Nonlinear Dynamics of Grains in a Liquid-Saturated Soil

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Abstract. A new kind of nonlinearity of inertial type caused by accelerated motion of interacting particles is described. The model deals with an ensemble of grains immersed into a vibrating fluid. First, the nonlinear vibration of two connected grains is studied. The temporal behaviours of displacement and velocity, as well as spectrum of vibration, are analysed. Numerical simulations are performed. Then an infinite chain of grains is considered and the corresponding differential-difference equation is derived. For the continuum limit the inhomogeneous nonlinear wave equation is solved and temporal profiles are calculated. A new resonant phenomenon is described and the resonant curves are constructed.

Keywords: Vibration, wave, acoustics, nonlinear interaction, mathematical model, analytical solution, grainy media, liquid saturated media, grain interaction, grain dynamics.

1. Introduction

Among numerous mathematical models describing nonlinear vibrations and waves of different physical nature, the corresponding models related to geological media and geophysical processes are of particular interest today. For experts in geophysics and mechanics of real inhomogeneous and multi-phase media (like fractured or molten rocks, fluid-saturated grounds or sediments), the nonlinear phenomena offer new possibilities for remote testing of such media, for example, to detect an embedded crack or localize oil-and-gas carrying region by seismic exploration. For experts in nonlinear dynamics, the complicated geophysical systems having unusual properties are sources of definition of new nonlinear problems. The corresponding nonlinear models can often be solved by classical methods developed and approved earlier [1–3].

One of the unusual properties of geological media is their very high nonlinearity. Known models with huge nonlinear response of grainy media are based on nonlinear stress-strain relationships typical for solids containing mesoscopic inhomogeneities or defects in their structure (see, for example, reviews in [4–6]). Such nonlinear behaviour is well defined at significant local deformation caused by high applied load, even if the load does not vary in time. However, there exist a different type of nonlinearity which manifests itself due to inertial forces between grains. Such forces appear in a moving noninertial frame of reference, in particular, if a small system of interacting particles is placed into a vibrating fluid. High spatial gradients of internal forces are determined by the nonuniform distribution of mass. These gradients varying in time can excite the internal degrees of freedom. So, even at harmonic vibration of the fluid caused by sound, the nonlinear internal dynamics can be responsible for generation of higher harmonics. A model of grainy medium is developed here as an example of inertial nonlinearity

caused by accelerated motion of particles [7]. The model deals with an ensemble of grains immersed into a vibrating fluid. The inertial attractive forces have hydrodynamic origin, and the repulsive forces are caused by deformation of colliding grains. Dynamic and stochastic motion is studied, resonances are discovered and nonlinear response is evaluated.

2. Nonlinear Vibration of a Pair of Interacting Grains

Let us consider a body immersed in an ideal liquid. The acoustic wave propagating through the liquid produces an oscillating motion around this obstacle. The stream-line flow initiates an oscillating hydrodynamic force on the body. The characteristic size of the disturbed region of flow is assumed to be small in comparison with the wavelength.

According to the known d'Alembert paradox, the resistance to potential flow must be equal to zero. However, for nonsteady-state flow around objects caused by sound waves this paradox is not true, and the body will move in the oscillating fluid.

The equation of motion of a body having the velocity $\bar{u}$, mass m and density $\rho = m/V$, can be written as (see, for example, [8])

$$m \frac{d\bar{u}}{dt} = m_0 \frac{d\bar{v}}{dt} - \hat{M} \frac{d}{dt}(\bar{u} - \bar{v}) + \bar{F}. \quad (1)$$

Here $\bar{v}$ is the velocity of the fluid, $m_0 = \rho_0 V$ is the mass of fluid displaced by the body, $\bar{F}$ is the external applied force, and $\hat{M}$ is the symmetric tensor of the second rank, so-called 'associated mass' tensor. For the simplest case of spherical body [8]

$$M_{ik} = \frac{2}{3} \pi R^3 \rho_0 \cdot \delta_{ik} = \frac{1}{2} m_0 \cdot \delta_{ik}. \quad (2)$$

