

Semi-Infinite Linear Programming: A Unified Approach to Digital Filter Design with Time- and Frequency-Domain Specifications

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Abstract—Using the recently developed semi-infinite linear programming techniques and Caratheodory's dimensionality theory, we present a unified approach to digital filter design with time and/or frequency-domain specifications. Through systematic analysis and detailed numerical design examples, we demonstrate that the proposed approach exhibits several salient features compared to traditional methods: 1) using the unified approach, complex responses can be handled conveniently without resorting to discretization; 2) time-domain constraints can be included easily; and 3) any filter structure, recursive or nonrecursive, can be employed, provided that the frequency response can be represented by a finite-complex basis. More importantly, the solution procedure is based on the numerically efficient simplex extension algorithms. As numerical examples, a discrete-time Laguerre network is used in a frequency-domain design with additional group-delay specifications, and in a \mathcal{H}_∞ -optimal envelope constrained filter design problem. Finally, a finite impulse response phase equalizer is designed with additional frequency domain \mathcal{H}_∞ robustness constraints.

Index Terms—Digital filter design, Laguerre filters, linear programming, semi-infinite programming.

I. INTRODUCTION

THE DESIGN of a finite impulse response (FIR) filter with arbitrary phase response is closely related to the design of a nonuniformly spaced sensor array or beamformer [1], [2]. The essential similarity is the finite-dimensional nature of the complex approximating functions. The design and applications of FIR filters with nonlinear phase have previously been extensively studied, see, e.g., [3]–[6]. In this contribution, we consider a wider class of digital filters for which the frequency response can be represented by a finite-complex basis. The discrete-time Laguerre networks constitute a practically important example of a recursive low-order alternative to FIR filters, which falls into this category (see, e.g., [7]–[13]).

For both beamformers and FIR filters, the design problem is often cast as a finite-dimensional complex approximation problem [2], [3]. Classical least squares approximation methods can, in many cases, be used to obtain a desired solution [1], [10]. However, when the filter specification is given as a

bound on the complex design error (in frequency and/or space), the problem is naturally converted to a complex Chebyshev approximation problem.

It was shown in [2] and [14] that the (nonlinear) complex Chebyshev approximation problem can be reformulated as an equivalent real semi-infinite linear program. However, in [2], the complex error was approximated by a finitization in order to solve the problem numerically. Other possible methods to find the Chebyshev solution include quadratic programming [15] (true complex error and finite response domain) and a functional inequality approach [16] (true complex error, infinite response domain, and approximation by numerical evaluation of integrals).

The complex Chebyshev approximation problem for the design of FIR filters has been intensively studied over the last few years (see, e.g., [3]–[6]). Earlier approaches, such as in [3], approximate the optimum solution by finitization, and employ conventional finite-dimensional linear programming. However, it was shown in [17] that the semi-infinite linear program corresponding to the uniform approximation problem can be solved by using numerically efficient simplex extension algorithms. These results were later exploited for the design of FIR filters with the complex Chebyshev error criterion (see, e.g., [6]).

Finitization can, in principle, give an arbitrarily accurate approximation of the complex Chebyshev solution using conventional linear programming. However, this approach becomes excessively memory intensive as the grid spacing decreases. The problem becomes awkward when the number of filter coefficients is large. The semi-infinite simplex extension is much more computationally efficient, since the constraint set can be represented in functional form rather than stored in memory as numerical values. Furthermore, the semi-infinite formulation deals directly with the true complex error and not an approximation thereof. The quadratic programming [15] and functional inequality approaches [16] are also, in general, more computationally intensive than the semi-infinite linear programming approach.

It was shown in [17] that the celebrated Remez exchange algorithm employed for the design of linear-phase FIR filters can be put into the framework of semi-infinite linear programming. In this context, it is noted that some other recent approaches to complex FIR filter design, such as a generalized Remez algorithm [18] and Tang's algorithm [5], [19] also employ simplex extension algorithms.

Manuscript received May 4, 1998; revised February 10, 1999. This paper was recommended by Associate Editor P. S. R. Diniz.

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Publisher Item Identifier S 1057-7130(99)04893-4.

The linear programming approach to linear-phase FIR filters is known to be advantageous due to its flexibility with respect to the design specification and the numerical efficiency of the simplex algorithm [20]. However, the new results on semi-infinite linear programming [17] have significantly increased the scope of applications for the simplex algorithm in digital filter design.

It is the aim of this paper to give a comprehensive description of semi-infinite linear programming (SIP) as a unified approach to digital filter design with time and/or frequency-domain specifications. Furthermore, since no commercial software for SIP is available (to the author's knowledge), it is our objective to provide the reader with sufficient information to implement the proposed design technique.¹ For a complete theory/description of linear programming and SIP, we refer to [17] and [21].

The rest of the paper is organized as follows. In Section II, the filter design problem is formulated as a semi-infinite linear program involving time and/or frequency-domain specifications. The discrete-time (first-order) Laguerre network is introduced as an example of a suitable filter structure. Section III gives a comprehensive description of semi-infinite linear programming and the extended simplex algorithm as implemented for the design of digital filters. Section IV gives three numerical examples: Frequency-domain design with specifications on group delay [3], the \mathcal{H}_∞ -optimal envelope constrained (EC) filter design problem [12], [13], [22]–[24], and equalizer design [4], [25] with additional frequency domain \mathcal{H}_∞ robustness constraints. A summary and conclusions are given in Section V.

II. PROBLEM FORMULATION

It is assumed that the frequency response can be represented by a finite-complex expansion of the following form:

$$H(\omega) = \sum_{n=0}^{N-1} \xi_n \phi_n(\omega) = \xi^T \phi(\omega) \quad (1)$$

where ξ is a real $N \times 1$ vector containing the filter coefficients ξ_n , and $\phi(\omega)$ a complex vector of basis functions $\phi_n(\omega)$.

Examples of digital filters with complex response are the N -tap FIR filter with basis functions

$$\phi_n(\omega) = e^{-j\omega n} \quad (2)$$

and the N th-order digital Laguerre filter [8], [10] defined by

$$\phi_n(\omega) = \sqrt{1-a^2} \frac{(e^{-j\omega} - a)^n}{(1 - a \cdot e^{-j\omega})^{n+1}} \quad (3)$$

where a is an unknown pole parameter $-1 < a < 1$. The associated discrete-time Laguerre network ($z^{-1}H(z)$) is depicted in Fig. 1, where s and ψ denote the input and output signals, respectively. Note that the FIR basis in (2) corresponds to the case where $a = 0$.

Some applications concerning communication systems employing narrowband signals may require digital filters with

¹Our algorithm in MATLAB will be available for noncommercial use by a request on e-mail: sven.nordebo@isb.hk-r.se.

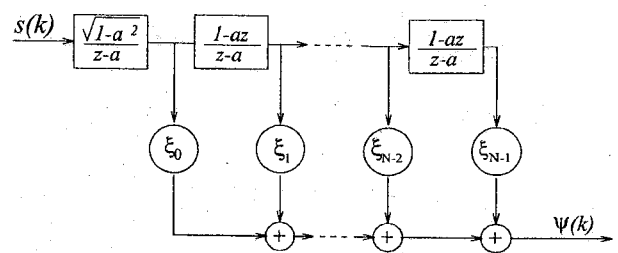


Fig. 1. Discrete-time Laguerre network.

complex coefficients $\xi_c = \xi_R + j\xi_I$, see, e.g., [6]. This case is covered by the description (1) by defining $H(\omega) = \xi_c^T \phi_c(\omega) = \xi^T \phi(\omega)$, where $\xi^T = [\xi_R^T \ \xi_I^T]$ and $\phi^T(\omega) = [\phi_c^T(\omega) \ j\phi_c^T(\omega)]$. In this case, N is even and the number of complex filter coefficients is $N/2$.

