Fourier decomposition of a plane nonlinear sound wave and transition from Fubini's to Fay's solution of Burgers' equation

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Abstract. Burgers' equation describes plane sound wave propagation through a thermoviscous fluid. If the boundary condition at the sound source is given as a pure sine wave, the exact solution is given by the Cole-Hopf transformation as a quotient between two Fourier series. Two approximate Fourier series representations of this solution are known: Fubini's (1935) solution, neglecting dissipation and valid at short distance from the sound source, and Fay's (1931) solution, valid far from the source. In the present investigation a linear system of equations is found, from which the coefficients in a series expansion of each Fourier coefficient can be derived one by one. The Fourier coefficients turn out to be power series in $\exp(-es)$, where e is a dimensionless measure of dissipation and s is a dimensionless measure of distance from the boundary. Curves of the Fourier coefficients as functions of s are given for $s > 0.9$. They join smoothly to Fubini's solution (valid for $s < 1$ and corrected for dissipation) and to Fay's solution (valid for $s \gg 1$). Maxima for the Fourier coefficients of the higher harmonics as functions of s are given. These maxima are situated in a region where neither Fubini's nor Fay's solution is valid.

INTRODUCTION

The aim of this work is to use the exact solution of the problem of finding the Fourier coefficients of an original sine wave to connect the wellknown Fubini and Fay solutions of Burgers' equation. It will be shown how this exact solution is found and, by curves, how it joins to the abovementioned approximate solutions. It will also be shown how the maxima of the fourier coefficients of the exact solution move from the source with increasing harmonic number and increasing viscosity.

FOURIER DECOMPOSITION OF THE WAVE FROM A SINUSOIDAL SOURCE

Plane nonlinear sound waves are described by Burgers' equation

$$\frac{\partial v}{\partial x} - \frac{\beta}{c_0^2} v \frac{\partial v}{\partial \tau} - \frac{b}{2c_0^3 \rho_0} \frac{\partial^2 v}{\partial \tau^2} = 0, \tag{1}$$

where v is the fluid velocity, x is the distance from the source, c_0 is the sound velocity in the undisturbed fluid, $\tau = t - \frac{x}{c_0}$ is retarded time, b is viscosity, the number β is defined as $\beta = \frac{\gamma+1}{2}$, where $\gamma = \frac{c_p}{c_v}$, i.e. the ratio between the heat capacities at constant pressure and volume respectively, and ρ_0 is the density of the undisturbed fluid.

By the transformations

$$V = \frac{v}{v_0},\ \theta = \omega\tau,\ \sigma = \frac{\beta\omega v_0 x}{c_0^2},\ \epsilon = \frac{b\omega}{\beta c_0 v_0 \rho_0} \tag{2}$$

equation (1) becomes dimensionless:

$$\frac{\partial V}{\partial \sigma} - V\frac{\partial V}{\partial \theta} = \epsilon \frac{\partial^2 V}{\partial \theta^2}. \tag{3}$$

With the boundary condition at $\sigma = 0$

$$V(0, \theta) = \sin\theta \tag{4}$$

the exact solution of equation (1) is [1]

$$V(\sigma, \theta) = -4\epsilon \frac{\sum_{k=1}^{\infty} I_k(\frac{1}{2\epsilon}) \exp(-k^2\epsilon\sigma)(-1)^k k \sin k\theta}{I_0(\frac{1}{2\epsilon}) + 2\sum_{k=1}^{\infty} I_k(\frac{1}{2\epsilon}) \exp(-k^2\epsilon\sigma)(-1)^k \cos k\theta}, \tag{5}$$

where I_n is the modified Bessel function.

We want to replace (5) by a single Fourier series

$$V(\sigma, \theta) = \sum_{n=1}^{\infty} b_n \sin n\theta. \tag{6}$$

Two approximate Fourier series of the type (6) are the corrected Fubini solution [2]

$$V(\sigma, \theta) = 2\sum_{m=1}^{\infty} \{ \frac{J_m(m\sigma)}{m\sigma} - \frac{\epsilon}{m} \sum_{r=1}^{\infty} r a_r [J_{m-r}(m\sigma) + J_{m+r}(m\sigma)] \} \sin m\theta, \tag{7}$$

where J_m are the Bessel functions and

$$a_r = r(1-\sigma)^{-\frac{1}{2}} \sigma^{-r} (1 - (1-\sigma^2)^{\frac{1}{2}})^r, \tag{8}$$

valid for $\sigma < 1$, and Fay's solution [3]

$$V(\sigma, \theta) = 2\epsilon \sum_{n=1}^{\infty} \frac{\sin n\theta}{\sinh n\epsilon\sigma}, \tag{9}$$

valid for $\sigma >> 1$.

