Now, because $f(x, y)$ is constant in each square of $M \times M$ pixels, $f(kM + l, y) = f(kM, y)$ for $l = 0, \dots, M - 1$, and we find that

$$F_x(u; y) = \left(\sum_{l=0}^{M-1} \exp(-j2\pi ul/N) \right) \times \sum_{k=0}^{K-1} f(kM, y) \exp(-j2\pi ukM/N). \quad (3)$$

The factor between brackets in this equation can be explicitly computed and equals

$$\frac{1 - \exp(-j2\pi uM/N)}{1 - \exp(-j2\pi u/N)} = w(u; K)/w(u; N) \quad (4)$$

where $w(u, K) = 1 - \exp(-j2\pi u/K)$ and where we have used the fact that $N/M = K$. The second sum in (3) is the 1-D-DFT of size $N/M = K$ w.r.t. x of the function $\chi_x(a, y) = f(aM, y)$. Let us call this DFT $X_x(u; y)$.

In conclusion, we have

$$F_x(u; y) = \frac{w(u; K)}{w(u; N)} X_x(u; y). \quad (5)$$

Note that this equation also holds for $u = 0$, provided that we interpret $w(0; K)/w(0; N)$ as $\lim_{u \rightarrow 0} w(0; K)/w(0; N)$.

To obtain $F(u, v)$, we must now compute the 1-D-DFT of $F_x(u; y)$ w.r.t. y . This is completely analogous to the previous derivations; therefore, we find that

$$F(u; v) = \frac{w(u; K)}{w(u; N)} \frac{w(v; K)}{w(v; N)} X(u; v). \quad (6)$$

We end with a small note on how to interpret the final equation. In this equation, $X(u, v)$ is evaluated for values $u, v = 0, \dots, N - 1$. First, note that $X(u, v)$ is defined for all of these values:

$$\begin{aligned} X(u; v) &= \sum_{a=0}^{K-1} \sum_{b=0}^{K-1} \chi(a, b) \exp(-j2\pi(ua + vb)/K) \\ &= \sum_{a=0}^{K-1} \sum_{b=0}^{K-1} \chi(a, b) \exp(-j2\pi(u'a + v'b)/K) \end{aligned} \quad (7)$$

where $u' = u \bmod K$ and $v' = v \bmod K$. From this, we see that $X(u, v) = X(u', v')$, which means that we only have to compute $X(u, v)$ for $u, v = 0, \dots, K - 1$.

Therefore, in order to compute $F(u, v)$, we first compute $X(u, v)$ for $u, v = 0, \dots, K - 1$. Next, we extend $X(u, v)$, $u, v = 0, \dots, K - 1$ periodically with period K to obtain $X(u, v)$, $u, v = 0, \dots, N - 1$. We then multiply this function with the (possibly precomputed) function $\frac{w(u; K)}{w(u; N)} \frac{w(v; K)}{w(v; N)}$.

REFERENCES

- [1] M. Anguh, "Quadtree and symmetry in FFT computation of digital images," *IEEE Trans. Signal Processing*, vol. 45, pp. 2896-2899, Dec. 1997.

Application of Infinite Dimensional Linear Programming to IIR Filter Design with Time Domain Constraints

Sven Nordebo and Zhuquan Zang

Abstract—Recently developed infinite dimensional linear programming techniques are used for the design of recursive filters with complex Chebyshev error criteria and time-domain constraints. The approach is applied to a numerical example that deals with the design of a Laguerre network as an equalization filter for a digital transmission channel.

I. INTRODUCTION

In signal processing, many filter design problems can often be cast as a constrained optimization problem where the constraints are defined by the specifications of the filter or the related output signal. These specifications can arise either from practical considerations or from the standards set by certain regulatory bodies (see, e.g., [1]). In this correspondence, we are concerned with the envelope-constrained (EC) filtering problem. Our objective is to design a digital filter to process a given input signal $s(k)$ that is corrupted by additive random noise $n(k)$; see Fig. 1(a). The noiseless output $\psi(k)$ is required to fit into a prescribed pulse shape envelope defined by the lower and upper boundaries $\varepsilon^-(k)$ and $\varepsilon^+(k)$; see Fig. 1(b). Previously [2], the L_2 optimal envelope-constrained filter was defined as the filter that minimizes the output noise power while satisfying the pulse shape constraints. Assuming that the random noise is white with constant power spectrum density, it can be verified that the output noise power is proportional to the squared L_2 norm of the filter to be designed. Hence, the L_2 optimal EC filtering problem can be posed as

$$\min \|H\|^2 \text{ subject to } \varepsilon^-(k) \leq \psi(k) \leq \varepsilon^+(k) \quad (1)$$

where $k = 0, 1, \dots, K - 1$.