We now pose the following design criterion:

$$\min_{\xi \in \mathbb{R}^N} \max_{\omega \in \Omega_1} v(\omega) |H_d(\omega) - H(\omega)|, \quad \text{subject to} \quad (4)$$

$$|H(\omega)| \leq \sigma(\omega), \quad \omega \in \Omega_2 \quad (5)$$

$$\mathbf{P}\xi \leq \mathbf{p} \quad (6)$$

where $H_d(\omega)$ is the desired complex response, $v(\omega)$ a strictly positive weighting function, $\sigma(\omega)$ a strictly positive magnitude bound, \mathbf{P} an $M \times N$ constraint matrix, and \mathbf{p} an $M \times 1$ constraint vector. It is assumed that the domains Ω_1 and Ω_2 are closed and bounded subsets of $[-\pi, \pi]$ and that all the associated frequency functions are continuous.

The design criterion (4) can be used if, for example, the complex specification is given in the form

$$|H_d(\omega) - H(\omega)| \leq \varepsilon(\omega), \quad \omega \in \Omega_1 \quad (7)$$

where $\varepsilon(\omega)$ is a prescribed (strictly positive) upper bound. With $v(\omega) = 1/\varepsilon(\omega)$, the specification (7) will be satisfied if and only if the optimum objective value in (4) is less than or equal to one.

The purpose to incorporate the extra constraints (5) and (6) with the design formulation (4) is to allow for additional time and/or frequency-domain specifications. Simple examples are constraints on the weight magnitudes $|\xi_n| \leq \sigma_n$ (in case of numerically ill-conditioned problems or due to implementation considerations), frequency point (notch) constraints $\xi^T \phi(\omega_i) = 0$ and additional stopband requirements $|H(\omega)| \leq \sigma(\omega)$. In Section IV, we will give detailed numerical examples of filter design with time and/or frequency-domain specifications employing the design criterion (4)–(6).

We will now convert the design formulation (4)–(6) into a linear approximation problem. According to the *real rotation theorem* [25], a magnitude inequality in the complex plane can be expressed in the following equivalent form:

$$|z| \leq \delta \Leftrightarrow \Re\{ze^{j\theta}\} \leq \delta \quad \forall \theta \in [0, 2\pi] \quad (8)$$

where z is a complex number, δ a real and positive number, and $\Re\{\cdot\}$ denotes the real part.

By making use of (8), the nonlinear approximation problem (4)–(6) can be reformulated as the following continuous semi-

infinite linear program:

$$\begin{cases} \min \delta \\ v(\omega) \Re\{(H_d(\omega) - H(\omega)) \cdot e^{j\theta}\} \leq \delta, \quad \omega \in \Omega_1, \theta \in [0, 2\pi] \\ \Re\{H(\omega) \cdot e^{j\theta}\} \leq \sigma(\omega), \quad \omega \in \Omega_2, \theta \in [0, 2\pi] \\ \mathbf{P}\xi \leq \mathbf{p} \end{cases} \quad (9)$$

where δ is an additional real variable. For simplicity, we employ the same notation for variables (ω, θ) in the first and second (independent) constraint rows above.

The linear program (9) is called (continuous) semi-infinite since the constraint set is infinite (uncountable) and the number of variables is finite.

Define the $(N + 1) \times 1$ vectors \mathbf{y} (variable) and \mathbf{b} (constant) by

$$\mathbf{y} = \begin{pmatrix} \xi \\ \delta \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \quad (10)$$

where $\mathbf{0}$ is a zero vector of compatible dimension. Further, let the $N \times 1$ vector function $\mathbf{a}(\omega, \theta)$ and the scalar function $c(\omega, \theta)$ be defined by

$$\mathbf{a}(\omega, \theta) = \Re\{\phi(\omega) \cdot e^{j\theta}\} \quad (11)$$

$$c(\omega, \theta) = v(\omega) \Re\{H_d(\omega) \cdot e^{j\theta}\}. \quad (12)$$

The linear program (9) is now restated in the following form:

$$\begin{cases} \min \mathbf{b}^T \mathbf{y} \\ \begin{bmatrix} v(\omega) \mathbf{a}^T(\omega, \theta) & 1 \\ -\mathbf{a}^T(\omega, \theta) & 0 \\ -\mathbf{P} & 0 \end{bmatrix} \mathbf{y} \geq \begin{bmatrix} c(\omega, \theta) \\ -\sigma(\omega) \\ -\mathbf{p} \end{bmatrix} \end{cases} \quad (13)$$

where $\mathbf{0}$ is an $M \times 1$ vector of zeros. Note that the first and second constraint rows in (13) are independent as in (9).

The design problem (4)–(6) can now be compactly formulated as the following (dual) continuous semi-infinite linear program:

$$(D) \begin{cases} \min \mathbf{b}^T \mathbf{y} \\ \mathbf{A}_\alpha^T \mathbf{y} \geq c_\alpha \end{cases} \quad (14)$$

where \mathbf{A}_α^T and c_α are the rows of the left-hand side constraint matrix and the elements of the right-hand side constraint vector in (13), respectively. Here, α is an index belonging to a compact index set $\mathcal{A} \subset R^k$, which has a one-to-one correspondence with the constraint rows of (13). The set \mathcal{A} can be constructed as follows: let $\alpha = (\omega, \theta, i, l)$ where $l = 1, 2, \text{ or } 3$ since there are three types of constraints in (13). If $l = 1$, then $(\omega, \theta) \in \Omega_1 \times [0, 2\pi]$ and $i = 1$. If $l = 2$, then $(\omega, \theta) \in \Omega_2 \times [0, 2\pi]$ and $i = 1$. If $l = 3$, then $\omega = 0, \theta = 0$, and $i = 1, \dots, M$. The formulation (14) is now in a form which has been thoroughly investigated [17]. We refer to (14) as the dual formulation since it corresponds to the dual of a linear program in standard form [17], [21].

III. SEMI-INFINITE LINEAR PROGRAMMING

The application of simplex extension algorithms for the solution of (9) is based on two important theorems, given in [17], concerning strong duality and finite dimensionality. These theorems will be quoted and explained below. We will also give a comprehensive description of the extended simplex algorithm as implemented for the filter design problem under consideration.

A. Duality and Dimensionality

The primal linear programming formulation corresponding to (14) is given by

$$(P) \begin{cases} \max \int c_\alpha dx_\alpha \\ \int \mathbf{A}_\alpha dx_\alpha = \mathbf{b} \\ x_\alpha \geq 0 \end{cases} \quad (15)$$

where the maximization is with respect to the space of regular Borel measures x_α on \mathcal{A} (dual of the space \mathcal{C} of continuous functions c_α on \mathcal{A}), cf. [17]. The formulation (15) is said to be in standard form.

Finite dimensional linear programs satisfy a condition known as *strong duality*: if either of the primal and dual formulations has a finite optimum solution, so does the other, and the corresponding optimum values are equal [21]. Strong duality does not necessarily hold for semi-infinite linear programs. Examples can easily be constructed for which there exists a *duality gap* between the optimal values of the dual programs in (14) and (15), cf. [17].

Fortunately, the following result applies to the filter design problem (4)–(6) under consideration.