If the plane longitudinal wave propagates along the X -axis and $\bar{F} = 0$, the body will execute a motion in the same direction. The velocity of this vibration is connected with the velocity of fluid by the simple relation

$$u(t) = \frac{3m_0}{2m + m_0} v(t). \quad (3)$$

If the densities of both the immersed body and liquid are equal, i.e. $\rho = \rho_0$, the velocities are equal as well: $u(t) = v(t)$. A body denser than the liquid ($\rho > \rho_0$) moves more slowly than the liquid ($|u| < |v|$), and a less dense one ($\rho < \rho_0$) moves faster ($|u| > |v|$).

Two immersed bodies being in contact during part of the period exhibits a strongly non-linear behaviour. For simplicity we consider here two spherical bodies with different densities ρ_1, ρ_2 and volumes V_1, V_2 . The coordinates x_1, x_2 indicating their center of mass satisfy the coupled system of equations following from (1) and (2):

$$(\rho_0 + 2\rho_1) \frac{d^2 x_1}{dt^2} = 3\rho_0 \frac{dv}{dt} - \frac{2}{V_1} [F(x_2 - x_1) - F_0], \quad (4)$$

$$(\rho_0 + 2\rho_2) \frac{d^2 x_2}{dt^2} = 3\rho_0 \frac{dv}{dt} + \frac{2}{V_2} [F(x_2 - x_1) - F_0]. \quad (5)$$

Here F is a repulsive force depending on distance between centers, and F_0 is a constant external holding (clamping) force.

This system has a simple first integral

$$(\rho_0 + 2\rho_1)V_1x_1 + (\rho_0 + 2\rho_2)V_2x_2 = 3\rho_0(V_1 + V_2) \int v \, dt. \quad (6)$$

For the difference $\xi = x_2 - x_1$ one can derive from (4)–(5) the inhomogeneous nonlinear equation

$$\frac{d^2\xi}{dt^2} - a[F(\xi) - F_0] = b \frac{dv}{dt}, \quad (7)$$

where

$$a = 2 \frac{V_1(\rho_0 + 2\rho_1) + V_2(\rho_0 + 2\rho_2)}{V_1V_2(\rho_0 + 2\rho_1)(\rho_0 + 2\rho_2)}, \quad b = 6\rho_0 \frac{\rho_1 - \rho_2}{(\rho_0 + 2\rho_1)(\rho_0 + 2\rho_2)}. \quad (8)$$

According to (8), at equal densities of the bodies, $\rho_1 = \rho_2$, the right-hand side of Equation (7), describing the driving force, equals to zero.

The repulsive force appearing at the contact between two elastic spheres was calculated by H. Hertz in 1882. His results are reproduced in many textbooks (see, for example, [9]). The idea to replace the real sand-like medium by a model of packed spherical grains was used many years ago to describe the properties of seismic waves propagating in geological structures. An exhaustive review of these works is given by White [10]. More recent studies devoted to nonlinear properties of such models are reviewed in [6]. However, interaction of spherical grains immersed in a vibrating fluid, has not been considered earlier.

According to Hertz's theory, the repulsive force between deformed spheres is

$$F = E \sqrt{\frac{R_1R_2}{R_1 + R_2}} [(R_1 + R_2) - (x_1 - x_2)]^{3/2}, \quad (9)$$

where E is the effective Young's modulus which depends on the modules E_1 and E_2 of the materials of the spheres and their Poisson coefficients σ_1 and σ_2 :

$$\frac{1}{E} = \frac{3}{4} \left(\frac{1 - \sigma_1^2}{E_1} + \frac{1 - \sigma_2^2}{E_2} \right). \quad (10)$$

Taking into account the concrete type of force (9), the equations of motion (4) and (5) can be reduced to one nonlinear equation:

$$\frac{d^2h}{dt^2} + \alpha^2 h^{3/2} \Theta(h) - aF_0 = -b \frac{dv}{dt}, \quad (11)$$

where

$$h = R_1 + R_2 - (x_2 - x_1), \quad \alpha^2 = aE \sqrt{\frac{R_1R_2}{R_1 + R_2}}, \quad (12)$$

and Θ is the Heaviside step-function. The set of Equation (11) and the integral (6) give the possibility to calculate the law of motion $x_1(t)$ and $x_2(t)$.

