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Minimum Norm Design of Two-Dimensional Weighted Chebyshev FIR Filters

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Abstract—The weighted Chebyshev design of two-dimensional FIR filters is in general not unique since the Haar condition is not generally satisfied. However, for a design on a discrete frequency domain, the Haar condition might be fulfilled. The question of uniqueness is, however, rather extensive to investigate. It is therefore desirable to define some simple additional constraints to the Chebyshev design in order to obtain a unique solution. The weighted Chebyshev solution of minimum Euclidean filter weight norm is always unique, and represents a sensible additional constraint since it implies minimum white noise amplification. This unique Chebyshev solution can always be obtained by using an efficient quadratic programming formulation with a strictly convex objective function and linear constraints. An example where a conventional Chebyshev solution is nonunique is discussed in the brief.

I. INTRODUCTION

A number of methods have been developed for the solution of the two-dimensional (2-D) weighted Chebyshev FIR filter design problem [1]–[4]. The problem is, however, generally nonunique since the Haar condition [1], [2] may not hold due to the 2-D nature of the approximating functions.

In case of nonuniqueness, some additional constraints must be defined in order to make the Chebyshev design unique. The concept of strict approximation [1], [5] was defined for this purpose. Basically, the strict approximation is based on the requirement that the maximum deviation should be as small as possible when the Chebyshev deviation points are removed.

The process of strict approximation is, however, quite complicated and may require several repetitions. Furthermore, the question of uniqueness of a given Chebyshev solution is rather extensive to investigate.

A simple and applicable choice of additional constraint is the requirement of minimum Euclidean filter weight norm. This requirement leads to a unique weighted Chebyshev solution which can be obtained by using a well-conditioned quadratic programming formulation. With this design approach, the extensive uniqueness-investigation is avoided. Further, the minimum norm requirement implies minimum white noise amplification, less sensitivity to coefficient quantization errors, and less sensitivity to model imperfections in applications such as 2-D beamformer design [6], [7].

II. THE WEIGHTED CHEBYSHEV DESIGN PROBLEM

The frequency response of a 2-D FIR filter with real impulse response $h(n_1, n_2)$ can be written as

$$H(\omega_1, \omega_2) = \mathbf{h}^T \mathbf{d} = \sum_{n_1} \sum_{n_2} h(n_1, n_2) e^{-j\omega_1 n_1 - j\omega_2 n_2} \quad (1)$$

where $\mathbf{h}$ is a vector of FIR filter coefficients $h(n_1, n_2)$ and $\mathbf{d}$ is the response vector of corresponding complex exponentials

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$e^{-j\omega_1 n_1 - j\omega_2 n_2}$. Let $H_d(\omega_1, \omega_2)$ denote the specified desired response of the FIR filter frequency function.

The discussion will be restricted to zero phase FIR filters with four-fold symmetry [4] given by

$$h(n_1, n_2) = h(-n_1, n_2) = h(n_1, -n_2). \quad (2)$$

This restriction is merely for convenience, and does not imply any loss in generality. All results below will be valid for any symmetry condition implying zero or linear phase frequency response.

With four-fold symmetry, the frequency response is given by

$$H(\omega_1, \omega_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} a(n_1, n_2) \cos(\omega_1 n_1) \cos(\omega_2 n_2) \quad (3)$$

where $a(0, 0) = h(0, 0)$, $a(n_1, 0) = 2h(n_1, 0)$, $a(0, n_2) = 2h(0, n_2)$ and $a(n_1, n_2) = 4h(n_1, n_2)$ for $n_1 \geq 1$ and $n_2 \geq 1$. With $\varphi_1(\omega_1) = [1 \cos(\omega_1) \cdots \cos(\omega_1(N_1 - 1))]^T$ and $\varphi_2(\omega_2) = [1 \cos(\omega_2) \cdots \cos(\omega_2(N_2 - 1))]^T$, the FIR filter frequency response can be written as

$$H(\omega_1, \omega_2) = \boldsymbol{\phi}^T(\omega_1, \omega_2) \mathbf{a} \quad (4)$$

where $\boldsymbol{\phi}(\omega_1, \omega_2) = \varphi_1(\omega_1) \otimes \varphi_2(\omega_2)$, $N = N_1 N_2$ and $\mathbf{a}$ is an $N \times 1$ real coefficient vector containing the elements $a(n_1, n_2)$ corresponding to the elements $\cos(\omega_1 n_1) \cos(\omega_2 n_2)$ of $\boldsymbol{\phi}$. The symbol $\otimes$ denotes the Kronecker product.

