A Multimode Mean Field Annealing Technique to Design Recursive Digital Filters

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***Abstract*—The multi-mode mean field annealing (MM-MFA) approach to combinatorial optimization is introduced as a tool to design recursive infinite-impulse response (IIR) digital filters with discrete coefficients. As an application example demonstrating the potential of the method we consider the design of structurally passive IIR digital filters realized as the sum of two all-pass functions. The new design technique facilitates the solution of nontrivial filter design problems such as satisfying a general frequency specification by solving a combinatorial optimization problem over discrete coefficients and a max-norm cost. The final solution is not guaranteed to be a globally optimal solution but the convergence time is short enough to allow interactive design even for large problems.**

***Index Terms*—Combinatorial optimization, hardware constraints, IIR filter, quantized coefficients.**

I. Introduction

The problem of designing digital infinite-impulse response (IIR) filters with a general specification, quantized coefficients and hardware constraints is very computationally intensive. This brief discusses a new method that has the potential to solve some of the above problems in a time short enough to allow interactive design of digital filters.

The theoretical foundation of the multi-mode mean field annealing (MM-MFA) algorithm originates from thermodynamics and statistical mechanics which states that the probability distribution of states (or configurations) in a physical system (or optimization problem) in thermal equilibrium is given by the Boltzmann distribution [1]. At high temperature the states are disordered and are all equally probable, but when the temperature is close to zero the states are highly ordered and optimum or near optimum states (or solutions) have the highest probability.

Simulated annealing (SA) is a well-known [1] optimization algorithm that relies on the above facts to find a good solution to hard problems and the MFA algorithm [2]–[4] is a close relative of SA. By applying a technique known as the mean field approximation [5] the mean value of binary decision variables are analytically estimated directly from the Boltzmann distribution as opposed to SA where the values of the decision variables changes randomly.

In this brief, we introduce the MM-MFA algorithm for combinatorial optimization applied to the design of structurally passive digital filters [6], [7]. The MM-MFA algorithm can be regarded as a generalization of the MFA procedure in the sense that the MM-MFA handles decision variables with general multiple levels whereas the MFA is usually described using binary decisions such as $\{-1, +1\}$ or $\{0, 1\}$. Furthermore, the novelty of the MM-MFA algorithm described herein is also the deduction using the concept of conditional expectations. This concept makes it very straightforward to extend the algorithm to combinatorial optimization problems over any discrete (and finite) variable space. However, for practical implementations, the number of levels of each decision variable should be reasonably low. The total number of states, or possible combinations, can be very large.

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In order to demonstrate the potential of the new method we consider the design of recursive (IIR) digital filters realized as the sum of two all-pass functions [6], [7]. This filter structure can be viewed as a variant of the celebrated wave lattice digital filters [8].

Digital filters realized as the sum of two all-pass functions possesses a number of attractive features such as: A minimum of multipliers is required, low-sensitivity with respect to coefficient quantization, simple stability requirements, simple signal scaling, low round-off noise and robustness with respect to limit cycles [6]. Since the structurally passive IIR digital filter implementation is stable regardless of the values of the filter parameters provided that these parameters are restricted to the range $(-1, 1)$ or $(-1, 0]$, we emphasize that this all-pass filter structure is particularly well-suited for combinatorial optimization over discrete coefficients with binary fixed point precision.

We also emphasize that the new MM-MFA design procedure facilitates a technique to solve nontrivial IIR digital filter design problems such as satisfying a general frequency specification. This is accomplished by solving a combinatorial optimization problem over discrete coefficients and a max-norm cost. With real (infinite precision) parameters, this design problem corresponds to a global nonconvex optimization problem which in general is very hard to solve. A two-step approach where the infinite precision problem is first solved and then quantized has been proposed by several authors [9], [10].

II. Problem Formulation

In order to make the design examples realistic, we will consider the low-sensitivity IIR digital filter implementation described in [6] together with a max-norm cost corresponding to a general magnitude response specification.