For a harmonic vibration of the fluid caused by a monochromatic acoustic wave, Equation (11) may be expressed in a nondimensional form:

$$\frac{d^2y}{d\tau^2} + \beta^2 [y^{3/2} \Theta(y) - 1] = D \cdot \cos(\tau). \quad (13)$$

Here

$$y = \frac{h}{h_0}, \quad \tau = \omega t, \quad \beta^2 = \frac{\alpha^2}{\omega^2} \sqrt{h_0} \quad \text{and} \quad D = \frac{bv_0}{\omega h_0}.$$

The constants v_0 and ω are the amplitude and the frequency of vibrational velocity of the fluid, and $h = h_0$ is the equilibrium position of the spheres defined by

$$\alpha^2 h_0^{3/2} = a F_0. \quad (14)$$

The equilibrium position in Equation (13) corresponds to $y = 1$. At weak force F_0 the vibration is strongly nonlinear. For small deviations from the equilibrium $y = 1 + y'$, $|y'| \ll 1$, the nonlinear equation (13) can be simplified:

$$\frac{d^2 y'}{d\tau^2} + \frac{3}{2} \beta^2 y' \left[1 + \frac{y'}{4} - \dots \right] = D \cdot \cos(\tau). \quad (15)$$

The natural frequency of linear vibration derived from (13)–(15) equals

$$\omega_0^2 = \frac{3}{2} \sqrt{h_0} \alpha^2 = \frac{3}{2} (\alpha^4 a F_0)^{1/3} = \frac{3}{2} a \left(E^2 F_0 \frac{R_1 R_2}{R_1 + R_2} \right)^{1/3}. \quad (16)$$

Estimation of the order of frequency $f_0 = \omega_0/2\pi$ using (16) is made below. Let the grains' mechanical properties be similar to those of rock, granite, marble or glass. For grains with radius about 1 mm the magnitude of the frequency f_0 is of order $10^5 F_0^{1/6}$, here the force F_0 is measured in Newtons (N). At high amplitudes of displacement the frequency of free vibration equals

$$f = \frac{a^{1/2} F_0^{5/6}}{2 h_{\max} E^{1/3}} \frac{R_1 + R_2}{R_1 R_2}. \quad (17)$$

For the displacement $h = h_{\max}$ comparable with the size of grain, the frequency (17) is two to three orders less than that for small vibrations (16).

It is important to emphasize that Equation (13) contains two kinds of nonlinearities. The first one is connected with the nonlinear dependence of repulsive force on distance between centers of spheres (power law 3/2 in (13)). The second one appears because of difference in repulsive and holding forces (Heaviside function in (13)). Evidently, it is important to understand the role of each kind of nonlinearity by itself. To do this, in parallel with the basic equation (13), a more simple equation will be considered

$$\frac{d^2 y}{d\tau^2} + \beta^2 [y \Theta(y) - 1] = D \cdot \cos(\tau), \quad (18)$$

which takes into account the only nonlinearity of the second kind. For comparison, in Figure 1 the potentials are shown corresponding to both models (13) (curve 1) and (18) (curve 2):

$$\frac{U_1}{\beta^2} = \frac{2}{5} y^{5/2} \Theta(y) - y, \quad \frac{U_2}{\beta^2} = \frac{1}{2} y^2 \Theta(y) - y. \quad (19)$$

The total energy W of vibration is determined by the following relation:

$$W = \frac{1}{2} \left(\frac{dy}{d\tau} \right)^2 + U(y) - U(1). \quad (20)$$

Figure 1. The interaction potential for two grains. Curve 1 corresponds to the model ((13) and curve 2 to the model (18).

So, according to (19) and (20), at zero energy $W = 0$ the vibrating system is at rest and the displacement is $y = 1$.

