QUADRATIC PROGRAMMING FOR THE DESIGN OF TWO-DIMENSIONAL WEIGHTED CHEBYSHEV FIR FILTERS WITH INCOMPLETE SPECIFICATIONS

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1. INTRODUCTION

The two-dimensional weighted Chebyshev FIR filter design problem [1, 2] is considered when the filter specification is incomplete. The incompleteness is with respect to two different situations. Firstly, the specification may be incomplete in the sense that the Chebyshev solution is non-unique. This occurs when the basis for the two-dimensional approximating function fails to satisfy the Haar condition [1].

Secondly, the specification may be incomplete in the sense that the filter is specified only on a fraction of the two-dimensional Fourier-plane. Conventional filter design methods may then yield anomalous designs in the sense that the filter coefficients may become excessively large. An application when this occurs is when a delay-and-sum beamformer [3] is specified only in space and frequency. With large filter weight values, the frequency response of the resulting beamformer may be severely degraded due to inaccuracies in sensor positions and/or sensor frequency response.

The remedy for both situations described above is given by a numerically well-conditioned quadratic programming formulation. In the first situation, the quadratic program is used to obtain the unique *minimum norm* Chebyshev solution. In the second situation, a trade off is made between a small Euclidean filter weight norm and a small Chebyshev deviation.

2. THE WEIGHTED CHEBYSHEV DESIGN PROBLEM

A number of methods have been developed for the solution of the two-dimensional weighted Chebyshev FIR filter design problem [1, 2], [4]–[8]. The problem is, however, generally non-unique since the Haar condition [1, 4] may not hold due to the two-dimensional nature of the approximating functions.

In case of non-uniqueness, some additional constraints must be defined in order to make the Chebyshev design unique. The concept of strict approximation [4, 9] was defined for this purpose. Basically, the strict approximation is based on the requirement that the maximum deviation should be as small as possible when the Chebyshev deviation points are removed.

The process of strict approximation is, however, quite complicated and may require several repetitions. Furthermore, the question of uniqueness of a given Chebyshev solution is rather extensive to investigate.

A simple and applicable choice of additional constraint is the requirement of minimum Euclidean filter weight norm. This requirement leads to a unique weighted Chebyshev solution which can be obtained by using a well-conditioned quadratic programming formulation. With this design approach, the extensive uniqueness-investigation is avoided. Further, the minimum norm requirement

where δ is an additional real variable and the v_i s are the same weighting coefficients as in (5), cf. [6]–[8]. For an optimal solution $(\mathbf{a}_o, \delta_o)$ to (6), $\mathbf{a}_o$ will be referred to as a Chebyshev solution, and $\delta_o = \|\mathbf{H}_o - \mathbf{H}_d\|$ as the optimal Chebyshev deviation.

The optimization problem in (5) or (6) is in general not unique since the N -dimensional linear space of approximating functions $H(\omega_{1i}, \omega_{2i}) = \phi_i^T \mathbf{a}$ does not generally satisfy the Haar condition [1, 4]. This is due to the two-dimensional nature of the basis functions for the linear space above. The question of uniqueness of a given Chebyshev solution $\mathbf{a}_o$ is rather complicated and requires an extensive investigation in order to determine whether the solution is unique or not.

Uniqueness is guaranteed if there is a critical point set $\mathcal{C}$ of $N + 1$ extremal points related to the given Chebyshev solution, cf. [1, 4, 9]. An extremal point is a frequency point $(\omega_{1i}, \omega_{2i})$, where the weighted error magnitude attains the Chebyshev deviation. A critical point set is a set of $p + 1$ extremal points such that the origin belongs to the convex hull of the corresponding signed and weighted characteristic vectors $\sigma_i v_i \phi_i$, and if any point were removed from the set, the origin would not belong to the convex hull of the reduced set [1, 4]. Here $\sigma_i = +1$ if $\phi_i^T \mathbf{a} - H_{di} > 0$ and $\sigma_i = -1$ if $\phi_i^T \mathbf{a} - H_{di} < 0$.

