Nonlinear Standing Waves in a Layer Excited by the Periodic Motion of its Boundary

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Abstract. Simplified nonlinear evolution equations describing nonsteady-state forced vibrations in an acoustic resonator having one closed end and the other end periodically oscillating are derived. An approach is used based on a nonlinear functional equation. This approach is shown to be equivalent to the version of the successive approximation method developed in 1964 by Chester. It is explained how the acoustic field in the cavity is described as a sum of counterpropagating waves with no cross-interaction. The nonlinear Q-factor and the nonlinear frequency response of the resonator are calculated for steady-state oscillations of both inviscid and dissipative media. The general expression for the mean intensity of the acoustic wave in terms of the characteristic value of a Mathieu function is derived. The process of development of a standing wave is described analytically for three different types of periodic motion of the wall: harmonic excitation, sawtooth-shaped motion and ”inverse saw motion”.

INTRODUCTION AND BASIC EQUATIONS

Resonance is known as one of the most interesting phenomena in the physics of vibrations and waves. It manifests itself when the amplitude of a forced oscillation dependence on frequency (i.e. the frequency response) has a sharp maximum. The ratio of the central frequency ω_0 to the characteristic width of its spectral line is called Q-factor and is used as a measure of the ”quality” of a resonant system. This work is devoted to the analysis of the frequency response and Q-factor of a nonlinear acoustic resonator. The treatment is based on Kuznetsov’s equation [1]

$$\frac{\partial^2\Phi}{\partial t^2} - c^2\Delta\Phi = \frac{\partial}{\partial t}\left[(\mathrm{grad}\,\Phi)^2 + \frac{1}{c^2}(\varepsilon-1)\left(\frac{\partial\Phi}{\partial t}\right)^2 + \frac{b}{\rho}\Delta\Phi\right] , \tag{1}$$

written for the potential Φ of the particle velocity $\vec{u} = -\nabla\Phi$. Here c is the sound velocity, ρ is the density of the medium, b is the effective viscosity and ε is the nonlinearity parameter, defined, for example, in ref. [2].

The equation (1) is applied to a one-dimensional system of length L with the boundary conditions formulated for the particle velocity $u = -\frac{\partial\Phi}{\partial x}$:

$$u(x=0,t) = 0 \tag{2}$$

$$u(x=L,t) = Af(\omega t), \tag{3}$$

where A is an amplitude constant and ω is an imposed frequency. For $b = 0$ the solution of Eq. (1) can be written as a sum of two travelling Riemann waves:

$$u = u_1 + u_2 = F_1(\omega t - \kappa x + \frac{\varepsilon}{c^2}\omega x F_1) + F_2(\omega t + \kappa x + \frac{\varepsilon}{c^2}\omega x F_2). \tag{4}$$

On the other hand, for the first-approximation solution of Eq. (1), the velocity is written as the sum of two waves travelling in opposite directions [3]

$$u^{(1)}(x,t) = F_1(\eta_1 = t - \frac{x}{c}) + F_2(\eta_2 = t + \frac{x}{c}), \tag{5}$$

where from (2) we obtain $F_1 = F_2 = F$.

Two appropriate assumptions are made. First, the resonator length must be small in comparison to the shock formation length [2]:

$$L << \frac{c^2}{\varepsilon\omega|F|_{max}}, \tag{6}$$

where $|F|_{max}$ is the maximum amplitude of the function F.

Second, the frequency ω of the vibration of the righthand boundary must differ only slightly from a resonant frequency $n\omega_0$:

$$\omega - n\omega_0 = \frac{\Delta}{\pi}\omega_0,\ \Delta << 1, \tag{7}$$

where Δ is the discrepancy and $\omega_0 = \frac{\pi c}{L}$ is the frequency corresponding to the fundamental eigenmode $n = 1$.

Now the boundary condition (3), applied both to a form of (4), generalized to the case $b \neq 0$, and to (5), after some manipulations gives the "inhomogeneous Burgers equation" with "slow time" T and "fast time" ξ:

$$\frac{\partial U}{\partial T} + \Delta\frac{\partial U}{\partial\xi} - \pi\varepsilon U\frac{\partial U}{\partial\xi} - D\frac{\partial^2 U}{\partial\xi^2} = -\frac{M}{2}f(\xi - \pi) \tag{8}$$

with the notation

$$U = \frac{F}{c},\ M = \frac{A}{c},\ \xi = \omega t + \pi,\ T = \frac{\omega t}{\pi},\ D = \frac{b\omega^2 L}{2c^3\rho} << 1. \tag{9}$$

Thus it is shown that Chester's approach is equivalent to an approach based on a nonlinear functional equation approach using (4).