In standards, the performance of a digital link is often specified in terms of a mask applied to the received signal [1], [3], [4]. The envelope-constrained filter design problem is directly applicable, and the input signal $s(k)$ would correspond to the test signal specified in the standard.

In this contribution, using the recently developed infinite dimensional linear programming techniques¹ [5], [6], we shall design an orthonormal Laguerre network with frequency response $H(\omega)$ such that its L_∞ norm, which is defined as $\|H\|_\infty = \max_{\omega \in [0, \pi]} |H(\omega)|$, is minimized subject to the same time-domain constraints as specified in (1). However, the proposed design technique is valid for any filter structure described by a finite dimensional basis of complex frequency functions. The Laguerre network is used as an example of a simple recursive filter structure, offering a low-order alternative to FIR filters; see, e.g., [2] and [7].

Manuscript received February 10, 1998; revised December 14, 1998. The associate editor coordinating the review of this paper and approving it for publication was Dr. Hitoshi Kiya.

S. Nordebo is with the Department of Signal Processing, University of Karlskrona/Ronneby, Ronneby, Sweden.

Z. Zang is with the Australian Telecommunications Research Institute (ATRI), Curtin University of Technology, Perth, Australia.

Publisher Item Identifier S 1053-587X(99)04689-9.

¹Our algorithm in MATLAB will be available for noncommercial use by a request on e-mail: sven.nordebo@isb.hk-r.se.

Fig. 1. Envelope-constrained filtering problem: (a) Block diagram. (b) Pulse shape envelope.

Fig. 2. Laguerre filter network.

II. FILTER DESIGN FORMULATION

The frequency response of an N th-order Laguerre filter is given by

$$H(\omega) = \sum_{n=0}^{N-1} \xi_n \phi_n(\omega) = \xi^T \phi(\omega) \quad \omega \in [0, \pi] \quad (2)$$

where ξ is a real $N \times 1$ vector containing the Laguerre coefficients ξ_n , and $\phi(\omega)$ is a complex vector of basis functions

$$\phi_n(\omega) = \sqrt{1-a^2} \frac{(1-a \cdot e^{j\omega})^n}{(e^{j\omega} - a)^{n+1}} \quad (3)$$

where a is an unknown pole parameter $-1 < a < 1$. The corresponding Laguerre filter network is depicted in Fig. 2.

The discrete-time L_∞ optimal EC filter design problem can be formulated as follows.

Given an input signal $s(k)$, find a Laguerre filter $H(\omega)$ that solves the constrained optimization problem

$$\min_H \|H\|_\infty^2 \text{ subject to } \varepsilon^-(k) \leq \psi(k) \leq \varepsilon^+(k) \quad (4)$$

where $\psi(k) = h(k) * s(k)$, $k = 0, 1, \dots, K-1$, $h(k)$ is the impulse response of the filter, and $*$ denotes convolution. Note that compared with the L_2 formulation of the EC filtering problem (1), minimizing the squared L_∞ norm of the filter is equivalent to minimizing the output noise power corresponding to the worst case input noise n (see e.g., [8]). Therefore, in the situation where the power spectrum of the exogenous input noise is bounded but otherwise unknown, the L_∞ formulation of the EC filtering problem is directly applicable.

In order to obtain a nontrivial solution [i.e., $H(\omega) \neq 0$] we assume that there exists at least one k ($0 \leq k \leq K-1$) such that $\varepsilon^-(k)\varepsilon^+(k) > 0$. Furthermore, we assume that $\varepsilon^+(k) > \varepsilon^-(k)$, ($k = 0, 1, \dots$).

