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Central idempotents in group-graded rings

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Let *G* be a group and let *R* be a *G*-graded ring. We show that every nonzero central idempotent in *R* has finite support group in two broad settings: when *G* is abelian and when *G* is arbitrary but the grading satisfies a certain one-sided non-annihilation condition on nonzero homogeneous elements. In particular, under the respective hypotheses, if *G* is torsion-free, then every central idempotent lies in the principal component of the grading. Our results generalize earlier results by Bass, Burns, and Bovdi–Mihovski, from group rings and crossed products, to non-commutative, possibly non-unital, group-graded rings. We demonstrate the utility of our results by applying them to semigroup-graded rings, Leavitt path rings, fractional skew monoid rings, partial skew group rings, and algebraic Cuntz–Pimsner rings.

1 Introduction

Let G be a group with identity element e. Recall that a (possibly non-unital) ring R is said to be *G-graded* if there is a collection of additive subgroups $\{R_g\}_{g\in G}$ of R such that $R = \oplus_{g\in G} R_g$, and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. If $R_g R_h = R_{gh}$ holds for all $g, h \in G$, then R is said to be *strongly G-graded*. The *support* of an element $r = \sum_{g\in G} r_g \in R$ is the finite set $\mathbf{Supp}(r) := \{g \in G \mid r_g \neq 0\}$. For a nonzero $r \in R$, the subgroup of G that is generated by $\mathbf{Supp}(r)$ is said to be *the support group of r*. Although finitely generated, the support group of an element need not be finite.

A group ring is a prototypical example of a strongly group-graded ring. In 1970, Burns [6] established the following result regarding the support group of central idempotents in group rings, building on earlier work by Rudin and Schneider [19].

> **Theorem** (Burns). Let A be a unital ring and let G be a group. If f is a nonzero central idempotent in the group ring $A[G]$, then f has finite support group.

In fact, Burns proved the theorem for non-unital group rings, but we only cite the unital version here since it will be enough for our purposes and also consistent with how group rings are usually defined

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nowadays. Independently, Bovdi and Mihovski [4, 5] showed that the above conclusion holds for any (unital) G-crossed product. Later on, Bass [3, Lem. 6.7] provided a short and elegant proof that every idempotent in a commutative group ring $A[G]$ is contained in the subring $A[T]$, where T denotes the torsion subgroup of G.

In this article, we use Burns's theorem as a lever, allowing us to prove similar statements in the more general context of group-graded rings. In Section 2, we establish our first main result that generalizes Bass's result from the setting of commutative group rings to the setting of group-graded rings that are allowed to be both non-commutative and non-unital.

Theorem A. Let G be an abelian group and let R be a G-graded ring. If f is a nonzero central idempotent in R, then f has finite support group. In particular, the support of a nonzero central idempotent in R is contained in the torsion subgroup of G.

As a direct consequence, we obtain a partial generalization of a result of Kirby [11, Thm. 1] that was proved in the setting of strongly group-graded commutative rings.

Corollary A. Let G be a torsion-free abelian group and let R be a G-graded ring. Then every central idempotent in R is contained in R_e.

Recall that an element of a G-graded ring R is said to be *homogeneous* if it is contained in the set $\cup_{g\in G}R_g$. Also recall that we write $\mathbf{Supp}(R) := \{g \in G \mid R_g \neq \{0\}\}$. In Section 3, we establish our second main result that is the following generalization of Burns's theorem.

Theorem B. Let G be a group and let R be a G-graded ring. Suppose that at least one of the following two assertions holds:

(i) $R_g r \neq \{0\}$ for all nonzero homogeneous $r \in R$ and all $g \in \mathbf{Supp}(R)$.
(ii) $rR_g \neq \{0\}$ for all nonzero homogeneous $r \in R$ and all $g \in \mathbf{Supp}(R)$.

If f is a nonzero central idempotent in R, then f has finite support group.

We point out that Theorems A and B cannot be generalized to hold for arbitrary idempotents (see Example 3.9). We also demonstrate that the conclusions of Theorems A and B may fail if we weaken the assumptions (see Example 3.10).

