Environmental and Ecological Statistics
https://doi.org/10.1007/s10651-026-00703-y

A dynamical regression model for double-bounded time series based on the reflected unit Burr XII distribution

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Received: 16 July 2025 / Revised: 3 January 2026 / Accepted: 4 January 2026

Abstract

This paper introduces a new time series model based on the reflected unit Burr XII (RUBXII) distribution that is an alternative to the Kumaraswamy autoregressive moving average and Beta autoregressive moving average models for time series analysis taking values in the standard unit interval. The proposed model describes the conditional median of RUBXII-distributed discrete-time series by a dynamic structure that includes autoregressive and moving average (ARMA) terms, a set of regressors, and a link function. We perform the model's parameter estimation using the conditional maximum likelihood method. Closed-form expressions for the score vector and observed information matrix are presented. We propose and discuss techniques of diagnostic and forecasting for the new model. A Monte Carlo simulation study is carried out to evaluate the finite sample performance of the conditional maximum likelihood estimator. Finally, the proportion of stored hydroelectric energy in Northern Brazil is analyzed through the proposed model. The results evidence that the introduced RUBXII-ARMA model is suitable for describing the dynamics of the data and provides more accurate forecasts for the proportion of stored energy in Northern Brazil than those from competitors' models.

Keywords Dynamical model · Forecasts · Rates and proportions · Unit regression models

1 Introduction

In many practical situations, there is interest in studying phenomena and experiments that assume values on the standard unit interval. Hence, several models have been proposed to analyze random variables such as rates, proportions, and indices in different social, hydro-environmental, health, and educational contexts. For example,

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Published online: 17 February 2026

the proportion of atheists by country was modeled by Cribari-Neto and Souza (2013) and Souza and Cribari-Neto (2018) using Beta regression analysis (Ferrari and Cribari-Neto 2004). Sagrillo et al. (2021) introduced two new probability distributions to model the proportion of the useful volume of Brazilian water reservoirs. In the context of public health, mortality rates due to COVID-19 in the United States were analyzed by Ribeiro et al. (2021), Santana-e-Silva et al. (2022). In the educational area, a new regression model was developed by Ribeiro et al. (2022) to model the dropout proportion of Brazilian undergraduate courses.

On the other hand, it is common for variables of this kind to exhibit serial correlation, especially in environmental studies. In such cases, models for time series analysis must be considered. Although conventional autoregressive integrated moving average (ARIMA) models (Box et al. 2011) can be used, the Gaussianity assumption for inference on model parameters is too restrictive for many applications (Bayer et al. 2017). In addition, in terms of prediction, these models may yield forecasts outside the natural bounds of the data. To overcome the issue of analyzing non-Gaussian time series, Benjamin et al. (2003) pioneered the generalized autoregressive moving average (GARMA) models for random variables in the canonical form of the exponential family. Recent works have focused on proposing models for non-Gaussian time series; see, for example, Bayer et al. (2020), Stone et al. (2023), and de Araújo et al. (2024).

In the context of double-bounded time series in the unit interval, Rocha and Cribari-Neto (2009) introduced the Beta autoregressive moving average (βARMA) model. Later, following a similar approach, Bayer et al. (2017) proposed the Kumaraswamy autoregressive moving average (KARMA) model. In the first approach, the conditional mean of a Beta-distributed dependent variable is modeled, while in the second model, the conditional median is modeled and the dependent variable follows the Kumaraswamy distribution. In general, when the variable of interest exhibits asymmetric behavior and atypical observations, it is recommended to model the median instead of the mean. The median tends to be more robust than the mean and provides better results in such cases (John 2015; Lemonte and Bazán 2016). Alternative formulations have been developed for bounded and median-based time series models. For instance, Aknouche and Dimitrakopoulos (2023) introduced the autoregressive conditional proportion model, a multiplicative error approach suitable for (0, 1)-valued processes, while Liu and Zhu (2024) proposed an asymmetric exponential power Bayesian median autoregression, which emphasizes robustness to outliers and skewness. Recent works have considered similar structures from the GARMA model with random components involving the Beta distribution; see, for example, Bayer et al. (2023); Scher et al. (2024). However, in both cases, the conditional mean of the interest of variable is modeled.

The βARMA and KARMA models have been applied in various fields. Recently, they were employed by Melchior et al. (2021) to examine mortality rates due to occupational accidents in the Southern region of Brazil, and by Palm and Bayer (2017), Scher et al. (2020), Cribari-Neto et al. (2023) to forecast stored hydroelectric energy. Given the relevance of time series analysis constrained to the interval (0, 1), there is a need to develop alternative dynamic models that can better accommodate their specific characteristics. Some generalizations of these models, incorporating features such as inflation, seasonality, and non-fixed precision parameters, have been explored by Scher et al. (2024), Bayer et al. (2025), Stefanan et al. (2025).

Models based on the Burr XII (BXII) distribution have received considerable attention in the literature. Given the flexibility of its probability density function, several extensions have been proposed; see, for example, Paranaíba et al. (2011), Guerra et al. (2021, 2023). Besides, transformations of BXII-distributed random variables have been considered recently for analyzing data in the standard unit interval. Korkmaz and Chesneau (2021) modified the BXII distribution through an inverse exponential scheme, defining the unit Burr XII (UBXII) distribution and its associated quantile regression. By taking the complement of a UBXII random variable, Ribeiro et al. (2021) introduced the reflected unit Burr XII (RUBXII) distribution and its associated quantile regression model. More recently, another UBXII quantile regression model based on a different reparameterization was proposed by Ribeiro et al. (2022).

Specifically, the RUBXII distribution is quite flexible for modeling variables with different types of skewness and heavy tails. Its density can assume tilde-shaped forms, that are not accommodated by traditional regression models for unit data (Ribeiro et al. 2021). In this paper, we include autoregressive and moving average terms in the systematic component of the RUBXII quantile regression model, thus defining the RUBXII autoregressive and moving average (RUBXII-ARMA) model. However, unlike the quantile regression framework, the conditional median is modeled instead of the quantile. Thus, the new dynamical regression model is an alternative to the KARMA and βARMA models which may be useful for analyzing double-bounded time series, particularly in environmental applications.

Using the RUBXII-ARMA model, we analyze the proportion of stored hydroelectric energy in Northern Brazil.This application empirically demonstrates the ability of the new model to capture the dynamics of hydro-environmental time series. The remainder of the paper unfolds as follows. Section 2 introduces the new time series model based on the RUBXII distribution for conditional variables taking values in the standard unit interval. Parameter estimation for the RUBXII-ARMA model via the conditional likelihood method is presented and discussed in Section 3. This section also provides closed forms for the conditional score vector. Section 4 discusses large-sample inference and presents closed-form expressions for the observed information matrix. Based on the properties of the conditional maximum likelihood estimators, we also present asymptotic confidence intervals and hypothesis tests in this section. Section 5 presents diagnostic analysis and model selection criteria and discusses techniques for obtaining out-of-sample forecasts from the RUBXII-ARMA model. The Monte Carlo simulation results are given in Section 6. Section 7 illustrates the model's applicability to the proportion of stored hydroelectric energy in Northern Brazil. Finally, concluding remarks are presented in Section 8.

2 The proposed model

This section aims to introduce a new time series model based on the RUBXII distribution, which was recently introduced by Ribeiro et al. (2021). We use a similar approach to the GARMA (Benjamin et al. 2003), βARMA (Rocha and Cribari-Neto 2009), and KARMA (Bayer et al. 2017) models to define the RUBXII autoregressive and moving average model. The proposed model relates the conditional median of a discrete-time series to a dynamical linear predictor through a strictly monotonic and twice differentiable link function $g(\cdot)$, where $g : (0, 1) \to \mathrm{I\!R}$.

Let $\{y_t\}_{t\in\mathbb{Z}}$ be the discrete-time observations of a random variable Y_t, where each y_t assumes values in the unit interval and $\mathcal{F}_{t-1}$ be the set of observations up to time $t-1$. Assume that, conditionally to the information set $\mathcal{F}_{t-1}$, each Y_t is distributed according to the RUBXII distribution. Considering the quantile-based parametrization proposed in Ribeiro et al. (2021), we have that the conditional density of Y_t, given $\mathcal{F}_{t-1}$, is

$$f(y_t|\mathcal{F}_{t-1}) = \frac{\log(1-\tau)^{-c}\log^{c-1}(1-y_t)^{-1}}{(1-y_t)\log\left[1+\log^{c}(1-q_t)^{-1}\right]}\left[1+\log^{c}(1-y_t)^{-1}\right]^{\frac{\log(1-\tau)}{\log\left[1+\log^{c}(1-q_t)^{-1}\right]}-1}, \quad (1)$$

where $0 < y_t < 1$, $0 < \tau < 1$, $0 < q_t < 1$ is the $(\tau \times 100)$th conditional quantile of Y_t, and $c > 0$ is a shape parameter. Here, we set $\tau = 0.5$ and, thus, q_t represents the conditional median of Y_t. The cumulative distribution and quantile functions of Y_t are

$$F(y_t|\mathcal{F}_{t-1}) = 1 - \left[1+\log^{c}(1-y_t)^{-1}\right]^{\frac{\log(1-\tau)}{\log\left[1+\log^{c}(1-q_t)^{-1}\right]}}, \quad (2)$$

and

$$Q(u|\mathcal{F}_{t-1}) = 1 - \exp\left\{-\left[(1-u)^{\log\left[1+\log^{c}(1-q_t)^{-1}\right]/\log(1-\tau)} - 1\right]^{1/c}\right\}, \quad 0 < u < 1, \quad (3)$$

respectively.