A. Structurally Passive Digital Filters

Consider the recursive filters $H(z)$ defined as the sum of the two all-pass filters $A_1(z)$ and $A_2(z)$ according to

$$H(z) = \frac{1}{2}\left(A_1(z) + A_2(z)\right) \tag{1}$$

where

$$A_1(z) = \frac{s_0 + z^{-1}}{1 + s_0 z^{-1}} \prod_{i=1}^{m} \frac{s_{2i-1} + s_{2i}(1 + s_{2i-1})z^{-1} + z^{-2}}{1 + s_{2i}(1 + s_{2i-1})z^{-1} + s_{2i-1}z^{-2}} \tag{2}$$

$$A_2(z) = \prod_{i=m+1}^{m+n} \frac{s_{2i-1} + s_{2i}(1 + s_{2i-1})z^{-1} + z^{-2}}{1 + s_{2i}(1 + s_{2i-1})z^{-1} + s_{2i-1}z^{-2}} \tag{3}$$

where $m = (M-1)/2$ and $n = N/2$.

As an example, we consider a low-pass filter $H(z)$ of odd order L, obtained as the sum of two all-pass functions as in (1), (2), and (3) of order M (odd) and N (even), respectively. For strictly stable filters with $|z_0| < 1$, we conclude that $-1 < s_i < 1$ for $i = 0, 2, 4, \ldots$ and $-1 < s_i \leq 0$ for $i = 1, 3, 5, \ldots$.

As a discrete coefficient space we consider a set of signed, fixed point digits with a resolution of 2^{-b}. Hence

$$s_i \in \mathcal{N} \subseteq \left\{n \cdot 2^{-b} \mid -2^b < n < 2^b\right\}, \qquad i \text{ even} \tag{4}$$

$$s_i \in \mathcal{N}^- \subseteq \left\{n \cdot 2^{-b} \mid -2^b < n \leq 0\right\}, \qquad i \text{ odd} \tag{5}$$

where n denotes an integer.

B. Optimization Formulation

As an example, we assume that $H(z) = (A_1(z) + A_2(z))/2$ is a low-pass filter of odd order L; other cases are similar.

Consider the general magnitude response specification

$$H_l(\omega) \leq |H(e^{j\omega})| \leq 1, \qquad \omega \in \Omega_p \subset [0, \pi] \tag{6}$$
$$0 \leq |H(e^{j\omega})| \leq H_u(\omega), \qquad \omega \in \Omega_s \subset [0, \pi] \tag{7}$$

where Ω_p and Ω_s denote the passband and stopband regions, respectively, $H_l(\omega)$ is a lower magnitude bound (in the passband), $0 < H_l(\omega) < 1$ and $H_u(\omega)$ is an upper magnitude bound (in the stopband), $0 < H_u(\omega) < 1$. It is assumed that the passband and stopband regions Ω_p and Ω_s are disjoint and closed subsets of the interval $[0, \pi]$.

Defining the desired magnitude response

$$H_d(\omega) = \begin{cases} 1, & \omega \in \Omega_p \\ 0, & \omega \in \Omega_s \end{cases} \tag{8}$$

and the weighting function

$$w(\omega) = \begin{cases} \dfrac{1}{1 - H_l(\omega)}, & \omega \in \Omega_p \\ \dfrac{1}{H_u(\omega)}, & \omega \in \Omega_s \end{cases} \tag{9}$$

it is straightforward to show that the specifications (6) and (7) are equivalent to the requirement

$$w(\omega) \cdot \big||H(e^{j\omega})| - H_d(\omega)\big| \leq 1, \qquad \omega \in \Omega \tag{10}$$

where $\Omega = \Omega_p \cup \Omega_s$.

The cost function $E(\mathbf{s})$ is now defined as

$$E(\mathbf{s}) = \max_{\omega \in \Omega} w(\omega) \cdot \big||H(e^{j\omega})| - H_d(\omega)\big| \tag{11}$$

where $\mathbf{s}$ is an $L \times 1$ vector of variables s_i.

