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Antenna Array Design using Dual Nested Complex Approximation

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Abstract—This paper presents a new practical approach to complex Chebyshev approximation by semi-infinite linear programming. The approximation problem may be general with arbitrary complex basis functions.

By the new front-end technique, the associated semi-infinite linear programming problem is solved exploiting the finiteness of the related Lagrange multipliers by adapting finite-dimensional linear programming to the dual semi-infinite problem, and thereby taking advantage of the numerical stability and efficiency of conventional linear programming software packages. Furthermore, the optimization procedure is simple to describe theoretically and straightforward to implement in computer coding. The new design technique is therefore highly accessible. The new algorithm is formally introduced as the linear Dual Nested Complex Approximation (DNCA) algorithm.

The DNCA algorithm is versatile and can be applied to a variety of applications such as narrow-band as well as broad-band beamformers with any geometry, conventional Finite Impulse Response (FIR) filters, analog and digital Laguerre networks, and digital FIR equalizers.

The proposed optimization technique is applied to several numerical examples dealing with the design of a narrow-band base-station antenna array for mobile communication. The flexibility and numerical efficiency of the proposed design technique are illustrated with these examples where hundreds of antenna elements are optimized without numerical difficulties.

I. Introduction

The array pattern synthesis of a non-uniformly spaced sensor array or beamformer [1], [2] is closely related to the design of an FIR filter with arbitrary or non-linear phase response. The essential similarity is the finite-dimensional nature of the complex approximating functions, see for example [3]-[6].

For beamformers as for FIR filters the design problem is often cast as a finite-dimensional complex approximation problem [2], [3]. It was shown that the (non-linear) complex Chebyshev approximation problem can be reformulated as an equivalent real semi-infinite linear program [2], [7]. Classical least squares approximation methods can in many cases be used to obtain a desired solution [1]. However, when the design specification is given as a bound on the complex design error, the problem is naturally converted to a complex Chebyshev approximation problem.

Other possible methods to find the Chebyshev solution include quadratic programming [9]. It was shown that the semi-infinite linear program corresponding to the (real) Chebyshev approximation problem can be solved by using numerically efficient simplex extension algorithms [8]. In this context, it is also noted that some other recent approaches to complex FIR filter design such as a generalized Remez algorithm [10], and Tang’s algorithm [5], [11] in fact also employ simplex extension algorithms. These results were later exploited for the design of digital FIR filters and digital Laguerre networks with complex Chebyshev error criteria, see for example [6] and [12].

Finitization, see for example [3], can in principle give an arbitrarily accurate approximation of the complex Chebyshev solution but becomes exceedingly memory intensive as the grid spacing decreases. The semi-infinite formulation deals directly with the true complex error and not an approximation thereof. The general complex approximation problems require an optimization formulation such as with Semi-Infinite Programming (SIP), see e.g. [6],[9], or Second Order Cone Programming (SOCP), see e.g. [13], [14]. With SIP, a finite number of non-linear constraints are transformed to an infinite number of linear constraints, i.e. a linear combination of continuous functions. The problem can then be solved efficiently using algorithms which are extensions of the standard linear and quadratic programming algorithms such as the simplex algorithm etc., see e.g. [8], [6], [9]. With SOCP, the problem is solved without linearizing the constraints, and by using newly developed interior point methods, see e.g. [13], [14], [15].

An obstacle with the semi-infinite simplex extension algorithm as described in [6], [8], [12] is the lack of commercially available software for efficient and reliable numerical solution of general complex approximation problems. In order to overcome the difficulties mentioned above, we present in this paper an applied semi-infinite front-end technique for complex Chebyshev approximation which is based on conventional finite-dimensional linear programming subroutines. The essence of the new technique, justified by the Caratheodory dimensionality theorem [18], is to exploit the finiteness of the related Lagrange multipliers by adapting conventional finite-dimensional linear programming to the semi-infinite linear programming problem.

By the proposed front-end technique, the complex Chebyshev approximation problem can be solved taking advantage of the numerical stability and efficiency of the given linear programming software package. Furthermore, the optimization procedure is simple to describe theoretically and straightforward to implement in computer coding. The new design technique should therefore be highly accessible for most design engineers.