The period of the free vibrations for the model (13) is given by

$$T(W) = \sqrt{2} \int_{y_1(W)}^{y_2(W)} \frac{dy}{\sqrt{W - \frac{3}{5}\beta^2 + \beta^2 y - \frac{2}{5}\beta^2 y^{5/2} \Theta(y)}}. \quad (21)$$

In (21) the integration limits $y_{1,2}(W)$ are determined as roots of the following equation:

$$W - \frac{3}{5}\beta^2 + \beta^2 y - \frac{2}{5}\beta^2 y^{5/2} \Theta(y) = 0. \quad (22)$$

The corresponding expression for the model (18) is calculated analytically. The period equals

$$T(W) = \frac{2}{\beta} \left(\sqrt{\frac{2W}{\beta^2} - 1} + \frac{\pi}{2} + \arcsin \sqrt{\frac{\beta^2}{2W}} \right), \quad W > \frac{\beta^2}{2}. \quad (23)$$

At small energies $W < \beta^2/2$ the period does not depend on energy and $T(W) = 2\pi/\beta$. At large energies the period equals

$$T(W) = \frac{2\sqrt{2}}{\beta^2} \sqrt{W}. \quad (24)$$

Its duration is determined mainly by the slow motion of the particle entrained by hydrodynamic force in the time interval between two fast collisions. It is possible to derive the analytical solution for the normalized distance $y(\tau)$ between two spheres. Let the initial conditions be

$$\frac{dy(0)}{d\tau} = 0, \quad y(0) = y_1(W) = \frac{1}{2} - \frac{W}{\beta^2}. \quad (25)$$

It means that the system is at rest at $\tau = 0$ and the initial displacement is negative and has the maximum absolute value. It follows from Equation (18) that displacement-on-time dependence for negative displacement has the parabolic form:

$$y(\tau) = \frac{1}{2} - \frac{W}{\beta^2} + \frac{1}{2}\beta^2 \tau^2, \quad -\sqrt{\frac{2W}{\beta^2} - 1} < \beta\tau < \sqrt{\frac{2W}{\beta^2} - 1}. \quad (26)$$

Figure 2. Nonlinear vibration time signal described by the solution (26)–(28) at different values of energy $2W/\beta^2 = 0.8, 2, 5, 10$ and 17 .

Positive values of $y(\tau)$ can be expressed through the trigonometric functions:

$$y(\tau) = 1 - \cos\left(\beta\tau - \sqrt{\frac{2W}{\beta^2} - 1}\right) + \sqrt{\frac{2W}{\beta^2} - 1} \sin\left(\beta\tau - \sqrt{\frac{2W}{\beta^2} - 1}\right) \quad (27)$$

inside the variation interval

$$\sqrt{\frac{2W}{\beta^2} - 1} < \beta\tau < \pi + \sqrt{\frac{2W}{\beta^2} - 1} + 2 \arcsin \sqrt{\frac{\beta^2}{2W}}. \quad (28)$$

One can see that solutions (26) and (27) are sewed together and both the function $y(\tau)$ and its first derivative are continuous at

$$\beta\tau = \pm \sqrt{\frac{2W}{\beta^2} - 1}.$$

The form of nonlinear vibration (26)–(28) is shown in Figure 2 for different values of normalized energy $2W/\beta^2$ equal to $0.8, 2, 5, 10$ and 17 which are indicated at each curve.

It is possible also to derive the analytical expression for the Fourier spectrum of this nonlinear vibration. It is done below for the velocity of vibration which is an odd function. Its definition over one-half of period in accordance with (26)–(28) is

$$\frac{1}{\beta} \frac{dy}{d\tau} = \beta\tau, \quad 0 < \beta\tau < \sqrt{\frac{2W}{\beta^2} - 1}, \quad (29)$$

$$\begin{aligned} \frac{1}{\beta} \frac{dy}{d\tau} &= \sin\left(\beta\tau - \sqrt{\frac{2W}{\beta^2} - 1}\right) + \sqrt{\frac{2W}{\beta^2} - 1} \cos\left(\beta\tau - \sqrt{\frac{2W}{\beta^2} - 1}\right), \\ &\sqrt{\frac{2W}{\beta^2} - 1} < \beta\tau < \frac{\pi}{2} + \sqrt{\frac{2W}{\beta^2} - 1} + \arcsin \sqrt{\frac{\beta^2}{2W}}. \end{aligned} \quad (30)$$

