Interaction between low and high-frequency modes in a nonlinear system: gas-filled cylinder covered by movable piston

O. V. RUDENKO * and C. M. HEDBERG
Department of Mechanics, Blekinge Institute of Technology, 371 79 Karlskrona, Sweden
Laboratory of Nonlinear Physics and Group Analysis
* *also at: Research Center ALGA: Advances in Lie Group Analysis*

Abstract. A simple mechanical system containing a low-frequency vibration mode and set of high-frequency acoustic modes is considered. The frequency response is calculated. Nonlinear behaviour and interaction between modes is described by system of functional equations. Two types of nonlinearities are taken into account. The first one is caused by the finite displacement of a movable boundary, and the second one is the volume nonlinearity of gas. New mathematical models based on nonlinear equations are suggested. Some examples of nonlinear phenomena are discussed on the base of derived solutions.

Keywords: vibration, wave, acoustics, nonlinear interaction, mathematical model, analytical solution, Lie group analysis

1. Introduction

Many multi-mode systems are known to demonstrate nonlinear behaviour. Interesting phenomena have been observed in high-Q acoustical resonators [1] - [3], namely, shock front formation in temporal profiles of standing waves, amplitude-dependent absorption of energy, distortion of frequency response, and nonlinear decrease in magnitude of Q-factor which does not depend on linear dissipative properties of a system at high levels of excitation. Another paradoxical phenomenon is the increase in Q-factor, when appropriately organized energy outflow from a resonator cavity results not in the expected attenuation of nonlinear vibrations but in their noticeable enhancement [4].

Some resonant systems have two different sets of modes with the frequencies lying in different frequency ranges. One of the best-known examples is a gas bubble oscillating in a liquid. The fundamental mode of radial oscillation of the bubble has the resonant frequency [1]

$$f = \frac{1}{2\pi}\sqrt{\frac{3\rho_g}{\rho_l}}\,\frac{c_g}{R} \tag{1}$$

where ρ_g and ρ_l are the densities of gas and liquid, c_g is the velocity of sound propagation in gas, and R is the radius of the bubble. For an air bubble in water one can easily evaluate the radius $R \approx 3.3 \cdot 10^{-4}$ cm corresponding to a resonant frequency of 1 MHz. Because the wavelength λ of ultrasound is much longer than R (for 1 MHz λ equals to 1.5 mm), this mode can be referred to as a low-frequency one. So, the low-frequency spectrum of nonlinear oscillations of the bubble is formed by multiples of fundamental harmonic (1) and by the combination frequencies $nf_1 \pm mf_2$, where f_1 and f_2 are two different low frequencies, and n and m are integers.

In addition to the low-frequency spectrum of bubble oscillation, a high-frequency one exists. This spectrum is formed by modes of wave in the gas inside the bubble. One can determine the natural frequencies of such spherical resonator from the equation [5]

$$\frac{2\pi R}{\lambda} = \tan(\frac{2\pi R}{\lambda}) \ , \tag{2}$$

which has the approximate solution:

$$\frac{R}{\lambda} = \frac{2n+1}{4} - \frac{1}{\pi^2(2n+1)} \ . \tag{3}$$

One can see from (3), that for high-frequency modes the wavelength is of the order of the radius of the bubble.

Another system having two sets of modes lying in different frequency ranges is a spherical cavity in a rubber-like solid whose shear elastic modulus is small in comparison with the bulk modulus. In other words, the velocity c_l of longitudinal waves is much higher than the velocity c_t of transverse waves. Resonant frequency of the low-frequency fundamental mode for such cavity equals to [6]

$$f = \frac{c_t}{\pi R}\sqrt{1 + \frac{3}{4}\frac{\rho_g c_g^2}{\rho_s c_t^2}} \approx \frac{c_t}{\pi R} \ , \tag{4}$$

where ρ_s is density of the solid. If such cavity performs a spherically symmetric pulsation, it radiates a longitudinal wave, and its wavelength

$$\lambda = \pi\frac{c_l}{c_t} \gg R \ . \tag{5}$$

is much longer than the radius. Again, the standing waves in the gas inside the cavity have wavelengths of order of R; these modes form the high-frequency branch of the spectrum. A list with further analogous examples can be made. However, high-frequency modes were studied separately from low-frequency modes in preceding works, and the interaction between high- and low-frequency modes were not taken into account. The aim of this paper is to derive equations describing these modes simultaneously. This kind of mathematical model offers possibility to describe new types of nonlinear interactions. The simplest one-dimensional system having physical properties discussed above is shown in Fig.1.

