Complex Spreading Sequence Design

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Abstract

The performance of a code division multiple access system depends on the correlation properties of the employed spreading code. Low cross-correlation values between spreading sequences are desired to suppress multiple access interference. An auto-correlation function with a distinct peak enables proper synchronization and suppresses intersymbol interference. However, these requirements contradict each other and a trade-off needs to be established. In this paper, a global two dimensional optimization method is proposed to minimize auto-correlation with cross-correlation being allowed to lie within a fixed region. This approach is applied to design sets of complex valued spreading sequences.

1 Introduction

In direct sequence code division multiple access (DS-CDMA) systems, each user has its own spreading sequence which distinguishes it from other users. Since mobile stations are spatially separated, the signals arrive at the base station with different delays. Thus, it is not possible to maintain spreading sequences to be synchronized among each other. As a result, multiple access interference (MAI) will be experienced. In addition, multipath propagation causes multiple copies of a transmitted signal and generates intersymbol interference (ISI). Therefore, sequences in a spreading code should have low cross-correlation (CC) values to suppress MAI. The auto-correlation (AC) function of a spreading sequence on the other hand, should have a narrow peak to avoid ISI and to enable proper synchronization. Unfortunately, both correlation requirements contradict each other. A trade-off can be obtained by properly optimizing respective correlation measures of the spreading code in a joint manner.

This paper focuses on the design of spreading sequences with optimized correlation properties for asynchronous CDMA systems. Major focus is given to a family of complex valued polyphase spreading sequences, which offers a wide range of correlation properties. An optimization problem is formulated to minimize average mean square AC, R_{ac}, for a certain range of average mean square CC, R_{cc}. This problem can be formulated as a two dimensional optimization problem with two parameters m and n. A modified bridging method is proposed to minimize R_{ac} for a certain range of R_{cc}.

2 Correlation Measures

Let N denote the length of the spreading sequences. For a pair of sequences $u_k = [u_k(0), \ldots, u_k(N-1)]$ and $u_l = [u_l(0), \ldots, u_l(N-1)]$, the aperiodic correlation between u_k and u_l is defined as

$$C_{k,l}(i) = \begin{cases} \frac{1}{N}\sum\limits_{v=0}^{N-1-i} u_k(v)u_l^*(v+i), & 0 \le i \le N-1 \\ \frac{1}{N}\sum\limits_{v=0}^{N-1+i} u_k(v-i)u_l^*(v), & 1-N \le i < 0 \\ 0 \text{ otherwise} \end{cases} \quad (1)$$

where '*' denotes the complex conjugate of a complex variable.

As far as performance analysis for wireless communication systems is concerned, the focus has been changed from periodic to aperiodic correlation measures, e.g. [1, 2]. Instead of worst case scenarios extracted from maximum side lobes of periodic correlations, it seems to be more reasonable to consider mean square values of AC and CC of spreading sequences. Especially, the latter measure is much more appropriate for asynchronous CDMA systems in which the average bit error rate (BER) and eventually the capacity of the system depends on the average interference caused by aperiodic CC properties of the employed set of spreading sequences. Thus, this work is focused on optimizing average mean square aperiodic AC and average mean square aperiodic CC characteristics of spreading sequences.

Denote S as the number of sequences taken from a spreading code. The average mean square AC for S sequences $u_1, \cdots, u_S$ is defined as

$$R_{ac} = \frac{1}{S}\sum_{k=1}^{S}\sum_{i=1-N,i\neq 0}^{N-1} |C_{k,k}(i)|^2, \quad (2)$$

and the average mean square CC is given by

$$R_{cc} = \frac{1}{S(S-1)}\sum_{k=1}^{S}\sum_{l=1,l\neq k}^{S}\sum_{i=1-N}^{N-1} |C_{k,l}(i)|^2. \quad (3)$$

3 Complex Spreading Sequences

A family of complex spreading sequences with a wide range of correlation properties has been proposed in [1]. For $1 \leq k \leq N-1$, the v^{th} element $u_k(v)$ of a sequence u_k is defined by:

$$u_k(v) = (-1)^{k(v+1)} \exp\left[\frac{j\pi(k^m(v+1)^p + (v+1)^n)}{N}\right], \quad (4)$$

where $0 \leq v \leq N-1$ and parameters m, n and p take real values. For different combinations of m, n and p, the sequences have different correlation properties.

The mean square AC and the mean square CC are defined for a fixed combination of m, n and p. Thus, they are functions of the parameters m, n and p. The following theorems give some properties for the correlations of the sequences given in (4).