We wish to put the L_∞ optimal EC filter design problem (4) in a more general setting, which may also include a desired frequency response of the digital filter designed. For this purpose, we pose the

design criterion

$$\min_{\xi \in R^N} \max_{\omega \in \Omega} v(\omega) |H_d(\omega) - H(\omega)| \quad (5)$$

$$\text{subject to } \varepsilon^- \leq S\xi \leq \varepsilon^+ \quad (6)$$

where

- Ω closed subset of the interval $[0, \pi]$;
- $v(\omega)$ strictly positive weighting function;
- $H_d(\omega)$ desired complex response;
- S constraint matrix;
- ε^- vector of lower bound $\varepsilon^-(k)$;
- ε^+ vector of upper bound $\varepsilon^+(k)$.

The constraint matrix S is defined so that $\psi = S\xi$, where ψ is the vector containing the noiseless output signal $\psi(k)$, $k = 0, 1, \dots, K-1$. Hence, $S_{kn} = s(k) * l_n(k)$, where $l_n(k)$ is the time-domain sequence corresponding to the (frequency-domain) Laguerre basis function $\phi_n(\omega)$. The L_∞ EC design problem (4) corresponds to the case with $\Omega = [0, \pi]$, $v(\omega) = 1$ and $H_d(\omega) = 0$.

According to the *real rotation theorem* [9], a magnitude inequality in the complex plane can be expressed in the equivalent form

$$|z| \leq \delta \Leftrightarrow \Re\{ze^{j\theta}\} \leq \delta \quad \forall \theta \in [0, 2\pi] \quad (7)$$

where

- z complex number;
- δ real and positive number;
- $\Re\{\cdot\}$ real part.

By making use of (7), the nonlinear approximation problem (5), (6) can be reformulated as the continuous semi-infinite linear program

$$\begin{cases} \min \delta \\ v(\omega) \Re\{(H_d(\omega) - H(\omega)) \cdot e^{j\theta}\} \leq \delta \\ P\xi \leq p \end{cases} \quad (8)$$

where δ is an additional real variable, $\omega \in \Omega$, $\theta \in [0, 2\pi]$, $P^T = [S^T | -S^T]$, and $p^T = [\varepsilon^{+T} | -\varepsilon^{-T}]$.

The linear program (8) is called (continuous) semi-infinite since the constraint set is infinite (uncountable) and the number of variables is finite.

Define the $(N+1) \times 1$ vectors y (variable) and b (constant) by

$$y = \begin{pmatrix} \xi \\ \delta \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (9)$$

where 0 is a vector of zeros. Further, let the $N \times 1$ vector function $a(\omega, \theta)$ and the scalar function $c(\omega, \theta)$ be defined by

$$a(\omega, \theta) = v(\omega) \Re\{\phi(\omega) \cdot e^{j\theta}\} \quad (10)$$

$$c(\omega, \theta) = v(\omega) \Re\{H_d(\omega) \cdot e^{j\theta}\}. \quad (11)$$

The linear program (8) is now restated in the form

$$\begin{cases} \min b^T y \\ \begin{bmatrix} a^T(\omega, \theta) & 1 \\ -P & 0 \end{bmatrix} y \geq \begin{bmatrix} c(\omega, \theta) \\ -p \end{bmatrix} \end{cases} \quad (12)$$

where 0 is a $2M \times 1$ vector of zeros.

The filter design problem (5), (6) can now be compactly formulated as the (dual) continuous semi-infinite linear program

$$(D) \begin{cases} \min b^T y \\ A_\alpha^T y \geq c_\alpha \end{cases} \quad (13)$$

where A_α^T and c_α are the rows of the left-hand side constraint matrix and the elements of the right-hand side constraint vector in (12), respectively. Here, α is an index belonging to a closed index set $\mathcal{A} \subset R^4$, which has a one-to-one correspondence with the constraint rows of (12). The set $\mathcal{A}$ can be constructed as follows. Let $\alpha = (\omega, \theta, i, l)$, where $l = 1$ or 2 since there are two types of constraints in (12). If $l = 1$, then $(\omega, \theta) \in \Omega \times [0, 2\pi]$ and $i = 1$. If $l = 2$, then $\omega = 0$, $\theta = 0$, and $i = 1, \dots, 2K$. The formulation (13) is now in a form that has been thoroughly investigated [5]. We refer to (13) as the dual formulation since it corresponds to the dual of a

linear program in standard form [5], [10].