Theorem B yields the following three corollaries that we prove at the end of Section 3. Recall that a ring R is said to be *s-unital* if $r \in rR \cap Rr$ for every $r \in R$. The following corollary applies in particular to any unital strongly G-graded ring.

Corollary B.1. Let G be a group and let R be an s-unital strongly G-graded ring. If f is a nonzero central idempotent in R, then f has finite support group. In particular, if G is torsion-free, then every central idempotent in R is contained in R_e.

Note that the preceding corollary and the next corollary both apply to any G-crossed product (see, e.g., [16] and [17]).

Corollary B.2. Let G be a group and let $A * G$ be a crystalline G-graded ring. If f is a nonzero central idempotent in $A * G$, then f has finite support group. In particular, if G is torsion-free, then every central idempotent in $A * G$ is contained in A.

A G-graded ring R is said to be *right non-degenerately G-graded* (resp. *left non-degenerately G-graded*) if, for each $g \in G$ and each nonzero $r_g \in R_g$, we have $r_g R_{g^{-1}} \neq \{0\}$ (resp. $R_{g^{-1}} r_g \neq \{0\}$). Notably, the next corollary generalizes [18, Thm. 6.2(iii)].

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Corollary B.3. Let G be a group and let R be a G-graded ring. Suppose that R_e is a prime ring and that R is right (or left) non-degenerately G-graded. If f is a nonzero central idempotent in R, then f has finite support group. In particular, if G is torsion-free, then every central idempotent in R is contained in R_e.

In Section 4, we apply our results to semigroup-graded rings, Leavitt path rings, fractional skew monoid rings, partial skew group rings, and algebraic Cuntz–Pimsner rings.

2 Proof of Theorem A

Given a unital ring A and a group G, the group ring $A[G]$ is defined as the free left (and right) A-module with $\{u_g\}_{g\in G}$ as its basis. Multiplication is defined by biadditively extending the rule $(au_g)(bu_h) := abu_{gh}$, for $a, b \in A$ and $g, h \in G$. Note that $R := A[G]$ carries a canonical G-grading that is given by $R_g := Au_g$, for $g \in G$.

The proof of Theorem A uses a trick based on the following graded embedding that is inspired by [14, p. 327].

Lemma 2.1. Let G be a group and let R be a unital G-graded ring. Equip the group ring $R[G]$ with its canonical G-grading. There is an injective identity-preserving ring homomorphism $\phi : R \to R[G]$ satisfying $\mathbf{Supp}(r) = \mathbf{Supp}(\phi(r))$ for every $r \in R$.

Proof. Let $r = \sum_{g\in G} r_g \in R$, with $r_g \in R_g$ zero for all but finitely many $g \in G$. We define $\phi : R \to R[G]$ by $\phi\left(\sum_{g\in G} r_g\right) := \sum_{g\in G} r_g u_g$. The map ϕ is clearly well defined, additive and injective.
Take $g, h \in G$, $a \in R_g$, and $b \in R_h$. Using that $ab \in R_gR_h \subseteq R_{gh}$, we note that

$$\phi(ab) = abu_{gh} = (au_g)(bu_h) = \phi(a)\phi(b).$$

It follows that ϕ is multiplicative. By [16, Prop. 1.1.1], $\phi(1_R) = 1_R u_e = 1_{R[G]}$. Clearly, $\mathbf{Supp}(r) = \mathbf{Supp}(\phi(r))$. ∎

For each $g \in G$, we define the map $(\cdot)_g : R \to R_g$, $\sum_{h\in G} r_h \mapsto r_g$. We let $Z(R)$ denote the center of R.

Lemma 2.2. Let G be an abelian group and let R be a G-graded ring. If $f \in Z(R)$, then $f_g \in Z(R)$, for every $g \in G$.

