Optimum Window Design by Semi-Infinite Quadratic Programming

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***Abstract*—This letter presents a new extended active set strategy for optimum finite impulse response (FIR) window design by semi-infinite quadratic programming. The windows may be asymmetric corresponding to frequency responses with general nonlinear phase. The optimality criterion is to minimize the sidelobe energy (L_2-norm) subject to a peak sidelobe magnitude-constraint (L_∞-constraint). Additional linear constraints are used to form the mainlobe (unity DC gain). Numerical examples involving group delay specifications are used to illustrate the usefulness of the algorithm.**

I. Introduction

MANY signal processing applications involving filter design depend on both minimax and least squares methods. The least squares criterion is relevant when the exogenous input noise is random with known power spectrum whereas the minimax criterion is relevant when the disturbing signal is sinusoidal with unknown frequency. These two criteria correspond to a noise gain specification and a magnitude specification, respectively. The most flexible design-approach in a practical situation is to consider the combination of these methods, i.e., the tradeoff between the minimax and the least squares errors.

A new optimal window was defined in [1] providing the optimum tradeoff between the peak sidelobe level and the sidelobe energy. It was demonstrated that the classical minimax and least squares methods (Dolph–Chebyshev/prolate-spheriodal windows) are both fundamentally inefficient with respect to the other design criterion (the minimax window has the highest sidelobe energy etc.).

The optimum window [1] minimizes the sidelobe energy subject to a peak sidelobe magnitude-constraint together with some additional linear constraints (which can be used to form the mainlobe etc.). The solution for linear-phase FIR filters (symmetric windows) can be found by conventional quadratic programming methods [2] since the magnitude can be represented by a real amplitude function [3]. An efficient multiple exchange algorithm is proposed in [1] for the case of symmetric windows.

In this letter, we consider the general case with asymmetric windows and complex response. The optimum window design is formulated as a semi-infinite quadratic program. A new extended active set strategy is proposed for the solution of the optimization problem. Numerical examples involving group delay specifications are used to demonstrate the efficiency of the new algorithm.

We emphasize that the presented design technique may be useful for complex approximation with any filter structure having finite complex basis such as the digital Laguerre networks [4] and beamformers [5].

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II. Problem Formulation

The frequency response of a real N-tap finite impulse response (FIR) window is given by

$$H(\omega) = \sum_{n=0}^{N-1} h_n e^{-j\omega n} = \mathbf{h}^T \phi(\omega) \tag{1}$$

where $\mathbf{h}$ is a real $N \times 1$ vector containing the filter coefficients h_n, and $\phi(\omega)$ a complex vector of basis functions $e^{-j\omega n}$, $n = 0, \cdots, N-1$.

We pose the following design criterion:

$$\min_{\mathbf{h}\in R^N} \frac{1}{2\pi} \int_\Omega |H(\omega)|^2 \, d\omega, \quad \text{subject to} \tag{2}$$

$$|H(\omega)| \leq \sigma(\omega), \quad \omega \in \Omega_1 \tag{3}$$

$$\mathbf{Ph} = \mathbf{p} \tag{4}$$

where the stopband region Ω is a closed and bounded subset of $[-\pi, \pi]$, Ω_1 a finite subset of Ω, $\sigma(\omega)$ is a strictly positive magnitude bound, $\mathbf{P}$ an $M \times N$ constraint matrix and $\mathbf{p}$ an $M \times 1$ constraint vector.

The design objective in (2) is to minimize the sidelobe energy together with a peak sidelobe magnitude constraint (3). The linear constraints (4) can be used to form the mainlobe by choosing $\mathbf{P} = [1 \cdots 1]$ and $\mathbf{p} = 1$ (unity DC gain $H(0) = 1$).

We will now convert the design formulation (2)–(4) into a semi-infinite quadratic programming problem. According to the *real rotation theorem* [3], a magnitude inequality in the complex plane can be expressed in the equivalent form

$$|z| \leq \sigma \Leftrightarrow \Re\{ze^{j\theta}\} \leq \sigma, \quad \forall\theta \in [0, 2\pi] \tag{5}$$

where z is a complex number, σ a real and positive number, and $\Re\{\cdot\}$ denotes the real part.