Theorem 1 *The aperiodic correlation is a continuous function of m, n and p for all k, l and i.*

Proof: This follows from (1) since $u_k(v)$, $u_l(v+i)$, $u_k(v-i)$ and $u_l(v)$ are continuous functions of m, n and p.

Theorem 2 *Average mean square AC and average mean square CC are continuous functions of m, n and p.*

Proof: This follows from (2) and (3) since $C_{k,l}(i)$ is a continuous function of m, n and p for all k, l and i.

Theorem 3 *For a fixed combination of (m,n,p), all the sequences have the same AC magnitude. This magnitude depends only on n if $p=1$, and is given $\forall\, k,i$ by [1]*

$$|C_{k,k}(i)| = \left|\frac{1}{N}\sum_{v=0}^{N-1-i}\exp\left[\frac{j\pi}{N}((v+1)^n - (v+i+1)^n)\right]\right|. \quad (5)$$

Note that $i=0$ gives

$$|C_{k,k}(0)| = 1.$$

Proof: For $1 \leq k \leq N-1$, the auto-correlation $C_{k,k}(i)$ for a sequence u_k is derived by substituting $l=k$ and $p=1$ to (1) and (4). We have

$$\begin{aligned} C_{k,k}(i) &= \tfrac{1}{N}\sum_{v=0}^{N-1-i} u_k(v)u_k^*(v+i) \\ &= \tfrac{1}{N}\sum_{v=0}^{N-1-i} (-1)^{2kv+2k+ki} \\ &\times \exp\left[\tfrac{j\pi}{N}(k^m(v+1) + (v+1)^n)\right] \\ &\times \exp\left[\tfrac{-j\pi}{N}(k^m(v+i+1) + (v+i+1)^n)\right] \end{aligned} \quad (6)$$

for $0 \leq i \leq N-1$ and

$$\begin{aligned} C_{k,k}(i) &= \tfrac{1}{N}\sum_{v=0}^{N-1+i} u_k(v-i)u_k^*(v) \\ &= \tfrac{1}{N}\sum_{v=0}^{N-1+i} (-1)^{2kv+2k-ki} \\ &\times \exp\left[\tfrac{j\pi}{N}(k^m(v-i+1) + (v-i+1)^n)\right] \\ &\times \exp\left[\tfrac{-j\pi}{N}(k^m(v+1) + (v+1)^n)\right] \end{aligned} \quad (7)$$

for $1-N \leq i < 0$.

It follows from (6) and (7) that

$$C_{k,k}(i) = \begin{cases} \frac{(-1)^{ki}}{N}\exp\left[\frac{-j\pi}{N}k^m i\right] \\ \times \sum_{v=0}^{N-1-i}\exp\left[\frac{j\pi}{N}((v+1)^n - (v+i+1)^n)\right] \\ \text{for } 0 \leq i \leq N-1, \\ \frac{(-1)^{ki}}{N}\exp\left[\frac{-j\pi}{N}k^m i\right] \\ \times \sum_{v=0}^{N-1+i}\exp\left[\frac{j\pi}{N}((v-i+1)^n - (v+1)^n)\right] \\ \text{for } 1-N \leq i < 0. \end{cases} \quad (8)$$

Thus, the auto-correlation magnitude depends only on the parameter n and is given by

$$|C_{k,k}(i)| = \left|\frac{1}{N}\sum_{v=0}^{N-1-i}\exp\left[\frac{j\pi}{N}((v+1)^n - (v+i+1)^n)\right]\right| \quad (9)$$

with $0 \leq i \leq N-1$ and

$$|C_{k,k}(i)| = |C_{k,k}(-i)|, \;\; 1-N \leq i < 0. \quad (10)$$

For $i=0$, we have $|C_{k,k}(0)| = 1$.

In view of Theorem 3, we will concentrate only on the case $p=1$.

4 Problem Formulation

Sequences with low mean square AC approximate a delta function in time domain. Thus, the frequency spectrum of these sequences is wide and flat as desired for CDMA systems. This characteristic comes at the expense of the mean square CC values, which are large between different sequences in the set when the mean square AC is low. In most situations, however, an acceptable region for R_{cc} can be defined for sufficient cross-correlation. Then, the problem becomes:

For a fixed region of R_{cc}, obtain a set of parameters m and n so that R_{ac} is minimized with the corresponding R_{cc} lying within a bound α.

The respective optimization problem is formulated as

$$\begin{cases} \min\limits_{m,n} R_{ac} \\ \text{subject to } R_{cc} \leq \alpha. \end{cases} \tag{11}$$

Note that (11) is a two dimensional optimization problem with two continuous variables m and n. The problem has a non-linear continuous cost function and a non-linear continuous constraint. A modified approach based on the idea of using the bridge function for one dimensional problem [3, 4] is proposed to solve this two dimensional optimization problem with a non-convex constraint.