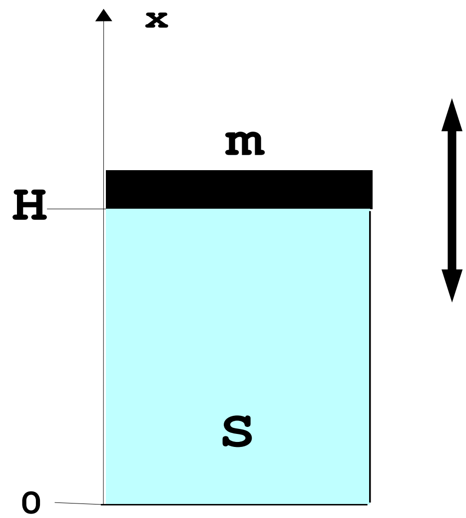

FIGURE 1. Gas filled cylinder covered by a movable piston: a system where vibration and acoustic modes can interact.

2. Governing equations and frequency response

Let the cross-section of the cylinder shown in Fig. 1 be S, its bottom is immovable and is located at $x = 0$. The coordinate axis x is directed along the vertical. Equilibrium position of piston having mass m is $x = H$, and $\zeta(t)$ is its displacement. So, the current coordinate of piston is $x = H + \zeta(t)$. The equation of motion is nonlinear

$$m\frac{d^2\zeta}{dt^2} + \nu\frac{d\zeta}{dt} = S \cdot p'(x = H + \zeta(t), t) + F(t) \ , \tag{6}$$

because the finite piston displacement is presented in the argument of pressure deviation p' from its equilibrium value. The friction ν and the given external force $F(t)$ are taken into account in the Eq.(6). Let the gas motion between the bottom and the piston be described by equations of continuous medium. Nonlinear and viscous terms in these equations are not considered, and a one-dimensional solution for pressure deviation p' and velocity u is given by the formulas:

$$p'(x,t) = p_+(t - x/c) + p_-(t + x/c) \ , \quad \rho c u(x,t) = p_+(t - x/c) - p_-(t + x/c) \ , \tag{7}$$

ρ is the density, and c is the sound velocity in the gas, the subscript $''g''$ is omitted, for simplicity. Here p_+ and p_- are two unknown functions describing the temporal profiles of waves propagating in positive and negative directions of x-axis, correspondingly. It follows that $p_+(t) = p_-(t) = p(t)$ from the boundary condition $u(x=0,t)=0$ on the immovable bottom. The second boundary condition

$$\rho c \frac{d\zeta}{dt} = p(t - \frac{H+\zeta(t)}{c}) - p(t + \frac{H+\zeta(t)}{c}) \tag{8}$$

follows from the equality of gas and piston velocities on the surface of the piston. The equation describing the piston vibration with account for the gas motion (7) takes the form

$$\frac{m}{S}\frac{d^2\zeta}{dt^2} + \frac{\nu}{S}\frac{d\zeta}{dt} = p(t - \frac{H+\zeta(t)}{c}) + p(t + \frac{H+\zeta(t)}{c}) + \frac{F(t)}{S} \ . \tag{9}$$

So, the model to be considered consists of two ordinary differential equations (8) and (9) with two unknown functions ζ and p. The difficulties in their treatment are in the arguments of function p , containing both "linear" $\pm H/c$ and "nonlinear" $\pm\zeta(t)/c$ shifts. We start with the analysis of linearized system (8), (9). After neglecting the shift $\pm\zeta(t)/c$ one can seek for the solution in the form

$$\zeta = \zeta_0 \exp(-i\omega t) \ , \quad p = p_0 \exp(-i\omega t) \ , \quad F = F_0 \exp(-i\omega t) \ . \tag{10}$$

Substitution of (10) into the linearized system of differential-difference equations reduces this system to an algebraic one for the constant amplitudes ζ_0 and p_0 :

$$p_0 \cos\frac{\omega H}{c} + \zeta_0(\frac{m\omega^2}{2S} + i\frac{\nu\omega}{2S}) = -\frac{F_0}{2S} \ , \tag{11}$$

$$p_0 \sin\frac{\omega H}{c} + \zeta_0\frac{\rho c\omega}{2} = 0 \ . \tag{12}$$

If we put $F_0 = \nu = 0$ and require the determinant of (11), (12) to be zero, we derive the equation for natural frequencies:

$$(kH)\cdot\tan(kH) = (K_0H)^2 = \frac{m_g}{m} \ , \tag{13}$$

where $k = \omega/c$, $K_0 = \Omega_0/c$, and

$$\Omega_0 = \sqrt{\frac{\rho c^2 S}{mH}} = \frac{c}{H}\sqrt{\frac{m_g}{m}} \tag{14}$$

is the frequency of piston vibration, when high-frequency modes are frozen. Here m_g is the mass of the gas inside the cylinder. This frequency (14) can be obtained from a general equation at the limiting transition $kH \to 0$. Another limiting case $K_0H \to 0$ corresponds to a heavy immovable piston, when

$$\tan(kH) = 0 \ , \quad \omega_n = \pi\frac{c}{H}n \ , \quad H = n\frac{\lambda}{2} \ , \quad (n = 1,\ 2,\ 3,\ \ldots) \ . \tag{15}$$