By making use of (5), the design problem (2)–(4) can be reformulated as the semi-infinite quadratic program

$$\begin{cases}\min \frac{1}{2}\mathbf{h}^T\mathbf{Q}\mathbf{h} \\ \mathbf{a}^T(\omega,\theta)\mathbf{h} \le \sigma(\omega), \quad \omega\in\Omega_1, \quad \theta\in[0,2\pi] \\ \mathbf{P}\mathbf{h}=\mathbf{p}\end{cases} \tag{6}$$

where

$$\mathbf{Q} = \frac{1}{\pi}\int_\Omega \phi(\omega)\phi^H(\omega)\,d\omega \tag{7}$$

$$\mathbf{a}(\omega,\theta) = \Re\{\phi(\omega)\cdot e^{j\theta}\}. \tag{8}$$

Note that Ω_1 is assumed to be finite, whereas the phase variable θ belongs to an infinite set.

In our examples below $\Omega = [-\pi, -\omega_s] \cup [\omega_s, \pi]$ where ω_s is the stopband frequency. Thus, the elements q_{mn} of $\mathbf{Q}$ are calculated as follows:

$$q_{mn} = \begin{cases} -\dfrac{2\sin(\omega_s(m-n))}{\pi(m-n)} & m\ne n \\ \dfrac{2(\pi-\omega_s)}{\pi} & m=n. \end{cases} \tag{9}$$

III. Semi-Infinite Quadratic Programming

A. Kuhn–Tucker Conditions

The necessary and sufficient Kuhn–Tucker conditions related to (6) are given by [6]

$$\mathbf{Q}\mathbf{h} + \int_\mathcal{D} \mathbf{a}(\omega,\theta)\,d\Lambda + \mathbf{P}^T\boldsymbol{\mu} = 0 \tag{10}$$

$$\int_\mathcal{D} (\mathbf{a}^T(\omega,\theta)\mathbf{h} - \sigma(\omega))\,d\Lambda = 0 \tag{11}$$

$$\mathbf{a}^T(\omega,\theta)\mathbf{h} - \sigma(\omega) \le 0, \quad (\omega,\theta)\in\mathcal{D} \tag{12}$$

$$\mathbf{P}\mathbf{h} = \mathbf{p} \tag{13}$$

$$\Lambda \ge 0 \tag{14}$$

where Λ is a regular Borel measure, $\mathcal{D} = \Omega_1 \times [0,2\pi]$ and $\boldsymbol{\mu}$ is an $M\times 1$ vector of Lagrange multipliers.

It can be shown [6], [7] that the optimum (nonunique) measure Λ satisfying (10)–(14) can always be represented by a measure with finite support (atomic measure) at no more than N points. The Kuhn–Tucker conditions may therefore be written

$$\mathbf{Q}\mathbf{h} + \sum_{i=1}^{r} \lambda_i \mathbf{a}(\omega_i,\theta_i) + \mathbf{P}^T\boldsymbol{\mu} = 0 \tag{15}$$

$$\lambda_i(\mathbf{a}^T(\omega_i,\theta_i)\mathbf{h} - \sigma(\omega_i)) = 0, \quad i=1,\cdots,r \tag{16}$$

$$\mathbf{a}^T(\omega,\theta)\mathbf{h} - \sigma(\omega) \le 0, \quad (\omega,\theta)\in\mathcal{D} \tag{17}$$

$$\mathbf{P}\mathbf{h} = \mathbf{p} \tag{18}$$

$$\lambda_i \ge 0 \tag{19}$$

where $(\omega_i,\theta_i)\in\mathcal{D}$, $i=1,\cdots,r\le N$ and λ_i are the values of the atomic measure. The proof follows the same technique as given in [7, pp. 73–76] by an application of Caratheodory's theorem.

The implication of the conditions (15)–(19) is extremely useful since it allows us to solve the problem (6) using a *finite* active set in very much the same way as with the active set strategy for finite-dimensional quadratic programs.

B. Extended Active Set Strategy

Below, an extended active set strategy is given to solve the semi-infinite quadratic program (6). The algorithm is an extension of the finite-dimensional version given in [2].

It is assumed that an initial feasible solution $\mathbf{h}_0$ to (6) is available. This solution must satisfy the constraints in (3) and (4) and can be found by solving the semi-infinite linear programming problem

$$\begin{cases}\min \delta \\ \frac{1}{\sigma(\omega)}\mathbf{a}^T(\omega,\theta)\mathbf{h} \le \delta, \quad \omega\in\Omega_1, \quad \theta\in[0,2\pi] \\ \mathbf{P}\mathbf{h}=\mathbf{p}\end{cases} \tag{20}$$

where δ is an additional real variable. The problem (20) can be solved by using simplex extension algorithms; see e.g., [7], [8]. If the optimum objective $\delta_0 \le 1$, the corresponding solution $\mathbf{h}_0$ is feasible. If $\delta_0 > 1$, a feasible solution does not exist.