III. SEMI-INFINITE LINEAR PROGRAMMING

The primal linear programming formulation corresponding to (13) is given by

$$(P) \begin{cases} \max \int c_\alpha dx_\alpha \\ \int A_\alpha dx_\alpha = b \\ x_\alpha \geq 0 \end{cases} \quad (14)$$

where the maximization is with respect to the space of regular Borel measures x_α on $\mathcal{A}$ (dual of the space $\mathcal{C}$ of continuous functions c_α on $\mathcal{A}$); cf. [5]. The formulation (14) is said to be in standard form.

By introducing the semi-infinite matrix and vector notations $A = (A_\alpha)$, $c = (c_\alpha)$ and $x = (x_\alpha)$ and with the interpretation of linear functionals on $\mathcal{C}$ as in (14), the primal and dual problem (14) and (13) can be formulated as

$$(P) \begin{cases} \max c^T x \\ Ax = b \\ x \geq 0 \end{cases} \quad (D) \begin{cases} \min b^T y \\ A^T y \geq c. \end{cases} \quad (15)$$

The notation for the dual pair in (15) is now in the same form as for finite-dimensional linear programs [10].

The duality theorem for finite-dimensional linear programs states that if x and y are feasible for (P) and (D), respectively, then $c^T x = b^T y$ iff x is optimal for (P) and y is optimal for (D); cf. [10]. This condition is also known as *strong duality*. Strong duality does not necessarily hold for semi-infinite linear programs. Examples can easily be constructed for which there exists a *duality gap* between the optimal values of the dual programs in (15); cf. [5]. Fortunately, the semi-infinite linear program corresponding to the *uniform approximation problem* (8) satisfies strong duality. This is readily proven by applying [5, th. 4.4c].

The success of the semi-infinite linear programming approach relies on the property of strong duality but on the equally important *dimensionality theorem* ([5, th. 4.8]) as well. This dimensionality theorem states that if there is a feasible solution to (P) in (14) with value z_0 , then there is also a feasible solution of finite support (atomic measure) consisting of at most $N + 1$ points, which achieves the same value. The implication of this theorem is extremely useful since it allows the simplex algorithm to work with finite basic feasible solutions in very much the same way as in the case of finite-dimensional linear programming.

The duality theorem enables us to solve the primal (P) rather than the dual (D) in (15). When the optimum solution to (P) is obtained, we will also be in possession of the optimum solution to (D). The advantage of this procedure is that the primal (P) is in standard form for solution by the simplex algorithm employing a finite basis. The problem (P) is thus much easier to solve than (D).

Given a basis $B = [A_{\alpha_1}, \dots, A_{\alpha_m}]$, where $\alpha_i \in \mathcal{A}$ for $i = 1, \dots, m = N + 1$ and a basic feasible solution $x_B \geq 0$ satisfying $Bx_B = b$. The dimension of B and x_B are $m \times m$ and $m \times 1$, respectively. Let the matrix A be partitioned as $A = [B \ N]$, where $N = (A_\alpha)$ consists of those columns of A , which is not included in B . Define the corresponding partition of the vectors $c^T = [c_B^T \ c_N^T]$ and $x^T = [x_B^T \ x_N^T]$. The primal cost is $c^T x = c_B^T x_B = c_B^T B^{-1} b$.

The extended revised simplex algorithm proceeds as follows; cf. [5], [10].

- 1) Calculate the dual variables by solving $B^T y = c_B$, that is, $y^T = c_B^T B^{-1}$.

The dual cost is $y^T b = c_B^T B^{-1} b = c^T x$. If y is feasible for (D), then the current solution x_B is optimal for (P) according to the duality theorem. Stop.