It is concluded that the specification (6) and (7), or equivalently (10), is satisfied iff $E(\mathbf{s}) \leq 1$. The filter design problem can thus be regarded as an optimization problem where the objective is to minimize the cost function $E(\mathbf{s})$.

III. MM-MFA Algorithm

Now, the MM-MFA algorithm is introduced for combinatorial optimization applied to the design of structurally passive digital filters as described above.

Consider the state-space $\mathcal{S}$ consisting of all vectors $\mathbf{s}$ with elements $s_i, i = 0, \ldots, L-1$ defined as in (4) and (5). Consider also the global cost $E(\mathbf{s})$ connected with each state as defined by (11).

The idea behind the proposed algorithm is to put the problem into a statistical mechanics framework and then apply some powerful results originating from that discipline. From the theory of statistical mechanics it is well-known that for a (physical) system in (thermal) equilibrium with respect to an external control parameter (temperature) c the probability of a particular state $\mathbf{s}$ is given by the Boltzmann distribution

$$P(\mathbf{s}) = \frac{e^{-E(\mathbf{s})/c}}{\sum_{\mathcal{S}} e^{-E(\mathbf{s})/c}} \tag{12}$$

and the mean of the variable s_i, denoted $\langle s_i \rangle$, is given by

$$\langle s_i \rangle = \frac{\sum_{\mathcal{S}} s_i e^{-E(\mathbf{s})/c}}{\sum_{\mathcal{S}} e^{-E(\mathbf{s})/c}}. \tag{13}$$

Under the condition that the system is kept in equilibrium at all times, from (12) we have

$$\lim_{c \to \infty} P(\mathbf{s}_{\text{opt}}) = 1 \tag{14}$$

where $\mathbf{s}_{\text{opt}}$ is a state corresponding to an optimal solution.

The mean value of the variable s_i for a system in equilibrium is given by (13). For most practical problems the sums in (13) are impossible to compute in a reasonable amount of time. However, a good approximation can be found by first rewriting (13) using a conditional expectation of the variable s_i as

$$\langle s_i \rangle = \frac{\sum_{\mathcal{S}} e^{-E(\mathbf{s})/c} \left\{ \dfrac{\sum_{s_i} s_i e^{-\frac{1}{c}E(\mathbf{s})|_{s_i}}}{\sum_{s_i} e^{-\frac{1}{c}E(\mathbf{s})|_{s_i}}} \right\}}{\sum_{\mathcal{S}} e^{-E(\mathbf{s})/c}} \tag{15}$$

or

$$\langle s_i \rangle = \left\langle \frac{\sum_{s_i} s_i e^{-\frac{1}{c}E(\mathbf{s})|_{s_i}}}{\sum_{s_i} e^{-\frac{1}{c}E(\mathbf{s})|_{s_i}}} \right\rangle \tag{16}$$

where the sums over s_i above extends over all possible outcomes of s_i and then approximating (16) by applying the expectation operator $\langle\cdot\rangle$ inside the argument of $E(\cdot)$ as follows:

$$\langle s_i \rangle \approx \frac{\sum_{s_i} s_i e^{-(1/c)E(\langle \mathbf{s} \rangle)|_{s_i}}}{\sum_{s_i} e^{-(1/c)E(\langle \mathbf{s} \rangle)|_{s_i}}}. \tag{17}$$

By introducing the vector $\mathbf{v}$, with elements $v_i \in (-1, 1)$, for $i = 0, 2, 4, \ldots$ and $v_i \in (-1, 0]$ correspondingly for $i = 1, 3, 5, \ldots$ and by defining the cost function $E(\mathbf{v})$ consistent with the definition in (11) we may then formulate the approximation in (17) as the following real valued fixed point equation:

$$v_i = \frac{\sum_{s_i} s_i e^{-(1/c)E(\mathbf{v})|_{v_i = s_i}}}{\sum_{s_i} e^{-(1/c)E(\mathbf{v})|_{v_i = s_i}}} \tag{18}$$

where $v_i = \langle s_i \rangle$ and $\mathbf{v} = \langle \mathbf{s} \rangle$.