For practical purposes, the approximation domain $\mathcal{D}$ for the four-fold symmetrical FIR filter is made discrete and consists of I frequency points $(\omega_{1k}, \omega_{2l})$ defined on a dense grid in the 2-D Fourier-domain $[0, \pi] \times [0, \pi]$. Here $k = 0, 1, \dots, K-1$ and $l = 0, 1, \dots, L-1$. Let $H_{di} = H_d(\omega_{1k}, \omega_{2l})$ be the desired response specified on $\mathcal{D}$, and let $\boldsymbol{\phi}_i = \boldsymbol{\phi}(\omega_{1k}, \omega_{2l})$ be the corresponding characteristic vector [1], [2]. The index i constitutes a unique numbering of the frequency points $(\omega_{1k}, \omega_{2l})$.

In the weighted Chebyshev approximation problem considered here, cf. [1], [2], the optimal coefficient vector $\mathbf{a}_o$ is chosen to minimize the max-norm

$$\|\mathbf{H} - \mathbf{H}_d\| = \max_{i \in \{1, \dots, I\}} v_i |\boldsymbol{\phi}_i^T \mathbf{a} - H_{di}| \quad (5)$$

where $\mathbf{H}$ and $\mathbf{H}_d$ are real vectors with elements $\boldsymbol{\phi}_i^T \mathbf{a}$ and H_{di} respectively, and the v_i s are real and positive frequency weights. The quantity $\|\mathbf{H} - \mathbf{H}_d\|$ will be referred to as the Chebyshev deviation related to $\mathbf{H}$.

A linear program with the same optimal solutions as the Chebyshev approximation in (5) is

$$\begin{cases} \text{minimize } \delta, \text{ subject to} \\ v_i (\boldsymbol{\phi}_i^T \mathbf{a} - H_{di}) - \delta \leq 0 & \text{for } i = 1, \dots, I \\ -v_i (\boldsymbol{\phi}_i^T \mathbf{a} - H_{di}) - \delta \leq 0 & \text{for } i = 1, \dots, I \end{cases} \quad (6)$$

where δ is an additional real variable and the v_i s are the same weighting coefficients as in (5), cf. [3]. For an optimal solution $(\mathbf{a}_o, \delta_o)$ to (6), $\mathbf{a}_o$ will be referred to as a Chebyshev solution, and $\delta_o = \|\mathbf{H}_o - \mathbf{H}_d\|$ as the optimal Chebyshev deviation.

The optimization problem in (5) or (6) is in general not unique since the N -dimensional linear space of approximating functions $H(\omega_{1k}, \omega_{2l}) = \boldsymbol{\phi}_i^T \mathbf{a}$ does not generally satisfy the Haar condition [1], [2]. This is due to the 2-D nature of the basis functions for

the linear space above. The question of uniqueness of a given Chebyshev solution $\mathbf{a}_o$ is rather complicated and requires an extensive investigation in order to determine whether the solution is unique or not.

The question of uniqueness could, of course, be determined by directly checking the Haar condition, which is a necessary and sufficient condition for uniqueness. This would, however, require an exhaustive calculation of the rank of all subcollections of N vectors among all I characteristic vectors. The Haar condition is fulfilled if and only if all these $\binom{I}{N}$ subcollections are linearly independent.

III. MINIMUM NORM CHEBYSHEV SOLUTION OBTAINED BY QUADRATIC PROGRAMMING

The squared Euclidean filter weight norm is given by

$$\|\mathbf{h}\|^2 = \mathbf{h}^T \mathbf{h} = \sum_{n_1} \sum_{n_2} h^2(n_1, n_2) = \mathbf{a}^T \mathbf{C} \mathbf{a} \quad (7)$$

where $\mathbf{C}$ is a diagonal matrix determined by the four-fold symmetry defined by (2) so that the diagonal elements $(\mathbf{C})_{ii}$ are either 1, 1/2, or 1/4, see (3).

Consider the following quadratic program:

$$\begin{cases} \text{minimize } \mathbf{a}^T \mathbf{C} \mathbf{a}, \text{ subject to} \\ v_i(\phi_i^T \mathbf{a} - H_{di}) \leq \delta_o & \text{for } i = 1, \dots, I \\ -v_i(\phi_i^T \mathbf{a} - H_{di}) \leq \delta_o & \text{for } i = 1, \dots, I \end{cases} \quad (8)$$

where the v_i s are the same weighting coefficients as in (6), and δ_o is the optimal Chebyshev deviation related to any Chebyshev solution. Since the objective function $\|\mathbf{h}\|^2 = \mathbf{a}^T \mathbf{C} \mathbf{a}$ is strictly convex [8], it follows from (8) that there always exists a unique Chebyshev solution $\mathbf{a}'_o$ to (8) with minimum filter weight norm $\|\mathbf{h}\|$.