These are the usual acoustical modes, corresponding to standing waves between two immovable walls. Resonance in such a system happens if the length H of the resonator is equal to an integer number of half wavelengths. The frequency response of the system calculated by means of (11), (12), is

$$p_0 = \frac{\frac{m_g}{2m}\frac{F_0}{S}}{(kH)\sin(kH) - \frac{m_g}{m}\cos(kH) + i\frac{\nu H}{mc}\sin(kH)} \tag{16}$$

$$\zeta_0 = -\frac{\frac{H^2}{c^2}\frac{F_0}{m}\frac{\sin(kH)}{(kH)}}{(kH)\sin(kH) - \frac{m_g}{m}\cos(kH) + i\frac{\nu H}{mc}\sin(kH)} \ . \tag{17}$$

Two curves illustrating the normalized responses for pressure P_0 and displacement Σ_0 of piston

$$P_0 = \frac{2m}{m_g}\frac{S}{F_0}|p_0| \ , \qquad \Sigma_0 = \frac{mc^2}{H^2 F_0}|\zeta_0|$$

are shown in Fig.2 for the following parameters values: $\frac{m_g}{m} = 0.4$, $\frac{\nu H}{mc} = 0.2$, left, and $\frac{m_g}{m} = 4$, $\frac{\nu H}{mc} = 0.8$, right. The resonant curves demonstrate an interesting behaviour. All spectral peaks for pressure response are well defined. On the other hand, in the piston displacement curve only one peak is clearly defined, namely, the low-frequency one. The acoustic resonances do not significantly excite the piston.

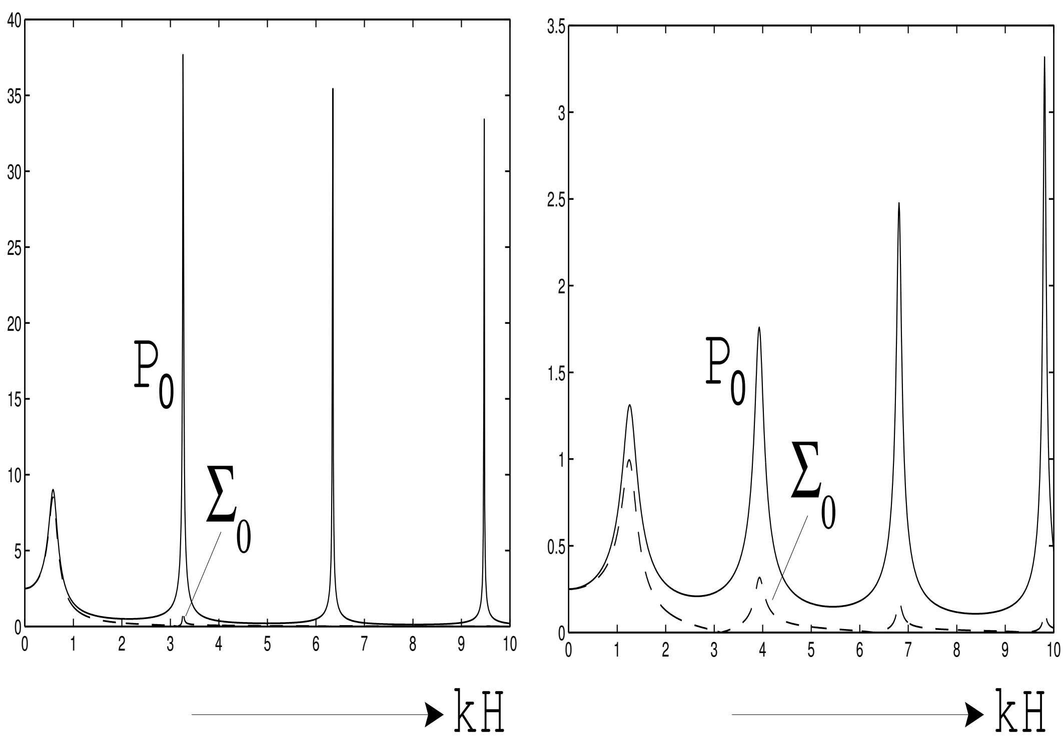

FIGURE 2. Frequency response for pressure (solid curve) and displacement of piston (dashed curve).