In the class of RUBXII regression models (Ribeiro et al. 2021), the quantile q_t is related to a linear predictor, η_t, through a strictly monotonic and twice differentiable function link, $g(\cdot)$, which maps (0, 1) to $\mathrm{I\!R}$. Common choices for $g(\cdot)$ include the logit, probit, and complementary log-log functions. Other link functions have also been proposed in recent studies, such as those discussed by Wang et al. (2025). The systematic component of the RUBXII regression model (Ribeiro et al. 2021) is

$$g(q_t) = \eta_t = \boldsymbol{x}_t^{\top}\boldsymbol{\beta}, \quad (4)$$

where $\boldsymbol{x}_t = (x_{t1}, \dots, x_{tk})^{\top} \in \mathrm{I\!R}^k$, with $t = 1, \dots, n$, is a nonstochastic k-dimensional vector of covariates and the vector $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k)^{\top} \in \mathrm{I\!R}^k$ is composed by a set of unknown parameters. The proposed RUBXII-ARMA(p, q) additively includes autoregressive (AR) and moving average (MA) terms in (4), following a similar approach to Benjamin et al. (2003), Rocha and Cribari-Neto (2009), Bayer et al. (2017), Ribeiro et al. (2024). Thus, the conditional median q_t is specified as the following dynamic structure:

$$g(q_t) = \eta_t = \alpha + \mathrm{x}_t^\top \boldsymbol{\beta} + \sum_{i=1}^{p} \phi_i \left[g(y_{t-i}) - \mathrm{x}_{t-i}^\top \boldsymbol{\beta} \right] + \sum_{j=1}^{q} \theta_j \, r_{t-j}, \qquad (5)$$

where $\alpha \in \mathrm{I\!R}$ is an intercept, $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)^\top \in \mathrm{I\!R}^p$ is a set of AR parameters, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_q)^\top \in \mathrm{I\!R}^q$ is a set of MA parameters, *p* and *q* are the orders of the model, and r_t is the moving average error term. Several choices are available for the error r_t; see Benjamin et al. (2003), Kedem and Fokianos (2005) for more details. Here, since our interest lies in modeling the conditional median, we define r_t on the predictor scale, that is, $r_t = g(y_t) - g(q_t)$. Therefore, the RUBXII-ARMA(*p*, *q*) model is fully specified by the conditional density in (1) (random component) and the dynamic predictor in (5) (systematic component).

3 Parameter estimation

The estimation of the RUBXII-ARMA(*p*, *q*) model parameters can be carried out using the conditional maximum likelihood method as in Rocha and Cribari-Neto (2009), Bayer et al. (2017, 2020), Palm et al. (2021). Let $\boldsymbol{\gamma} = (\alpha, \boldsymbol{\beta}^\top, \boldsymbol{\phi}^\top, \boldsymbol{\theta}^\top, c)^\top$ be the $(p+q+k+2)$-dimensional parameter vector that indexes the RUBXII-ARMA(*p*, *q*) model. Based on a sample $(y_1, \boldsymbol{x_1}), \ldots, (y_n, \boldsymbol{x_n})$ of length *n*, satisfying the specification given in (1) and (5), the log-likelihood function for the parameter vector $\boldsymbol{\gamma}$ can be expressed as

$$\ell(\boldsymbol{\gamma}) = \sum_{t=m+1}^{n} \log f(y_t | \mathcal{F}_{t-1}) = \sum_{t=m+1}^{n} \ell_t(q_t, c), \qquad (6)$$

where $m = \max(p, q) < n$ and $\ell_t(q_t, c)$ is the logarithm of $f(y_t|\mathcal{F}_{t-1})$ given in (1). That is,

$$\begin{aligned} \ell_t(q_t, c) = {} & \log(1 - y_t)^{-1} - \log\left[w(q_t) \right] + \log\left[\log(1-\tau)^{-c} \right] \\ & + \log\left[\log^{c-1}(1-y_t)^{-1} \right] + \left[\log(1-\tau)/w(q_t) - 1 \right] w(y_t), \end{aligned}$$

where $w(x) = \log\left[1 + \log^c(1-x)^{-1} \right]$.

The conditional score vector ($\boldsymbol{U}(\boldsymbol{\gamma})$) is composed of the partial derivatives of $\ell(\boldsymbol{\gamma})$ with respect to each component of $\boldsymbol{\gamma}$. That is,

$$\boldsymbol{U}(\boldsymbol{\gamma}) = \frac{\partial \ell}{\partial \boldsymbol{\gamma}^\top} = \left(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \boldsymbol{\beta}^\top}, \frac{\partial \ell}{\partial \boldsymbol{\phi}^\top}, \frac{\partial \ell}{\partial \boldsymbol{\theta}^\top}, \frac{\partial \ell}{\partial c} \right)^\top = (U_\alpha(\boldsymbol{\gamma}), \boldsymbol{U}_{\boldsymbol{\beta}}(\boldsymbol{\gamma}), \boldsymbol{U}_{\boldsymbol{\phi}}(\boldsymbol{\gamma}), \boldsymbol{U}_{\boldsymbol{\theta}}(\boldsymbol{\gamma}), U_c(\boldsymbol{\gamma}))^\top.$$

Let γ_j be the *j*th component of $\boldsymbol{\gamma}$, where $\gamma_j \neq c$. The $(k+p+q+1)$ first components of the conditional score vector are obtained using the chain rule as

$$U_{\gamma_j}(\boldsymbol{\gamma}) = \frac{\partial \ell}{\partial \gamma_j} = \sum_{t=m+1}^{n} \frac{\partial \ell_t(q_t, c)}{\partial q_t} \frac{\mathrm{d}q_t}{\mathrm{d}\eta_t} \frac{\partial \eta_t}{\partial \gamma_j}. \tag{7}$$

The first two derivatives in (7) reduce to

$$\frac{\partial \ell_t(q_t, c)}{\partial q_t} = a_t, \quad \text{and} \quad \frac{\mathrm{d}q_t}{\mathrm{d}\eta_t} = \frac{1}{g'(q_t)},$$

where

$$a_t = -\frac{c\log^{c-1}(1-q_t)^{-1}}{(1-q_t)w(q_t)\exp[w(q_t)]} + \frac{\log(1-\tau)^{-c}\log^{c-1}(1-q_t)^{-1}w(y_t)}{(1-q_t)[w(q_t)]^2\exp[w(q_t)]}.$$

The partial derivatives, $\partial\eta_t/\partial\gamma_j$, are computed recursively as

$$\begin{aligned}
\frac{\partial \eta_t}{\partial \alpha} &= 1 - \sum_{j=1}^{q} \theta_j \frac{\partial \eta_{t-j}}{\partial \alpha}, \\
\frac{\partial \eta_t}{\partial \beta_l} &= x_{tl} - \sum_{i=1}^{p} \phi_i x_{(t-i)l} - \sum_{j=1}^{q} \theta_j \frac{\partial \eta_{t-j}}{\partial \beta_l}, \quad \text{for } l = 1, \ldots, k, \\
\frac{\partial \eta_t}{\partial \phi_i} &= g(y_{t-i}) - \boldsymbol{x}_{t-i}^{\top}\boldsymbol{\beta} - \sum_{j=1}^{q} \theta_j \frac{\partial \eta_{t-j}}{\partial \phi_i}, \quad \text{for } i = 1, \ldots, p,
\end{aligned}$$

and

$$\frac{\partial \eta_t}{\partial \theta_j} = r_{t-j} - \sum_{v=1}^{q} \theta_v \frac{\partial \eta_{t-v}}{\partial \theta_j}, \quad \text{for } j = 1, \ldots, q.$$

The last component of the conditional score vector, $U_c(\gamma)$, follows from direct differentiation of (6):

$$\frac{\partial \ell}{\partial c} = \sum_{t=m+1}^{n} \frac{\partial \ell_t(q_t, c)}{\partial c} = \sum_{t=m+1}^{n} b_t,$$

where

$$b_t = \frac{1}{c} + s(y_t) + \frac{s(y_t)\log^c(1-y_t)^{-1}[\log(1-\tau)/w(q_t) - 1]}{\exp[w(y_t)]}$$
$$- \frac{\log^c(1-q_t)^{-1}s(q_t)}{w(q_t)\exp[w(q_t)]} - \frac{\log(1-\tau)s(q_t)\log^c(1-q_t)^{-1}w(y_t)}{[w(q_t)]^2\exp[w(q_t)]},$$

with $s(x) = \log\left[\log\left(1-x\right)^{-1}\right]$.

Let $\boldsymbol{M}$, $\boldsymbol{P}$, $\boldsymbol{R}$ be matrices with dimensions $(n-m)\times k$, $(n-m)\times p$ and $(n-m)\times q$, respectively. The (i, j)th elements of those matrices are given by

$$\boldsymbol{M}_{i,j} = \frac{\partial \eta_{i+m}}{\partial \beta_j}, \quad P_{i,j} = \frac{\partial \eta_{i+m}}{\partial \phi_j}, \quad \text{and} \quad R_{i,j} = \frac{\partial \eta_{i+m}}{\partial \theta_j},$$

respectively.