Theorem 1—Strong Duality: The semi-infinite linear program corresponding to the *uniform approximation problem* (9) satisfies strong duality. \square

Theorem 1 is readily proven by applying [17, Theorem 4.4c]. This theorem states that if (D) has finite optimum value and there are no $\mathbf{y} \neq \mathbf{0}$, such that $\mathbf{b}^T \mathbf{y} = 0$ and $\mathbf{A}_\alpha^T \mathbf{y} \geq 0, \forall \alpha \in \mathcal{A}$, then (D) and (P) have the same optimum value. This sufficient condition is fulfilled since there are no $\mathbf{H}(\omega) = \xi^T \phi(\omega)$, with $v(\omega) \Re\{H(\omega)e^{j\theta}\} \geq 0, \forall \theta \in [0, 2\pi], \forall \omega \in \Omega_1$ except for $\mathbf{H}(\omega) = 0$, or equivalently $\xi = 0$, since the basis functions $\phi_n(\omega)$ are assumed to be linearly independent on Ω_1 .

Note that (D) has finite optimum value whenever the set of additional constraints (5) and (6) define a nonempty closed and bounded subset of R^N .

The success of the semi-infinite linear programming approach relies on the property of strong duality, but also on the equally important *Dimensionality Theorem*.

Theorem 2—Finite Dimensionality: If there is a feasible solution to (P) in (15) with value z_0 , then there is also a feasible solution of finite support (atomic measure) consisting of at most $N + 1$ points which achieves the same value. \square

The dimensionality theorem [17, Theorem 4.8] is based on an application of Caratheodory's theorem, a fundamental dimensionality result in convexity theory [26]. The implication

of Theorem 2 is extremely useful, since it allows the simplex algorithm to work with finite basic feasible solutions in very much the same way as in the case of finite-dimensional linear programming. The primal formulation (15) can thus be rewritten as

$$(P) \begin{cases} \max \sum_{i=1}^{N+1} c_{\alpha_i} x_{\alpha_i} \\ \sum_{i=1}^{N+1} \mathbf{A}_{\alpha_i} x_{\alpha_i} = \mathbf{b} \\ x_{\alpha_i} \geq 0 \\ \alpha_i \in \mathcal{A} \end{cases} \quad (16)$$

where the maximization is with respect to the points $\alpha_1, \dots, \alpha_{N+1}$ in \mathcal{A} .

The duality theorem enables us to solve the primal (P) in (16) rather than the dual (D) in (14). When the optimum solution to (P) is obtained we will also be in possession of the optimum solution to (D). The advantage of this procedure is that the primal (P) is in standard form for solution by the simplex algorithm employing a finite basis. The problem (P) is thus much easier to solve than (D).

B. The Extended Simplex Algorithm

In order to simplify the algorithmic description and emphasize the similarity to standard finite-dimensional linear programming techniques, we introduce the following semi-infinite matrix and vector notations $\mathbf{A} = (\mathbf{A}_{\alpha})$, $\mathbf{c} = (c_{\alpha})$, and $\mathbf{x} = (x_{\alpha})$. Matrix and vector products of the form $\mathbf{A}\mathbf{x}$ and $\mathbf{c}^T\mathbf{x}$ will then have a meaning for atomic measures as in (16).

Given a basis $\mathbf{B} = [\mathbf{A}_{\alpha_1}, \dots, \mathbf{A}_{\alpha_m}]$ where $\alpha_i \in \mathcal{A}$ for $i = 1, \dots, m = N + 1$ and a basic feasible solution $\mathbf{x}_B \geq 0$ satisfying $\mathbf{B}\mathbf{x}_B = \mathbf{b}$. The dimension of \mathbf{B} and \mathbf{x}_B are $m \times m$ and $m \times 1$, respectively.

Let the matrix \mathbf{A} be partitioned as $\mathbf{A} = [\mathbf{B} \ \mathbf{N}]$ where $\mathbf{N} = (\mathbf{A}_{\alpha})$ consists of those columns of \mathbf{A} which are not included in \mathbf{B} . Define the corresponding partition of the vectors $\mathbf{c}^T = [\mathbf{c}_B^T \ \mathbf{c}_N^T]$ and $\mathbf{x}^T = [\mathbf{x}_B^T \ \mathbf{x}_N^T]$ (basic and nonbasic variables). The primal cost is $\mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$.

The revised simplex algorithm proceeds with four basic steps as described below, cf. [21].

- 1) Calculate the dual variables by solving $\mathbf{B}^T\mathbf{y} = \mathbf{c}_B$, that is $\mathbf{y}^T = \mathbf{c}_B^T\mathbf{B}^{-1}$. The dual cost is $\mathbf{y}^T\mathbf{b} = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} = \mathbf{c}^T\mathbf{x}$. If \mathbf{y} is feasible for (D), then the current solution \mathbf{x}_B is optimal for (P) according to the duality theorem. Stop.
- 2) Determine a column $\mathbf{A}_{\alpha'}$ to enter the basis by choosing an index α' so that $\mathbf{y}^T\mathbf{A}_{\alpha'} < c_{\alpha'}$ [constraint violation for (D)]. The standard pivoting rule is to choose

$$\alpha' = \arg \max_{\alpha} \{c_{\alpha} - \mathbf{y}^T\mathbf{A}_{\alpha}\}. \quad (17)$$

- 3) Determine a column \mathbf{A}_{α_j} to leave the basis by the ratio test

$$j = \arg \min_{i: d_i > 0} \left\{ \frac{x_{B_i}}{d_i} \right\} \quad (18)$$

where the x_{B_i} 's are the elements of the basic feasible solution \mathbf{x}_B and the d_i 's are the coordinates of the entering column in terms of the old basis. Thus, $\mathbf{B}\mathbf{d} = \mathbf{A}_{\alpha'}$ where $\mathbf{d} = (d_i)$.

- 4) Update the basis matrix \mathbf{B} by replacing the column \mathbf{A}_{α_j} for $\mathbf{A}_{\alpha'}$. \square

C. Implementation Issues

The representation of the vector \mathbf{b} in the new basis is given by

$$\mathbf{B}(\mathbf{x}_B - t \cdot \mathbf{d}) + t \cdot \mathbf{A}_{\alpha'} = \mathbf{b} \quad (19)$$

where $t \geq 0$ is the minimum ratio in (18) in step 3) above. The ratio test ensures that the new basic solution is feasible, that is, $\mathbf{x}_B - t \cdot \mathbf{d} \geq 0$ with equality for at least one component (j). If there are no index i for which $d_i > 0$, then there are no restrictions on the positive parameter t , and the new basic feasible solution to (P) is unbounded. The *degenerate* case is when $t = 0$.

A general basic solution \mathbf{x}_B is determined by the relation $\mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}$. The corresponding primal cost is given by

$$z(\mathbf{x}_N) = \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_N^T\mathbf{x}_N = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} + \mathbf{r}^T\mathbf{x}_N \quad (20)$$

where $\mathbf{r}^T = \mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}$ is the vector of relative cost coefficients [21].

The new basic feasible solution is given by $\mathbf{x}_N^T = [0 \dots t \dots 0]$ (atomic measure), with t at the position corresponding to the index α' . With this choice of nonbasic variables, the increment in primal cost is

$$\mathbf{r}^T\mathbf{x}_N = t \cdot (c_{\alpha'} - \mathbf{y}^T\mathbf{A}_{\alpha'}). \quad (21)$$

Since $t \geq 0$ and, according to the conditions of step 2) above, $\mathbf{r}^T\mathbf{x}_N \geq 0$, the primal cost will constitute a monotonically increasing sequence. Convergence is therefore guaranteed, except for the possibility of *cycling* of degenerate basic solutions (in which case $t = 0$). However, cycling is a rarely occurring phenomenon, and there are several well-established remedies (see, e.g., Bland's rule [21]).