The Fourier series of the velocity (29)–(30) at energies $W > \beta^2/2$ contains an infinite number of harmonics

$$\frac{1}{\beta} \frac{dy}{d\tau} = \sum_{n=1}^{\infty} b_n \sin(n\beta\tau), \quad (31)$$

and their amplitudes are given by formula

$$b_n = \frac{\beta T}{\pi^2 n^2} \frac{\sin(nQ) - (nQ) \cos(nQ)}{1 - (\frac{2\pi n}{\beta T})^2}, \quad Q = \frac{2\pi}{\beta T} \sqrt{\frac{2W}{\beta^2} - 1}. \quad (32)$$

The period $T(W)$ (23) of vibration increases with increase of energy W , and magnitudes of all frequencies (of fundamental and higher ones) move to low-frequency region. At the same time, amplitudes of harmonics increase with increase in energy W .

At $W \rightarrow \beta^2/2$ the period $T(W)$ tends to $2\pi/\beta$, and an undeterminate of 0/0 type appears in formula (32) for the fundamental harmonic $n = 1$. One can resolve it and obtain $b_1 = 1$. Evidently, at lower energies $W \leq \beta^2/2$ only the fundamental harmonic is nonzero: $b_1 = 1$, and $b_n = 0$ ($n = 2, 3, \dots$); consequently, at such energies the vibration described by (18) is linear.

Several interesting phenomena were observed at the computer simulation of the system described by the nonlinear dynamic model (13). In Figure 3 the position and velocity are shown as well as the spectral distribution of the velocity. Parameter values of $\beta = 0.35$ and $D = 2$ were used for the computer simulation. In Figure 4 analogous results are shown for parameters $\beta = 1$ and $D = 2$. One can see that the usual nonlinear process of generation of higher harmonics takes place. However, for some magnitudes of parameters, a more interesting effect of appearance of low-frequency spectra is well pronounced, particularly in Figure 4. These low frequencies do not depend on the driving frequency of the acoustic wave. The efficient transformation of energy to low frequencies is connected with the high amplitude of negative displacement (Figure 2) and the long time spent by particles in free motion between two subsequent collisions. In reality, after the short-time collisions each particle moves slowly away from its neighbour and then it moves back. In the moment of collision the velocities of colliding particles are unpredictable if the amplitude is large enough. This random interaction between colliding particles leads to the appearance of stochastic oscillations with a very broad spectrum. Such spectral broadening for a harmonic input signal during its propagation through grainy media is a well-known phenomenon.

3. Waves in a Distributed Nonlinear Model of Fluid-Saturated Grainy Medium

Now we go from a localized vibrating system to a more general distributed one-dimensional nonlinear model which includes a spatial coordinate. The considered structure has periodically located nonlinear elements which form an infinite discrete chain. Each element of this chain is an aggregate of two spherical particles, like those studied before. However, now both densities and radii of the particles will be considered as approximate. The Lagrange function for this chain is

$$L = \sum_n \left\{ \frac{m}{2} \left(\frac{d\xi_n}{dt} \right)^2 - \frac{2}{5} E \sqrt{\frac{R}{2}} [\Delta - (\xi_{n+1} - \xi_n)]^{5/2} + \frac{3}{2} m_0 \xi_n \frac{dv_n}{dt} \right\}. \quad (33)$$

Figure 3. The position x , the velocity u , and the frequency content of the velocity u for the parameter values $\beta = 0.35$ and $D = 2$.

Here R is radius of particle, $m = \rho V + m_0/2$ is mass of sphere with account for the associated mass $m_0 = \rho_0 V$ of the liquid,

$$\Delta = \left(\frac{2F_0^2}{RE^2} \right)^{1/3} \quad (34)$$

is static compression caused by a constant force F_0 , ξ_n is displacement of particle from equilibrium position, v_n is velocity of liquid flowing around n -th particle.