Then, we can compactly write the score vector's components of $\boldsymbol{\gamma}$ as

$$U_\alpha(\boldsymbol{\gamma}) = \boldsymbol{\nu}^\top \boldsymbol{T}\,\boldsymbol{a},$$
$$\boldsymbol{U}_{\boldsymbol{\beta}}(\boldsymbol{\gamma}) = \boldsymbol{M}^\top \boldsymbol{T}\,\boldsymbol{a},$$
$$\boldsymbol{U}_{\boldsymbol{\phi}}(\boldsymbol{\gamma}) = \boldsymbol{P}^\top \boldsymbol{T}\,\boldsymbol{a},$$
$$\boldsymbol{U}_{\boldsymbol{\theta}}(\boldsymbol{\gamma}) = \boldsymbol{R}^\top \boldsymbol{T}\,\boldsymbol{a},$$
$$U_c(\boldsymbol{\gamma}) = \boldsymbol{b}^\top \mathbf{1},$$

where $\boldsymbol{\nu} = (\partial\eta_{m+1}/\partial\alpha, \ldots, \partial\eta_n/\partial\alpha)^\top$, $\boldsymbol{T} = \text{diag}\{1/g'(q_{m+1}), \ldots, 1/g'(q_n)\}$ is a diagonal matrix, $\boldsymbol{a} = (a_{m+1}, \ldots, a_n)^\top$, $\boldsymbol{b} = (b_{m+1}, \ldots, b_n)^\top$, and $\mathbf{1}$ is an $(n-m)$ -dimensional vector of ones.

By setting $\boldsymbol{U}(\boldsymbol{\gamma}) = \mathbf{0}$, where $\mathbf{0}$ denotes a vector of zeros with dimension $p+q+k+2$, and solving the resulting system of equations, we obtain the conditional maximum likelihood estimator (CMLE) of $\boldsymbol{\gamma}$, $\hat{\boldsymbol{\gamma}} = (\hat{\alpha}, \hat{\boldsymbol{\beta}}^\top, \hat{\boldsymbol{\phi}}^\top, \hat{\boldsymbol{\theta}}^\top, \hat{c})^\top$. However, this system is nonlinear and cannot be solved explicitly. Therefore, it is necessary to use nonlinear optimization algorithms such as Newton–Raphson or quasi-Newton (Nocedal and Wright 1999). We adopt the quasi-Newton algorithm, the so-called Broyden-Fletcher-Goldfarb-Shanno (BFGS) method (Press et al. 1992), including the analytical conditional score vector $\boldsymbol{U}(\boldsymbol{\gamma})$. This method is an iterative optimization algorithm, and thus, it requires initialization. We compute the starting values for α, $\boldsymbol{\beta}$, and $\boldsymbol{\phi}$ from an ordinary least squares estimate by considering a linear regression, where the response vector is

$$\boldsymbol{Y} = (g(y_{m+1}), \ldots, g(y_n))^\top,$$

and the matrix of covariates is expressed as

$$\boldsymbol{X} = \begin{bmatrix} 1 & x_{(m+1)1} & \cdots & x_{(m+1)k} & g(y_m) & g(y_{m-1}) & \cdots & g(y_{m-p+1}) \\ 1 & x_{(m+2)1} & \cdots & x_{(m+2)k} & g(y_{m+1}) & g(y_m) & \cdots & g(y_{m-p+2}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{nk} & g(y_{n-1}) & g(y_{n-2}) & \cdots & g(y_{n-p}) \end{bmatrix}.$$

For the moving average parameters $\boldsymbol{\theta}$, the starting values are set to zero as in Bayer et al. (2017) and Bayer et al. (2018).

4 Large sample inference

This section aims to introduce large-sample inferences for the RUBXII-ARMA(p, q) model parameters. From the asymptotic properties of conditional likelihood estimators, under the usual regularity conditions (Andersen 1970), the CLMEs, $\hat{\boldsymbol{\gamma}}$, are consistent and

$$\hat{\boldsymbol{\gamma}} \xrightarrow{d} \mathcal{N}_{(k+p+q+2)}\left(\boldsymbol{\gamma}, \boldsymbol{K}^{-1}(\boldsymbol{\gamma})\right),$$

where $\xrightarrow{d}$ denotes convergence in distribution, $\mathcal{N}_r$ denotes the r-variate normal distribution, and $\boldsymbol{K}^{-1}(\boldsymbol{\gamma})$ is the inverse of the conditional expected information matrix.

4.1 Conditional observed information matrix

For some models, the computation of the expected information matrix can be cumbersome. Moreover, obtaining $\boldsymbol{K}(\boldsymbol{\gamma})$ is not always possible. However, the expected information matrix is asymptotically equivalent to the observed information matrix (Pawitan 2001) and it can be consistently estimated by the latter, substituting the parameters by their corresponding CMLEs. Furthermore, according to Efron and Hinkley (1978) and Pawitan (2001), when the distribution is not in the exponential family, the observed matrix is preferable for hypothesis testing.

The conditional observed information matrix, denoted by $\boldsymbol{J}(\boldsymbol{\gamma})$, is obtained as

$$\boldsymbol{J}(\boldsymbol{\gamma}) = -\frac{\partial \ell^2(\boldsymbol{\gamma})}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\gamma}^\top}.$$

In this case, the second derivatives are obtained by deriving (7) with respect to γ_i, where $\gamma_i \neq c$ and $i = 1, 2, \ldots, p + q + k + 1$, as

$$\frac{\partial^2 \ell}{\partial \gamma_j \partial \gamma_i} = \sum_{t=m+1}^{n} \frac{\partial}{\partial \gamma_i} \left(\frac{\partial \ell_t(q_t, c)}{\partial q_t} \frac{\mathrm{d} q_t}{\mathrm{d} \eta_t} \frac{\partial \eta_t}{\partial \gamma_j} \right). \tag{8}$$

By applying the chain rule in (8), we have that

$$\frac{\partial^2 \ell}{\partial \gamma_j \partial \gamma_i} = \sum_{t=m+1}^{n} \left\{ \left[\frac{\partial^2 \ell_t(q_t, c)}{\partial q_t^2} \left(\frac{\mathrm{d}q_t}{\mathrm{d}\eta_t} \right)^2 + \frac{\partial \ell_t(q_t, c)}{\partial q_t} \frac{\mathrm{d}}{\mathrm{d}q_t} \left(\frac{\mathrm{d}q_t}{\mathrm{d}\eta_t} \right) \frac{\mathrm{d}q_t}{\mathrm{d}\eta_t} \right] \left(\frac{\partial \eta_t}{\partial \gamma_i} \frac{\partial \eta_t}{\partial \gamma_j} \right) + \frac{\partial \ell_t(q_t, c)}{\partial q_t} \frac{\mathrm{d}q_t}{\mathrm{d}\eta_t} \frac{\partial^2 \eta_t}{\partial \gamma_i \partial \gamma_j} \right\}. \tag{9}$$

Note that the derivatives $\frac{\partial \ell_t(q_t,c)}{\partial q_t}$, $\frac{\mathrm{d}q_t}{\mathrm{d}\eta_t}$, $\frac{\partial \eta_t}{\partial \gamma_i}$ and $\frac{\partial \eta_t}{\partial \gamma_j}$ are given in Section 3. In what follows, we compute the other derivatives given in (9). Specifically, $\frac{\partial^2 \ell_t(q_t,c)}{\partial q_t^2}$, $\frac{\mathrm{d}^2 q_t}{\mathrm{d}\eta_t^2}$, and $\frac{\partial^2 \eta_t}{\partial \gamma_i \partial \gamma_j}$.

The second derivative of the conditional log-likelihood function with respect to q_t is

$$\frac{\partial^2 \ell_t(q_t, c)}{\partial q_t^2} := \Delta_t^{qq},$$

where

$$\begin{aligned} \Delta_t^{qq} = \frac{c^2 \exp^{2c-2}(s(q_t))}{(1-q_t)^2 \exp^2(w(q_t))\, w(q_t)} \Bigg\{ & \left[1 + w(q_t)^{-1}\right]^{-1} - \frac{c-1}{c} \exp^{-c}(s(q_t)) \exp(w(q_t)) \\ & - \frac{1}{c} \exp^{-c+1}(s(q_t)) \exp(w(q_t)) + 2\log(1-\tau)\, w(q_t)^{-2}\, w(y_t) \\ & + \log(1-\tau)\, w(q_t)^{-1}\, w(y_t) - h(c) \exp^{-c}(s(q_t))\, l(q_t)\, w(y_t) \\ & - \frac{h(c)}{c-1} \exp^{-c+1}(s(q_t))\, l(q_t)\, w(y_t) \Bigg\} \end{aligned}$$

with

$$[\exp(x)]^a := \exp^a(x), \;\; h(x) = \frac{(x-1)}{x} \log(1-\tau), \;\; l(x) = \frac{\exp(w(x))}{w(x)}.$$

Observe that

$$\frac{\mathrm{d}}{\mathrm{d}q_t} \left(\frac{\mathrm{d}q_t}{\mathrm{d}\eta_t} \right) = -\frac{g''(q_t)}{[g'(q_t)]^2},$$

since $\mathrm{d}q_t/\mathrm{d}\eta_t = 1/g'(q_t)$. Hence, the fourth term in (9) is given by

$$\kappa_t := \frac{\mathrm{d}}{\mathrm{d}q_t} \left(\frac{\mathrm{d}q_t}{\mathrm{d}\eta_t} \right) \frac{\mathrm{d}q_t}{\mathrm{d}\eta_t} = -\frac{g''(q_t)}{[g'(q_t)]^3}.$$