In order to illustrate the flexibility and numerical efficiency of the proposed design technique we have included several design examples concerning the optimization of a narrow-band base-station antenna array for mobile communication in the 450 MHz band.
II. Problem Formulation

To demonstrate the versatility of the complex approximation technique the optimization is performed from an application point of view. The design examples are taken from the mobile communication base-station area, or more precisely, design of an antenna array in the 450 MHz band. As a numerical example we consider the shielded planar hexagonal antenna array where the sensor elements are evenly distributed in sections on a hexagon, and with the incident wave front propagating in the same plane as the array, see Fig. 1. The rather unusual configuration also illustrates the generality and the applicability for arbitrary basis functions (array response) for arbitrary geometries.

![Fig. 1. A planar hexagonal antenna array with shield for the far-field where the outer '●' denote the sensor elements, the solid hexagon the ground-plane shield and the inner '●' denote the mirror source. The passband (main-lobe look direction) is defined by the angle $\varphi_p$ ($\varphi_p = 0$ in the figure) and the stopband (side-lobe region) by the interval $[\varphi_p + \varphi_s, \varphi_p - \varphi_s + 360^\circ]$. The ground-plane sections are positioned in direction of $\varphi_y = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ]$.

We consider the far-field and narrow-band case where the phase of the wave front is given by $e^{j(2\pi f_0 t - k r)}$ where $f_0$ is the frequency, $t$ the time, $k = -\frac{2\pi f_0}{c}$ ($\cos \varphi, \sin \varphi$) the wave vector, $c$ the speed of wave propagation, $\varphi$ the angle of incidence and $r$ the evaluated spatial point, see Fig. 1. The hexagonal sectioned antenna array consists of antenna elements distributed along the ground-plane sections. One antenna element and the ground-plane together form a dipole. The dipoles have the spatial positions $(r_m, \varphi_m)$ where $m = 0, \ldots, M - 1$ and $r_m$ and $\varphi_m$ are the distance and angle, respectively, between the middle of the hexagon to the dipole center. The phase center and origin of coordinates are located in the middle of the hexagon and the total array response $H(\varphi)$ is given as

$$H(\varphi) = \sum_{m=0}^{M-1} w_m a_m(\varphi) e^{j2\pi f_0 r_m \cos(\varphi_m - \varphi)}$$

(1)

where $M$ is the total number of used/active dipoles in the antenna and $w_m$ is a complex weight. The radiation characteristics for a dipole located in $(r_m, \varphi_m)$ is denoted $a_m(\varphi)$. We only use three of the six array-sections at a time and consequently a subset of all dipoles are used. Depending on the angle of incidence $\varphi$ the three nearest heading sections with angle $\varphi_y$ are selected.

The complex antenna array response for the angle $\varphi$ using vector notation is given by

$$H(\varphi) = w^H d(\varphi)$$

$$= \tilde{w}^T \tilde{d}(\varphi)$$

$$= \begin{bmatrix} \Re\{\tilde{w}\} \\ \Im\{\tilde{w}\} \end{bmatrix}^T \begin{bmatrix} d(\varphi) \\ -j \tilde{d}(\varphi) \end{bmatrix}$$

(4)

where the complex array vector $w = \Re\{\tilde{w}\} + j\Im\{\tilde{w}\}$ is an $M \times 1$ array vector of complex coefficients $w_m$ and $d(\varphi)$ the corresponding array response vector of complex continuous and linearly independent transfer functions $d_m(\varphi) = a_m(\varphi) e^{2\pi f_0 r_m \cos(\varphi_m - \varphi)}$, $m = 1, \ldots, M - 1$. The $\tilde{w}$ and $\tilde{d}(\varphi)$ in (3) are defined as in (4), respectively. Consequently, $\tilde{w}$ is an $N \times 1$ real vector and $\tilde{d}(\varphi)$ an $N \times 1$ complex vector where $N = 2M$.