Evidently, the general solution of linearized system (8), (9) can be written as a sum of a partial solution of the inhomogeneous equations and the general homogeneous solution:

$$p = \frac{F_0}{2S} \frac{\cos(\omega t)}{\frac{m}{m_g}(kH)\sin(kH) - \cos(kH)} + \sum_{n=0}^{\infty} [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)] \ . \tag{18}$$

The solution is written here for a harmonic external force $F = F_0 \cos(\omega t)$ and for the absorption $\nu = 0$. The natural frequencies ω_n are determined by equation (13). If the frequency ω approaches one of natural frequencies ω_l, the amplitude of this selected mode $n = l$ will increase in time:

$$p = -\frac{F_0}{S} \frac{c}{2H} t \sin(\omega_l t) \frac{\frac{m_g}{m}\sqrt{(\frac{\omega_l H}{c})^2 + (\frac{m_g}{m})^2}}{(\frac{\omega_l H}{c})^2 + (\frac{m_g}{m})^2 + \frac{m_g}{m}} \ . \tag{19}$$

Near these resonances or at high level of excitation of the system the nonlinear phenomena must be considered.

3. Mode interaction caused by finite displacement of movable piston

The system (8), (9) is too difficult for an exact analytical solution. However, it can be simplified at small, but moderate, displacements of the piston (in comparison with the wavelength). In this case, the pressure in the right-hand-sides (r.h.s.) of equations (8) and (9) can be expanded in series with account for only first and second (linear on ζ) terms:

$$\rho c \frac{d\zeta}{dt} - [p(t - \frac{H}{c}) - p(t + \frac{H}{c})] = -\frac{m}{cS} \zeta \frac{d^3\zeta}{dt^3} \ , \tag{20}$$

$$\frac{m}{S}\frac{d^2\zeta}{dt^2} - [p(t - \frac{H}{c}) + p(t + \frac{H}{c})] = -\rho\zeta \frac{d^2\zeta}{dt^2} \ , \tag{21}$$

Again, external force and absorption are neglected. One can see, that l.h.s. of equations (20) and (21) are linear, but r.h.s. contain small quadratic nonlinear terms. Now we can combine these equations to exclude the variable $\zeta(t)$ in linear l.h.s. terms and reduce this system to the single nonlinear equation:

$$\frac{d}{dt}[p(t - \frac{H}{c}) - p(t + \frac{H}{c})] - \frac{H}{c}\Omega_0^2 [p(t - \frac{H}{c}) + p(t + \frac{H}{c})] = \frac{\rho c}{H} \frac{1}{\Omega_0^2} \frac{d}{dt}(\zeta \frac{d^3\zeta}{dt^3}) \ . \tag{22}$$

The second term in the r.h.s. of (22) has been omitted because it is much smaller than the kept one.

We consider now the interaction between disturbances having three different frequencies connected by the relation

$$\omega_2 = \omega_1 + \Omega \ . \tag{23}$$

We seek for the solution of equation (22) in the form

$$p = \frac{1}{2}(p_1 \exp(-i\omega_1 t) + p_2 \exp(-i\omega_2 t) + P \exp(-i\Omega t) + c.c.) \ , \tag{24}$$

$$\zeta = \frac{1}{2}(\zeta_1 \exp(-i\omega_1 t) + \zeta_2 \exp(-i\omega_2 t) + \Sigma \exp(-i\Omega t) + c.c.) \ , \tag{25}$$

Here *c.c.* means "complex conjugated" terms which are not written, for example, *c.c.* to the first term in (24) is $p_1^* \exp(i\omega_1 t)$. Other notations are as follows: $p_{1,2}$ and $\zeta_{1,2}$ are amplitudes of pressure and displacement of acoustic modes, and P, Σ are corresponding amplitudes of the low-frequency mode. Frequencies $\omega_{1,2}$ relate to acoustic modes and Ω is low frequency. All amplitudes can depend slowly on time. For example,

$$p_{1,2} = p_{1,2}(t_1 = \mu t) \ , \tag{26}$$

where t_1 is "slow time" (distinct from "fast" time describing oscillations, see exponential terms in (24), (25)), $\mu \ll 1$ is a small parameter proportional to the ratio of nonlinear term (r.h.s. of (22)) to any linear one. The order of smallness increases with increase in order of derivative:

$$\frac{d^2 p_{1,2}}{dt^2} = \mu^2 \frac{d^2 p_{1,2}}{dt_1^2} \ll \frac{dp_{1,2}}{dt} = \mu \frac{dp_{1,2}}{dt_1} \ll p_{1,2} \ . \tag{27}$$

The version of multiple scale method [7], [8] used here is traditionally applied to wave problems in dispersive media and to multi-mode oscillations (see, for example, books [1], [7]-[9]). The standard procedure in deriving simplified equations for complex amplitudes of interacting modes is the following. After substitution of (24), (25) into equation (22) terms of zero and first order (μ^0 and μ^1) are kept, but terms of higher orders are neglected. Consequently, second derivatives of amplitudes (27) disappear. Terms of zero order cancel each other in accordance with equations (11), (12) for natural frequencies. As a result, we derive an equation containing only μ^1 terms.