It is noted that the fixed point (18) forms an update rule for the variables v_i and thereby constitutes the core of the MM-MFA algorithm. The actual values for s_i are given by a suitably selected subset, or possibly all, of $\mathcal{N}$ and $\mathcal{N}^-$ for even and odd i, respectively. One useful subset, $\hat{\mathcal{N}}$, would be, e.g., all coefficients, $\hat{s}_i$, that can be represented using two signed power-of-two terms

$$\hat{s}_i = d_1 2^{-p_1} + d_2 2^{-p_2} \tag{19}$$

where $d_{1,2} = \{-1, 0, +1\}$ and $p_{1,2} = \{0, \ldots, b\}$.

Starting with a high value of c and iterating the update rule (18) while decreasing c will force the $v_i \to s_i \in \mathcal{N}$. For infinite c the trivial fixed points are

$$v_i = \frac{1}{|\mathcal{N}|} \sum_{s_i \in \mathcal{N}} s_i \tag{20}$$

where $|\mathcal{N}|$ denotes the size of set $\mathcal{N}$. As the control parameter c decreases and $c \to 0$ the properties of the Boltzmann distribution (12) ensure that a solution with a cost close to the global minimum is reached, provided that the approximation error in (17) is sufficiently small.

The MM-MFA algorithm can be described as:

1) set initial v_i and initial value of c;
2) calculate v_i from (18);
3) decrease the control parameter c geometrically;
4) repeat steps 2 and 3 until all v_i are close to an $s_i \in \mathcal{N}$.

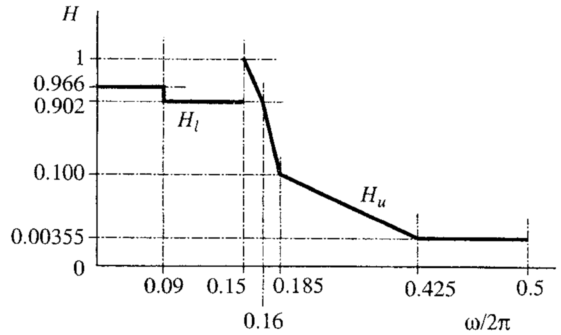

Fig. 1. The filter specification is a channel filter in a mobile communications system with a passband $0 \leq \omega/2\pi \leq 0.15$ and stopband $0.15 < \omega/2\pi \leq 0.5$. H_l denotes the maximum ripple in the passband and H_u the minimum attenuation in the stopband.

IV. Design Examples

As a design example using the MM-MFA algorithm, we consider the design of a structurally passive digital filter as described in Section II. The example specification is taken as the channel filter in a mobile communications system.[1]

The filter is intended for multiplierless very large scale integration-implementation and in order to conserve chip-area and reduce power consumption, while still allowing high sampling frequencies the coefficients are limited to a precision of 2^{-b}.

The example parameters are as follows: The filter order is $L = 5$, the number of bits is $b = 4$, and the filter specification is given in Fig. 1 where the different levels correspond to a passband ripple of 0.3 and 0.9 dB and a stopband attenuation of 0.9, 20, and 49 dB.

The MM-MFA solution was obtained in $K = 36$ iterations with a starting control parameter of $c = 800$ and a geometric decay factor $\lambda = 0.8$. The control parameter as a function of iteration index was thus given by

$$c(k) = 800 \cdot \lambda^{k-1}. \quad (21)$$

Fig. 3 shows the MM-MFA variables v_i as a function of iteration index k.

The solution found by the MM-MFA algorithm with the above parameters is $s = 2^{-4} \cdot [10, -14, 9, -9, 12]$, $E(s) = 0.963$ and the corresponding magnitude response $20 \log |H(\omega)|$ is shown in Fig. 2 together with the filter specification $20 \log H_u(\omega)$ and $20 \log H_l(\omega)$.