In order to make the passband extremely narrow, the passband in this formulation was restricted to a single point $\varphi_p$. The stopband is defined as $\Phi = [\varphi_p + \varphi_s, \varphi_p + 2\pi - \varphi_s]$ in radians.

A. The Design Specification

Consider the following design specification

$$\begin{cases} |H(\varphi)| \leq \sigma(\varphi), \; \varphi \in \Phi \\ H(\varphi_p) = 1 \end{cases}$$

(5)

where $\sigma(\varphi)$ is a prescribed strictly positive magnitude bound which leads to the minimax design formulation

$$\begin{cases} \min \max_{\varphi \in \Phi} |H(\varphi)| \\ H(\varphi_p) = 1 \end{cases}$$

(6)

where $v(\varphi) = \frac{1}{\sigma(\varphi)}$.

It is concluded that a solution to the specification (5) exists if and only if the optimal objective value in (6) is less than or equal to one. Hence, the optimization formulation (6) will give us the answer to the question if there exists a feasible solution to (5), and furthermore, if a solution to (5) exists we will obtain the solution which is furthest away from the upper bound in a “logarithmic” minimax sense.

To elaborate on this last property, let $\delta$ denote the objective value in (6). The optimal solution to (6) will give us the smallest $\delta$ such that $|H(\varphi)| \leq \delta \sigma(\varphi), \forall \varphi \in \Phi$ or

$$20 \log |H(\varphi)| \leq 20 \log \delta + 20 \log \sigma(\varphi), \; \forall \varphi \in \Phi.$$  

(7)

Thus, if a feasible solution to (5) exists, the optimal solution to (6) will give us the solution to (5) which maximizes the minimum distance to the specification $\sigma(\varphi)$ in decibels (dB). If a feasible solution to (5) does not exist, the solution to (6) will give us the solution which minimizes the maximum constraint violation in (5) in decibels (dB).
III. SEMI-INFINITE LINEAR PROGRAMMING

The optimal solution to the minimax formulation in (6) is given by the equivalent formulation

\[
\begin{align*}
\min & \delta \\
\text{s.t.} & \quad v(\varphi)|H(\varphi)| - \delta \leq 0, \quad \varphi \in \Phi \\
& \quad H(\varphi_p) = 1
\end{align*}
\]

(8)

where \( \delta \) is an additional real variable.

A. Semi-Infinite Linear Programming Formulation

The problem (8) corresponds to a non-linear optimization problem which is very difficult to treat as it stands. We will therefore convert (8) into a semi-infinite linear programming problem. According to the real rotation theorem [17], a magnitude inequality in the complex plane can be expressed in the equivalent form

\[
|z| \leq \sigma \iff \Re\{ze^{j\theta}\} \leq \sigma \quad \forall \theta \in [0, 2\pi]
\]

(9)

where \( z \) is a complex number and \( \sigma \) a real and positive number.

By making use of (9), the design problem (8) is now reformulated as

\[
\begin{align*}
\min & \quad \delta \\
\text{s.t.} & \quad v(\varphi)^{\Re}(H(\varphi)e^{j\theta}) - \delta \leq 0, \quad (\varphi, \theta) \in \Phi \times \Theta \\
& \quad H(\varphi_p) = 1
\end{align*}
\]

(10)

where \( \Theta = [0, 2\pi] \). In order to emphasize the linear structure of this formulation we finally rewrite (10) as the following semi-infinite linear program

\[
\begin{align*}
\min & \quad \delta \\
\text{s.t.} & \quad \mathbf{a}^T(\varphi, \theta)\mathbf{w} - \delta \leq 0, \quad (\varphi, \theta) \in \Phi \times \Theta \\
& \quad \mathbf{p}^T = \mathbf{p}
\end{align*}
\]

(11)

where \( \mathbf{a}(\varphi, \theta) = v(\varphi)^{\Re}(\mathbf{d}(\varphi)e^{j\theta}) \), \( \mathbf{P} \) is an \( L \times N \) constraint matrix and \( \mathbf{p} \) an \( L \times 1 \) constraint vector. The main-lobe constraint \( H(\varphi_p) = 1 \) in (10) is obtained by choosing \( \mathbf{P}^T = \{\Re\{\mathbf{d}(\varphi_p)\} \} \) and \( \mathbf{p}^T = [1, 0] \).