To illustrate in more detail the derivation of simplified differential equations, consider the group of linear terms μ^1 at the exponential factor $\exp(i\omega_1 t)$:

$$2i \sin(k_1 H) \frac{dp_1}{dt} + 2\frac{c}{H}[(k_1 H)\sin(k_1 H) - (K_0 H)^2 \cos(k_1 H)]p \ .$$

Let here $(k_1 H)$ be equal approximately to (kH), where (kH) satisfies the condition (13), i.e. the frequency ω_1 is close to one of the resonant frequencies ω, namely $\omega_1 = \omega - \delta_1$. where $\delta_1 \ll \omega$ is a small discrepancy. So, linear terms are transformed to

$$2i \sin(k_1 H) \frac{dp_1}{dt} - 2\delta_1 (\frac{\omega}{\Omega_0})^2 \sin(kH) \ .$$

At simplification, the inequality $(K_0 H) \ll 1$ was also taken into account. Now, collecting both linear and nonlinear terms at $\exp(-i\omega_1 t)$, one gets the following equation:

$$\frac{dp_1}{dt} + i\delta_1 (\frac{\omega}{\Omega_0})^2 p_1 = i\frac{\omega^3}{2H\Omega_0^2} \frac{\sin(k_2 H)}{\sin(k_1 H)} p_2 \Sigma^* \ , \tag{28}$$

By analogy, terms at the second exponential factor $\exp(-i\omega_2 t)$ form the equation

$$\frac{dp_2}{dt} - i\delta_2(\frac{\omega}{\Omega_0})^2 p_2 = i\frac{\omega^3}{2H\Omega_0^2}\frac{\sin(k_1H)}{\sin(k_2H)}p_1\Sigma \ . \tag{29}$$

Here δ_2 is introduced as $\omega_2 = \omega + \delta_2$, $\delta_2 \ll \omega$. That $\delta_1 + \delta_2 = \Omega$ follows from equation (23).

The third equation describing the evolution of the complex amplitude of the low-frequency vibration is

$$\frac{d\Sigma}{dt} - i2(\Omega - \Omega_0)\Sigma = i\frac{6}{(\rho c)^2 H\Omega_0}\sin(k_1H)\sin(k_2H)p_1^* p_2 \ . \tag{30}$$

Now one can use the system (28)-(30) to solve different problems of three-mode interaction, caused by the mobility of boundary, or, in other words, by finite displacement of the piston from its equilibrium position. An interesting problem is connected with the influence of boundary vibration on high-frequency acoustic modes inside the cavity. Let the external source excite a high-amplitude vibration of the piston. The approximation of pre-determined pump [9] is used based on the assumption that the amplitude is approximately constant. Physically, it means that energy outflow weakly influencies the pump, linear "beating" is negligible ($\Omega = \Omega_0$). In this case the pair of coupled equations (28),(29) reduces a linear system which can be easily solved for all values of parameters. In the simplest case $\delta_1 = \delta_2 = \delta$ this system reduces to the equation for a harmonic oscillator

$$\frac{d^2p_1}{dt^2} + \Omega_*^2 p_1 = 0 \ ,$$
$$\Omega_*^2 = \delta^2(\frac{\omega}{\Omega_0})^4 + (\frac{\omega^3}{2H\Omega_0})^2|\Sigma|^2 \quad . \tag{31}$$

With no restriction on generality the amplitude Σ is considered as a real quantity; phases of high-frequency pressure waves are measured from zero phase of Σ. The partial solution

$$p_2 = p_0[\cos(\Omega_* T) + i\frac{\delta}{\Omega_*}(\frac{\omega}{\Omega_0})^2\sin(\Omega_* t)] \ , \qquad p_1 = ip_0\beta\sin(\Omega_* t) \ , \tag{32}$$

where

$$\beta = \frac{\omega^3\Sigma}{2H\Omega_0^2\Omega_*}\frac{\sin(k_2H)}{\sin(k_1H)} \ ,$$

describes the "beating" of acoustic modes. At the moment $t = 0$ the amplitude of the ω_2-wave equals p_0, and the amplitude of the ω_1-wave is zero. With increase in time, the amplitudes (32) behave differently as p_2 decreases and p_1 increases. If the frequency ω_1 is close to the resonant one (13), the denominator in β (32) is small and amplitude p_1 may exceed the initial amplitude p_0 of another high-frequency wave many times. So, the piston vibrating with low frequency can excite a new intensive acoustic mode inside the cavity in the presence of a weak second wave.