When the basic feasible solution \mathbf{x}_B to (P) is optimal, the dual variables \mathbf{y} are feasible for (D) and, thus, optimal according to the duality theorem. This follows from the fact that nonpositive relative cost coefficients $\mathbf{r}^T = \mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N} \leq 0$ imply feasibility for the dual

$$\begin{aligned} \mathbf{y}^T\mathbf{A} &= \mathbf{c}_B^T\mathbf{B}^{-1}[\mathbf{B} \ \mathbf{N}] \\ &= [\mathbf{c}_B^T \ \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}] \\ &\geq [\mathbf{c}_B^T \ \mathbf{c}_N^T] \\ &= \mathbf{c}^T. \end{aligned} \quad (22)$$

Since the number of constraints is infinite, the algorithm will, in general, not be able to find the optimum solution in a finite number of steps. However, the algorithm will converge to any accuracy level $\varepsilon > 0$ in a finite number of steps as follows. Consider first the case without the additional constraints (5) and (6). The duality theorem ensures that $\delta = \mathbf{y}^T\mathbf{b} = \mathbf{c}^T\mathbf{x} \leq \delta_o \leq \delta_m$ where δ_o is the optimum

objective value and δ_m is the Chebyshev error of the current solution $\delta_m = \max_{\omega} v(\omega)|H_d(\omega) - H(\omega)|$. If the algorithm is terminated when $\delta_m - \delta \leq \delta \cdot \epsilon$, the relative error in objective value will satisfy $(\delta_o - \delta)/\delta \leq \epsilon$.

Now consider the case with additional constraints (5) and (6) included. Define a tolerance limit $\epsilon > 0$ for these constraints. When a solution ξ satisfies these relaxed constraints $|H(\omega)| \leq \sigma(\omega) + \epsilon(\omega)$, $\mathbf{P}\xi \leq \mathbf{p} + \epsilon$, and $\delta_m - \delta \leq \delta \cdot \epsilon$, the relative error in objective value will satisfy $(\delta_o - \delta)/\delta \leq \epsilon$, where δ_o is the optimum objective value corresponding to the relaxed constraints.

Note that the existence of an optimal solution with a bounded objective δ_o , together with the duality theorem, ensures that $\mathbf{c}^T \mathbf{x} \leq \delta_o$ for any feasible solution \mathbf{x} . Boundedness of the incremental cost in (21) implies that the variable t in the representation (19) must be bounded. Hence, there must exist at least one $d_i > 0$ in the ratio test in step 3) above.

An important property of the simplex algorithm is the nonsingularity of the basis. If the basis matrix \mathbf{B} is nonsingular, then so is the new basis (even at degeneracy when $t = 0$). To see this, consider the relation $\mathbf{Bd} = \mathbf{A}_{\alpha'}$, where $d_j > 0$. The vector $\mathbf{A}_{\alpha'}$ cannot be written as a linear combination of the remaining columns of \mathbf{B} when \mathbf{A}_{α_j} is excluded. Thus, $\mathbf{A}_{\alpha'}$ is linearly independent of these columns and the new basis is nonsingular.

Since the new and old basis differ in one column only, there is no need to perform a complete matrix inversion of \mathbf{B} at each iteration [complexity $\mathcal{O}(m^3)$]. A common technique used with commercial linear programming softwares is to employ the Sherman-Morrison rank-one update formula (complexity $\mathcal{O}(m^2)$), cf. [27].

Suppose that there are no additional constraints (5) and (6) to the problem (4). The column vector \mathbf{A}_{α} is then given by $\mathbf{A}_{\alpha}^T = [v(\omega)\mathbf{a}^T(\omega, \theta)1]$ and the scalar $c_{\alpha} = c(\omega, \theta)$. Using (17) in step 2 above and the definitions given in (11) and (12), the explicit relations for the variables (ω', θ') corresponding to the column entering the basis are given by

$$\omega' = \arg \max_{\omega} \{v(\omega)|H_d(\omega) - H(\omega)|\} \quad (23)$$

$$\theta' = -\arg\{H_d(\omega') - H(\omega')\}. \quad (24)$$

The relations (23) and (24) display the infinite-dimensional nature of the simplex extension algorithm, and are easily extended to include additional linear constraints as in (13).

The revised simplex algorithm as outlined above assumes that an initial basic feasible solution to (P) is available. This is phase two in the two-phase simplex algorithm [21]. In phase one, a basic feasible solution to (P) is found by solving the so-called artificial minimization problem

$$\begin{cases} \min \mathbf{1}^T \mathbf{z} \\ \mathbf{A}\mathbf{x} + \mathbf{z} = \mathbf{b} \\ \mathbf{x} \geq 0 \\ \mathbf{z} \geq 0 \end{cases} \quad (25)$$

where \mathbf{z} is an $m \times 1$ vector of artificial variables and $\mathbf{1}$ is a vector of ones.

The formulation (25) is in standard form and can be solved by the simplex algorithm. The initial basis is the identity matrix $\mathbf{B} = \mathbf{I}$, and the initial basic feasible solution is $\mathbf{z} = \mathbf{b} \geq 0$.

If there is a basic feasible solution to (P), then the optimal solution to (25) has the objective value zero. Phase one is terminated when all artificial variables have left the basis.

Since the initial basic feasible solution \mathbf{b} is highly degenerate, cycling may occur for this case. It is therefore advantageous to apply the following simple anticycling rule: once an artificial variable has left the basis, it cannot enter again.

IV. NUMERICAL EXAMPLES

In the numerical studies below we have employed the digital Laguerre filter as a simple example of a recursive low-order alternative to FIR filters. The properties of the Laguerre filter versus the FIR filter are investigated in order to demonstrate the use of semi-infinite linear programming as a unified approach to digital filter design. The digital Laguerre network is depicted in Fig. 1 and the corresponding frequency response is given by (1) and (3).

We will consider three design examples each of which can be treated as a special case of the design criterion (4)–(6).

- 1) *Frequency-Domain Design*: where H_d is given, $v = 1/\epsilon$ and additional group-delay specifications (example: channel filter in a mobile communication system).
- 2) *EC Filter Design*: where $H_d = 0$, $v = 1$, and additional output pulse-shaped envelope specifications (example: equalization filter in a communication system).
- 3) *Frequency-Domain Equalizer Design*: where $H_d = e^{-j\omega\tau_d}$ (flat, linear-phase response), $v = 1$, and additional frequency domain \mathcal{H}_{∞} robustness constraints (example: design of a robust phase equalizer).

A. Frequency-Domain Design

A special case of the criterion (4)–(6) is considered where $H_d(\omega)$ is a given desired response and $v(\omega) = 1/\epsilon(\omega)$. Here, $\epsilon(\omega)$ is a given design tolerance as in (7). Furthermore, additional linear constraints as in (6) will be used to control the group delay of the resulting filter. Equation (5) will not be employed in this problem.

As a specific numerical example we will consider the design of a low-pass channel filter in a mobile communication system.² The filter specification is given as a magnitude specification in the frequency domain

$$H_l(\omega) \leq |H(\omega)| \leq H_u(\omega), \quad \omega \in [0, \pi] \quad (26)$$

where $H_l(\omega)$ and $H_u(\omega)$ are nonnegative lower and upper bounds, respectively. In addition, a group-delay specification (approximative linear-phase response) is also considered with this design problem.