The linear program (11) is called semi-infinite since the number of variables (unknowns) are finite but the constraint set is infinite. For practical purposes in the implementation of the optimization algorithm, it is assumed that the set \( \Phi \) is finite. Note however, that the total corresponding approximation problem is with respect to the true complex error since the phase parameter \( \theta \) belongs to the infinite set \( \Theta = [0, 2\pi] \).

B. The DNCA-LP Optimization Algorithm

The proposed semi-infinite Dual Nested Complex Approximation (DNCA) Linear Programming (LP) algorithm to solve the semi-infinite problem (10) proceeds with the following basic steps:

1. Given a reference set \( \mathcal{R}_k = \{(\varphi_1, \theta_1), \ldots, (\varphi_r, \theta_r)\} \subset \mathcal{D} = \Phi \times \Theta \), let \( \tilde{\mathbf{w}}_k, \delta_k \) and \( H_k(\varphi) = \tilde{\mathbf{w}}_k^T \mathbf{d}(\varphi) \) denote the optimal solution to the subproblem

\[
\begin{align*}
\min & \quad v(\varphi_i)^{\Re}(H(\varphi_i)e^{j\theta_i}) - \delta \leq 0, \quad (\varphi_i, \theta_i) \in \mathcal{R}_k \\
& \quad H(\varphi_p) = 1
\end{align*}
\]

(12)

and let \( \lambda_1, \ldots, \lambda_r \) denote the corresponding Lagrange multipliers. The subproblem (12) is solved by using conventional finite-dimensional linear programming techniques.

2. Define the entering index \((\varphi_e, \theta_e)\) by

\[
(\varphi_e, \theta_e) = \arg \max_{\varphi, \theta} \{ v(\varphi)^{\Re}(H_k(\varphi)e^{j\theta}) \}
\]

(13)

where

\[
\varphi_e = \arg \max_{\varphi} \{ v(\varphi)|H_k(\varphi)| \}
\]

(14)

\[
\theta_e = -\arg(H_k(\varphi_e))
\]

(15)

and calculate \( \|H_k\|_{\infty} = \max\{v(\varphi)|H_k(\varphi)|\} = v(\varphi_e)^{\Re}(H_k(\varphi_e)e^{j\theta_e}) \).

3. Stop if

\[
\|H_k\|_{\infty} < \delta_k(1 + \varepsilon)
\]

(16)

where \( \varepsilon \) is a predefined tolerance parameter. If (16) is satisfied then \( \|H_k\|_{\infty} < \delta_0(1 + \varepsilon) \) where \( \delta_0 \) is the optimal Chebyshev deviation related to the problem (10).

4. Define the leaving indices by

\[
\mathcal{R}_k = \{(\varphi_i, \theta_i) \in \mathcal{R}_k \mid \lambda_i = 0\}
\]

(17)

which essentially consists of the inactive constraints in (12).

5. Define the new reference set by

\[
\mathcal{R}_{k+1} = (\mathcal{R}_k \setminus \mathcal{R}_l) \cup \mathcal{R}_e
\]

(18)

and return to step 1.

The optimal variable \( \delta_o \) given by the optimization process yields the amplitude margin with respect to the desired amplitude function \( \sigma(\varphi) \). Three different cases are possible:

1. \( \delta_o < 1 \) \( \Rightarrow \) the solution satisfies the specification given by

\[
20 \log \sigma(\varphi) \quad [\text{dB}]
\]

with a (maximized) minimum amplitude margin of \( 20 \log \delta_o \quad [\text{dB}] \).

2. \( \delta_o = 1 \) \( \Rightarrow \) the solution satisfies the specification given by

\[
20 \log \sigma(\varphi) \quad [\text{dB}]
\]

exactly, without margin.

3. \( \delta_o > 1 \) \( \Rightarrow \) the solution violates the specification given by

\[
20 \log \sigma(\varphi) \quad [\text{dB}]
\]

with a (minimized) maximum of \( 20 \log \delta_o \quad [\text{dB}] \).