The equation of motion for l -th particle of the chain according to (33) has the following form:

$$m \frac{d^2 \xi_l}{dt^2} = -E \sqrt{\frac{R}{2}} [(\Delta - \xi_{l+1} + \xi_l)^{3/2} - (\Delta - \xi_l + \xi_{l-1})^{3/2}] + \frac{3}{2} m_0 \xi_n \frac{dv_l}{dt}. \quad (35)$$

Figure 4. The position x , the velocity u , and the frequency content of the velocity u for the parameter values $\beta = 1$ and $D = 2$.

Only small displacements will be considered with account for linear and quadratic nonlinear terms. Expanding the right-hand side of (35) in power series, one can get the equation

$$m \frac{d^2 \xi_l}{dt^2} = -\frac{3}{2} \left(\frac{R}{2} E^2 F_0 \right)^{1/3} \left\{ 2\xi_l - \xi_{l+1} - \xi_{l-1} + \frac{1}{4\Delta} [(\xi_{l+1} - \xi_l)^2 - (\xi_l - \xi_{l-1})^2] \right\} + \frac{3}{2} m_0 \frac{dv_l}{dt}. \quad (36)$$

One can pass now to the continuum putting

$$\xi_l = \xi(x), \quad \xi_{l \pm m} = \xi(x \pm mL), \quad (37)$$

where L is the spatial period of the chain. Substituting (37) into (36) and keeping the remaining main terms in L -power series expansions of both linear and nonlinear parts, reduces (33) to an inhomogeneous nonlinear wave equation

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \left(1 - 2\epsilon \frac{\partial \xi}{\partial x} \right) \frac{\partial^2 \xi}{\partial x^2} + \frac{3m_0}{2m} \frac{\partial v}{\partial t}. \quad (38)$$

The velocity c of wave propagation and the nonlinear parameter ϵ in Equation (38) are given by

$$c = L \left(\frac{3}{2m} \right)^{1/2} \left(\frac{R}{2} E^2 F_0 \right)^{1/6}, \quad \epsilon = \frac{L}{4} \left(\frac{R E^2}{2 F_0^2} \right)^{1/3}. \quad (39)$$

At clamping force F_0 tending to zero the velocity c also tends to zero but the nonlinear parameter ϵ tends to infinity. So, at small F_0 we deal with a slow but strongly nonlinear wave.

Let us now pass to one connected problem of special interest for geophysics and nonlinear wave theory. Let the vibration of surrounding liquid be caused by a propagating acoustic wave. In this case the velocity in Equation (38) varies according to the law $v = v(t - x/c_0)$, where c_0 is the velocity of sound. One can control the velocity c (39) of wave propagation through the chain of grains by varying the clamping force F_0 . By increasing F_0 , one can increase c to be approximately equal to c_0 . If the condition $c \approx c_0$ is fulfilled the acoustic wave will excite a wave in the chain of grains most efficiently (this is a so-called wave resonance phenomenon [11]). Because only one of the two counter-propagating waves described by Equation (38) will be excited efficiently (namely, the wave propagating along the x -axis to the right), the solution has to be sought for in form

$$\xi = \xi(\zeta = t - x/c_0, x_1 = \mu x). \quad (40)$$

Here μ is a small parameter having the order of small acoustic Mach number and small discrepancy in wave propagation velocities:

$$\mu \sim \frac{1}{c} \left| \frac{\partial \xi}{\partial t} \right|, \quad \mu \sim \frac{|c_0 - c|}{c_0}. \quad (41)$$

After substitution of (40) into Equation (38) and change of variables we eliminate all terms having the order of smallness μ^2 and of higher orders. So, all terms kept in the resulting equation

$$\frac{\partial u}{\partial x} + \frac{c_0 - c}{c_0^2} \frac{\partial u}{\partial \zeta} - \frac{\epsilon}{c_0^2} u \frac{\partial u}{\partial \zeta} = \frac{3m_0}{4mc_0} \frac{dv}{d\zeta}, \quad u \equiv \frac{\partial \xi}{\partial \zeta} = \frac{\partial \xi}{\partial t} \quad (42)$$

are of the same order of μ . The method used above for derivation of simplified Equation (42) is one of modifications of multiple scale method described in detail by Nayfeh [12, 13].