Proof. Suppose that $f \in Z(R)$. Take $h \in G$ and $r_h \in R_h$. Then, for any $g \in G$, we get that $r_h f_g = (r_h f)_{hg} = (fr_h)_{gh} = f_g r_h$. It now follows that $f_g \in Z(R)$, for every $g \in G$. ∎

Remark 2.3. Let D be the Dorroh unitization of R. Recall that $D := R \times \mathbb{Z}$ is an associative and unital ring with addition defined componentwise and multiplication defined by $(r, n)(s, m) := (rs + ns + mr, nm)$ for all $r, s \in R$ and for all $n, m \in \mathbb{Z}$ (for more details, see Dorroh's original article [10]).
Note that there is an injective ring homomorphism $\psi : R \to D$ defined by $\psi(r) := (r, 0)$. Furthermore, note that if f is a nonzero central idempotent in R, then $\psi(f)$ is a nonzero central idempotent in D.
Suppose that R is a G-graded ring. Then D inherits a natural G-grading from R. Indeed, define $D_e := R_e \times \mathbb{Z}$ and $D_g := R_g \times \{0\}$ for every $g \in G \setminus \{e\}$. Then D becomes a unital G-graded ring. Importantly, note that the support group of f in the G-graded ring R equals the support group of f in the unital G-graded ring D. Therefore, for the purposes of proving Theorem A, after identifying R with its image in D, we may always view f as an element of a unital G-graded ring.

We are now ready to prove our first main result.

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Proof of Theorem A. By Remark 2.3, replacing R by its Dorroh unitization if necessary, we may assume that R is unital. Let f be a nonzero central idempotent in R. By Lemma 2.1, $\phi(f)$ is a nonzero idempotent in the group ring $R[G]$, and $\mathbf{Supp}(f) = \mathbf{Supp}(\phi(f))$.
Take $h \in G$ and $r \in R$. By Lemma 2.2 and the fact that G is abelian, we get that

$$ru_h\phi(f) = ru_h\left(\sum_{g\in G} f_g u_g\right) = \sum_{g\in G} rf_g u_{hg} = \left(\sum_{g\in G} f_g u_g\right) ru_h = \phi(f)ru_h.$$

From this we conclude that $\phi(f) \in Z(R[G])$.
It now follows immediately from Burns's theorem that $\phi(f)$, and hence also f, has finite support group. Since G is abelian, the set of torsion elements forms a subgroup T of G, and clearly the support group of f is contained in T. ■

3 Proof of Theorem B

Recall that a group is said to be an *FC-group* if each of its conjugacy classes is finite. Next, we establish the following generalization of [18, Prop. 6.1].

Lemma 3.1. Let G be a group and let R be a G-graded ring. Suppose that at least one of the following two assertions holds:

(i) $R_g r \neq \{0\}$ for all nonzero homogeneous $r \in R$ and all $g \in \mathbf{Supp}(R)$.
(ii) $rR_g \neq \{0\}$ for all nonzero homogeneous $r \in R$ and all $g \in \mathbf{Supp}(R)$.

If f is a nonzero central element in R, then the subgroup of G generated by $\mathbf{Supp}(f)$ is an FC-group.

