p conductive grains and $(1-p)$ isolated grains.

6 Vibration of grains

The typical nonlinear response of grainy media are based on nonlinear stress-strain relationships typical for solids containing mesoscopic inhomogeneities or defects in their structure (see, for example, chapters in this book and the reviews [20]-[26]). However, there exist a different type of nonlinearity which manifests itself due to inertial forces between grains, appearing if a system of interacting particles is placed into a vibrating fluid. Structural internal forces are caused by a non-uniform mass distribution. In the following, a model of grainy medium is developed which first deals with two grains, and then with an ensemble of grains, immersed into a vibrating fluid. The inertial attractive forces have hydrodynamic origin, and the repulsive forces are caused by deformation of colliding grains.

6.1 Vibration of a pair of grains

Let us consider two bodies immersed in an ideal fluid, being in contact during part of the period of an acoustic wave. We consider here two spherical bodies with different densities ρ_1, ρ_2 and volumes V_1, V_2 . The coordinates x_1, x_2 indicating their center of mass satisfy the coupled system of equations

$$(\rho_0 + 2\rho_1)\frac{d^2 x_1}{dt^2} = 3\rho_0 \frac{dv}{dt} - \frac{2}{V_1}[F(x_2 - x_1) - F_0] \quad (1)$$

$$(\rho_0 + 2\rho_2)\frac{d^2 x_2}{dt^2} = 3\rho_0 \frac{dv}{dt} + \frac{2}{V_2}[F(x_2 - x_1) - F_0] . \quad (2)$$

Here $\bar{v}$ is the velocity of the fluid, $m_0 = \rho_0 V$ is the mass of fluid displaced by the body, F is a repulsive force depending on distance between centers, and F_0

is a constant external holding (clamping) force. For the difference $\xi = x_2 - x_1$ one can derive from (1)-(2) the inhomogeneous nonlinear equation [27]

$$\frac{d^2\xi}{dt^2} - a[F(\xi) - F_0] = b\frac{dv}{dt} . \quad (3)$$

where

$$a = 2 \frac{V_1(\rho_0 + 2\rho_1) + V_2(\rho_0 + 2\rho_2)}{V_1 V_2 (\rho_0 + 2\rho_1)(\rho_0 + 2\rho_2)} , \quad b = 6\rho_0 \frac{\rho_1 - \rho_2}{(\rho_0 + 2\rho_1)(\rho_0 + 2\rho_2)} . \quad (4)$$

According to Hertz's theory, and for a harmonic vibration of the fluid caused by a monochromatic acoustic wave, is obtained in a non-dimensional form: [27]

$$\frac{d^2y}{d\tau^2} + \beta^2[y^{3/2}\Theta(y) - 1] = D \cdot \cos(\tau) . \quad (5)$$

Here $y = h/h_0$, $\tau = \omega t$, $\beta^2 = \frac{\alpha^2}{\omega^2} \sqrt{h_0}$, $D = \frac{bv_0}{\omega h_0}$, $h = R_1 + R_2 - (x_2 - x_1)$, $\alpha^2 = aE \sqrt{\frac{R_1 R_2}{R_1 + R_2}}$, and Θ is the Heaviside step-function. The constants v_0 and ω are the amplitude and the frequency of vibrational velocity of the fluid, and $h = h_0$ is the equilibrium position of the spheres defined by the equation

$$\alpha^2 h_0^{3/2} = aF_0 . \quad (6)$$

In Figure 4 the position, velocity and the amplitude of the frequency content of the velocity are shown for $\beta = 1$ and $D = 2$. The usual nonlinear generation of higher harmonics takes place, and the appearance of low-frequency spectra is well pronounced. These low frequencies do not depend on the driving frequency of the acoustic source. The transformation of energy to low frequencies is connected with the high amplitude of negative displacement and the long time spent by particles in free motion between collisions. After each short collision the particles move slowly away from its neighbour and then back, leading to a very broad spectrum. Such spectral broadening for a harmonic input signal during its propagation through grainy media is a well-known phenomenon.

FIGURE 4. The position x , the velocity u , and the frequency content of the velocity u for the parameter values $\beta = 1$ and $D = 2$.

7 A distributed fluid-saturated grainy medium

We consider a more general distributed one-dimensional nonlinear model. The structure has periodically located nonlinear elements which form an infinite discrete chain. Each element of this chain is an aggregate of two spherical particles, like those studied in the preceding section. One can pass now to the continuum leading to an inhomogeneous nonlinear wave equation in the displacement ξ : [28]

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \left(1 - 2\epsilon \frac{\partial \xi}{\partial x}\right) \frac{\partial^2 \xi}{\partial x^2} + \frac{3m_0}{2m} \frac{\partial v}{\partial t} . \quad (7)$$

The velocity c of wave propagation and the nonlinear parameter ϵ in equation (7) are given by

$$c = L(\frac{3}{2m})^{1/2}(\frac{R}{2}E^2F_0)^{1/6}, \quad \epsilon = \frac{L}{4}(\frac{RE^2}{2F_0})^{1/3}. \quad (8)$$

Here R is radius of particle, $m = \rho V + m_0/2$ is mass of sphere with account for the associated mass, $m_0 = \rho_0 V$ of the liquid, and v_n is velocity of liquid flowing around n -th particle. At clamping force F_0 tending to zero the velocity c also tends to zero but the nonlinear parameter ϵ tends to infinity. So, at small F_0 we deal with a slow but strongly nonlinear wave.

Let us now return to one problem of special interest for geophysics and nonlinear wave theory. Let the vibration of surrounding liquid be caused by a propagating acoustic wave. In this case the velocity in equation (7) varies according to the law $v = v(t - x/c_0)$, where c_0 is the velocity of sound. One can control the velocity c (8) of wave propagation through the chain of grains by varying the clamping force F_0 . By increasing F_0 , one can increase c to be equal approximately to c_0 . If the condition $c \approx c_0$ is fulfilled the acoustic wave will excite a wave in the chain of grains most efficiently (this is a so-called wave resonance phenomenon [29]). The wave resonance phenomenon described here is of interest for geophysical applications. Let a wave propagate in water along a horizontal bottom surface. The sub-bottom layers consist of water-saturated sand or other grains moved around by a vibrating liquid. The pressure pressing the grains together increases with depth into the sediment. Therefore, at a specific depth the clamping force - the pressure increasing with depth - will lead to the equality of velocity of wave propagation (8) and the speed of sound at that depth. Consequently, wave resonance occurs at this depth, and waves in the sediment will be excited with high efficiency only in this layer. The existence of resonant frequencies of vibration of grains was discovered more than 20 years ago (see, for example, review [30]).

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