The second-order derivatives of $\frac{\partial \eta_t}{\partial \boldsymbol{\gamma}}$ with respect to $\boldsymbol{\gamma}$ ($\gamma_i \neq \theta_j$ and $\gamma_i \neq c$) are equal to zero, where $i = 1, 2, \ldots, (p + k + 2)$. The remaining derivatives of $\frac{\partial \eta_t}{\partial \boldsymbol{\gamma}}$ with respect to θ_j are given by

$$\frac{\partial^2 \eta_t}{\partial \alpha \partial \theta_j} = -\frac{\partial \eta_{t-j}}{\partial \alpha} - \sum_{v=1}^{q} \theta_v \frac{\partial^2 \eta_{t-v}}{\partial \alpha \partial \theta_j},$$
$$\frac{\partial^2 \eta_t}{\partial \beta_l \partial \theta_j} = -\frac{\partial \eta_{t-j}}{\partial \beta_l} - \sum_{v=1}^{q} \theta_v \frac{\partial^2 \eta_{t-v}}{\partial \beta_l \partial \theta_j}, \quad \text{for } l = 1, \ldots, k,$$
$$\frac{\partial^2 \eta_t}{\partial \phi_i \partial \theta_j} = -\frac{\partial \eta_{t-j}}{\partial \phi_i} - \sum_{v=1}^{q} \theta_v \frac{\partial^2 \eta_{t-v}}{\partial \phi_i \partial \theta_j}, \quad \text{for } i = 1, \ldots, p.$$

Additionally,

$$\frac{\partial^2 \eta_t}{\partial \theta_k \partial \theta_j} = -\frac{\partial \eta_{t-j}}{\partial \theta_k} - \frac{\partial \eta_{t-v}}{\partial \theta_j} - \sum_{v=1}^{q} \theta_v \frac{\partial^2 \eta_{t-v}}{\partial \theta_k \partial \theta_j}, \quad \text{for } k, j = 1, \ldots, q.$$

By deriving (7) with respect to c, we obtain

$$\frac{\partial^2 \ell}{\partial \gamma_j \partial c} = \sum_{m+1}^{n} \frac{\partial^2 \ell}{\partial q_t \partial c} \frac{\mathrm{d} q_t}{\mathrm{d} \eta_t} \frac{\partial \eta_t}{\partial \gamma_j} = \sum_{m+1}^{n} \frac{\partial^2 \ell}{\partial q_t \partial c} \frac{1}{g\prime(q_t)} \frac{\partial \eta_t}{\partial \gamma_j}.$$

The first

$$\frac{\partial^2 \ell_t(q_t, c)}{\partial q_t \, \partial c} := \Delta_t^{qc},$$

where

$$\begin{aligned}
\Delta_t^{qc} = -\frac{c\, s(q_t)\, \exp^{c-1}(s(q_t))}{(1-q_t)\, \exp(w(q_t))\, w(q_t)} \Bigg\{ & 1 + \frac{1}{c\, s(q_t)} - \exp^c(s(q_t))\, \exp^{-1}(w(q_t)) \\
& - \exp^c(s(q_t))\, \exp^{-1}(w(q_t))\, w(q_t)^{-1} + \log(1-\tau) \Bigg[s(q_t)^{-1}\, s(y_t)\, \exp^c(s(y_t)) \\
& \times w(q_t)^{-1}\, \exp^{-1}(w(y_t)) + \frac{1}{c\, s(q_t)}\, w(y_t)\, w(q_t)^{-1} + w(y_t)\, w(q_t)^{-1} \\
& - 2\, \exp^c(s(q_t))\, w(y_t)\, \exp^{-1}(w(q_t))\, w(q_t)^{-2} \\
& - \exp^c(s(q_t))\, w(y_t)\, \exp^{-1}(w(q_t))\, w(q_t)^{-1} \Bigg] \Bigg\}.
\end{aligned}$$

Finally, the second-order derivative of $\ell(\gamma)$ with respect to c is

$$\frac{\partial^2 \ell_t(q_t, c)}{\partial c^2} := \Delta_t^{cc}$$

where

$$\Delta_t^{cc} = -\frac{s(q_t)\log(1-\tau)\, s(y_t)\, \exp^c(s(q_t))\, \exp^c(s(y_t))}{\exp(w(q_t))\, w(q_t)^2\, \exp(w(y_t))}\Bigg\{2 - \frac{s(q_t)}{\log(1-\tau)\, s(y_t)} \\ \times \exp^{-c}(s(y_t))\, [\exp^c(s(q_t)) - w(q_t)] \exp^{-1}(w(q_t))\, \exp(w(y_t)) \\ + \frac{s(q_t)\, w(y_t)\, \exp(w(y_t))}{s(y_t)\, \exp^c(s(y_t))} - \frac{s(q_t)\, \exp^c(s(q_t))\, w(y_t)\, \exp(w(y_t))\, \exp^{-1}(w(q_t))}{s(y_t)\, \exp^c(s(y_t))} \\ - 2 \cdot \frac{s(q_t)\, \exp^c(s(q_t))\, \exp^{-1}(w(q_t))\, w(y_t)\, \exp(w(y_t))\, w(q_t)^{-1}}{s(y_t)\, \exp^c(s(y_t))}\Bigg\} \\ - c^{-2} + h^\star(q_t)\, s(y_t)^2\, \exp^c(s(y_t))\, \exp^{-1}(w(y_t))\, \left[1 - \exp^c(s(y_t))\, \exp^{-1}(w(y_t))\right],$$

with $h^\star(x) = \frac{\log(1-\tau)}{w(x)} - 1$.

To simplify the matrix notation, we introduce some auxiliary vectors and matrices. Define $\boldsymbol{D} = \mathrm{diag}\{\Delta_{m+1}^{qc}, \ldots, \Delta_n^{qc}\}$, $\boldsymbol{C} = \mathrm{diag}\{\Delta_{m+1}^{c}, \ldots, \Delta_n^{c}\}$, $w_t = \Delta_t^{qq} T_{t,t}^2 + a_t \kappa_t$, and $\boldsymbol{W} = \mathrm{diag}\{w_{m+1}, \ldots, w_n\}$. For the second-order derivatives of $\partial\eta_t/\partial\boldsymbol{\gamma}$ with respect to $\boldsymbol{\gamma}$ that are not null, we define the following matrices $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}^s$, $\boldsymbol{\mathcal{M}}^s$, $\boldsymbol{\mathcal{P}}^s$, and $\boldsymbol{\mathcal{R}}^s$ of dimensions $(n-m)\times q$, $(n-m)\times p$, $(n-m)\times q$, $(n-m)\times q$, and $(n-m)\times q$, respectively, whose (i, j)-th elements are given by

$$\mathcal{A}_{i,j} = \frac{\partial^2\eta_{i+m}}{\partial\alpha\partial\theta_j},\ \mathcal{B}_{i,j}^r = \frac{\partial^2\eta_{i+m}}{\partial\beta_r\partial\phi_j},\ \mathcal{M}_{i,j}^r = \frac{\partial^2\eta_{i+m}}{\partial\beta_r\partial\theta_j},\ \mathcal{P}_{i,j}^r = \frac{\partial^2\eta_{i+m}}{\partial\phi_r\partial\theta_j},\ \text{and } \mathcal{R}_{i,j}^r = \frac{\partial^2\eta_{i+m}}{\partial\theta_r\partial\theta_j}.$$

Based on these quantities, the following matrices are defined: $\boldsymbol{\mathcal{I}}$, $\boldsymbol{\mathcal{J}}$, $\boldsymbol{\mathcal{K}}$, and $\boldsymbol{\mathcal{L}}$ with dimensions $k\times p$, $k\times q$, $p\times q$, and $q\times q$, respectively, given by

$$\boldsymbol{\mathcal{I}} = \begin{bmatrix} \boldsymbol{a}^\top\boldsymbol{T}\boldsymbol{\mathcal{B}}^1 \\ \vdots \\ \boldsymbol{a}^\top\boldsymbol{T}\boldsymbol{\mathcal{B}}^k \end{bmatrix},\ \boldsymbol{\mathcal{J}} = \begin{bmatrix} \boldsymbol{a}^\top\boldsymbol{T}\boldsymbol{\mathcal{M}}^1 \\ \vdots \\ \boldsymbol{a}^\top\boldsymbol{T}\boldsymbol{\mathcal{M}}^k \end{bmatrix},\ \boldsymbol{\mathcal{K}} = \begin{bmatrix} \boldsymbol{a}^\top\boldsymbol{T}\boldsymbol{\mathcal{P}}^1 \\ \vdots \\ \boldsymbol{a}^\top\boldsymbol{T}\boldsymbol{\mathcal{P}}^p \end{bmatrix},\ \text{and } \boldsymbol{\mathcal{L}} = \begin{bmatrix} \boldsymbol{a}^\top\boldsymbol{T}\boldsymbol{\mathcal{R}}^1 \\ \vdots \\ \boldsymbol{a}^\top\boldsymbol{T}\boldsymbol{\mathcal{R}}^q \end{bmatrix}.$$