Key observation 1: Since the number of variables is \( N + 1 \), we note that an optimization software will usually give us a total of \( N + 1 \) Lagrange multipliers greater than zero including the multipliers in \( \mu \). Therefore, the size of the so-called reference set \( \mathcal{R}_k \) is in fact \( r \leq N + 1 - L \).

Key observation 2: The dual formulation suggests that the primal problem can be solved by considering a sequence of subproblems as in with increasing minimum cost and which is based only on finite subsets \( \mathcal{R}_k = \{(\varphi_1, \theta_1), \ldots, (\varphi_r, \theta_r)\} \) consisting of no more than \( N + 1 - L \) points of \( \mathcal{D} = \Phi \times \Theta \). This observation constitutes the foundation for the development of the DNCA optimization algorithm.

Key observation 3: The number of variables is \( N + 1 \) and the size of the reference set \( \mathcal{R}_k \) is only \( r \leq N + 1 - L \). The constraint index \( \mathcal{R}_e = (\varphi_e, \theta_e) \) which is chosen to enter the basis \( \mathcal{R}_k \) is usually defined by the maximum constraint violation. Hence, this entering constraint \( \mathcal{R}_e \) is very likely to be independent of the small reference set \( \mathcal{R}_k \). This is the primary reason ensuring that the DNCA itself is a highly numerical stable procedure.
IV. Convergence Proof

In this section we give a convergence proof for the optimization algorithm described in the previous section. We show that this can be accomplished without explicit reference to the conceptually abstract dual formulation.

A. The Complex Approximation Problem

The complex approximation problem considered in this paper is max-norm optimization and some auxiliary convex constraints

\[
\begin{align*}
\min & \max_{\omega_1} v_1(\varphi) |H(\varphi) - H_d(\varphi)| d\varphi \\
& \max_{\omega_2} v_2(\varphi) |H(\varphi) - H_d(\varphi)| \leq \varepsilon \\
& \hat{w} \in \hat{W}
\end{align*}
\]

(19)

where \(H_d(\varphi)\) is the desired complex function, \(H(\varphi)\) the complex array response as defined in (1), \(\hat{W}\) an \(N \times 1\) vector of real coefficients \(\hat{w}_n\), \(\hat{W}\) a convex set, \(v_1(\varphi)\) and \(v_2(\varphi)\) prescribed positive weighting functions, \(\varepsilon\) a prescribed upper bound, and \(\varphi \in \Omega_1\) and \(\varphi \in \Omega_2\) where \(\Omega_1\) and \(\Omega_2\) are compact domains. All functions are assumed to be continuous on their respective domains. By employing the real rotation theorem (19) can be reformulated as a Semi-Infinite Programming (SIP) problem [17]

\[
|z| \leq \sigma \Leftrightarrow \Re \{z \cdot e^{j\theta}\} \leq \sigma, \quad \forall \theta \in \Theta = [0, 2\pi]
\]

(20)

where \(z\) is a complex number and \(\sigma\) a positive real number.

By using (20), the max-norm problem in (19) becomes

\[
\begin{align*}
\min & \delta \\
& \sum_{\omega} \Re \left\{ \left[ \phi^T(\varphi) \hat{w} - H_d(\varphi) \right] \cdot e^{j\theta} \right\} - \delta \leq \varepsilon \\
& \hat{w} \in \hat{W}
\end{align*}
\]

(21)

where \(\delta\) is a scalar defined by the expression in (19) and \((\varphi, \theta) \in \Phi_2 \times \Theta\). The problem in (21) is referred to as Semi-Infinite Linear Programming (SILP) Problem and can be described using the general semi-infinite programming formulation

\[
\begin{align*}
\min & f(\hat{w}) \\
g_a(\hat{w}) & \leq 0, \quad \alpha \in A \subset R^k \\
\hat{w} & \in \hat{W} \subset R^n
\end{align*}
\]

(22)

where \(\hat{w}\) is an \(N \times 1\) variable vector, \(f(\hat{w})\) a convex continuous function \(\hat{W}\) a convex restriction set, \(A\) an infinite index set as a compact subset of Euclidean \(k\)-space, and \(g_a(x)\) a continuous constraint function which is convex for any fixed index \(\alpha\).