Proof. Let $f = \sum_{s\in G} f_s$ be a nonzero central element in R, and let H be the subgroup of G that is generated by the set $\mathbf{Supp}(f)$. Write $\Omega := \mathbf{Supp}(f) \cup \{e\} \cup \mathbf{Supp}(f)^{-1}$. We begin by showing that Ω is invariant under conjugation by elements from Ω. The element e is obviously fixed under conjugation by any element, and any element is fixed under conjugation by e.
Take $s, g \in \mathbf{Supp}(f) \setminus \{e\}$. Consider $r := f_s$.
Case A. (i) holds:
Choose a nonzero $a \in R_g$ such that $af_s \neq 0$. Note that $gs \in \mathbf{Supp}(af) = \mathbf{Supp}(fa)$, because $af = fa$. Hence, there is some $t \in \mathbf{Supp}(f)$ such that $gs = tg$. In other words, $gsg^{-1} = t \in \mathbf{Supp}(f) \subseteq \Omega$.
We claim that $g^{-1}sg \in \mathbf{Supp}(f)$ also holds. The preceding paragraph shows that $g\,\mathbf{Supp}(f)g^{-1} \subseteq \mathbf{Supp}(f)$. Since conjugation by g is injective and $\mathbf{Supp}(f)$ is finite, this inclusion is equality. Hence, there is some $k > 0$ such that $g^k sg^{-k} = s$. If $k = 1$, then clearly $g^{-1}sg = s \in \mathbf{Supp}(f) \subseteq \Omega$. If $k > 1$, then we note that $g^{-1}sg = g^{k-1}sg^{-k+1} \in \mathbf{Supp}(f) \subseteq \Omega$.
From $gsg^{-1} \in \mathbf{Supp}(f)$, we get that $gs^{-1}g^{-1} \in \mathbf{Supp}(f)^{-1} \subseteq \Omega$. Furthermore, from $g^{-1}sg \in \mathbf{Supp}(f)$, we get that $g^{-1}s^{-1}g \in \mathbf{Supp}(f)^{-1} \subseteq \Omega$.
Case B. (ii) holds:
An argument completely analogous to that in Case A allows us to conclude that $g^{-1}sg \in \mathbf{Supp}(f) \subseteq \Omega$ and $gsg^{-1} \in \mathbf{Supp}(f) \subseteq \Omega$. It then follows that $g^{-1}s^{-1}g \in \mathbf{Supp}(f)^{-1} \subseteq \Omega$ and $gs^{-1}g^{-1} \in \mathbf{Supp}(f)^{-1} \subseteq \Omega$.
In both cases, we are able to establish that Ω is invariant under conjugation by elements from Ω. Every element of H is a word in Ω. Thus, Ω is invariant under conjugation by elements from H.
Finally, let h be an arbitrary element of H. Then there is some $m \geq 1$ such that $h = s_1 \cdots s_m$ with $\{s_1, \ldots, s_m\} \subseteq \Omega$. For any $w \in H$, we may write

$$whw^{-1} = (ws_1w^{-1}) \cdots (ws_mw^{-1}).$$

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By the preceding paragraph, each ws_iw^{-1} lies in the finite set Ω. Hence, the conjugacy class of h contains at most $|\Omega|^m$ elements. In particular, it is finite. Thus, H is an FC-group. ■

The following example shows that the two conditions in Lemma 3.1 are independent.

Example 3.2. Let $G := \mathbb{Z}$ and consider the triangular matrix ring

$$R := \begin{pmatrix} \mathbb{Q} & \mathbb{Q}[t] \\ 0 & \mathbb{Q}[t] \end{pmatrix}$$

with the usual addition and multiplication. Endow R with a G-grading by setting

- $R_n := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ for $n < 0$,
- $R_0 := \begin{pmatrix} \mathbb{Q} & 0 \\ 0 & \mathbb{Q} \end{pmatrix}$, and
- $R_n := \begin{pmatrix} 0 & \mathbb{Q}t^{n-1} \\ 0 & \mathbb{Q}t^n \end{pmatrix}$ for $n > 0$.

It is easy to verify that condition (ii) of Lemma 3.1 is satisfied. But note that for the nonzero homogeneous element $r := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R_0$ we have $R_1 r = \{0\}$. Thus, this grading fails to satisfy condition (i) of Lemma 3.1. Passing to the opposite ring with the induced G-grading gives an example satisfying condition (i) but not condition (ii).

We pause for a moment to record a consequence of Lemma 3.1 that is of independent interest. Recall that a group G is said to be an *ICC group* if each non-trivial element of the group has an infinite conjugacy class (see, e.g., [9]). Non-trivial abelian groups cannot be ICC. But there exist torsion ICC groups. From that perspective, the next corollary complements the picture provided by Corollary A.

Corollary 3.3. Let G be an ICC group and let R be a G-graded ring with $\mathbf{Supp}(R) = G$. Suppose that at least one of the following two assertions holds:

(i) $R_g r \neq \{0\}$ for all nonzero homogeneous $r \in R$ and all $g \in G$.
(ii) $rR_g \neq \{0\}$ for all nonzero homogeneous $r \in R$ and all $g \in G$.

If f is a central idempotent in R, then $f \in R_e$.