Thus, the conditional observed information matrix is given by

$$\boldsymbol{J}(\boldsymbol{\gamma}) = \begin{bmatrix} J_{(\alpha,\alpha)} & \boldsymbol{J}_{(\alpha,\beta)} & \boldsymbol{J}_{(\alpha,\phi)} & \boldsymbol{J}_{(\alpha,\theta)} & J_{(\alpha,c)} \\ \boldsymbol{J}_{(\beta,\alpha)} & \boldsymbol{J}_{(\beta,\beta)} & \boldsymbol{J}_{(\beta,\phi)} & \boldsymbol{J}_{(\beta,\theta)} & \boldsymbol{J}_{(\beta,c)} \\ \boldsymbol{J}_{(\phi,\alpha)} & \boldsymbol{J}_{(\phi,\beta)} & \boldsymbol{J}_{(\phi,\phi)} & \boldsymbol{J}_{(\phi,\theta)} & \boldsymbol{J}_{(\phi,c)} \\ \boldsymbol{J}_{(\theta,\alpha)} & \boldsymbol{J}_{(\theta,\beta)} & \boldsymbol{J}_{(\theta,\phi)} & \boldsymbol{J}_{(\theta,\theta)} & \boldsymbol{J}_{(\theta,c)} \\ J_{(c,\alpha)} & \boldsymbol{J}_{(c,\beta)} & \boldsymbol{J}_{(c,\phi)} & \boldsymbol{J}_{(c,\theta)} & J_{(c,c)} \end{bmatrix},$$

where $J_{(\alpha,\alpha)} = -\boldsymbol{\nu}^\top\boldsymbol{W}\boldsymbol{\nu}$, $\boldsymbol{J}_{(\alpha,\beta)} = \boldsymbol{J}_{(\beta,\alpha)}^\top = -\boldsymbol{\nu}^\top\boldsymbol{W}\boldsymbol{M}$, $\boldsymbol{J}_{(\alpha,\phi)} = \boldsymbol{J}_{(\phi,\alpha)}^\top = -\boldsymbol{\nu}^\top\boldsymbol{W}\boldsymbol{P}$, $\boldsymbol{J}_{(\alpha,\theta)} = \boldsymbol{J}_{(\theta,\alpha)}^\top = -\boldsymbol{\nu}^\top\boldsymbol{W}\boldsymbol{R} - \boldsymbol{a}^\top\boldsymbol{T}\boldsymbol{\mathcal{A}}$, $J_{(\alpha,c)} = J_{(c,\alpha)} = -\boldsymbol{\nu}^\top\boldsymbol{T}\boldsymbol{D}\boldsymbol{1}$, $\boldsymbol{J}_{(\beta,\beta)} = -\boldsymbol{M}^\top\boldsymbol{W}\boldsymbol{M}$, $\boldsymbol{J}_{(\beta,\phi)} = \boldsymbol{J}_{(\phi,\beta)}^\top = -\boldsymbol{M}^\top\boldsymbol{W}\boldsymbol{P} - \boldsymbol{\mathcal{I}}$, $\boldsymbol{J}_{(\beta,\theta)} = \boldsymbol{J}_{(\theta,\beta)}^\top = -\boldsymbol{M}^\top\boldsymbol{W}\boldsymbol{R} - \boldsymbol{\mathcal{J}}$, $\boldsymbol{J}_{(\beta,c)} = \boldsymbol{J}_{(c,\beta)}^\top = -\boldsymbol{M}^\top\boldsymbol{T}\boldsymbol{D}\boldsymbol{1}$, $\boldsymbol{J}_{(\phi,\theta)} = \boldsymbol{J}_{(\theta,\phi)}^\top = -\boldsymbol{P}^\top\boldsymbol{W}\boldsymbol{R} - \boldsymbol{\mathcal{K}}$, $\boldsymbol{J}_{(\phi,\phi)} = -\boldsymbol{P}^\top\boldsymbol{W}\boldsymbol{P}$, $\boldsymbol{J}_{(\phi,c)} = \boldsymbol{J}_{(c,\phi)}^\top = -\boldsymbol{P}^\top\boldsymbol{T}\boldsymbol{D}\boldsymbol{1}$, $\boldsymbol{J}_{(\theta,\theta)} = -\boldsymbol{R}^\top\boldsymbol{W}\boldsymbol{R} - \boldsymbol{\mathcal{L}}$, $\boldsymbol{J}_{(\theta,c)} = \boldsymbol{J}_{(c,\theta)}^\top = -\boldsymbol{R}^\top\boldsymbol{T}\boldsymbol{D}\boldsymbol{1}$ and $J_{(c,c)} = -\mathrm{tr}(\boldsymbol{C})$, where $\mathrm{tr}(\cdot)$ is the trace operator.

Based on these properties and obtained quantities, it is possible to perform large sample inferences from confidence intervals and hypothesis testing. In what follows, we construct an asymptotic confidence interval and propose an asymptotic hypothesis testing in the context of the RUBXII-ARMA(p, q) model.

4.2 Confidence interval and hypothesis testing

Let $\delta \in (0, 1/2)$ be the significance level of a confidence interval. By considering the asymptotic normality property of $\hat{\boldsymbol{\gamma}}$, we construct a $100(1-\delta)\%$ approximate confidence interval for each element of $\boldsymbol{\gamma}$ as

$$\left[\hat{\gamma}_i - z_{1-\delta/2}\sqrt{J(\hat{\boldsymbol{\gamma}})^{ii}}; \hat{\gamma}_i + z_{1-\delta/2}\sqrt{J(\hat{\boldsymbol{\gamma}})^{ii}}\right], \tag{10}$$

where $z_{1-\delta/2}$ is the standard normal quantile and $J(\hat{\boldsymbol{\gamma}})^{ii}$ is the (i, i)th element of the $\boldsymbol{J}^{-1}(\hat{\boldsymbol{\gamma}})$.

Let $\mathcal{H}_0 : \gamma_i = \gamma_i^0$ against $\mathcal{H}_1 : \gamma_i \neq \gamma_i^0$ be the null and alternative hypothesis, respectively, where γ_i^0 is a given hypothesized value for the γ_i. To test these hypotheses, we propose to use an asymptotic version for the signed square root of Wald's statistic (Wald 1943) as in Bayer et al. (2017). It can be computed as

$$Z = \frac{\hat{\gamma}_i - \gamma_i^0}{\sqrt{J(\hat{\boldsymbol{\gamma}})^{ii}}}.$$

Under $\mathcal{H}_0$, $Z \xrightarrow{d} \mathcal{N}(0, 1)$, where $\mathcal{N}$ denotes a univariate normal distribution. At the $100(\delta)\%$-significance level, we reject the null hypothesis, whenever the absolute observed value of Z

exceeds the quantile $z_{1-\delta/2}$.

5 Diagnostic analysis and forecasting

In this section, we present techniques for model selection and discuss residual analysis. We also discuss how to carry out in-sample and out-of-sample predictions from RUBXII-ARMA(p, q) model. Since a fitted model passes all proposed diagnostic checks, it can be considered for out-of-sample forecasting. We propose some measures based on the prediction errors to assess the suitability of these forecasts.

5.1 Model selection

The model selection process requires determining the orders p and q that provide a suitable fit to the data at hand. As in the classical ARMA models, the sample autocorrelation and partial autocorrelation functions (ACF and PACF, respectively) plots can be helpful in an initial graphical analysis. For example, a cut-off at lag q (the lag value is close to zero) in the sample ACF plot suggests that a RUBXII-MA(q) model can be suitable. If the sample PACF is cut-off at lag p, a RUBXII-AR(p) model can be an alternative. If there is no cut-off in the sample ACF and PACF, it is recommended to consider a structure ARMA(p, q) (Shumway and Stoffer 2017).

We adopt model selection criteria based on Akaike's information criterion (AIC) (Akaike 1973) to select the best RUBXII-ARMA(p, q) model. This criterion considers the conditional maximum log-likelihood function evaluated at the CMLEs, $\ell(\hat{\gamma})$. We consider its modified version (MAIC), proposed by Bayer et al. (2018) and defined as

$$\mathrm{MAIC}(\kappa) = 2\kappa - 2\hat{\ell}_*, \tag{11}$$

where $\hat{\ell}_* = \frac{n}{n-m}\ell(\hat{\gamma})$ and κ is the number of estimated parameters. This criterion does not incorrectly penalize models with larger values of m. This occurs since for different values of m, comparing several p and q orders, we can interpret $\hat{\ell}_*$ as the sum of n terms (Bayer et al. 2018). The modified version of the Bayesian information criterion (MBIC) (Akaike 1978) is obtained by replacing the term 2κ by $\kappa \log(n)$ in (11). The best model among a set of competitor models is one that presents the smallest MAIC and/or MBIC values.

5.2 Residual analysis

After selecting a candidate model, residual analysis is required to assess its suitability to the data. For diagnostic analyses in the RUBXII-ARMA(p, q), we adopt the quantile residuals (Dunn and Smyth 1996):

$$r_t^\star = \Phi^{-1}\left[F(y_t|\mathcal{F}_{t-1})\right],$$

where $\Phi^{-1}(\cdot)$ is the standard normal quantile function and $F(y_t|\mathcal{F}_{t-1})$, defined in (2), is evaluated in the CMLEs, $\hat{\gamma}$.

If the model is well-fitted, there should be no systematic pattern or autocorrelation in the residuals. The plots of residual ACF and PACF are useful for checking this assumption. Also, it can be employed the Ljung–Box hypothesis test (Ljung and Box 1978) for testing whether the residuals are independently distributed.

Moreover, recent developments in diagnostic analysis for time series models have introduced promising approaches. For instance, Scher et al. (2020) addressed the problem of portmanteau testing in Beta-based time series models, whereas Pei et al. (2024) proposed a new portmanteau statistic based on Chatterjee's rank correlation coefficient, which may provide useful tools for assessing model adequacy in bounded time series contexts.

5.3 Prediction and forecasting

Obtaining out-of-sample forecasts is one primary objective of time series analysis (Lütkepohl 2005). In forecasting, the goal is to predict future values of a given time series by using the data collected to the present (Shumway and Stoffer 2017).

We propose to obtain forecasts for the RUBXII-ARMA(p, q) model using the theory of traditional ARIMA models (Box et al. 2011) as proposed by Bayer et al. (2017). In what follows, we explain how to compute the in-sample and out-of-sample forecasting of the conditional median of a RUBXII-ARMA(p, q) model.