B. Dual Nested Complex Approximation

The Dual Nested Complex Approximation (DNCA) algorithm to solve (22) is outlined below. Let \(A^{(k)}\) denote a sequence of finite subsets of the infinite index set \(A\) and initialize the algorithm with the subset \(A^{(0)}\).

1. Given \(A^{(k)} \subset A\), solve the subproblem

\[
\begin{align*}
(P^{(k)}) \left\{ \begin{array}{l}
\min f(\hat{w}) \\
g_a(\hat{w}) \leq 0, \quad \alpha \in A^{(k)} \\
\hat{w} \in \hat{W} \subset R^n
\end{array} \right. \quad \text{yielding the solution vector } \hat{w}_k \text{ and the Lagrange multiplier vector } \lambda_k.
\end{align*}
\]

(23)

2. Reduce the subset by the inactive constraints

\[
A_R^{(k)} = A^{(k)} \setminus \{ \alpha \in A^{(k)} | (\lambda_k)_{\alpha} = 0 \}
\]

(24)

3. Define the entering index

\[
\hat{a}_k = \arg \max_\alpha g_a(\hat{w}_k), \quad A^{(k+1)} = \hat{A}^{(k)} \cup \{ \hat{a}_k \}
\]

(25)

and return to step 1 above.

Since the reduced subset \(A^{(k)}\) yields the same solution \(\hat{w}_k\) as the subset \(A^{(k)}\) we have \(f(\hat{w}_k) \leq f(\hat{w}_{k+1}) \leq f_{\text{opt}}\) and the sequence \(f(\hat{w}_k)\) converges. However, it remains to be shown that the sequence \(\hat{w}_k\) is not stuck in any state of cycling.

Convergence can be proved straightforwardly when the cost function \(f(\hat{w})\) is strictly convex. Assume that \(\hat{w}_k\) is not optimal, then \(g_a(\hat{w}_k) > 0\). The claim is that there is a strict ascent, \(f(\hat{w}_{k+1}) > f(\hat{w}_k)\), so that cycling cannot occur. Assume on the contrary that \(f(\hat{w}_{k+1}) \leq f(\hat{w}_k)\). Define \(\hat{W}_k = \{ \hat{w} | g_a(\hat{w}) \leq 0, \alpha \in A^{(k)} \}\) and \(\hat{W}_{k+1} = \hat{W}_k \cap \{ \hat{w} | g_a(\hat{w}) \leq 0 \}\). Obviously \(\hat{w}_k \in \hat{W}_k, \hat{w}_{k+1} \in \hat{W}_{k+1} \in \hat{W}_{k+1} \cap \hat{W}_k \notin \hat{W}_{k+1}\). Thus, \(\hat{w}_k \neq \hat{w}_{k+1}\) and \(f(t\hat{w}_k + (1-t)\hat{w}_{k+1}) < tf(\hat{w}_k) + (1-t)f(\hat{w}_{k+1}) \leq f(\hat{w}_{k+1}) + (1-t)f(\hat{w}_k) \) for any \(0 < t < 1\) which contradicts the fact that \(\hat{w}_k\) is optimum on \(\Omega_k\).

If the cost function \(f(\hat{w})\) is not strictly convex, strict ascent can still be obtained by requiring that the condition \(f(\hat{w}_k) > f(\hat{w}_{k-1})\) be satisfied in order to execute step 2 (Eq. (24)) in the DNCA algorithm.

Since the reduced subset \(A^{(k)}\) yields the same solution \(\hat{w}_k\) as the subset \(A^{(k)}\) we have \(f(\hat{w}_k) \leq f(\hat{w}_{k+1}) \leq f_{\text{opt}}\) and the sequence \(f(\hat{w}_k)\) converges. However, convergence to the optimum value \(f_{\text{opt}}\) remains to be shown. If \(\hat{w}_k\) is the optimum solution since it is necessary and sufficient for an optimum solution that there exists Lagrange multipliers defined on a finite subset of \(A\).