Equation (42) will be used to calculate the resonant curve, i.e. the dependence of magnitude of wave in grainy medium on discrepancy between c and c_0 . To simplify all expressions in this calculation, Equation (42) is written in dimensionless form:

$$\frac{\partial U}{\partial X} + \delta \frac{\partial U}{\partial \theta} - U \frac{\partial U}{\partial \theta} = \sin \theta. \quad (43)$$

Here

$$X = \left(\frac{4}{3\epsilon} \frac{m}{m_0} \frac{c_0^3}{\omega^2 v_0} \right)^{-1/2} x, \quad U = \left(\frac{3}{4} \frac{m_0}{m\epsilon} c_0 v_0 \right)^{-1/2} u, \quad (44)$$

$$\delta = \left(\frac{4}{3\epsilon} \frac{m}{m_0} \frac{c_0}{v_0} \right)^{1/2} \frac{c_0 - c}{c_0}, \quad \theta = \omega \zeta. \quad (45)$$

The hydrodynamic velocity in (42) is assumed in form of a harmonic wave: $v = -v_0 \cos \theta$.

Equation (42) is known as ‘Inhomogeneous Riemann’s wave equation with discrepancy’. Adding a viscous term proportional to the second derivative transforms (43) to the inhomogeneous Burgers equation. It was studied in connection with the problems of nonlinear hyper-sound excitation at stimulated Raman scattering [14] and acoustic wave excitation by a moving laser beam [15]. Results given below are related to the steady-state solution reached at large distances, $X \rightarrow \infty$, $\partial U / \partial X \rightarrow 0$. The integral of (43) is simple:

$$U^2 - 2\delta U - C^2 - 2\cos \theta = 0. \quad (46)$$

However, one has to solve the quadratic equation (46) with two additional conditions

$$\overline{U} = \frac{1}{2\pi} \int_{-\pi}^{\pi} U \, d\theta = 0, \quad \overline{U^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} U^2 \, d\theta = C^2. \quad (47)$$

The first condition means that there is no constant flow in the wave propagating through the grainy medium, and the second one identifies the constant C^2 with the averaged (over the period) intensity $\overline{U^2}$ of the wave.

One can see that at large discrepancies, $|\delta| > 4/\pi$, only one of two branches of the solution (46) can be realized, namely

$$U = \delta - \sqrt{\delta^2 + C^2 + 2\cos \theta}. \quad (48)$$

The first condition (47) applied to (48) gives the connection between discrepancy δ and intensity $\overline{U^2}$:

$$\frac{\pi}{2} \frac{\delta}{\sqrt{\delta^2 + \overline{U^2} + 2}} = E \left(\frac{4}{\delta^2 + \overline{U^2} + 2} \right). \quad (49)$$

Here E is the complete elliptic integral of the second kind [16]. Solution (49) is the necessary connection between $\overline{U^2}$ and δ at large discrepancies when the nonlinear distortion of wave profile is weak. At $|\delta| = 4/\pi$ a bifurcation occurs, and at $|\delta| < 4/\pi$ the wave profile is constructed of two branches of the solution of Equation (46), namely

$$U = \delta \pm \sqrt{\delta^2 + C^2 + 2\cos \theta}. \quad (50)$$

Because only a shock wave of compression can exist in quadratic-nonlinear and hardening ($\epsilon > 0$) media and a rarefaction shock is prohibited (it is unstable and disappears through unlimited broadening), the two branches in (50) must have one common point. It is possible if

$$C^2 = \overline{U^2} = 2 - \delta^2. \quad (51)$$

Using solutions (49) and (51), one can describe the behavior of wave magnitude $\sqrt{\overline{U^2}}$ defined as the square root of the averaged intensity. This behavior is shown in Figure 5. Evidently, the accepted definition of resonant curve as $\sqrt{\overline{U^2}} = \sqrt{\overline{U^2}}(\delta)$ differs from the usual definition for

Figure 5. Resonant curves dependent on the discrepancy of maximum (peak) magnitude, and of the square root of the mean intensity.

a linear wave $A = A(\delta)$ (A is amplitude) because a nonlinear wave contains many harmonics and amplitude-on-discrepancy dependence is meaningless.