A characterization theorem for Chebyshev approximation [9] states that $\mathbf{H}_o$ is a Chebyshev solution if and only if the set $\mathcal{E}$ of extremal points contains a critical point set $\mathcal{C}$ of $p + 1$ points, where $p \leq N$. If $p = N$, the given Chebyshev solution $\mathbf{a}_o$ is unique, and if $p < N$ the solution *might* be non-unique [1, 4].

The question of uniqueness could, of course, be determined by directly checking the Haar condition, which is a necessary and sufficient condition for uniqueness. This would, however, require an exhaustive calculation of the rank of all subcollections of N vectors among all I characteristic vectors. The Haar condition is fulfilled if and only if all these $\binom{I}{N}$ subcollections are linearly independent.

3. MINIMUM NORM CHEBYSHEV DESIGN USING QUADRATIC PROGRAMMING

The squared Euclidean filter weight norm is given by

$$\|\mathbf{h}\|^2 = \mathbf{h}^T \mathbf{h} = \sum_{n_1} \sum_{n_2} h^2(n_1, n_2) = \mathbf{a}^T \mathbf{C} \mathbf{a}, \quad (7)$$

where $\mathbf{C}$ is a diagonal matrix determined by the four-fold symmetry defined by (2) so that the diagonal elements $(\mathbf{C})_{ii}$ are either 1, 1/2 or 1/4, see (3).

Consider the following quadratic program

$$\begin{cases} \text{minimize } \mathbf{a}^T \mathbf{C} \mathbf{a}, \text{ subject to} \\ v_i(\phi_i^T \mathbf{a} - H_{di}) \leq \delta_o & \text{for } i = 1, \dots, I \\ -v_i(\phi_i^T \mathbf{a} - H_{di}) \leq \delta_o & \text{for } i = 1, \dots, I \end{cases} \quad (8)$$

where the v_i s are the same weighting coefficients as in (6), and δ_o is the optimal Chebyshev deviation related to any Chebyshev solution. Since the objective function $\|\mathbf{h}\|^2 = \mathbf{a}^T \mathbf{C} \mathbf{a}$ is strictly convex [13], it follows from (8) that there always exists a unique Chebyshev solution $\mathbf{a}'_o$ to (8) with minimum filter weight norm $\|\mathbf{h}\|$.

The minimum norm Chebyshev solution $\mathbf{a}'_o$ to (8) can be obtained by solving the modified quadratic program given by

$$\begin{cases} \text{minimize } w \cdot \mathbf{a}^T \mathbf{C} \mathbf{a} + \delta^2, \text{ subject to} \\ v_i(\phi_i^T \mathbf{a} - H_{di}) - \delta \leq 0 & \text{for } i = 1, \dots, I \\ -v_i(\phi_i^T \mathbf{a} - H_{di}) - \delta \leq 0 & \text{for } i = 1, \dots, I. \end{cases} \quad (9)$$

An example where a conventional Chebyshev solution is non-unique is described below. An 81-tap two-dimensional lowpass FIR filter was designed. Four-fold symmetry with $N_1 = N_2 = 5$ was used. Fig. 1 shows the passband and stopband regions, defined by ellipses on a rectangular grid in the first quadrant with spacing 0.15×0.15 . This yielded a total of $I = 397$ frequency points.

Fig. 1. Approximation domain defined by ellipses. The major and minor axes defining the passband and stopband regions are (2,1) and (2.5,1.25) respectively. The frequency weighting in the passband and stopband regions are $v_i = 1$ and $v_i = 10$ respectively.

The resulting weighted Chebyshev design is non-unique. A critical point set was found, consisting of $p+1 = 6$ points. The filter weight norm obtained by using the linear program (6) and the modified quadratic program (9) was $\|\mathbf{h}_o\| = 0.4132$, and $\|\mathbf{h}_o''\| = 0.4009$ respectively. The Chebyshev deviation in both cases was $\delta_o = \delta_o'' = 0.5885$.

Fig. 2 shows the Chebyshev deviation and the filter weight norm as obtained with the modified quadratic program (9) when the parameter w is varied. This figure illustrates the fact that when the parameter w is chosen sufficiently small, the solution to the modified quadratic program is consistent with the minimum norm Chebyshev solution.