STEADY-STATE VIBRATIONS

The equilibrium state reached at $T \to \infty$ can be described by the ordinary differential equation, obtained by integration of Eq. (8) at $\frac{\partial U}{\partial T} = 0$, and with $f = \sin\xi$, i.e. harmonic vibration of the boundary:

$$D\frac{dU}{d\xi} + \frac{\pi\varepsilon}{2}(U^2 - C^2) - \Delta U = \frac{M}{2}\cos\xi. \tag{10}$$

From (10) follows that the constant C can be interpreted as the normalized intensity of one of two counterpropagating waves:

$$\overline{U^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} U^2 d\xi = C^2. \tag{11}$$

The mean value of U is assumed to be zero:

$$\overline{U} = \frac{1}{2\pi}\int_{-\pi}^{\pi} U d\xi = 0. \tag{12}$$

For negligible weak linear absorption, $D \to 0$, the solution of the quadratic equation corresponding to Eq. (10) is

$$U = \frac{\Delta}{\pi\varepsilon} \pm \sqrt{(\frac{\Delta}{\pi\varepsilon})^2 + C^2 + \frac{M}{\pi\varepsilon}\cos\xi}. \tag{13}$$

The jump between the branches in (13) gives a compression shock. In order to avoid a rarefaction shock, both branches of the solution (13) must have a common point in each period. This gives

$$C^2 = \frac{M}{\pi\varepsilon} - (\frac{\Delta}{\pi\varepsilon})^2. \tag{14}$$

For given eigenvalue (14) the solution (13) reduces to

$$U = \frac{\Delta}{\pi\varepsilon} \pm \sqrt{\frac{2M}{\pi\varepsilon}}|\cos\frac{\xi}{2}|. \tag{15}$$

By use of (12) the position ξ_{SH} of the jump is calculated:

$$\sin(\frac{\xi_{SH}}{2}) = \frac{\Delta}{2}\sqrt{\frac{\pi}{2\varepsilon M}}. \tag{16}$$

The frequency response is analysed not for the frequency dependence of the amplitude, as is customary for linear vibration, but for the frequency dependence of the root-mean-square (rms) particle velocity $\sqrt{\overline{U^2}}$. In analogy with linear theory we have for the nonlinear Q-factor Q_{NL}, defined in two ways:

$$Q_{NL} = \frac{1}{\Delta} = \frac{\pi}{2\sqrt{2}}\frac{1}{\sqrt{M\pi\varepsilon}}$$

$$Q_{NL} = \frac{c(\sqrt{\overline{U^2}})_{\Delta=0}}{A} = \frac{c}{A}\sqrt{\frac{M}{\pi\varepsilon}} = \frac{1}{\sqrt{M\pi\varepsilon}}, \tag{17}$$

which two results differ slightly.

For $D \neq 0$ we have to study equation (10). Using the transformation

$$U = \frac{2D}{\pi\varepsilon}\frac{d}{d\xi}\ln W, \tag{18}$$

we obtain from (10) for zero discrepancy ($\Delta = 0$) the Mathieu equation in canonical form

$$\frac{d^2W}{dz^2} + [-(\frac{\pi\varepsilon}{D})^2 C^2 - \frac{\pi\varepsilon M}{D^2}\cos 2z]W = 0,\ z = \frac{\xi}{2}. \tag{19}$$

Because of (12) the function W must be periodic and thus W can be written in terms of Mathieu functions [4]:

$$W = ce_0(z, q = \frac{\pi\varepsilon M}{2D^2}). \tag{20}$$

The intensity (11) of the wave (18), (20) is determined by the characteristic value $a_0(q)$ of the Mathieu function ce_0 [4]:

$$\overline{U^2} = C^2 = -(\frac{D}{\pi\varepsilon})^2 a_0(q). \tag{21}$$

Approximate values of $a_0(q)$ for $q << 1$ and $q >> 1$ are respectively

$$a_0 \approx -\frac{q^2}{2} + \frac{7q^4}{128},\ q << 1,\ \ a_0 \approx -2q + 2\sqrt{q} - \frac{1}{4} - \frac{1}{32\sqrt{q}},\ q >> 1. \tag{22}$$

For arbitrary values of dissipation D the Q-factor equals to

$$Q = Q_{NL}\sqrt{-d^2 a_0(q = \frac{1}{2d^2})},\ d = \frac{D}{\sqrt{\pi\varepsilon M}}. \tag{23}$$

For the case $\Delta \neq 0$ the solution to (10) cannot be expressed through Mathieu functions. Approximate solutions to (10) can then be sought for by perturbation theory in the cases $q << 1$ and $q >> 1$.

The approach to the nonlinear equilibrium state in the development of standing waves is described by the evolution equation (8). For some special types of boundary vibration, namely sawtooth-shaped periodic motion, harmonic vibration and inverse saw motion this equation can be solved for the case $\Delta = 0$.

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