For better understanding of this phenomenon numerical estimations are given below describing a simple laboratory experiment. Let the air-filled cylinder have height 10 cm and be covered by an organic glass piston of 1 mm in thickness - so the ratio m_g/m is about 0.1. The fundamental acoustic mode has frequency 1700 Hz, and the oscillation frequency is 170 Hz; the condition $\Omega/\omega \ll 1$ is valid. The modulation period (at $\delta \to 0$) of the acoustic amplitudes

$$T_{mod} = \frac{4}{\pi^2}\frac{m_g}{m}\frac{H^2}{c\Sigma}$$

shortens with increase in amplitude Σ of piston vibration. For example, at $\Sigma = 0.001 \cdot H$ the period T is approximately 0.01 s. Such modulation can easily be measured. Interactions of such a type are described in well-known books [10], [11], but for other nonlinear systems.

4. Interaction caused by nonlinearity of gas

A second principal kind of nonlinearity having great influence on the behaviour of waves inside the cavity is the gas volume nonlinearity. Effects of such nonlinearity can be studied by the simplified approach to nonlinear standing waves given in Ref. [3]. The vibration of gas between walls is described in [3] as the sum of two Riemann or Burgers counter-propagating waves [1]. Each of these two waves can be distorted significantly by a nonlinear self-action, which results in the formation of a sawtooth-shaped profile. There is, however, no significant energy exchange between these waves, and cross-interaction phenomena for periodic waves are negligible.

However, the piston motion was described in [3] by a given periodic function. In this paper the piston displacement $\zeta(t)$ is unknown, as is the acoustic pressure $p(x,t)$ in the gas. These functions are connected by the system

$$\rho c\frac{d\zeta}{dt} = p(t - \frac{H}{c} + \frac{\epsilon H}{c^3\rho}p) - p(t + \frac{H}{c} - \frac{\epsilon H}{c^3\rho}p) \ , \tag{33}$$

$$\frac{m}{S}\frac{d^2\zeta}{dt^2} = p(t - \frac{H}{c} + \frac{\epsilon H}{c^3\rho}p) + p(t + \frac{H}{c} - \frac{\epsilon H}{c^3\rho}p) + \frac{F(t)}{S} \ . \tag{34}$$

Here ϵ is the coefficient of nonlinearity, defined by $\epsilon = (\gamma+1)/2$ where γ is the adiabatic coefficient in the equation of state of the gas [1]. The system (33), (34) differs from (8), (9) only in phase shifts in arguments of the function $p(t)$. These shifts were determined earlier (see (8), (9)) by the finite piston displacement, and now they are caused by the volume nonlinearity. Equations (33), (34) can be considered as a generalization of the mathematical model in [3] with account for the new unknown variable $\zeta(t)$.

It is possible to eliminate ζ and reduce the system (33), (34) to one equation

$$\frac{d}{dt}[p(t - \frac{H}{c} + \frac{\epsilon H}{c^3\rho}p) - p(t + \frac{H}{c} - \frac{\epsilon H}{c^3\rho}p)]$$

$$= \Omega_0^2 \frac{H}{c}[p(t - \frac{H}{c} + \frac{\epsilon H}{c^3 \rho} p) + p(t + \frac{H}{c} - \frac{\epsilon H}{c^3 \rho} p) + \frac{F(t)}{S}] \ . \tag{35}$$

Let $F(t)$ be a known periodic external force. The pressure $p(t)$ is to be found. The nonlinear functional equation (35) is too complicated for analytical treatment, but it can be simplified by the method used in [12]. This method is effective if the frequency of vibration is near one of the resonant frequencies (15), for example near the frequency of the first ($n = 1$) acoustic mode:

$$\omega \frac{H}{c} = \omega_1 \frac{H}{c} + \Delta \approx \pi + \Delta \ , \quad \Delta = \pi \frac{\omega - \omega_1}{\omega_1} \ll 1 \ . \tag{36}$$