The actual and desired frequency responses are represented in the form

$$H(\omega) = A(\omega)e^{j\theta(\omega)} \quad (27)$$

$$H_d(\omega) = A_d(\omega)e^{j\theta_d(\omega)} \quad (28)$$

²Filter specification from Ericsson mobile communications AB, mobile phones, Lund, Sweden.

where $A(\omega)$ and $A_d(\omega)$ are real amplitudes and $\theta(\omega)$ and $\theta_d(\omega)$ are continuous phase responses, cf. [25].

The frequency domain $\Omega_1 = [0, \pi]$ is divided into a passband \mathcal{P} and a stopband \mathcal{S} where $H_l(\omega) = 0$ for $\omega \in \mathcal{S}$.

Now, define $A_u(\omega) = H_u(\omega)$ for $\omega \in [0, \pi]$, $A_l(\omega) = H_l(\omega)$ for $\omega \in \mathcal{P}$ and $A_l(\omega) = -H_u(\omega)$ for $\omega \in \mathcal{S}$. Further, let

$$\varepsilon(\omega) = \frac{A_u(\omega) - A_l(\omega)}{2} \quad (29)$$

$$A_d(\omega) = \frac{A_l(\omega) + A_u(\omega)}{2} \quad (30)$$

$$H_d(\omega) = A_d(\omega)e^{-j\omega\tau_d} \quad (31)$$

where τ_d is the desired group delay. Note that $H_d(\omega) = 0$ and $\varepsilon(\omega) = H_u(\omega)$ for $\omega \in \mathcal{S}$.

By employing the inequality $\|H\| - \|H_d\| \leq \|H - H_d\|$, it is readily verified that

$$|H(\omega) - H_d(\omega)| \leq \varepsilon(\omega) \Rightarrow H_l(\omega) \leq |H(\omega)| \leq H_u(\omega) \quad (32)$$

for all $\omega \in [0, \pi]$.

Note that the above definitions may render the desired response $H_d(\omega)$ discontinuous. However, this does not imply any difficulties if the approximation problem is solved on a finite domain $\Omega_1 \subset [0, \pi]$ (any function on a discrete domain is continuous). Furthermore, it is advantageous to employ a finite frequency domain Ω_1 since the pivoting process in (23) becomes very simple. The main purpose with the semi-infinite programming formulation is to efficiently solve the design problem employing the true complex error. That is, even though the frequency domain Ω_1 is discrete, the argument θ in (9) does not have to be finitized and the pivoting rule (24) may be considered on the infinite domain $[0, 2\pi]$.

Linear constraints on the group-delay error can be obtained by using an approximative formula as in [3] (the FIR case in [3] can easily be extended to more general cases such as with the present digital Laguerre filter). The derivation is as follows.

It is assumed that the desired real amplitude is constant in the passband, that is $A_d(\omega) = A_d > 0$ for $\omega \in \mathcal{P}$. It is further assumed that the design is successful so that the design error $\delta \ll A_d$. Hence, by employing the approximations $\theta_d(\omega) - \theta(\omega) \approx \sin(\theta_d(\omega) - \theta(\omega))$ and $H(\omega) \approx A_d e^{j\theta(\omega)}$, we obtain the formula

$$\theta_d(\omega) - \theta(\omega) \approx \frac{1}{A_d} \Im \left\{ H^*(\omega) e^{j\theta_d(\omega)} \right\}, \quad \omega \in \mathcal{P} \quad (33)$$

where $\Im\{\cdot\}$ denotes the imaginary part.

The actual and desired group delays are given by $\tau(\omega) = -\theta'(\omega)$ and $\tau_d(\omega) = -\theta'_d(\omega)$, respectively. By employing the linear relationship $H(\omega) = \phi^T(\omega)\xi$ and differentiating both sides of (33), we obtain an approximate expression for the group-delay error $e(\omega)$

$$e(\omega) = \tau(\omega) - \tau_d(\omega) \approx \mathbf{k}^T(\omega)\xi \quad (34)$$

where $\mathbf{k}(\omega)$ is an $N \times 1$ real vector defined by

$$\mathbf{k}(\omega) = \frac{1}{A_d} \Im \left\{ \frac{d}{d\omega} \left\{ \phi^*(\omega) e^{j\theta_d(\omega)} \right\} \right\}. \quad (35)$$

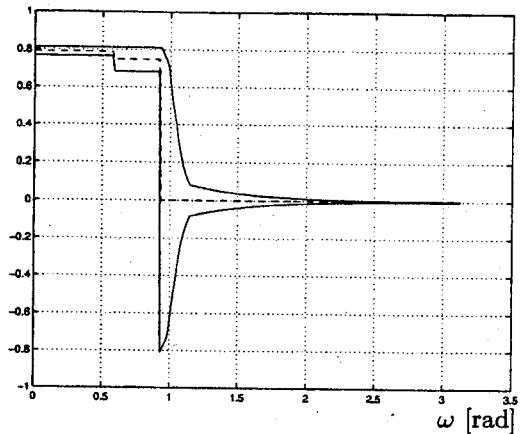


Fig. 2. Solid line: real amplitude specifications $A_l(\omega)$ and $A_u(\omega)$. Dashed line: desired real amplitude response $A_d(\omega)$.

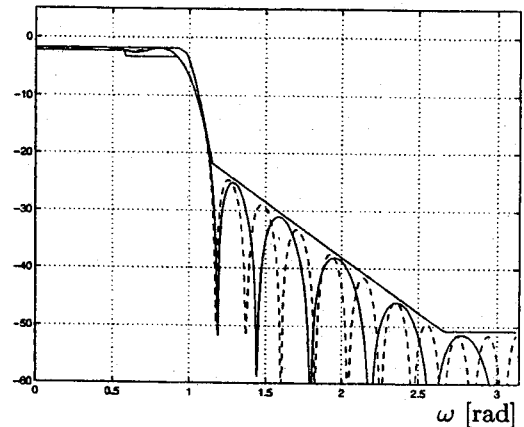


Fig. 3. Upper solid line: magnitude specification in dB. Lower solid line: response of Laguerre filter ($N = 24$). Dashed line: response of FIR filter ($N = 29$).

By substituting the expression for the Laguerre basis functions (3) into (35), we finally arrive at the formula

$$k_n(\omega) = \frac{1}{A_d} \Re \left\{ \phi_n^*(\omega) e^{j\theta_d(\omega)} \left(\frac{ne^{j\omega}}{e^{j\omega} - b} + \frac{(n+1)be^{j\omega}}{1 - be^{j\omega}} - \tau_d(\omega) \right) \right\} \quad (36)$$

where $k_n(\omega)$ are the elements of $\mathbf{k}(\omega)$, $n = 0, \dots, N-1$, and $\omega \in \mathcal{P}$.

In the numerical examples below we have employed additional linear constraints (6) of the form

$$\begin{cases} \mathbf{k}^T(\omega_i)\xi \leq e \\ -\mathbf{k}^T(\omega_i)\xi \leq e \end{cases} \quad (37)$$

where $\omega_i \in \mathcal{P}$, $\theta_d(\omega) = -\omega\tau_d$ and e a fixed upper bound on the group-delay error.