Using Kuhn–Tucker conditions [8], and provided that the positive parameter w below is chosen sufficiently small, it can be shown that the minimum norm Chebyshev solution $\mathbf{a}'_o$ to (8) can be obtained by solving the modified quadratic program given by

$$\begin{cases} \text{minimize } w \cdot \mathbf{a}^T \mathbf{C} \mathbf{a} + \delta^2, \text{ subject to} \\ v_i(\phi_i^T \mathbf{a} - H_{di}) - \delta \leq 0 & \text{for } i = 1, \dots, I \\ -v_i(\phi_i^T \mathbf{a} - H_{di}) - \delta \leq 0 & \text{for } i = 1, \dots, I. \end{cases} \quad (9)$$

Here δ is an additional real variable, w is a real and positive parameter, and the v_i s are the same weighting coefficients as in (6). The optimal solution $(\mathbf{a}''_o, \delta''_o)$ to (9) is unique since the objective function $w \cdot \mathbf{a}^T \mathbf{C} \mathbf{a} + \delta^2$ is strictly convex [8].

With a strictly convex objective function, quadratic programs with linear constraints, like the formulations in (8) and (9), can be efficiently solved by using all-purpose optimization algorithms such as the active set method [8].

In order to obtain the minimum norm Chebyshev solution using (8), the right side of the inequalities must be *exactly* δ_o . If the right side of these inequalities is less than δ_o , the quadratic program in (8) has no solution. In contrast, the quadratic program in (9) *always* has a solution, which can be advantageous for numerical reasons.

An example where a conventional Chebyshev solution is nonunique is described below. An 81-tap 2-D lowpass FIR filter was designed. Four-fold symmetry with $N_1 = N_2 = 5$ was used. The passband and stopband regions were defined by ellipses on a rectangular grid in the first quadrant with spacing 0.15×0.15 . The major and minor axes defining the passband and stopband regions were (2,1) and (2.5, 1.25), respectively. This yielded a total of $I = 397$ frequency points. The frequency weighting in the passband and stopband regions were $v_i = 1$ and $v_i = 10$, respectively.

Fig. 1. Filter weight norm $\|\mathbf{h}_o''\|$ and Chebyshev deviation δ_o'' as a function of the parameter w in the modified quadratic program. Here $(\mathbf{h}_o, \delta_o)$ corresponds to a Chebyshev solution obtained by solving the linear program.

20 log $|H_o(\omega_1, \omega_2)|$ [dB]

Fig. 2. Frequency response of an 81-tap 2-D Chebyshev FIR filter with four-fold symmetry, and passband and stopband regions defined by ellipses. The Chebyshev FIR filter is obtained by solving the linear program. Chebyshev deviation $\delta_o = 0.5885$, filter weight norm $\|\mathbf{h}_o\| = 0.4132$.

The resulting weighted Chebyshev design is nonunique. The filter weight norm obtained by using the linear program (6) and the modified quadratic program (9) was $\|\mathbf{h}_o\| = 0.4132$, and $\|\mathbf{h}_o''\| = 0.4009$, respectively. The Chebyshev deviation in both cases was $\delta_o = \delta_o'' = 0.5885$.

Fig. 1 shows the Chebyshev deviation and the filter weight norm as obtained with the modified quadratic program (9) when the parameter w is varied. This figure illustrates the fact that when the parameter w is chosen sufficiently small, the solution to the modified quadratic program is consistent with the minimum norm Chebyshev solution.

Figs. 2 and 3 show the frequency response of the FIR filters with optimal Chebyshev deviation obtained by using the linear program and the modified quadratic program, respectively. The latter corresponds to the unique minimum norm Chebyshev solution.

IV. SUMMARY AND CONCLUSIONS

In the design of 2-D weighted Chebyshev FIR filters, the solution is generally not unique. This is due to the 2-D nature of the approximating functions, which implies that the Haar condition may not be satisfied. However, for a design on a discrete frequency domain, the Haar condition might be fulfilled. The question of uniqueness is, however, rather extensive to investigate. It is therefore desirable to

Fig. 3. Frequency response of the unique 81-tap minimum norm 2-D Chebyshev FIR filter with fourfold symmetry, and passband and stopband regions defined by ellipses. The minimum norm Chebyshev FIR filter is obtained by solving the modified quadratic program. Chebyshev deviation $\delta''_o = 0.5885$, filter weight norm $\|\mathbf{h}''_o\| = 0.4009$.

define some simple additional constraints to the Chebyshev design in order to obtain a unique solution.

The weighted Chebyshev solution with minimum Euclidean filter weight norm is always unique, and represents a sensible additional constraint since it implies minimum white noise amplification and less sensitivity to model imperfections for some applications.

A quadratic programming formulation is given for this purpose, and it is shown that the formulation can always be made consistent with the unique minimum norm Chebyshev solution. The quadratic formulation is well-conditioned since it consists of a strictly convex objective function and linear constraints, and since it always has a feasible solution.

An important quality of the proposed method is that it always yields a unique solution which means that the extensive uniqueness-investigation is avoided.

An example where a conventional Chebyshev solution is nonunique is included in order to demonstrate the properties of the proposed quadratic program.

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