Proof. Suppose that f is a central idempotent in R. If $f = 0$, then $f \in R_e$. Now, suppose that f is nonzero. The same argument as in Lemma 3.1 applies with $g \in \mathbf{Supp}(R) = G$, since the hypotheses supply a homogeneous element of degree g whose product with f_s is nonzero on the appropriate side. Thus, the finite set $\Omega := \mathbf{Supp}(f) \cup \{e\} \cup \mathbf{Supp}(f)^{-1}$ is invariant with respect to conjugation by every $g \in G$. Using that G is an ICC group, we conclude that Ω contains no non-trivial element. Hence, $\mathbf{Supp}(f) \subseteq \{e\}$, that is, $f \in R_e$. ■

Recall that a group is said to be *locally normal* if every finite subset is contained in a finite normal subgroup. We recall the following result from [6, Lem. 1].

Lemma 3.4 (Burns). Any FC-group G is isomorphic to a subdirect product of a torsion-free abelian group A with a locally normal group B.

Remark 3.5. Suppose that a group G is a subdirect product of A and B, for groups A and B. After identifying G with its image in $A \times B$, there are surjective group homomorphisms $\pi_A : G \to A,\ (a, b) \mapsto a$ and $\pi_B : G \to B,\ (a, b) \mapsto b$.

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Consider $N_A := \ker(\pi_A)$, which is a normal subgroup of G. By the first isomorphism theorem, we have $G/N_A \cong A$. Also note that N_A is isomorphic to a subgroup of B.

Remark 3.6. We recall some standard notation from [16, Sec. 1.2]. Let R be a G-graded ring. For any non-empty subset X of G, we will write $R_X := \oplus_{x\in X} R_x$. If H is a subgroup of G, then R_H is an H-graded subring of R. If N is a normal subgroup of G, then we may view R as a G/N-graded ring. For any $C \in G/N$ we set $R_C := \oplus_{x\in C} R_x$. It is easy to see that $R = \oplus_{C\in G/N} R_C$, and $R_C R_D \subseteq R_{CD}$, for all $C, D \in G/N$. Note, in particular, that N is the identity element of G/N. When we view R as a G-graded ring, the so-called *principal component* will be the subring R_e. But when viewed as a G/N-graded ring, the principal component will be the subring R_N.

Next, we prove our second main result.

Proof of Theorem B. Let f be a nonzero central idempotent in R, and let H be the support group of f. Clearly, f is contained in the H-graded subring R_H. By Lemma 3.1, H is an FC-group. Using Lemma 3.4, we can view H as a subdirect product of A and B, where A is torsion-free abelian and B is locally normal. By Remark 3.5, there is a normal subgroup N_A of H such that $H/N_A \cong A$.
By the aforementioned isomorphism and Remark 3.6, we may view R_H as an A-graded ring with principal component R_{N_A}. Clearly, f is central in R_H. Hence, by Corollary A, f is contained in R_{N_A} since A is torsion-free abelian.
Note that subgroups of locally normal groups are again locally normal. Therefore, since B is locally normal, the subgroup N_A is also locally normal. By the preceding paragraph, $\mathbf{Supp}(f)$ is contained in N_A, and hence there is a finite subgroup of N_A containing $\mathbf{Supp}(f)$. Thus, the support group of f must be finite. ■

Remark 3.7. In Theorem A, G is abelian, and hence the finite support group obtained is normal in G. In Theorem B, the finite support group obtained is normal in G whenever $\mathbf{Supp}(R) = G$. Indeed, this is a direct consequence of the proof of Lemma 3.1. In general, however, the support group of a nonzero central idempotent need not be normal in G. To see this, let $G := S_3$ be the symmetric group on three elements and consider the subgroup K generated by the permutation (12). The group ring $R := \mathbb{Q}[K]$ has its canonical K-grading that can be extended to a G-grading by setting $R_g := \{0\}$ for $g \in G \setminus K$. The element $f := \frac{1}{2}(1 + u_{(12)})$ is a nonzero central idempotent in R. Clearly, K is the support group of f, but K is not normal in G.