Fig. 2. Filter weight norm $\|\mathbf{h}_o''\|$ and Chebyshev deviation δ_o'' as a function of the parameter w in the modified quadratic program. Here $(\mathbf{h}_o, \delta_o)$ corresponds to a Chebyshev solution obtained by solving the linear program.

Figs. 3 and 4 show the frequency response of the FIR filters with optimal Chebyshev deviation obtained by using the linear program and the modified quadratic program, respectively. The latter corresponds to the unique minimum norm Chebyshev solution.

Fig. 5. Broadband delay-and-sum beamformer. Here $X(f)$ is the Fourier transform of the input signal representing the plane wave front and $G(f, \theta)$ is the beamformer response. Each element n_1 of the sensor array is applied to a FIR filter where $h(n_1, n_2)$ denotes the tap weight n_2 .

By imposing appropriate symmetry on the coefficients $h(n_1, n_2)$, the response $H(\omega_1, \omega_2)$ can be designed to have exact linear phase. With odd number of sensors K , and odd number of FIR filter taps L for each sensor, a four-fold symmetry can be applied as in (2). The phase center is then chosen in the “middle” of the array, and the response has zero phase with the symmetries $H(\omega_1, \omega_2) = H(-\omega_1, \omega_2) = H(\omega_1, -\omega_2)$, and $G(f, \theta) = G(f, -\theta)$. With four-fold symmetry, the beamformer response is given by (3) and (4) with $N_1 = (K + 1)/2$ and $N_2 = (L + 1)/2$. The Chebyshev approximation problem can be solved using the linear programming formulation (6). The Euclidean filter weight norm is given by (7).

The approximation domain for the two-dimensional FIR filter representing the beamformer is given by (15). For $0 \leq f \leq 1/2T$ and $0 \leq \theta \leq \pi/2$, this is a triangular region in the ω_1 - ω_2 domain, as in Fig. 6. It is assumed here that $d/c = T$. The two dark shaded regions 1 and 2 in Fig. 6 are cut wedge regions corresponding to passband or stopband regions in the f - θ domain, defined by $(f_c \leq f \leq 1/2T, 0 \leq \theta \leq \theta_{c1})$ and $(f_c \leq f \leq 1/2T, \theta_{c2} \leq \theta \leq \pi/2)$, respectively.

Fig. 6. Two-dimensional FIR filter approximation domain corresponding to a far-field, linear and equispaced array.

A filter specification based on the f - θ domain only thus leaves a large region unspecified in the ω_1 - ω_2 Fourier plane. Computer experiments show that large unspecified frequency regions often imply large filter weight norms $\|\mathbf{h}\|$ when conventional filter design methods such as (6) are used. With large filter weight values, the beamformer response may be severely degraded due to inaccuracies in sensor positions and/or sensor frequency response [10]. A robust design will therefore refer to a filter design which prevents the filter weight values from becoming excessively large.

For the present far-field case, the problem can be overcome by using eight-fold symmetry [2], where in addition to (2), it is required that $h(n_1, n_2) = h(n_2, n_1)$, so that $H(\omega_1, \omega_2) = H(\omega_2, \omega_1)$.

Fig. 7 illustrates that with just a few dBs of increased Chebyshev deviation, the filter norm $\|\mathbf{h}\|$ will drastically decrease using (16), and the design will be robust with respect to array response perturbations.

Figs. 8 and 9 illustrate how the robust design (16) can be used to significantly reduce the filter weight norm $\|\mathbf{h}\|$, and hence the sensitivity to perturbations, without significantly affecting the nominal beamformer response, as compared with the ordinary Chebyshev design (6).

Fig. 7. Minimum filter norm $\|\mathbf{h}_r\|$ as a function of maximum weighted error α in a robust spatial LP-design. Here $(\mathbf{h}_o, \delta_o)$ corresponds to a Chebyshev solution obtained by solving a linear program (upper left corner).

Fig. 8. Nominal and perturbed beamformer response for a spatial LP-filter. One snapshot with a perturbation radius of 1%. Ordinary Chebyshev design: Chebyshev deviation $\|\mathbf{H}_o - \mathbf{H}_d\| = 0.0897$, $\|\mathbf{h}_o\|^2 = 69800$.

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