- 2) Determine a column $A_{\alpha'}$ to enter the basis by choosing an index α' so that $y^T A_{\alpha'} < c_{\alpha'}$ [constraint violation for (D)]. The standard pivoting rule is to choose

$$\alpha' = \max_{\alpha} \{c_\alpha - y^T A_\alpha\}. \quad (16)$$

- 3) Determine a column A_{α_j} to leave the basis by the ratio test

$$j = \arg \min_{i: d_i > 0} \left\{ \frac{x_{B_i}}{d_i} \right\} \quad (17)$$

where the x_{B_i} s are the elements of the basic feasible solution x_B , and the d_i s are the coordinates of the entering column in terms of the old basis. Thus, $Bd = A_{\alpha'}$, where $d = (d_i)$.

- 4) Update the basis matrix B by replacing the column A_{α_j} for $A_{\alpha'}$. $\square$

The representation of the vector b in the new basis is given by

$$B(x_B - t \cdot d) + t \cdot A_{\alpha'} = b \quad (18)$$

where $t \geq 0$ is the minimum ratio in (17) in step 3). The ratio test ensures that the new basic solution is feasible, that is, $x_B - t \cdot d \geq 0$ with equality for at least one component (j). The *degenerate* case is when $t = 0$. Convergence of the algorithm above is guaranteed [5], [10], except for the possibility of cycling of degenerate basic solutions (in which case, $t = 0$). However, cycling is a rarely occurring phenomenon, and there are several well-established remedies; see, e.g., Bland's rule [10].

Suppose that there are no additional constraints (6) to the problem (5). The column vector A_α is then given by $A_\alpha^T = [a^T(\omega, \theta) \ 1]$ and the scalar $c_\alpha = c(\omega, \theta)$. Using (16) in step 2) and the definitions given in (10) and (11), the explicit relations for the variables (ω', θ') corresponding to the column entering the basis are given by

$$\omega' = \arg \max_{\omega} \{v(\omega) | H_d(\omega) - H(\omega)\} \quad (19)$$

$$\theta' = -\arg \{H_d(\omega') - H(\omega')\}. \quad (20)$$

The relations (19) and (20) display the infinite-dimensional nature of the simplex extension algorithm and are easily extended to include additional linear constraints as in (12).

The revised simplex algorithm as outlined above assumes that an initial basic feasible solution to (P) is available. This is phase two in the two-phase simplex algorithm [10]. In phase one, a basic feasible solution to (P) is found by solving the so-called artificial minimization problem

$$\begin{cases} \min \mathbf{1}^T z \\ Ax + z = b \\ x \geq 0 \\ z \geq 0 \end{cases} \quad (21)$$

where z is an $m \times 1$ vector of *artificial* variables, and $\mathbf{1}$ is a vector of ones.

The formulation (21) is in standard form and can be solved by the simplex algorithm. The initial basis is the identity matrix $B = I$, and the initial basic feasible solution is $z = b \geq 0$.

If there is a basic feasible solution to (P), then the optimal solution to (21) has the objective value zero. Phase one is terminated when all artificial variables have left the basis.

IV. A DESIGN EXAMPLE

As a numerical example for the proposed L_∞ EC Laguerre filter design procedure, we consider the design of an equalization filter for a digital transmission channel consisting of a coaxial cable on which data is transmitted according to the DSX-3 standard [1], [3]. The design objective is to find an equalizing filter that takes a sampled impulse response of a coaxial cable with a loss of 30 dB at a

Fig. 3. Input signal s (dash-dotted line), specified output envelope ε (solid line), and resulting noiseless output signal ψ (dashed line).

Fig. 4. Resulting frequency response of equalizer for an L_∞ -design (solid line) and an L_2 -design (dashed line).

normalized frequency of $1/T$ as input and produces an output that lies within the envelope given by the DSX-3 pulse template [1], [3].

Fig. 3 shows the coaxial cable impulse response s , the output envelope ε , and the resulting noiseless output signal ψ . The solution corresponds to an L_∞ -design with $N = 8$ coefficients obtained with the semi-infinite simplex procedure. Fig. 4 shows the resulting frequency response of the equalizer designed when using both L_∞ and L_2 design techniques. Fig. 4 also shows a comparison with the common FIR filter ($a = 0, N = 20$), where the order N has been chosen to match the L_∞ -norm corresponding to the L_∞ -optimal Laguerre filter of order 8.