Figs. 2–6 show some numerical results on the filter design problem described above. Fig. 2 shows the real amplitude specifications $A_l(\omega)$ and $A_u(\omega)$, and the desired real amplitude response $A_d(\omega)$. The passband \mathcal{P} and stopband \mathcal{S} are given by

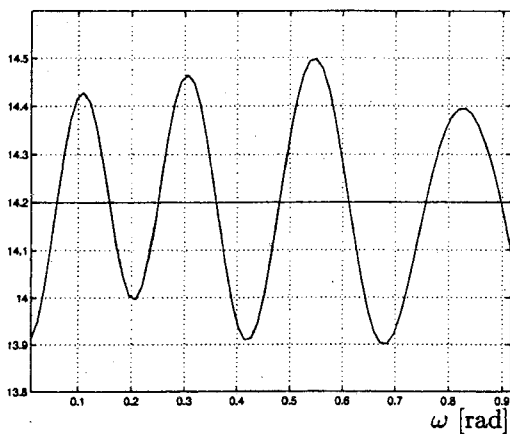


Fig. 4. Resulting group delay $\tau(\omega)$ of the Laguerre filter ($N = 24$). Desired group delay $\tau_d(\omega) = 14.2$, specified error $e = 0.3$.

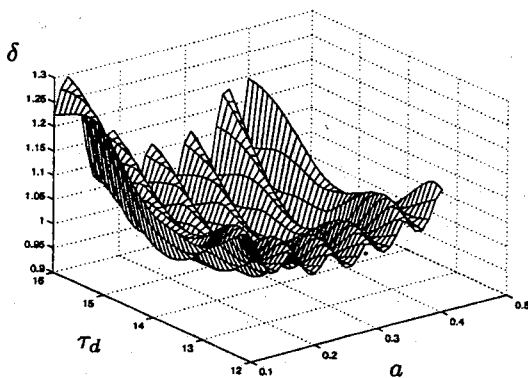


Fig. 5. Optimum objective value δ versus pole parameter and desired group delay (a, τ_d) for the Laguerre filter ($N = 24$).

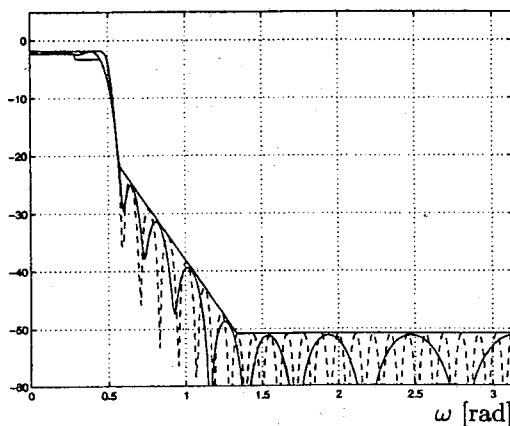


Fig. 6. Specification with oversampling. Upper solid line: magnitude specification in dB. Lower solid line: response of Laguerre filter ($N = 27$). Dashed line: response of FIR filter ($N = 55$).

76 and 180 uniformly spaced points in the intervals $[0, 0.92]$ and $[0.92, \pi]$, respectively.

The bound on group-delay error used is $e = 0.3$. The linear program (9) is solved by using the simplex method described in the previous section. The filter length N and parameters (a, τ_d) are fixed, and the specification is satisfied when $\delta \leq 1$, cf. (7) and (32).

Fig. 3 shows the magnitude specification H_l and H_u in dB, response of a minimum-length Laguerre filter ($N = 24, a = 0.31, \tau_d = 14.2$) and a minimum-length FIR filter ($N = 29, a = 0, \tau_d = (N - 1)/2$). Fig. 4 shows the resulting group delay of the Laguerre filter. The approximation (34) is highly accurate for this example, and the resulting group delay is consistent with the corresponding linear constraints used ($e = 0.3$).

The pole parameter a and desired group delay τ_d are unknown in the original filter specification, and should therefore be chosen to give as small an optimum objective value δ in (9) as possible. The corresponding optimization problem is very difficult since the optimum δ is a nonconvex function of (a, τ_d). However, due to the numerical efficiency of the simplex method, it is practically feasible to obtain a suboptimal solution by an exhaustive search. The time to solve one linear program [for fixed (a, τ_d)] is in the order of seconds for our examples on a standard PC. Fig. 5 shows the optimum objective value δ versus (a, τ_d) for the Laguerre filter of length $N = 24$ above.

We emphasize that it is possible to include the unknown parameters (a, τ_d) as variables in the minimax-optimization problem. One may approach the problem in many ways using one- or two-dimensional local or global optimization techniques (see, e.g., [21] and [28]). We believe that line-search strategies such as Fibonacci and Golden section search [21] could be used in combination with a bridging technique in order to obtain the global optimum. However, such an investigation is outside the scope of this paper and is an important topic for future research.

The virtue of using a Laguerre filter instead of a common FIR filter depends, of course, on the problem. The Laguerre filter is known to be robust with respect to the choice of sampling interval, as well as to the choice of filter order (cf. [8]). This robustness property is illustrated in Fig. 6 which shows the resulting Laguerre and FIR responses when the specification is subjected to an increase in sampling rate by a factor 2. The figure shows the magnitude specification in dB, response of a minimum-length Laguerre filter ($N = 27, a = 0.50, \tau_d = 29.2$) and a minimum-length FIR filter [$N = 55, a = 0, \tau_d = (N - 1)/2$].

B. EC Filter Design

We now consider a special case of the criterion (4)–(6) according to the \mathcal{H}_∞ -optimal EC filter design problem [12], [13], [22]–[24]. Here, $H_d(\omega) = 0, v(\omega) = 1$ and additional linear constraints as in (6) corresponding to a pulse-shaped envelope specification, see Fig. 7.

Our objective is to design a digital filter to process a given input signal $s(k)$ which is corrupted by additive random noise $n(k)$ [see Fig. 7(a)]. The noiseless output $\psi(k)$ is required to fit into a prescribed pulse-shaped envelope defined by the lower and upper boundaries $\varepsilon^-(k)$ and $\varepsilon^+(k)$ [see Fig. 7(b)].

As a specific numerical example, we consider the design of an equalization filter for a digital transmission channel consisting of a coaxial cable on which data is transmitted according to the DSX-3 standard [22], [23]. The design

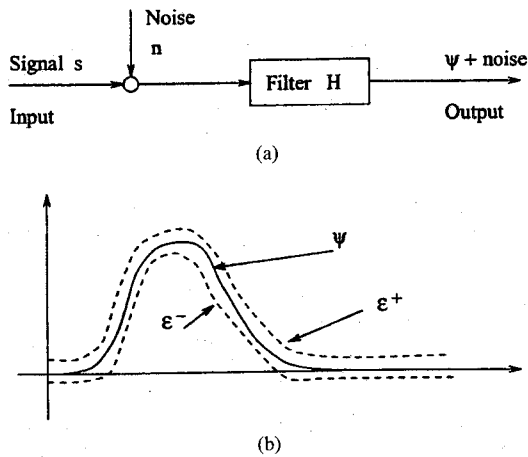


Fig. 7. Envelope-constrained filtering problem. (a) Block diagram. (b) Pulse-shaped envelope.