5 Global Two Dimensional Optimization Method

The main idea of the bridging method is to use the bridged functions in the search for the global minimum so that the method does not get stuck at a local minimum point. The bridging method finds a local minimum point of the bridged function starting from one end of the feasible region. The bridged function is then updated every time a local solution is obtained until the global minimum of the original cost function is found.

By incorporating a local search method such as the golden section search or the parabolic interpolation [5, 6], the modified bridging method can be improved so that no smoothing procedure is needed. Thus, it is applicable to cost functions which are only continuous instead of continuously differentiable. The modified algorithm is developed to solve the two dimensional global optimization problem (11) with a general form

$$\begin{cases} \min\limits_{m,n} f(m,n) \\ \text{subject to } g(m,n) \leq \alpha, \end{cases} \tag{12}$$

where $m \in [m_1, m_2]$, $n \in [n_1, n_2]$ and $f(m,n)$, $g(m,n)$ are continuous functions of m and n. For a fixed value of m, (12) becomes a one dimensional global optimization problem, which can be solved by extending the bridging method [3, 4] to include a non-linear constraint.

The algorithm starts by searching for the first feasible point (m_0, n_0) of the problem (12), i.e. $m_0 \in [m_1, m_2]$, $n_0 \in [n_1, n_2]$ and $g(m_0, n_0) \leq \alpha$. This is done as follows:

Procedure 1 *Search for feasible point from (m_0, n_0)*

1. *Specify step size h.*
2. *Fix the value of m_0 and search for the left most feasible solution of (12). This is done by finding the smallest value of ν such that*

$$g(m_0, n_0 + \nu h) \leq \alpha. \tag{13}$$

3. *If ν exists, set $n_0 = n_0 + \nu h$ and go to Step 4.*
   If ν does not exist, then
   - *If $m_0 + h \leq m_2$, set $m_0 = m_0 + h$, $n_0 = n_1$ and go back to Step 2.*
   - *If $m_0 + h > m_2$, the optimization is too restrictive and no feasible solution exists from (m_0, n_0). Go to Step 4.*
4. *Stop the procedure.*

The value of (m_0, n_0) will be used as the initial search point for problem (12). For each (m_0, n_0), define the following minimization problem:

$$\begin{cases} \min\limits_{n \in [n_0, n_2]} b(m_0, n) \\ \text{subject to } g(m_0, n) \leq \alpha, \end{cases} \tag{14}$$

where $b(m_0, n) = \min\{f(m_0, n), f(\hat{m}, \hat{n})\}$. The function $b(m_0, n)$ is called the bridged function and $f(\hat{m}, \hat{n})$ gives the value of the cost function $f(m,n)$ at the local minimum $(\hat{m}, \hat{n})$. The modified bridging algorithm incorporates Procedure 1 and is given as follows:

Procedure 2 *Modified bridging method*

1. *Initialize $(m_0, n_0) = (m_1, n_1)$.*
2. *Obtain the first feasible point from (m_0, n_0) by using Procedure 1. If this feasible point does not exist, go to Step 5. Otherwise, update (m_0, n_0) as this feasible point. If the optimum solution has not been initialized, then set $(\hat{m}, \hat{n}) = (m_0, n_0)$.*
3. *Find the left most local optimum solution $(\hat{m}, \hat{n})$ of problem (14). This is done by searching for the smallest value of ν such that*

$$\begin{aligned} b(m_0, n_0 + \nu h) &< b(m_0, n_0 + (\nu+1)h), \\ n_0 + (\nu+1)h &\leq n_2 \end{aligned} \tag{15}$$

   or

$$g(m_0, n_0 + \nu h) > \alpha. \tag{16}$$

4. *If ν exists, then*
   - *If (15) occurs, then $b(m_0, n_0 + \nu h) \leq b(m_0, n_0 + (\nu-1)h)$ and a local minimum with $m = m_0$ and $n \in [n_0 + (\nu-1)h, n_0 + (\nu+1)h]$ exists. The two dimensional local minimization method [5] can be used to locate the local minimum solution*

around this point. Update the local minimum $(\hat{m}, \hat{n})$. Set $n_0 = n_0 + (\nu+1)h$. Go back to Step 3.