The extended active set strategy to solve (6) proceeds as follows: Let $\mathbf{h}_k$ denote the solution at an iteration index k and $W_k = \{(\omega_1,\theta_1),\cdots,(\omega_r,\theta_r)\}$ the corresponding working set defined by r active constraints in the second row of (6). Note that the working set is identified by r *distinct* frequencies ω_i for which $|H_k(\omega_i)| = \sigma(\omega_i)$ and the corresponding phase angles uniquely given by $\theta_i = -\arg\{H_k(\omega_i)\}$. Note also that W_k may be empty (in particular, W_0 is empty if $\delta_0 < 1$).

Define

$$\mathbf{A}_k = \begin{bmatrix}\mathbf{a}^T(\omega_1,\theta_1)\\ \vdots \\ \mathbf{a}^T(\omega_r,\theta_r) \\ \mathbf{P}\end{bmatrix}, \quad \mathbf{b}_k = \begin{bmatrix}\sigma(\omega_1)\\ \vdots \\ \sigma(\omega_r) \\ \mathbf{p}\end{bmatrix} \tag{21}$$

where $(\omega_i,\theta_i)\in W_k$, $i=1,\cdots,r$. Thus $\mathbf{A}_k\mathbf{h}_k = \mathbf{b}_k$.

The Lagrangian multipliers related to the working set W_k are given by

$$\boldsymbol{\lambda}_k = -(\mathbf{A}_k\mathbf{Q}^{-1}\mathbf{A}_k^T)^{-1}\mathbf{b}_k. \tag{22}$$

Given an initial feasible solution $\mathbf{h}_0$ and the corresponding working set W_0, the algorithm proceeds with the following basic steps.

1) Solve the equality constrained quadratic program

$$\begin{cases}\min \frac{1}{2}\mathbf{h}^T\mathbf{Q}\mathbf{h} \\ H(\omega_i)\cdot e^{j\theta_i} = \sigma(\omega_i), \quad (\omega_i,\theta_i)\in W_k \\ \mathbf{P}\mathbf{h}=\mathbf{p}\end{cases} \tag{23}$$

where $H(\omega) = \mathbf{h}^T\phi(\omega)$, $\mathbf{h} = \mathbf{h}_k + \mathbf{d}_k$ and the optimization is performed with respect to the vector increment $\mathbf{d}_k$.

If $\mathbf{d}_k \ne \mathbf{0}$ proceed with step 2. If $\mathbf{d}_k = \mathbf{0}$ calculate

$$\lambda_0 = \min_{i\in\{1,\ldots,r\}} \lambda_k(i) \tag{24}$$

where $\lambda_k(i)$ are the elements of $\boldsymbol{\lambda}_k$ given by (22). Denote the minimizing index q. If $\lambda_0 \ge 0$ then $\mathbf{h}_k$ is optimal; stop. If $\lambda_0 < 0$, exclude (ω_q,θ_q) from W_k to form W_{k+1}. Set $k=k+1$ and return to step 1.

2) Calculate the step size parameter

$$\alpha_k = \min_{(\omega,\theta)\in\mathcal{D}_k} \frac{\sigma(\omega) - \mathbf{a}^T(\omega,\theta)\mathbf{h}_k}{\mathbf{a}^T(\omega,\theta)\mathbf{d}_k} \tag{25}$$

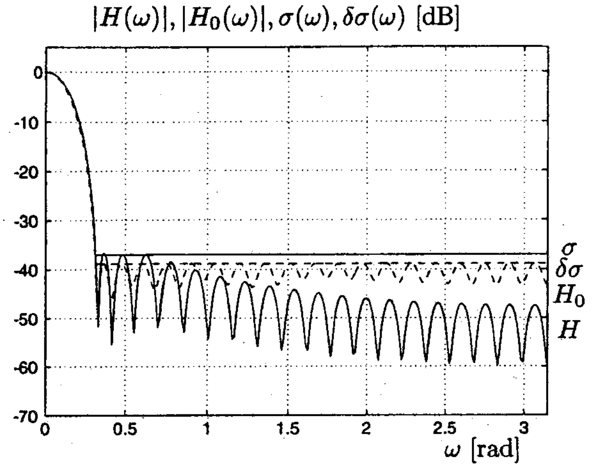

Fig. 1. Solid lines: Quadratic programming solution $|H(\omega)|$ and specified upper bound $\sigma(\omega)$ in dB. Dashed lines: Linear programming solution $|H_0(\omega)|$ and minimum bound $\delta\sigma(\omega)$ in dB. Specified group delay $\tau_d = 15$ for $\omega = 0$.