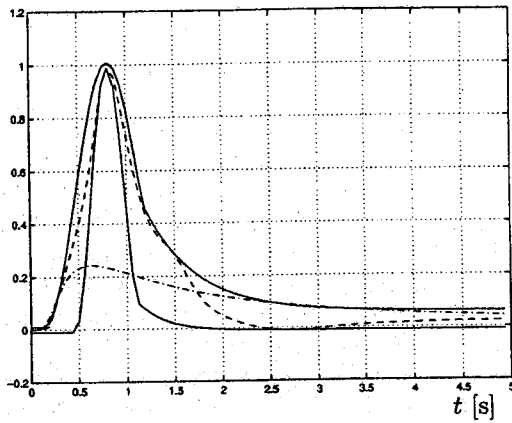


Fig. 8. Time-domain response of equalizer (Laguerre filter $N = 15$, $a = 0.38$). Solid lines: specified pulse-shaped envelope ϵ^- and ϵ^+ . Dashed-dotted line: input signal s . Dashed line: noiseless output signal ψ .

objective is to find an equalizing filter which takes a sampled impulse response of a coaxial cable with a loss of 30 dB at a normalized frequency of $1/T$ as input, and produces an output which lies within the envelope given by the DSX-3 pulse template [22], [23].

The discrete-time \mathcal{H}_∞ -optimal EC filter design problem can be formulated as follows.

Given an input signal $s(k)$, find a Laguerre filter $H(\omega)$ which solves the following constrained optimization problem:

$$\min_H \|H\|_\infty^2, \quad \text{subject to } \epsilon^-(k) \leq \psi(k) \leq \epsilon^+(k) \quad (38)$$

where $\psi(k) = h(k) * s(k)$, $k = 0, 1, \dots, K - 1$, $h(k)$ is the impulse response of the filter and $*$ denotes convolution. Note that compared to the \mathcal{H}_2 formulation of the EC filtering problem [12], minimizing the squared \mathcal{H}_∞ norm of the filter is equivalent to minimizing the output noise power corresponding to the worst case input noise n (see, e.g., [13]). Therefore, in the situation where the power spectrum of the exogenous input noise is bounded but otherwise unknown, the \mathcal{H}_∞ formulation of the EC filtering problem is directly applicable.

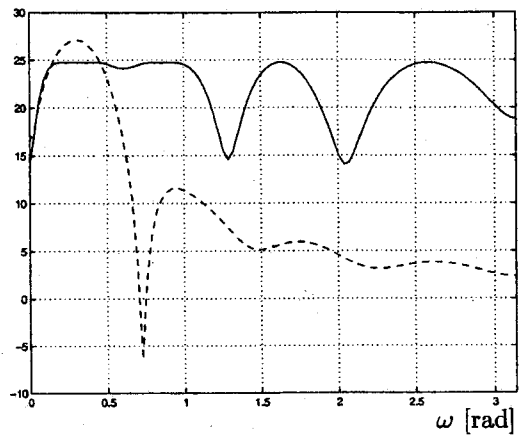


Fig. 9. Frequency-domain response of equalizer (Laguerre filter $N = 15$, $a = 0.38$). Solid line: min-max (\mathcal{H}_∞) design. Dashed line: least-squares (\mathcal{H}_2) design.

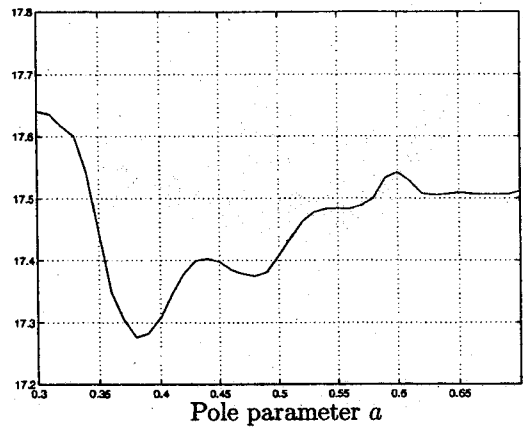


Fig. 10. Optimum objective value δ versus pole parameter a for the Laguerre filter ($N = 15$).

In order to obtain a nontrivial solution (i.e., $H(\omega) \neq 0$), we assume that there exists at least one $k(0 \leq k \leq K - 1)$ such that $\epsilon^-(k)\epsilon^+(k) > 0$. Furthermore, we assume that $\epsilon^+(k) > \epsilon^-(k)$, ($k = 0, 1, \dots$).

Problem (38) is now equivalent to the design criterion (4) with $\Omega_1 = [0, \pi]$, $H_d(\omega) = 0$, $v(\omega) = 1$, and (6), given in the form

$$\epsilon^- \leq \mathbf{S}\xi \leq \epsilon^+ \quad (39)$$

where \mathbf{S} is a constraint matrix, and ϵ^- and ϵ^+ are vectors of lower and upper bounds $\epsilon^-(k)$ and $\epsilon^+(k)$, respectively. The constraint matrix \mathbf{S} is defined so that $\psi = \mathbf{S}\xi$, where ψ is the vector containing the noiseless output signal $\psi(k)$, $k = 0, 1, \dots, K - 1$. Hence, $S_{kn} = s(k) * l_n(k)$, where $l_n(k)$ is the time-domain sequence corresponding to the (frequency-domain) Laguerre basis function $\phi_n(\omega)$. Hence, in (6), $\mathbf{P}^T = [\mathbf{S}^T \quad -\mathbf{S}^T]$ and $\mathbf{p}^T = [\epsilon^{+T} \quad -\epsilon^{-T}]$. Note that the (5) is not used in this problem.

Figs. 8–11 show some numerical results on the EC filter design problem described above. The domain Ω_1 was chosen as 101 uniformly spaced points in the interval $[0, \pi]$.

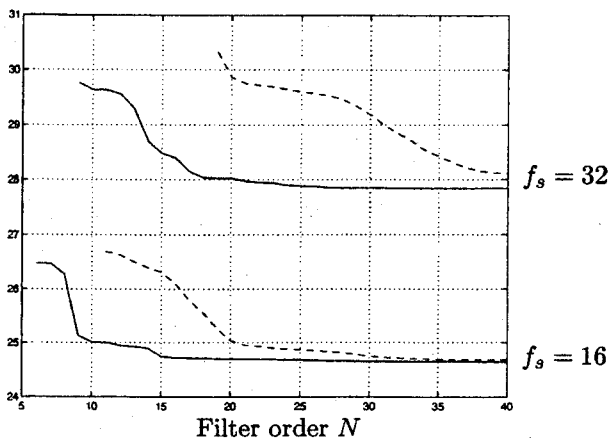


Fig. 11. Optimum objective δ versus filter order N . Solid lines: Laguerre filter (optimized over a). Dashed lines: FIR filter ($a = 0$). The upper solid and dashed plots correspond to a doubled sampling rate.

Fig. 8 shows the coaxial cable impulse response s , the output envelope ε , and the resulting noiseless output signal ψ . The solution corresponds to an \mathcal{H}_∞ -design, with $N = 15$ coefficients obtained with the semi-infinite simplex procedure. The pole parameter was chosen as $a = 0.38$. Fig. 9 shows the resulting frequency response of the equalizer designed. The frequency response of the corresponding \mathcal{H}_2 -design is also included for comparison.

The pole parameter a is unknown in the original specification and should be chosen to give as small optimum \mathcal{H}_∞ -norm in (38) as possible [optimum δ in (9)]. Again, this is a complicated optimization problem since the optimum \mathcal{H}_∞ -norm is a nonconvex function of the pole parameter a . However, due to the numerical efficiency of the simplex algorithm, we can readily obtain a good suboptimal solution by an exhaustive search. Fig. 10 shows the optimum objective value δ versus a for the Laguerre filter of length $N = 15$ above.