Initially, we shall compute the estimates $\{\hat{q}_t\}_{t=m+1}^{n}$ (predicted values or in-sample forecasts) for the conditional medians $\{q_t\}_{t=m+1}^{n}$ based on the CMLE $\hat{\gamma}$. The estimates are sequentially obtained as

$$\hat{q}_t = g^{-1}\left(\hat{\alpha} + \mathrm{x}_t^\top \hat{\boldsymbol{\beta}} + \sum_{i=1}^{p} \hat{\phi}_i \left[g(y_{t-i}) - \mathrm{x}_{t-i}^\top \hat{\boldsymbol{\beta}} \right] + \sum_{j=1}^{q} \hat{\theta}_j \, \hat{r}_{t-j} \right), \qquad (12)$$

where

$$\hat{r}_t = \begin{cases} 0 & \text{if } t \leq m, \\ g(y_t) - g(\hat{q}_t) & \text{if } m < t \leq n, \end{cases}$$

for $t = m+1, \ldots, n$. Thus, the predicted values from a fitted RUBXII-ARMA(p, q) are given by (12).

The main goal is to forecast the median h_0 steps forward using all the available previous information. Let $H = \{n+1, \ldots, n+h_0\}$ be the set of indices that defines the forecast horizon. Since the covariates $\boldsymbol{x}_t$ ($t = 1, \ldots, n$) are non-random, we assume their values are available or can be obtained for $\boldsymbol{x}_h$, with $h \in H$. For example, suppose that the covariates are deterministic functions of t, such as sines and cosines in harmonic analysis, dummy variables, or polynomial trends. In these cases, they can be easily computed for $t > n$. Hence, for $h \in H$, the forecast values are obtained sequentially as

$$\hat{q}_h = g^{-1}\left(\hat{\alpha} + \mathrm{x}_h^\top \hat{\boldsymbol{\beta}} + \sum_{i=1}^{p} \hat{\phi}_i \left[g^\star(y_{h-i}) - \mathrm{x}_{h-i}^\top \hat{\boldsymbol{\beta}} \right] + \sum_{j=1}^{q} \hat{\theta}_j \, \hat{r}_{h-j} \right),$$

where $\hat{r}_t = 0$, for $t \in H$, and

$$g^\star(y_t) = \begin{cases} g(\hat{q}_t) & \text{if } t \in H, \\ g(y_t) & \text{if } t \notin H. \end{cases}$$

The out-of-sample forecasts assessment can be conducted by removing the last h_0 observations of the sample and then fitting them using the RUBXII-ARMA(p, q) model with the $n - h_0$ remaining observations. Here, the set of indices is $H = \{n - h_0 + 1, \ldots, n\}$. For assessing their performance, we compute the root mean squared error (RMSE), mean absolute error (MAE), and mean absolute percentage error (MAPE) (Hyndman and Koehler 2006; Hyndman and Athanasopoulos 2018) given by

$$\text{RMSE}_{h_0} = \sqrt{\frac{1}{h_0}\sum_{h \in H} e_h^2}, \qquad \text{MAE}_{h_0} = \frac{1}{h_0}\sum_{h \in H} |e_h|, \quad \text{and} \quad \text{MAPE}_{h_0} = \frac{1}{h_0}\sum_{h \in H} |p_h|,$$

where $e_h := y_h - \hat{q}_h$ and $p_h := 100 \times e_h/y_h$ are the forecast error and percentage error, respectively. These measures are useful for comparing competing models. The more suitable forecasting model for the dataset is one that presents the smallest RMSE, MAE, and MAPE values. The `R accuracy` function from `forecast` package (Hyndman et al. 2015; Hyndman and Khandakar 2008) can be used to obtain these measures.

6 Monte Carlo simulation

This section presents a Monte Carlo simulation study to assess the performance of the developed conditional likelihood inference theory on finite sample sizes. `All simulations and numerical experiments were conducted using the R language` (R Core Team 2023).

The random component y_t of the model is generated from a RUBXII distribution with density (1), where q_t is obtained from the dynamical structure of a RUBXII-ARMA(p, q) model proposed in (5). We use the inversion method to generate pseudo-random realizations of $y_t|\mathcal{F}_{t-1}$ ~RUBXII-ARMA(p, q) using (3).

We set the number of Monte Carlo replications $R = 10,000$ and consider sample sizes $n \in \{100, 200, 300, 500\}$. We select three different simulation scenarios and orders p and q. The true parameter values are chosen so that the AR and MA characteristic polynomials do not have common roots, and the AR characteristic polynomial does not have unit roots, as in Bayer et al. (2020). The scenarios are as follows.

- Scenario 1: RUBXII-ARMA(1,1), with true parameter values $\alpha = -0.1$, $\phi_1 = \theta_1 = 0.3$, and $c = 5$.
- Scenario 2: RUBXII-AR(1), with true parameter values $\alpha = 0.2$, $\phi_1 = 0.5$, and $c = 6.0$.
- Scenario 3: RUBXII-MA(1), with true parameter values $\alpha = 0.3$, $\theta_1 = 0.2$, and $c = 3.0$.

For each scenario, we compute the percentual relative bias (RB%) defined as RB%$= (\text{bias}(\hat{\gamma})/\gamma \times 100)$,

where $\text{bias}(\hat{\gamma}) = \mathbb{E}(\hat{\gamma}) - \gamma$, and the mean square error (MSE) of the CMLEs, given by $\text{MSE} = \text{var}(\hat{\gamma}) + \text{bias}^2(\hat{\gamma})$. These measures are useful to numerically evaluate the CMLE's performance. Furthermore, for each Monte Carlo replication, we construct a confidence interval according to (10) and check whether the true parameter value lies within the interval . The coverage rate (CR) is then computed as the proportion of times the true parameters are inside the confidence interval. We consider a significance level of 5%, i.e., $\delta = 0.05$. Therefore, it is expected to obtain CRs close to 95%.

Table 1 presents the Monte Carlo simulation results for point and interval estimation. The overall performance of the CMLEs was good as, in all the scenarios

Table 1 Simulation results for point and interval estimation of the RUBXII-ARMA model parameters

n	Measure	$\hat{\alpha}$	$\hat{\phi}_1$	$\hat{\theta}_1$	$\hat{c}$
Scenario 1 - RUBXII-ARMA(1,1)					
100	RB%	− 1.4882	2.9790	− 2.6457	− 2.3401
	MSE	0.0030	0.0217	0.0223	0.1602
	CR	0.9428	0.9321	0.9178	0.9459
200	RB%	− 0.2651	1.4928	− 1.4843	− 1.1833
	MSE	0.0014	0.0101	0.0101	0.0733
	CR	0.9457	0.9388	0.9341	0.9476
300	RB%	0.0110	0.6503	− 0.6093	− 0.8021
	MSE	0.0009	0.0064	0.0064	0.0469
	CR	0.9447	0.9464	0.9427	0.9489
500	RB%	0.0623	0.3303	− 0.3646	− 0.5048
	MSE	0.0005	0.0037	0.0036	0.0278
	CR	0.9527	0.9530	0.9473	0.9483
Scenario 2 - RUBXII-AR(1)					
100	RB%	− 2.1908	4.1572	–	− 2.1275
	MSE	0.0018	0.0062	–	0.2926
	CR	0.9452	0.9474	–	0.9513
200	RB%	− 1.1786	3.0459	–	− 1.0971
	MSE	0.0008	0.0029	–	0.1356
	CR	0.9458	0.9488	–	0.9504
300	RB%	− 0.7153	2.6212	–	− 0.7134
	MSE	0.0005	0.0019	–	0.0887
	CR	0.9475	0.9474	–	0.9485
500	RB%	− 0.4323	2.3868	–	− 0.4635
	MSE	0.0003	0.0012	–	0.0507
	CR	0.9461	0.9482	–	0.9528
Scenario 3 - RUBXII-MA(1)					
100	RB%	− 0.5587	–	2.4316	− 1.9882
	MSE	0.0076	–	0.0069	0.0723
	CR	0.9440	–	0.9299	0.9493
200	RB%	− 0.3946	–	1.5353	− 1.0425
	MSE	0.0038	–	0.0031	0.0340
	CR	0.9439	–	0.9390	0.9527
300	RB%	− 0.1327	–	1.3659	− 0.6957
	MSE	0.0025	–	0.0019	0.0221
	CR	0.9447	–	0.9462	0.9516
500	RB%	− 0.0034	–	1.0440	− 0.4749
	MSE	0.0015	–	0.0011	0.0132
	CR	0.9498	–	0.9483	0.9496

studied, the RB% is smaller than 3%, and theMSE approaches zero as n increases. Moreover, the RB% also greatly decreases as the sample size increases in all situations. We observe that the parameter estimates with the biggest MSE values regard $\hat{c}$, estimates of the shape parameter. On the other hand, in Scenario 2, the model's intercept presents the smallest MSE for all sample sizes. In Scenario 1, the RB% and MSE of the autoregressive and moving averages parameter estimates are very similar. As expected, the CR tends to a confidence level of 95% as the sample size increases. For $n > 300$, the CRs are close to 95% in all scenarios. Therefore, from this simulation study, we obtain numerical evidence of the CMLE consistency of the proposed model parameters.

7 An application to stored hydroelectric energy data

In this section, we present an application to data on the monthly proportion of stored hydroelectric energy in Northern Brazil from April 2004 to December 2022, yielding a sample size of $n = 225$ months. The last ten observations are separated to assess the forecasting performance of the fitted models. These data are available at Operador Nacional do Sistema Elétrico (2023).

Adequately modeling data regarding Brazilian hydroelectric energy is quite relevant since this is the predominant energy source in the country (Luiz-Silva et al. 2021; Luiz-Silva and Garcia 2022). However, hydroelectric energy is quite vulnerable to climate changes, which directly affect Brazil's water availability (Luiz-Silva and Garcia 2022). Climate features tend to modify the hydrological cycle and the water regime and availability in the watersheds (Arora and Boer 2001; Allen and Ingram 2002; Asadieh and Krakauer 2015). According to Rocha et al. (2022), the temperature trends magnitudes are more accentuated in the northern and northeastern regions of Brazil in all Hydroelectric Power Plants analyzed.