A practical stopping criteria is therefore to stop the algorithm when \(g_a(\hat{w}_k) \leq \varepsilon\) where the tolerance parameter \(\varepsilon(\hat{w}_k) > 0\) and may depend on the current solution \(\hat{w}_k\).

The applicability of the algorithm above to complex approximation lies in the fact that (25) can easily be calculated when the approximation domain \(\Phi = \Phi_1 \cup \Phi_2\) is finite. Let e.g. \(g_a(x) = x = \hat{w}\) be given by

\[
g_a(x) = g_{\varphi, \theta}(x) = \Re \{ (a^T(\varphi) x + b(\varphi)) e^{j\theta} \} + c^T(\varphi) x + d(\varphi)
\]

(27)

where \(\alpha = (\varphi, \theta) \in A = \Phi \times \Theta, a(\varphi)\) and \(b(\varphi)\) are complex and \(c(\varphi)\) and \(d(\varphi)\) are real. (cf. (21). Due to the real rotation theorem (20) we can calculate (25) as \(\hat{a}_k = (\hat{\varphi}_k, \hat{\theta}_k)\).
where
\[
\hat{\varphi}_k = \arg \max_{\varphi} \left[ (a^T(\varphi)x_k + b(\varphi)) + c^T(\varphi)x_k + d(\varphi) \right]
\]
\[
\hat{\theta}_k = -\arg \left[ (a^T(\hat{\varphi}_k)x_k + b(\hat{\varphi}_k)) \right]
\]

V. Design Examples

As an application example we consider the planar hexagonal antenna array as defined in Section II. If the weighting function \(v(\varphi)\) is uniformly distributed we achieve an equiripple solution, that is we achieve the lowest possible side-lobe level in the stopband with respect taken to one or more linear constraints. The array response used is given by (1) and the corresponding real variables used, \(w_{in}\), are defined as in (3). The specification in (5) is used to state the desired array design. The solution is obtained by using the DPCA algorithm as described in Section III. The performance of using a hexagonal sectioned ground-plane shielded antenna array as in Fig. 1 is investigated.

The examples show the flexibility in design using the proposed semi-infinite front-end algorithm. The algorithm is capable of solving huge design problems with many antenna elements, see Fig. 2-3. Main-lobe steering, see Fig. 4, and side-lobe control by incorporating arbitrary weighting functions, can at the same time be taken into consideration during the optimization process as well. All figures and examples show the corresponding array pattern using configuration no. 1 as described in Section II.

All antenna array responses are designed for the frequency \(f_0 = 450\) MHz and the interspacing between antenna elements in the array is \(d = \lambda/2 \approx 0.3\) meter. The distance between the source element and the ground-plane is \(d_g = \frac{\lambda}{4} \approx 0.15\) meter. Note that by choosing \(f_0\) the design problem can easily be scaled or translated into different frequency bands.

Fig. 2 illustrates a typical equiripple solution for an antenna with \(M = 102\) elements. Each linear antenna section consists of 34 \((M = 3 \cdot 34 = 102)\) isotropic dipole elements. The antenna look direction for the main-lobe is \(\varphi_p = 0^\circ\), that is the main-lobe (passband) consists only of one point and is defined by one single point constraint \(H(\varphi_p) = H(0) = 1\). The radius of the antenna is approximately 9 meters. There are 102 complex weights \(w\), which implies that \(N = 204\) real variables \(\hat{w}\) are involved in the optimization process. The corresponding convergence behavior in Fig. 3 shows the advance of variable \(\delta\) which is continuously increasing to the optimal value \(\delta_o\). The maxnorm \(\|H\|\) has a more irregular but still overall decreasing behavior also approaching the optimal value \(\delta_o\). It is in accordance as described in Section III that \(\delta < \delta_o < \|H\|\) during the optimization process. In this design example the side-lobe (stopband) region starts at \(\pm 1.5^\circ\), and in all the other examples at \(\pm 5.5^\circ\) from the desired main-lobe direction.