We use this example to demonstrate the robustness of the Laguerre filter with respect to the choice of sampling interval and filter order. Fig. 11 shows the optimum objective value δ versus filter order N for the Laguerre and the FIR filter. For the Laguerre filter, the value of a has been optimized for each value of N , as in Fig. 10. Fig. 11 shows two cases of sampling frequencies f_s , 16 and 32 Hz. It is clearly seen that the Laguerre filter has superior convergence properties over the FIR filter, and that this superiority becomes essential as the sampling frequency increases.

For the EC filtering problem with sampling frequency $f_s = 16$ Hz, eighth-order Laguerre filter is adequate and the corresponding computation time was 0.4 s on a standard PC. The computation times for a length 20 and 40 Laguerre filter were 3.2 and 20.4 s, respectively (pole parameter $a = 0.4$).

We emphasize that the actual choice of filter structure should be made with respect to the total complexity of implementation and finite precision considerations. However, such an investigation is outside the scope of this paper and is an important topic for future research.

C. Frequency-Domain Equalizer Design

The design of digital equalizers is an important application for filter design in the complex domain [4], [25]. In this

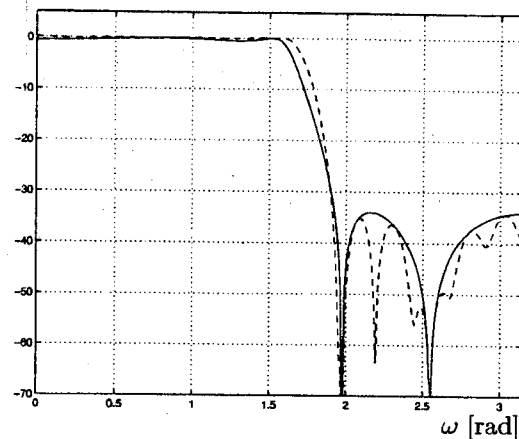


Fig. 12. Magnitude response of channel with/without phase equalization. Solid line: fourth-order elliptic filter G . Dashed line: total response GH , equalizer stopband constraint is 0 dB.

example, we consider the frequency-domain design of a phase equalizer similar to [25, pp. 126–129], and with additional frequency domain \mathcal{H}_∞ robustness constraints.

As a design criterion, the following special case of (4)–(6) is employed:

$$\min_{\xi \in \mathbb{R}^N} \max_{\omega \in \Omega_p} |H_d(\omega) - G(\omega) \cdot H(\omega)|, \quad \text{subject to} \quad (40)$$

$$|H(\omega)| \leq \sigma, \quad \omega \in \Omega_s \quad (41)$$

where $G(\omega)$ is the channel response to be equalized, $H_d(\omega) = e^{-j\omega\tau_d}$, τ_d the desired group delay, Ω_p passband, Ω_s stopband, and σ the \mathcal{H}_∞ robustness constraint. The \mathcal{H}_∞ robustness constraint is motivated by the assumption of bounded but otherwise unknown exogenous input noise in the stopband.

In this example, $G(\omega)$ is a fourth-order elliptic filter with passband ripple 0.5 dB, stopband attenuation greater than 34 dB, passband $\Omega_p = [0, 0.5\pi]$, and stopband $\Omega_s = [0.6\pi, \pi]$. In (40) and (41), the passband Ω_p and stopband Ω_s are discretized using 129 and 102 uniformly spaced points, respectively. The equalizer is a 31-tap FIR filter and the desired group delay is $\tau_d = 13$.

Fig. 12 shows the magnitude of the channel with/without equalization when the stopband robustness constraint is $20 \log \sigma = 0$ dB. Figs. 13 and 14 show the magnitude response of the equalizer and the passband group delay of the equalized channel, respectively, for two cases of robustness constraints: -10 and $+10$ dB. Fig. 14 also shows the group delay of the channel $G(\omega)$.

A comparison with a finite-dimensional linear programming (FLP) technique using discretization similar to [25, pp. 120–129] was performed for this problem. For this purpose, the phase variable θ in (9) was restricted to the set $0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4$. Thus, with this FLP technique, $8 \cdot (129 + 102) = 1848$ linear constraints have to be represented and stored in numerical form. With the SIP approach the infinite constraint set is represented in functional form and only $N + 1 = 32$ constraints have to be considered at one time. The computation times (on a standard PC) for the SIP and the FLP techniques were 2.4 and 8.8 s, respectively ($20 \log \sigma = 0$ dB). The SIP program was written

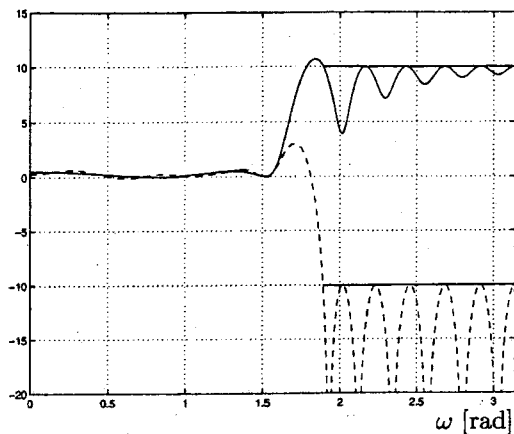


Fig. 13. Magnitude response of equalizer H . Solid line: stopband constraint $+10$ dB. Dashed line: stopband constraint -10 dB.

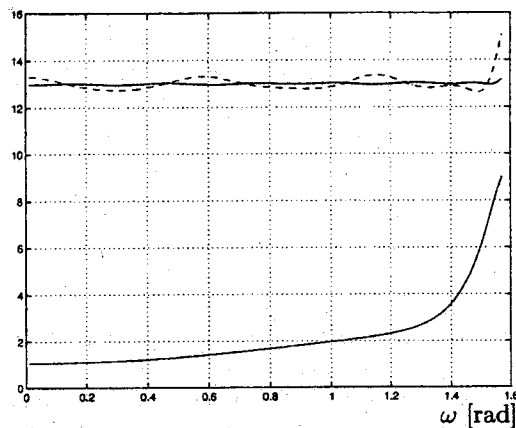


Fig. 14. Passband group delay. Lower solid line: fourth-order elliptic filter G . Upper solid line: total response GH , equalizer stopband constraint $+10$ dB. Dashed line: total response GH , equalizer stopband constraint -10 dB.

using standard Matlab code (not optimized) whereas the FLP solution was based on general purpose linear programming software from the Matlab optimization toolbox.

The resulting frequency responses in this comparison deviate with a maximum of 0.7 dB, cf. [25]. An increased accuracy with the FLP technique requires higher density in the finitization of θ and hence increased numerical complexity. In contrast, the numerical complexity using the SIP approach is independent of required accuracy.

V. SUMMARY AND CONCLUSION

A unified approach to digital filter design with time- and/or frequency-domain specifications has been presented. The corresponding optimization problem is cast as a semi-infinite linear program which can be solved by using numerically efficient simplex extension algorithms.

The semi-infinite linear programming formulation is inherently flexible with respect to design specifications, and enables the use of any filter structure with frequency response represented by a finite-complex basis. The proposed design technique has, thereby, an advantage over existing methods in terms of its ability to conveniently and efficiently handle

magnitude constraints on complex responses and simultaneously include various additional time and/or frequency-domain specifications.

As numerical examples, we have employed the digital Laguerre network in a frequency-domain design with additional group delay specifications, and in a \mathcal{H}_∞ -optimal envelope constrained filter design problem. Finally, a phase equalizer is designed with additional frequency domain \mathcal{H}_∞ robustness constraints.

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