An exact representation of the Fourier coefficients $b_m(\epsilon, \sigma)$ is given by the series

$$b_m = 4\epsilon \sum_{k=1}^{\infty} d_{mk} \exp(-k\sigma), \tag{10}$$

where the coefficients d_{mk} are determined by the recursion formula

$$\begin{aligned} I_0(\frac{1}{2\epsilon})d_{rs} &= I_1(d_{r-1,s-1} + d_{r+1,s-1}) - I_2(d_{r-2,s-4} + d_{r+2,s-4}) + \dots \\ &+(-1)^{n+1} I_n(d_{r-n,s-n^2} + d_{r+n,s-n^2}) + \dots \end{aligned} \tag{11}$$

In (11) we use the rules

$$d_{00} = 1,\ d_{0k} = d_{k0} = 0,\ k \neq 0,\ d_{-r,s} = -d_{r,s}. \tag{12}$$

The coefficients d_{mr} fulfil the relations

$$d_{mr} = 0,\ m > r \tag{13}$$

$$d_{mr} = 0,\ m + r = 2n + 1,\ n = 0, 1, 2... \tag{14}$$

The righthand side of equation (11) terminates with $(-1)^{r+1} r I_r$ if $s = r^2$. Otherwise it is broken for $n^2 > s$. In Fig. 1 the Fourier coefficients b_1,b_2, b_3, b_4 for $\epsilon = 0.05$ are shown as functions of σ for the corrected Fubini solution, Fay's solution and the present exact solution. They are compared with with the Fourier coefficients of the nondissipative solution ($\epsilon = 0$) [4]. It is also found that the maxima of $b_m(\epsilon, \sigma)$, $m = 2, 3, 4$, are displaced to the right for growing ϵ and growing m. This is shown in through the folowing numbers where the position of the maxima of higher harmonics increase with the number of harmonic and with the dissipation ϵ.

For $\epsilon = 0$ we have maxima of- $b_2 = 0.3609$ at $\sigma = 1.18$; $b_3 = 0.2268$ at $\sigma = 1.34$; and $b_4 = 0.1660$ at $\sigma = 1.41$

For $\epsilon = 0.05$ we have maxima of - $b_2 = 0.3355$ at $\sigma = 1.24$; $b_3 = 0.2043$ at $\sigma = 1.43$; and $b_4 = 0.1438$ at $\sigma = 1.53$

For $\epsilon = 0.1$ we have maxima of - $b_2 = 0.2957$ at $\sigma = 1.30$ $b_3 = 0.1639$ at $\sigma = 1.53$ $b_4 = 0.1032$ at $\sigma = 1.63$

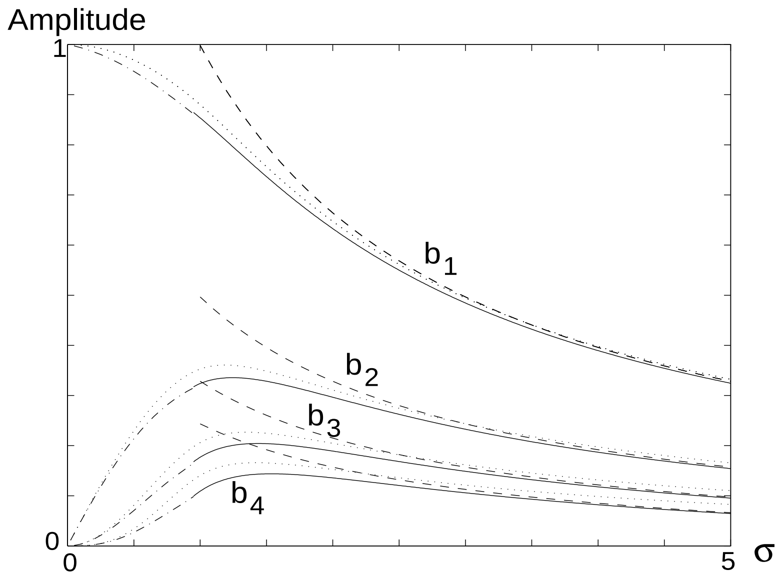

FIGURE 1. The first four Fourier coefficients $b_1 - b_n$ in the present solution - solid line, the corresponding coefficients in the Fay solution - dashed, and in the corrected Bessel-Fubini - dash-dotted; all for $\epsilon = 0.05$. The non-dissipative solution ($\epsilon = 0$) - dotted.

REFERENCES

1. J.D. Cole, *Quart. Appl. Math.* **9**, 225-236, (1951).
2. D.T. Blackstock, *J. Acoust. Soc. Am.* **36**, 534-542, (1964).
3. R. D. Fay, *J. Acoust. Soc. Am.* **3**, 222-241, (1931).
4. D.T. Blackstock, *J. Acoust. Soc. Am.* **39**, 411-413, (1966).