Taking into account the smallness of both the discrepancy Δ and the nonlinear phase shift, we can expand the unknown functions in the square brackets and reduce equation (35):

$$\frac{d}{dt}[p(\omega t - \pi) - (\Delta - \frac{\epsilon H \omega}{c^3 \rho} p) \frac{dp(\omega t - \pi)}{d(\omega t)} - p(\omega t + \pi) - (\Delta - \frac{\epsilon H \omega}{c^3 \rho} p) \frac{dp(\omega t + \pi)}{d(\omega t)}]$$

$$= \Omega_0^2 \frac{H}{c}[p(\omega t - \pi) - (\Delta - \frac{\epsilon H \omega}{c^3 \rho} p) \frac{dp(\omega t - \pi)}{d(\omega t)} + p(\omega t + \pi) + (\Delta - \frac{\epsilon H \omega}{c^3 \rho} p) \frac{dp(\omega t + \pi)}{d(\omega t)}]$$

$$+ \ \Omega_0^2 \frac{H}{c} \frac{F(\omega t)}{S} \ .$$

Because the function $p(\omega t)$ is quasi-periodic with period 2π, we can put in the last formula

$$p(\omega t - \pi) \approx p(\omega t + \pi) \ , \qquad p(\omega t - \pi) - p(\omega t + \pi) \approx -2\pi\mu \frac{\partial p}{\partial(\omega t_1)} \ , \tag{37}$$

where t_1 is "slow" time describing the temporal evolution of the "fast" vibration.

Taking into account (37), the equation is rewritten in the form:

$$\frac{\partial}{\partial \xi}[\frac{\partial U}{\partial T} + \Delta \frac{\partial U}{\partial \xi} - \pi \epsilon U \frac{\partial U}{\partial \xi}] = -\beta U + \frac{M}{2} \frac{d}{d\xi} f(\xi - \pi) \ . \tag{38}$$

The following dimensionless notations are used:

$$\xi = \omega t + \pi \ , \quad U = \frac{p}{c^2 \rho} \ , \quad T = \frac{1}{\pi} \omega t_1 \ , \quad \beta = \frac{1}{\pi} \frac{m_g}{m} \ , \quad M \frac{d}{d\xi} f(\xi - \pi) = -\beta \frac{F(\xi - \pi)}{S} \ . \tag{39}$$

It is important to compare the new nonlinear evolution equation (38) with another equation derived earlier (see (19) in Ref. [3]) where the piston motion is given by the function written in the r.h.s.:

$$\frac{\partial U}{\partial T} + \Delta \frac{\partial U}{\partial \xi} - \pi \epsilon U \frac{\partial U}{\partial \xi} = \frac{M}{2} f(\xi - \pi) \ . \tag{40}$$

In this paper the function $f(\xi-\pi)$ in equation (38) determines the temporal behaviour of the external force applied to the piston. This difference leads to a significant complication of the mathematical model. Now it is necessary to solve the inhomogeneous nonlinear equation (38) of second order, while the equation (40) is of the first one.

It is important to apply effective methods of mathematics on the new model (38) - first of all Lie group analysis [13]. Hopefully this will be done in future studies. For now we have to restrict ourselves to some approximate solutions.

The solution of equation (38) linearized for harmonic external force and zero initial condition:

$$f(\xi-\pi) = -\sin\xi \ , \qquad U(T=0,\xi) = 0$$

is given by the formula

$$U(T,\xi) = \frac{M}{4} T \frac{\sin(\frac{\beta-\Delta}{2}T)}{(\frac{\beta-\Delta}{2}T)} \sin(\xi + \frac{\beta-\Delta}{2}T) \ . \tag{41}$$

One can see, if the discrepancy Δ equals the dimensionless frequency shift β caused by the finite mass of the piston, the resonant growth of acoustic wave goes on according to a linear law $\sim T$. If $\Delta \neq \beta$, the amplitude of the wave oscillates, and its maximum magnitude

$$U_{max} = \frac{M}{2|\beta-\Delta|}$$

decreases at high difference between Δ and β.

The solution of equation (38) by successive approximation method is not given here, because anybody can perform these calculations by himself. This solution shows, that higher harmonics also grow in amplitude near the resonance. Therefore, the temporal profile of wave distorts and tends to a discontinuous one. Analogous behaviour was studied in Ref. [3] on the base of equation (40). At $T \to \infty$ the wave profile tends to its stationary form, which can be calculated from the ordinary differential equation

$$\frac{d^2}{d\xi^2}[\frac{\pi\epsilon}{2}U^2 - \Delta U] = \beta U - \frac{M}{2}\cos\xi \ . \tag{42}$$

This equation contains a harmonic source of energy inflow, but it does not contain dissipative terms. Consequently, the equilibrium in steady-state regime can be provided only by nonlinear absorption of energy on the shock fronts, and the wave profile is discontinuous. Further studies of both stationary waves and transient solutions on the base of (38), (42) calls for special attention and can be performed later by means of analytical [13] and numerical methods.