- *If (16) occurs and $f(m_0, n_0 + (\nu - 1)h) < f(\hat{m}, \hat{n})$, then update $(\hat{m}, \hat{n}) = (m_0, n_0 + (\nu - 1)h)$, set $n_0 = n_0 + \nu h$ and return to Step 2.*

If ν does not exist, then

- *If $m_0 + h \leq m_2$, assign $m_0 = m_0 + h$ and $n_0 = n_1$ and return to Step 2.*
- *If $m_0 + h > m_2$, $(\hat{m}, \hat{n})$ is the global solution and go to Step 5.*

5. *Stop the procedure.*

The solution obtained from Procedure 2 is the global minimum solution of the function $f(m, n)$ over the interval $[m_1, m_2] \times [n_1, n_2]$ for a small step size h. By introducing the bridged function at (m_0, n), we can eliminate local minima of the original cost function $f(m, n)$ with values larger than $f(\hat{m}, \hat{n})$.

The main advantage of this algorithm is that it only finds a small number of useful local minima. Once a global solution is obtained, the procedure moves quickly through the rest of the interval.

6 Design Example

Let us consider the design of a set of $S = 30$ complex spreading sequences with length $N = 31$ based on the construction rule given in (4) with parameter $p = 1$.

Results for this design example are shown in Table 1. The minimum value for R_{ac} and its corresponding parameters m and n are given for different bounds on R_{cc}. The first and the last rows are for the cases of optimum R_{cc} and R_{ac}, respectively. Depending on the requirements of a particular application, the table can be used to find a trade-off between R_{ac} and R_{cc}.

Figs. 1–4 plot the auto-correlation functions and the frequency spectra for the two extreme cases given in Table 1. Since we are only interested in the interaction of the spectrum for different sequences, all the spectrum plots are normalized. The spectrum for three sequences out of a complex spreading code are plotted, where (m, n) specifies the code.

Figs. 1 and 2 plot the auto-correlation function and the frequency spectra for three sequences taken from the spreading code with minimum R_{cc}. It can be seen that the auto-correlation function is broad, covering several lags while the overlap between the frequency spectra is low.

Figs. 3 and 4 plot the auto-correlation function and the frequency spectra for the extreme case with lowest possible $R_{ac} = 0.1107$ but very high $R_{cc} = 0.9984$. In practice, this characteristic would ease the synchronization task. The overlapping spectrum on the other hand, would cause some interference among users in an asynchronous CDMA system. This may result in poor system performance in terms of average bit error rate. Thus, given the system specification, the complex spreading sequence can be designed accordingly to obtain the best correlation properties for the particular application.

Table 1: Optimized R_{ac} for different upper bounds α on R_{cc} for $S = 30$ sequences of length $N = 31$.

R_{cc}	m	n	R_{ac}	R_{cc}
Optimum R_{cc}	1.0038	1.0000	19.6774	0.3419
$R_{cc} \leq 0.4$	1.0038	1.3000	17.9373	0.4000
$R_{cc} \leq 0.5$	1.0038	1.3837	15.2413	0.4900
$R_{cc} \leq 0.6$	1.0038	1.4468	11.9485	0.6000
$R_{cc} \leq 0.7$	1.0037	1.4966	8.9562	0.7000
$R_{cc} \leq 0.8$	1.0036	1.5573	5.9707	0.8000
$R_{cc} \leq 0.9$	1.0032	1.6807	2.9888	0.9000
Optimum R_{ac}	2.9200	2.0072	0.1107	0.9984

7 Conclusion

In this paper, we investigated the design of spreading sequences with minimum average mean square auto-correlation for an upper bound on the average mean square cross-correlation. This problem was formulated as a two dimensional global optimization problem with a non-linear cost function and a non-convex constraint. The modified bridging method was proposed for solving this optimization problem and was applied to the design of complex spreading sequences. The method may be used for sequence design in asynchronous CDMA systems.

References

[1] I. Oppermann and B.S. Vucetic, "Complex Spreading Sequences with a Wide Range of Correlation Properties," *IEEE Trans. Commun.*, vol. 45, no. 3, pp. 365–375, Mar. 1997.

[2] I. Oppermann, "Orthogonal Complex Valued Spreading Sequences with a Wide Range of Correlation Properties," *IEEE Trans. on Commun.*, vol. 45, no. 11, pp. 1379–1380, Nov. 1997.

[3] Y. Liu and K. L. Teo, "A Bridging Method for Global Optimization," in *Optmization Techniques and Applications*, vol. 2, pp. 558–565, 1998.

[4] H. H. Dam, K. L. Teo, Y. Liu, and S. Nordebo, "Optimum Pole Position for Digital Laguerre Network with Least Square Error Criterion," *Optimization Methods and Applications*, pp. 321–330, Boston: Kluwer Academic Publishers, 2000.

[5] Matlab: "Reference Guide," The Math Works, 1996.

[6] M. Minoux, "Mathematical Programming Theory and Applications," New York: Wiley, 1986.