The angular grid resolution is 0.25\(^\circ\) in this example and 0.5\(^\circ\) in all others.

In the examples concerning the main-lobe steering the look direction \(\varphi_p\) for the main-lobe is gradually increased from \(0^\circ\) to \(35^\circ\) in steps of 5\(^\circ\). The main-lobe look direction is defined by one single point constraint \(H(\varphi_p) = 1\), see Figs. 4. Each linear antenna section consists of 10 \((M = 3 \cdot 10 = 30)\) isotropic dipole elements. The unusual symmetry in the hexagonal antenna compared to a circular design is obvious. However, by using the proposed design method it is possible to apply lobe steering to direct the main-lobe in the angular domain \(\varphi\). The drawback is that several sets of weights must be used. Figs. illustrate the performance of the 3-sectioned hexagonal antenna using 30 antenna elements for main-lobe directions \(\varphi_p \in \{0^\circ, 5^\circ, 10^\circ, 15^\circ, 20^\circ, 25^\circ, 30^\circ, 35^\circ\}\). The radius of the antenna is approximately 3 meters. The loss in stopband attenuation performance between the \(0^\circ\) and \(30^\circ\) design is \(\sim 3\) dB. Note that lobe steering in the direction \(35^\circ\) degrees can be obtained by counterclockwise switching antenna sections and reuse a mirrored weight setup of the \(25^\circ\) design. In this way, the number of weight sets will be kept reasonably low.

The ability to incorporate a weighting function \(v(\varphi) = \frac{1}{\pi \sigma^2}\) is a useful option when designing antenna arrays. The array response in the angular domain can in that way be shaped in an arbitrary sense. The antenna look direction for the main-lobe and weighting function can be chosen arbitrarily.

The final value of variable \(\delta\) yields information about the amplitude margin with respect to the function \(\sigma(\varphi)\) and is plotted as \(\delta_o\) in the convergence behavior plots. This variable \(\delta\) is an important design parameter. If the uniform weighting function \(\sigma(\varphi) = 1\) \((0\) dB) is used we simply achieve the maximum side-lobe suppression by observing the final value of variable \(\delta\).

It is possible to add additional point constraints such as nulls in the stopband or constraints on the main-lobe.

VI. Summary

This paper solves antenna array Chebyshev approximation problems by exploiting Caratheodory dimensionality theorem in a conventional linear programming software front-end. The semi-infinite linear programming theory is fairly recent[8] and there is lack of commercial software available for efficient numerical solution of the problem (10). It is an extensive task to develop a specific software for semi-infinite linear programming to solve (10) if this software is also required to be stable and reliable with respect to numerical problems such as cycling phenomena, ill-conditioned matrix inversions, etc. On the other hand, the conventional finite-dimensional linear programming technique is well established and there is a lot of good, numerically reliable software available. The optimization technique proposed in this paper is therefore a convenient alternative which inherits the good numerical properties of the given linear programming subroutine. The computer code is significantly simplified in comparison with computer code which is tailored for semi-infinite linear programming. Moreover, the computational complexity is asymptotically equal. The proposed method is capable of solving large optimization problems such as huge antenna arrays and optimization variables. Extensive evaluations indicate the flexibility in design using the proposed front-end method.
Fig. 2. Minimax design of an hexagonal antenna using 3 sections containing 34 antenna elements each. Side-lobe suppression \(\approx 18.5\) dB. The design example consists of total \(M = 102\) complex weights i.e. \(N = 204\) real variables in the optimization.

Fig. 3. The monotonic convergence behaviour of \(\delta\) corresponding to the minimax design in fig. 2. The fluctuating maxnorm \(\|H\|\) and \(\delta\) will converge to the same optimal value \(\delta_o\).

Fig. 4. Main-lobe steering \(\varphi_p \in [0^\circ, 5^\circ, 10^\circ, \ldots, 35^\circ]\). Minimax design of an hexagonal antenna using 3 sections containing 10 antenna elements each using different passband angles \(\varphi_p\).

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