Brazil is responsible for one of the largest hydroelectric potentials in the world (Rocha et al. 2022). It has historically been very dependent on hydroelectricity, and due to climatic conditions, the power sector in Brazil has experienced multiple water crises . Proposing models for analyzing data of this kind can support decision-making related to electricity supply and crisis management (Hunt et al. 2018).

For the dataset at hand, besides we fit the proposed RUBXII-ARMA(p, q), we also adjust the βARMA and KARMA models. The systematic components of both models are similar to (5). The KARMA model is obtained by assuming that its random component $Y_t|\mathcal{F}_{t-1}$ has a Kumaraswamy (Kw) distribution, i.e. $Y_t|\mathcal{F}_{t-1} \sim$Kw$(q_t, \varphi)$. Thus, the conditional density is

$$f(y_t|\mathcal{F}_{t-1}) = \frac{\varphi \log 0.5}{\log(1-q_t^{\varphi})} y_t^{\varphi} (1-y_t^{\varphi})^{\log 0.5/\log(1-q_t^{\varphi})-1}, \quad y_t \in (0,1),$$

where $0 < q_t < 1$ is the median of $Y_t|\mathcal{F}_{t-1}$ and $\varphi > 0$ is a precision parameter. The random component of the βARMA model has a Beta distribution. Thus, in this case, $Y_t|\mathcal{F}_{t-1} \sim \mathrm{beta}(\mu_t, \nu)$ with conditional density function given by

$$f(y_t|\mathcal{F}_{t-1}) = \frac{\Gamma(\nu)}{\Gamma(\mu_t\nu)\Gamma((1-\mu_t)\nu)} y_t^{\mu_t\nu-1}(1-y_t)^{(1-\mu_t)\nu-1},$$

Table 2 Descriptive statistics of the monthly average proportions of stocked energy in Northern Brazil

Min	Median	Mean	Max	Var	Asymmetry	Exc. Kurtosis
0.1509	0.6205	0.6234	0.9968	0.0626	− 0.0551	− 1.2946

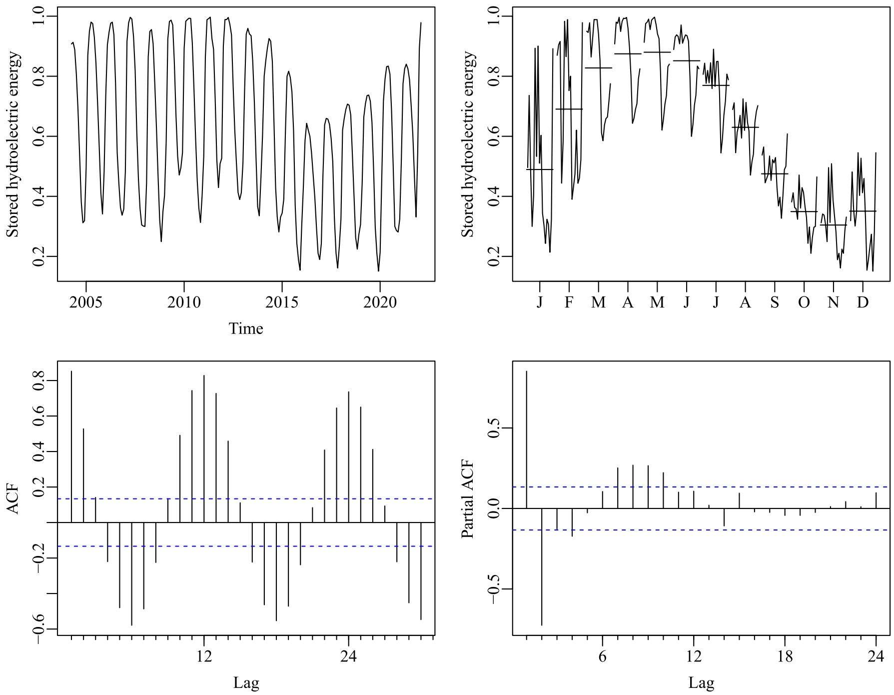

Fig. 1 Proportion of stored hydroelectric energy time series in Northern Brazil, its seasonal plot, sample autocorrelation, and partial autocorrelation plots

where $0 < \mu_t < 1$ is the mean of $Y_t|\mathcal{F}_{t-1}$, $\nu > 0$ is a precision parameter and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x$ is the complete gamma function. The dynamic component of this model is obtained by replacing q_t in (5) by the mean, μ_t.

Some descriptive statistics are presented in Table 2. There are several observations that are close to the upper bound of the standard unit interval limit. This indicates that modeling the conditional median may be more appropriate since it is more robust to atypical observations than the mean. Figure 1 displays the plot of the original time series, its seasonal plot, and corresponding sample ACF and PACF plots. A seasonal pattern is suggested in Figure 1, top-left panel. The ACF (Figure 1, bottom-left panel) also indicates seasonality in the time series. PACF (Figure 1, bottom-right panel) suggests that an order two autoregressive model can provide a good fit to these data as a cut-off is observed at lag two.

Since there is evidence of monthly seasonality, we consider a harmonic regression approach as in Bloomfield (2004). For this, we incorporate covariates into the models through trigonometric functions $x_{t1} = \cos(2\pi t/12)$ and $x_{t2} = \sin(2\pi t/12)$, with $t \in \{1, 2, \ldots, n\}$.

We consider the MAIC criterion to select the best model in each class. Further, we fit all models with autoregressive and moving average dynamics up to order three and adopt the logit link function. The smallest MAIC value is obtained for the RUBXII-AR(3), which presents the following dynamic component

$$\log\left(\frac{q_t}{1-q_t}\right) = \alpha + \boldsymbol{x}_t^\top \boldsymbol{\beta} + \sum_{i=1}^{3} \phi_i \left[\log\left(\frac{y_{t-i}}{1-y_{t-i}}\right) - \boldsymbol{x}_{t-i}^\top \boldsymbol{\beta}\right],$$

where $\alpha \in \mathbb{R}$ is the intercept of the model, $\boldsymbol{\beta} = (\beta_1, \beta_2)^\top \in \mathbb{R}^2$ is the parameter vector associated with the vector of seasonal covariates $\boldsymbol{x}_t^\top = (\cos(2\pi t/12), \sin(2\pi t/12))$, and $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)^\top \in \mathbb{R}^3$ are the AR coefficients. Table 3 displays the results of the fitted RUBXII-AR(3) model. Considering a significance level of 10%,all parameter estimates are statistically significant .

We also fitted the competing models considering the same order ($p = 3$ and $q = 0$), as well as their MAIC best-selected models KARMA(3, 3) and βARMA(2,3). The results of these fitted models are shown in Table 5 in the Appendix (A5). The estimates of θ_1 are not significant for the KARMA(3, 3) and βARMA(2, 3) models, and the intercept α is not significant for the βARMA(2, 3).

The Ljung-Box test (with 20 lags) is applied to the residuals associated with each fitted model. Considering a significance level of 5%, the null hypothesis of independence of the residuals is not rejected only for the RUBXII-AR(3), KARMA(3, 3), and KAR(3), as shown by the p-values in Tables 3 and A5. However, note that whether we adopt a significance level of 10% the null hypothesis is also rejected for the KAR(3) model.

In the residual ACF and PACF plots of the RUBXII-AR(3) (Figure 2, bottom-left and bottom-right panels), it is seen that most of the lags are within the limits of the confidence interval (dashed lines in blue), which suggests a convenient selection of the p and q values. Besides, no pattern is suggested by the residuals versus index plot (Figure 2, top-right panel). This indicates that there is no evidence of serial correlation in the residuals. The Ljung-Box test result confirms this finding; see Table 3.

Figure 2 (top-left panel) displays the plot of the observed and fitted time series. The forecasting from the RUBXII-AR(3) model for the ten observations separated is also

Table 3 Fitted RUBXII-AR(3) model for the monthly proportion of stored hydroelectric energy in Northern Brazil, MAIC, MBIC, and p-value of the associated Box-Ljung test

Parameter	Estimate	Std. Error	Z value	p-value
RUBXII-AR(3)				
α	0.0710	0.0369	1.9228	0.0545
β_1	0.8664	0.1084	7.9916	< 0.0001
β_2	1.3177	0.1332	9.8935	< 0.0001
ϕ_1	1.1656	0.0973	11.9800	< 0.0001
ϕ_2	-0.3532	0.1303		0.0067
ϕ_3	0.1279	0.0724	1.7669	0.0772
c	5.9798	0.3813	15.6818	< 0.0001
MAIC $= -479.4407$				
MBIC $= -455.8462$				
Box-Ljung (p-value)$= 0.3822$				

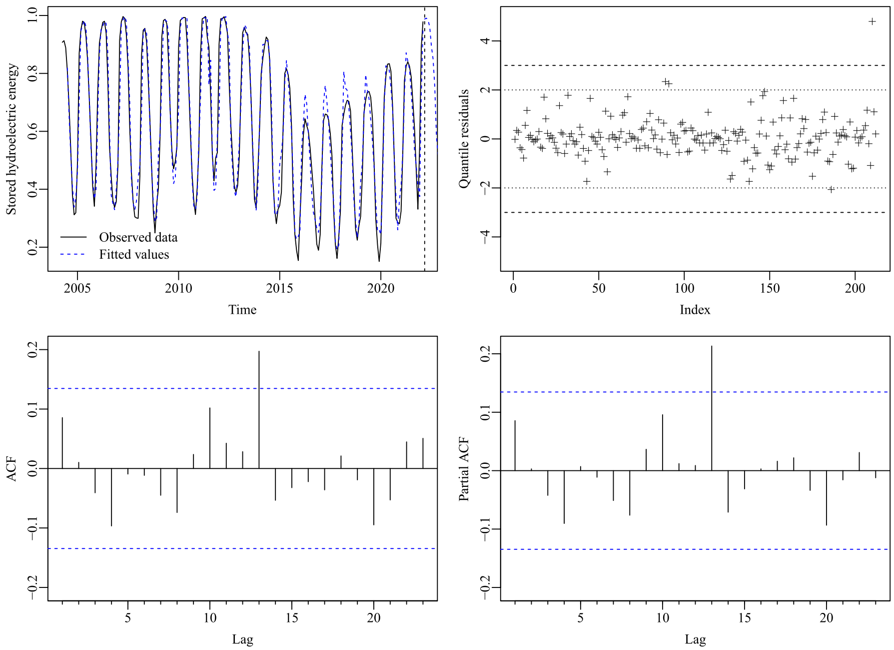

Fig. 2 Observed and fitted proportion of stored hydroelectric energy time series in Northern Brazil, residuals versus index plot, sample ACF and PACF plots of residuals of the RUBXII-AR(3) model

Table 4 Out-of-sample RMSE, MAE, and MAPE values, considering $h_0 = 10$ months from three fitted model (bold values indicate the lowest error for each performance measure).