V. CONCLUSIONS

The envelope-constrained (EC) IIR filter design problem has been formulated as a special case of a general frequency domain L_∞ optimization problem. The optimization problem is cast as a semi-infinite linear program that can be solved by using numerically efficient simplex extension algorithms. An orthonormal Laguerre network is used as an example of a recursive filter offering a low-order alternative to the conventional FIR filter. A numerical example

concerning the equalization of a digital transmission channel is included to demonstrate the efficiency of the design method.

REFERENCES

- [1] CCITT, "Physical/electrical characteristic of hierarchical digital interfaces," G.703, Fascicle III, 1991.
- [2] Z. Zang, B. Vo, A. Cantoni, and K. L. Teo, "Applications of discrete-time Laguerre networks to envelope constrained filter design," in *Proc. ICASSP*, May 1996, vol. 3, pp. 1363–1366.
- [3] Bell Commun., "DSX-3 isolated pulse template and equations," Tech. Ref. TR-TSY-000499, issue 2, pp. 9–17, Dec. 1988.
- [4] J. W. Lechleider, "A new interpolation theorem with application to pulse transmission," *IEEE Trans. Commun.*, vol. 39, pp. 1438–1444, 1991.
- [5] E. J. Anderson and P. Nash, *Linear Programming in Infinite-Dimensional Spaces*. New York: Wiley, 1987.
- [6] D. Burnside and T. W. Parks, "Optimal design of FIR filters with the complex Chebyshev error criteria," *IEEE Trans. Signal Processing*, vol. 43, pp. 605–616, Mar. 1995.
- [7] M. A. Masnadi-Shirazi and N. Ahmed, "Optimum Laguerre networks for a class of discrete-time systems," *IEEE Trans. Signal Processing*, vol. 39, pp. 2104–2108, Sept. 1991.
- [8] Z. Zang, A. Cantoni, and K. L. Teo, "Continuous-time envelope-constrained filter design via Laguerre filters and H_∞ optimization methods," in *Proc. ICASSP*, May 1997, vol. 1, pp. 55–58.
- [9] X. Chen and T. W. Parks, "Design of FIR filters in the complex domain," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 144–153, Feb. 1987.
- [10] D. G. Luenberger, *Linear and Nonlinear Programming*. Reading, MA: Addison-Wesley, 1984.

Comments on "A Weighted Least-Squares Method for the Design of Stable 1-D and 2-D IIR Digital Filters"

Phillip A. Regalia

Abstract—A recent iterative procedure by Lu *et al.* for designing IIR filters is recognized as the Sanathanan–Koerner or Steiglitz–McBride iteration, which was proposed originally in 1963 and 1965, respectively, for system identification purposes. We re-examine some claims and issues related to Lu *et al.*'s paper¹ in view of various properties that have been deduced for the Steiglitz–McBride iteration in the identification literature. In particular, the Steiglitz–McBride procedure is known not to minimize a least-squares distance measure, contrary to Lu *et al.*'s claim. It can, however, provide good models and remain stable during successive iterations under specific conditions which are reviewed within.

I. INTRODUCTION

In a recent paper [1], Lu *et al.* proposed an iterative least-squares procedure for designing IIR filters. In the one-dimensional (1-D) case, the iterative procedure seeks n th-order polynomials $D_k(z)$ and $N_k(z)$, with $D_k(z)$ monic, which minimize

$$J_k = \int_0^\pi \frac{W(\omega)}{|D_{k-1}(e^{j\omega})|^2} |F_d(\omega)D_k(e^{j\omega}) - N_k(e^{j\omega})|^2 d\omega \quad (1)$$

¹W. S. Lu *et al.*, *IEEE Trans. Signal Processing*, vol. 46, pp. 1–10, Jan. 1998.

Manuscript received March 30, 1998; revised June 19, 1998. The associate editor coordinating the review of this paper and approving it for publication was Dr. Ali H. Sayed.

The author is with the Département Signal et Image, Institut National des Télécommunications, Evry, France (e-mail: regalia@int-evry.fr).

Publisher Item Identifier S 1053-587X(99)04658-9.