One can define a nonlinear resonant curve as dependence of maximum (or peak) magnitude of disturbance $U_{\max}$ on δ . This curve is also shown in Figure 5. Its behavior differs from that for $\sqrt{U^2}$ in that the maximum of response is reached not at $\delta = 0$, but at certain positive δ . The analytical expression for the second curve is

$$\begin{aligned}
 U_{\max} &= -|\delta| + \sqrt{\delta^2 + \overline{U^2} + 2}, & -\infty < \delta < -4/\pi, \\
 U_{\max} &= -|\delta| + 2, & -4/\pi < \delta < 0, \\
 U_{\max} &= \delta + 2\sqrt{1 - \left(\frac{\pi}{4}\right)^2}, & 0 < \delta < 4/\pi, \\
 U_{\max} &= \delta - \sqrt{\delta^2 + \overline{U^2} - 2}, & 4/\pi < \delta < \infty.
 \end{aligned} \tag{52}$$

To derive expression (52), it is necessary to consider the temporal profile of the wave. In accordance with (46), (47), at $|\delta| > 4/\pi$ it is given by (48) where $C^2 = \overline{U^2}$ is calculated from Equation (49), and

$$U(\theta) = \delta \pm 2 \left| \cos \frac{\theta}{2} \right|, \quad -\pi \leq \theta \leq \pi, \quad |\delta| < \frac{4}{\pi}. \tag{53}$$

The jump from one branch of solution (53) to the other branch (from lower to higher magnitude of U) takes place at $\theta = \theta_{sh}$, where the position of shock in the wave profile is determined by the formula

$$\sin \frac{\theta_{sh}}{2} = \frac{\pi}{4} \delta. \tag{54}$$

It follows from the first condition (47) applied to the solution (53).

Profiles of one period of stationary wave $U(\theta)$ are shown in Figure 6 for different positive discrepancies $\delta = 0, 2/\pi, 4/\pi, 2.5$. One can see, the dependence of $U_{\max}$ on δ is a nonmonotonic one, as shown in Figure 5.

Figure 6. Profiles of stationary forced nonlinear waves at different discrepancies.

The wave resonance phenomenon [11] described here for a nonlinear problem can be of interest for geophysical applications. Let a wave propagate in water along a horizontal bottom surface. The sub-bottom layers consist of water-saturated sand or other grains moved around by a vibrating liquid. The pressure pressing the grains together increases with depth into the sediment. Therefore, at a specific depth the clamping force – the pressure increasing with depth – will lead to the equality of velocity of wave propagation (39) and the speed of sound at that depth. Consequently, wave resonance occurs at this depth, and waves in the sediment will be excited with high efficiency only in this layer.

4. Conclusion

Phenomena similar to the ones discussed above were observed in geophysics. Huge nonlinear parameters of grainy media were measured many times (see [17, 18] in addition to [4–6] cited earlier). The existence of resonant frequencies of vibration of grains was discovered more than 20 years ago (see, for example, review [19]). It was pointed out, that the resonant frequency depends on the content of water in space between grains, but resonant behaviour by itself was not explained. This paper is devoted to nonlinear phenomena, and details related to geophysical applications are out of this consideration. From the viewpoint of nonlinear vibrations and waves, new results are connected with the description of new type of nonlinearity which manifests itself at streamlining of grains by vibrating liquid. The temporal profiles and spectra of this nonlinear vibration are studied. The possibility of resonant excitation of wave in liquid-saturated grainy medium is studied using nonlinear inhomogeneous equations. Resonant curves and temporal profiles of travelling waves are calculated.

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