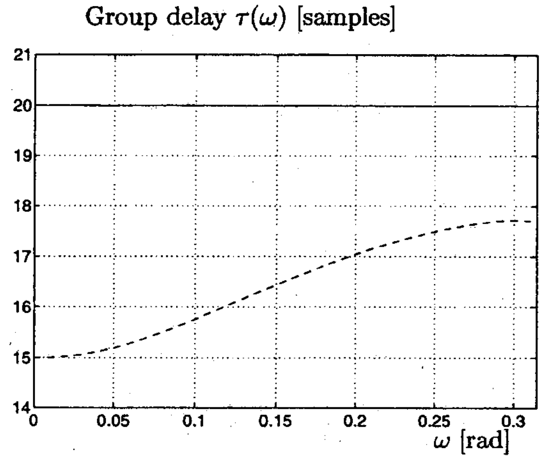

Fig. 2. Resulting group delay. Solid line: $\tau_d = 20$. Dashed line: $\tau_d = 15$.

where the minimization is performed over $\mathcal{D}_k = \{(\omega, \theta) : \mathbf{a}^T(\omega, \theta)\mathbf{d}_k > 0\}$. Let the minimizer be denoted (ω', θ'). If $\alpha_k < 1$, put $\mathbf{h}_{k+1} = \mathbf{h}_k + \alpha_k \mathbf{d}_k$ and $W_{k+1} = W_k \cup \{(\omega', \theta')\}$. If $\alpha_k \geq 1$, put $\mathbf{h}_{k+1} = \mathbf{h}_k + \mathbf{d}_k$ and $W_{k+1} = W_k$. Set $k = k + 1$ and return to step 1.

The minimization in (25) is performed in two steps. First calculate

$$\alpha_k(\omega) = \min_{\theta \in \Theta_k(\omega)} \frac{\sigma(\omega) - \Re\{H_k(\omega)e^{j\theta}\}}{\Re\{D_k(\omega)e^{j\theta}\}}, \quad \omega \in \Omega_1 \tag{26}$$

where $H_k(\omega) = \boldsymbol{\phi}^T(\omega)\mathbf{h}_k$, $D_k(\omega) = \boldsymbol{\phi}^T(\omega)\mathbf{d}_k$ and

$$\Theta_k(\omega) = \left\{\theta : -\arg(D_k(\omega)) - \frac{\pi}{2} < \theta < -\arg(D_k(\omega)) + \frac{\pi}{2}\right\}.$$

Next, calculate $\alpha_k = \min \alpha_k(\omega)$ where the minimization is performed over all $\omega \in \Omega_1$ such that $(\omega, \theta) \notin W_k$ for any θ. The minimizer is ω'.

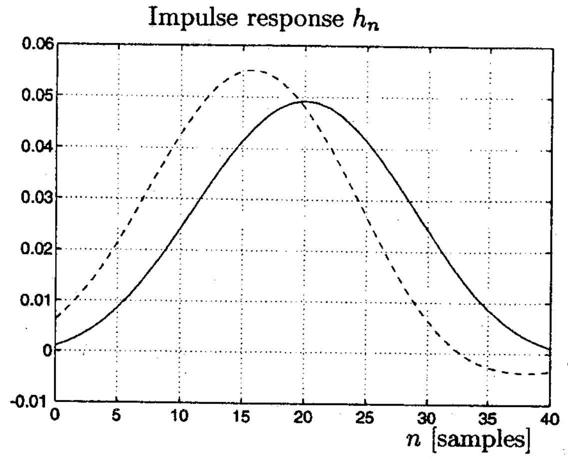

Fig. 3. Resulting impulse response. Solid line: $\tau_d = 20$. Dashed line: $\tau_d = 15$.