Model	RMSE	MAE	MAPE
RUBXII-AR(3)	**0.1104**	**0.0841**	**13.1864**
KARMA(3, 3)	0.1275	0.0969	15.1931
βARMA(2, 3)	0.1767	0.1453	20.8282
KAR(3)	0.2261	0.1650	26.4638
βAR(3)	0.2029	0.1629	23.2949

presented in this figure (after the vertical dashed line in black). It is possible to observe that the fitted values are quite close to the observed and that the forecasts follow a similar behavior of the observed time series.

Table 4 presents an out-of-sample forecasting performance comparison among the fitted models, where the best results are highlighted in bold. We provide out-of-sample forecasting results for the fitted βAR(3) and βARMA(2, 3) models just for purposes of comparison, since they present correlated residuals according to the performed test. However, the RMSE, MAE, and MAPE values for the RUBXII-AR(3) are substantially smaller than those of its competing models. These results indicate that the proposed RUBXII-ARMA(p, q) model provides more accurate forecasts of the monthly average proportion of stocked energy in Northern Brazil than the available competitors. Thus, it can be used as an alternative to the existing models in the unit time series context.

8 Concluding remarks

The reflected unit Burr XII (RUBXII) quantile regression model was recently introduced to analyze mortality rates by Covid-19 in the United States. It is useful for modeling independent random variables that assume values in the standard unit interval. Since the RUBXII distribution can take many density shapes depending on the parameter values, it is useful for modeling variables with different types of skewness and heavy tails. For example, its density takes tilde-shaped forms that classical distributions like Beta and Kumaraswamy do not assume. However, the RUBXII quantile regression model assumes the non-existence of serial correlation in the data.

This paper introduced an extension of the RUBXII quantile regression model, including additively a dynamic structure to its systematic component to analyze correlated data. The new dynamical regression model, called RUBXII autoregressive moving average (RUBXII-ARMA), proved useful for modeling and forecasting time series that assume values in the standard unit interval. Here, unlike quantile regression, the median is modeled through autoregressive and moving average terms. We carried out conditional maximum likelihood inference on the proposed model's parameters, obtaining closed-form expressions for the conditional score function and conditional observed information matrix. Based on the asymptotic properties from the conditional maximum likelihood estimators (CMLE), approximated confidence intervals and hypothesis tests were obtained.

Furthermore, diagnostic analysis tools for model selection and forecasting techniques were discussed. We also assessed the finite sample performance of the CMLE in the RUBXII-ARMA framework by a Monte Carlo simulation study. The results suggest numerical evidence that desirable properties, such as asymptotic consistency, are preserved. In future work, we intend to extend this framework to model conditional quantiles other than the median, allowing for a more comprehensive characterization of tail behavior and asymmetric dynamics in bounded time series.

We used the new model for forecasting the proportion of stored hydroelectric energy in Northern Brazil. To evaluate the forecast accuracy, we compared the forecasts from the RUBXII-ARMA model with those obtained by the well-known Kumaraswamy autoregressive and moving average and Beta autoregressive and moving average models. The findings of this analysis indicated that the RUBXII-ARMA model has the best forecasting and fit performance for the monthly proportion of stored hydroelectric energy in Northern Brazil. Therefore, the RUBXII-ARMA model is a flexible and useful alternative for modeling double-bounded time series.

Appendix

Table 5 displays the results from competing fitted models to the RUBXII-AR(3). It gives the parameter estimates from KAR(3), βAR(3), KARMA(3, 3), and β ARMA(2, 3) models, their associated standard errors, statistic Z values ($p-$values), MAIC, MBIC, and the p-value of the Ljung-Box test for the associated residuals.

Table 5 Fitted KAR(3), βAR(3), KARMA(3, 3), and βARMA(2, 3) models for the monthly proportion of stored hydroelectric energy in Northern Brazil, MAIC, MBIC, and p-value of the associated Ljung-Box test

Parameter	Estimate	Std. Error	Z value	p-value
KAR(3)				
α	0.1079	0.0353	3.0607	0.0022
β_1	0.8277	0.1043	7.9354	$<$ 0.0001
β_2	1.5421	0.1039	14.8428	$<$ 0.0001
ϕ_1	1.2435	0.0824	15.0998	$<$ 0.0001
ϕ_2	− 0.3586	0.1073	3.3425	0.0008
ϕ_3	0.1026	0.0614	1.6701	0.0949
φ	7.5337	0.4089	18.4248	$<$ 0.0001
MAIC $=$ − 560.4817				
MBIC $=$ − 536.8873				
Ljung-Box (p-value)$=$ 0.0941				
βAR(3)				
α	0.2011	0.0276	7.2886	$<$ 0.0001
β_1	1.1585	0.0651	17.7872	$<$ 0.0001
β_2	0.9395	0.0712	13.2011	$<$ 0.0001
ϕ_1	0.8623	0.0200	43.0759	$<$ 0.0001
ϕ_2	− 0.1096	0.0297	3.6970	0.0002
ϕ_3	− 0.0616	0.0209	2.9447	0.0032
ν	32.8173	3.1916	10.2823	$<$ 0.0001
MAIC $=$ − 543.5660				
MBIC $=$ − 519.9715				
Ljung-Box (p-value)$<$ 0.0001				
KARMA(3, 3)				
α	0.1659	0.0652	2.5429	0.0110
β_1	0.8915	0.0647	13.7841	$<$ 0.0001
β_2	1.3851	0.0733	18.8929	$<$ 0.0001
ϕ_1	1.2257	0.0491	24.9540	$<$ 0.0001
ϕ_2	− 1.2467	0.0325	38.3415	$<$ 0.0001
ϕ_3	0.9031	0.0483	18.6905	$<$ 0.0001
θ_1	− 0.0138	0.0706	0.1960	0.8446
θ_2	0.9876	0.0273	36.1268	$<$ 0.0001
θ_3	0.2475	0.0756	3.2742	0.0011
φ	7.7630	0.4231	18.3496	$<$ 0.0001
MAIC $=$ − 581.3892				
MBIC $=$ − 547.6828				
Ljung-Box (p-value)$=$ 0.1464				
βARMA(2, 3)				
α	0.0371	0.0252	1.4711	0.1413
β_1	1.1213	0.0619	18.1101	$<$ 0.0001
β_2	1.0046	0.0631	15.9284	$<$ 0.0001
ϕ_1	1.0944	0.1203	9.1000	$<$ 0.0001
ϕ_2	− 0.1819	0.0826	2.2025	0.0276
θ_1	− 0.1639	0.1218	1.3458	0.1784
θ_2	− 0.1808	0.0484	3.7324	0.0002

Table 5 (continued)

Parameter	Estimate	Std. Error	Z value	p-value
θ_3	-0.2627	0.0325	8.0808	< 0.0001
ν	33.0469	3.2149	10.2792	< 0.0001
MAIC $= -559.0402$				
MBIC $= -528.7044$				
Ljung-Box (p-value)$<$ 0.0001				

Acknowledgements We acknowledge the partial funding support received from Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), Fundação de Amparo à Pesquisa do Estado do Rio Grande do Sul (FAPERGS) (grant number 23/2551-0001595-1), Serrapilheira Institute (grant number 2211-41692), Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) (grant numbers 306274/2022-1, 308578/2023-6, and 200831/2024-0), Swedish-Brazilian Research and Innovation Centre (CISB), Saab AB, and Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) (grant number 23/02538-0), Brazil. We also thank three anonymous referees for comments and suggestions that led to a much improved manuscript.

Author contributions T.F.R. developed the methodology, implemented the model, carried out the simulations and application, and wrote the manuscript. F.M.B. and R.R.G. supervised the research and provided guidance throughout. F.A.P.R. contributed as co-supervisor and reviewed the theoretical developments. A.P.A. contributed to the development of the ARMA-based structure and manuscript revision. J.J.S.S. assisted in deriving the first- and second-order derivatives. All authors reviewed and approved the final version of the manuscript.

Funding The Article Processing Charge (APC) for the publication of this research was funded by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) (ROR identifier: 00x0ma614).

Data availability The data that support the findings of this study are publicly available from the Operador Nacional do Sistema Elétrico (ONS) at https://www.ons.org.br. All computer codes used in the application are available at https://github.com/tatianefribeiro/rubxiiarma.

Declarations

Conflict of interest The authors declare no conflict of interest.

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