The computing time τ is proportional to the total number of cost function evaluations and to the size of the set $\mathcal{N}$

$$\tau \sim KL|\mathcal{N}|. \quad (22)$$

In practice, to solve the problem using Matlab, 280 Mflops were performed in 20 s which means that the algorithm coded in e.g., C using optimized math libraries would execute in just a few CPU seconds. As a rule, combinatorial optimization methods require a large computational effort and e.g., the problem in [9] requires a CPU-hour to quantize the infinite precision solution.

From these test results we make the following conclusions. Despite its relation to the stochastic simulated annealing algorithm, the MM-MFA algorithm is deterministic and capable of reliably giving a good, reproducible solution in a very short time. In general, the obtained solution is suboptimal, which is due to the approximation in (17) where accuracy was traded for for speed, but that is not a problem if the filter fulfills the specification.

The MM-MFA algorithm should therefore be regarded as a complement rather than a replacement for other more time-consuming algorithms. The MM-MFA algorithm could be used to interactively investigate the effect of changing the problem parameters and, if neccessary, other methods requiring a large computational effort could be used to further improve the solution once the problem parameters are set.

[1]The example specification is obtained by Ericsson Mobile Communications AB, mobile phones, Lund, Sweden.

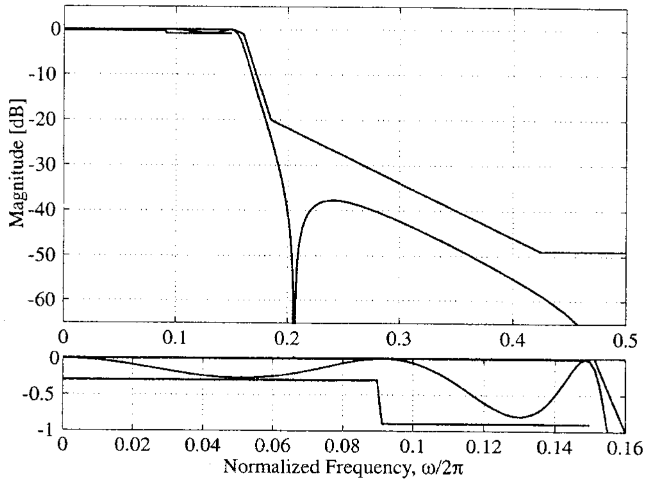

Fig. 2. The magnitude response $20 \log |H(\omega)|$ corresponding to the solution found by the MM-MFA algorithm together with the filter specification $20 \log H_u(\omega)$ and $20 \log H_l(\omega)$. The lower plot shows a zoom of the passband.

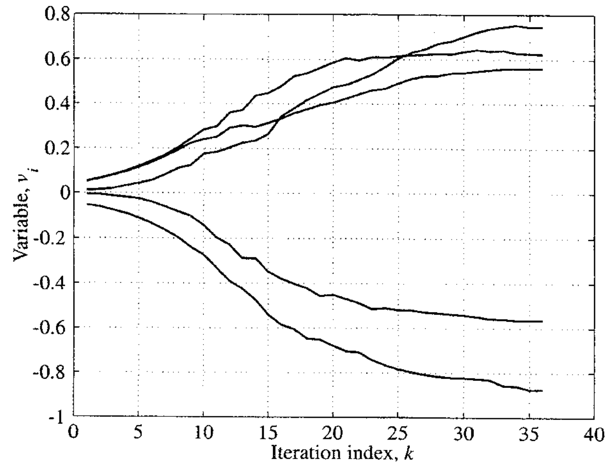

Fig. 3. The MM-MFA variables v_i as a function of iteration index, k.

V. Summary and Conclusions

A new MM-MFA technique is proposed for combinatorial optimization with applications to IIR digital filter design. The iterative algorithm is deterministic and deduced directly from the Boltzmann distribution. The MM-MFA algorithm shows very good potential for solving nontrivial IIR digital filter design problems such as satisfying a general frequency specification by solving a combinatorial optimization problem over discrete coefficients and a max-norm cost.

Generally, the MM-MFA technique is very fast and delivers a near optimal solution which makes the MM-MFA technique well suited as an interactive design tool.

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