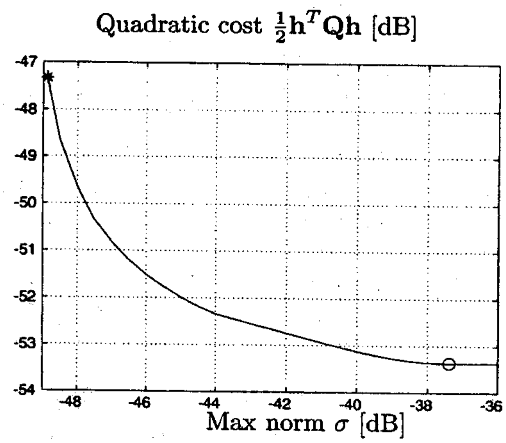

Fig. 4. Quadratic cost versus specified max norm σ ($N = 41$, $\tau_d = 20$). The $*$ and o denote the minimax and the eigenvector solutions, respectively.

An explicit formula for the minimizing phase θ in (26) can be found by differentiating the function

$$f(\theta) = \frac{\sigma - \Re\{ze^{j\theta}\}}{\Re\{we^{j\theta}\}} \tag{27}$$

with respect to θ. The differential is equated to zero and by standard trigonometric manipulations we arrive at

$$\theta = \sin^{-1}\left(\frac{xv - yu}{|w|\sigma}\right) - \arg(w) \tag{28}$$

where $z = x + jy$ and $w = u + jv$.

Formula (28) is used to calculate the function $\theta(\omega)$ of minimizing phase angles for all $\omega \in \Omega_1$ in (26). Thus, the minimizer in (25) is given by $(\omega', \theta') = (\omega', \theta(\omega'))$.

IV. Numerical Examples

As a numerical example we consider the design of an optimum FIR window as described in Section II. The filter length is $N = 41$, the stopband magnitude constraint $20 \log \sigma(\omega) = -37$ dB, the stopband frequency $\omega_s = 0.1\pi$ and the domain Ω_1 consists of 200 points evenly distributed in

$[\omega_s, \pi]$. In addition to the mainlobe constraint we also employ a linear group delay-constraint to define (4). This constraint approximates a group delay of τ_d samples for $\omega = 0$ and is given by

$$\sum_{n=0}^{N-1} h_n(n - \tau_d) = 0 \tag{29}$$

(see, e.g., [9]).

Figs. 1–4 show the resulting design when the specified group delay τ_d is 20 respective 15 samples. The design algorithm (the extended active set strategy as described in Section III) converged in two and six iterations, respectively. The case with $\tau_d = 15$ was chosen to illustrate the ability to design asymmetric windows with complex response.

Fig. 4 shows the tradeoff in quadratic cost versus max norm for the case with $\tau_d = 20$. Fig. 4 also shows the minimax solution given by (20) and the eigenfilter corresponding to the minimum eigenvalue of $\mathbf{Q}$.

V. Summary and Conclusions

This work presents a new extended active set strategy to solve the semi-infinite quadratic programming problem corresponding to an optimum FIR window design. The optimality criterion is to minimize the sidelobe energy subject to a max norm constraint and additional linear constraints (mainlobe DC gain etc.). The success of the approach relies on the finiteness of related Lagrange multipliers.

Design examples involving group delay constraints are included to demonstrate the flexibility (asymmetric windows and complex response) and numerical efficiency of the design algorithm.

Future work includes applications to general digital filter design, Laguerre filter design, and beamforming.

References

[1] J. W. Adams, "A new optimal window," *IEEE Trans. Signal Processing*, vol. 39, pp. 1753–1769, Aug. 1991.

[2] D. G. Luenberger, *Linear and Nonlinear Programming*. Reading, MA: Addison-Wesley, 1984.

[3] T. W. Parks and C. S. Burrus, *Digital Filter Design*. Wiley, 1987.

[4] B. Wahlberg, "System identification using Laguerre models," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 551–562, May 1991.

[5] S. Nordebo, I. Claesson, and S. Nordholm, "Weighted Chebyshev approximation for the design of broadband beamformers using quadratic programming," *IEEE Signal Processing Lett.*, vol. 1, pp. 103–105, July 1994.

[6] S. Ito, Y. Liu, and K. L. Teo, "A dual parametrization method for convex semi-infinite programming," in *Proc. ICOTA'98*, Perth, Australia, July 1998, pp. 550–557.

[7] E. J. Anderson and P. Nash, *Linear Programming in Infinite-Dimensional Spaces*. New York: Wiley, 1987.

[8] D. Burnside and T. W. Parks, "Optimal design of FIR filters with the complex Chebyshev error criteria," *IEEE Trans. Signal Processing*, vol. 43, pp. 605–616, Mar. 1995.

[9] X. Chen and T. W. Parks, "Design of FIR filters in the complex domain," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 144–153, Feb. 1987.