5. Conclusions

The problems studied above are important for several oscillatory systems of different physical nature. For example, oscillating bubbles are known to form spectra containing both low- and high-frequency modes and combination frequencies resulting from the interaction between linear modes. Cavitation spectra have been studied intensively up to now (see, for example, chapter 11 of the book [14], vol.2). A more exotic problem is connected with the low-frequency resonances of magmatic chamber containing gaseous melt of rocks which were discovered recently inside the Elbrus volcano [15].

Two types of nonlinearities, namely, those caused by the finite displacement of a vibrating boundary and by the volume nonlinearity of gas were taken into consideration. The simplest one-dimensional model in the form of a gas-filled cylinder covered by a movable piston was considered. Systems of functional equations (8), (9) and (33), (34) were derived with account for two above mentioned nonlinearities. These systems are rather complicated and, therefore, approximate methods were used for their analyses in some particular cases. The interaction between two high-frequency and one low-frequency vibrations was considered by reducing the functional system (8), (9) to three coupled ordinary differential equations for complex amplitudes.

A new partial differential equation (38) was derived to describe the forced standing waves excited near one of the acoustic resonant frequencies. This simplified evolution equation (38) differs in its structure from inhomogeneous nonlinear equations studied earlier and takes into account the term responsible for the influence of low-frequency motion on the acoustic wave profile in the cavity of resonator.

It is reasonable to continue the analytical and numerical studies of both general and simplified nonlinear models suggested here. A second promising direction of study will be the analysis of analogous problems for gas-filled cavities of spherical form which is more real and suitable for comparison between theoretical results and experimental data.

Acknowledgements

We are thankful to Professor Nail Ibragimov for fruitful discussions and for his idea to submit this paper. The work is partly supported by the grant "Nonlinear nondestructive evaluation of material conditions - resonance and pulse techniques" from Vetenskapsrådet, Sweden. We would like to thank the European Science Foundation for support of the network project NATEMIS (Nonlinear Acoustic Techniques for Micro-Scale Damage Diagnostics). The work is partly supported by RFBR.

References

1. O.V.Rudenko and S.I.Soluyan, *Theoretical foundations of nonlinear acoustics*, Plenum, Consultants Bureau, New York, 1977.
2. B.O.Enflo and C.M.Hedberg. *Theory of nonlinear acoustics in fluids*, Kluwer Academic Publishers, Dordrecht, 2002.
3. O.V.Rudenko, C.M.Hedberg, B.O.Enflo, 'Nonlinear standing waves in a layer excited by the periodic motion of its boundary', *Acoustical Physics*, **47**(4), 2001, 452-460.
4. O.V.Rudenko, A.L.Sobisevich, L.E.Sobisevich, and C.M.Hedberg, 'Enhancement of energy and Q-factor of a nonlinear resonator with an increase in its losses', *Doklady Physics*, **47**(3), 2002, 188-191.
5. M.A.Isakovich, *General acoustics*, Moscow, Nauka, 1973.
6. V.N.Alekseev, S.A., Rybak, *Bubble oscillations in elastic media*, Acoustical Physics, **45**(5), 1999, 535-540.
7. A.H.Nayfeh, *Perturbation methods*, John Wiley & Sons, New York, London, Toronto, 1976.
8. A.H.Nayfeh, *Introduction to perturbation techniques*, John Wiley & Sons, New York, Chichester, Brisbane, Toronto, 1981.
9. M.B.Vinogradova, O.V.Rudenko, A.P.Sukhorukov, *Theory of waves* (2nd edition), Moscow, Nauka, 1990.
10. A.H.Nayfeh and D.T.Mook, *Nonlinear oscillations*, Wiley-Interscience, New York, 1979.
11. A.H.Nayfeh, *Nonlinear Interactions: Analytical, Computational and Experimental Methods*, Wiley-Interscience, New York, 2000.
12. O.V.Rudenko, A.V.Shanin. 'Nonlinear phenomena accompanying the development of oscillations excited in a layer of a linear dissipative medium by finite displacements of its boundary', *Acoustical Physics*, **46**(3), 2000, 334-341.
13. N.H.Ibragimov, *Handbook of Lie group analysis of differential equations*, V.1, 2, 3. CRC Press, Boca Raton, Florida, 1994-1996.
14. *Nonlinear Acoustics at the Beginning of the 21st Century*, Eds. O.V.Rudenko and O.A.Sapozhnikov. Moscow State University, 2002.
15. A.L.Sobisevich, V.K.Milykov, A.V.Kozyreva, O.V.Rudenko and L.E.Sobisevich. 'On Resonant Properties of Magmatic Structures of the Elbrus Volkano'. In book [14], vol.1, pp.269-274.