We now present three examples, beginning with a familiar one.

Example 3.8. Let A be a unital ring and consider the polynomial ring $R := A[t]$ in one indeterminate t. Define a $\mathbb{Z}$-grading on R by $R_n := At^n$ for $n \geq 0$, and $R_n := \{0\}$ for $n < 0$. By Corollary A, every central idempotent in R must be contained in $R_0 = A$.

Burns's theorem, as well as the main results in this article, cannot be generalized to hold for arbitrary idempotents, as the next example illustrates.

Example 3.9. Consider the group ring $R := M_2(\mathbb{C})[\mathbb{Z}]$ and let t be a generator for $\mathbb{Z}$. The element $f := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u_e + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} u_t$ is a non-central idempotent in R. We have $\mathbf{Supp}(f) = \{e, t\}$. Hence, the support group of f is infinite.

The next example illustrates a limitation of possible generalizations of Theorem A and Theorem B.

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Example 3.10. Let $G := D_\infty = \langle s, t \mid s^2 = t^2 = e\rangle$ be the infinite dihedral group, and let $R := \mathbb{Q}^4$ be the commutative direct product $\mathbb{Q}$-algebra. Consider the four elements

$$1_R = (1,1,1,1),\ \ a := (1,1,-1,-1),\ \ b := (1,-1,0,0),\ \text{ and }\ c := (0,0,1,-1).$$

We define additive subgroups of R as follows:

$$R_e := \mathbb{Q}1_R \oplus \mathbb{Q}a,\ \ R_s := \mathbb{Q}b,\ \ R_t := \mathbb{Q}c,\ \text{ and }\ R_g := \{(0,0,0,0)\}\ \text{ for }\ g \in G \setminus \{e,s,t\}.$$

Note that $\{1_R, a, b, c\}$ is a basis for R as a $\mathbb{Q}$-vector space. Thus, $R = \oplus_{g\in G} R_g$.
We shall now make a few observations:

- $a^2 = 1_R \in R_e$, and hence $R_eR_e \subseteq R_e$.
- $ab = b$ and $ac = -c$, and hence $R_eR_s \subseteq R_s$ and $R_eR_t \subseteq R_t$.
- $b^2 = (1,1,0,0) = \frac{1}{2}(1+a) \in R_e$, and hence $R_sR_s \subseteq R_e = R_{s^2}$.
- $c^2 = (0,0,1,1) = \frac{1}{2}(1-a) \in R_e$, and hence $R_tR_t \subseteq R_e = R_{t^2}$.
- $bc = 0$, and hence $R_sR_t = \{0\} \subseteq R_{st}$.

We conclude that $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. This shows that R is G-graded. Note, however, that $R_sc = \{0\}$ and $cR_s = \{0\}$. Hence, neither assertion (i) nor assertion (ii) in Theorem B is satisfied. Furthermore, D_∞ is non-abelian and thus violates the requirement in Theorem A.
Now, consider the element

$$f := \frac{1}{2}1_R + \frac{1}{2}b + \frac{1}{2}c = \frac{1}{2}(1,1,1,1) + \frac{1}{2}(1,-1,0,0) + \frac{1}{2}(0,0,1,-1) = (1,0,1,0).$$

Note that f is a nonzero central idempotent in R. But the subgroup generated by $\mathrm{Supp}(f) = \{e,s,t\}$ is equal to the infinite group D_∞.

The following lemma generalizes [12, Lem. 2.10(b)].

Lemma 3.11. Let G be a group and let R be an s-unital G-graded ring. Then $r \in rR_e \cap R_er$ for every $r \in R$.

Proof. Take $r \in R$. If $r = 0$, then there is nothing to prove. Suppose therefore that $r \neq 0$. Write $\mathrm{Supp}(r) = \{g_1, \ldots, g_n\}$. Consider the finite set $\{r_{g_1}, \ldots, r_{g_n}\} \subseteq R$. By s-unitality and [20] (see also [12, Prop. 2.1]), there is some $a \in R$ such that $ar_{g_i} = r_{g_i}a = r_{g_i}$ for every $i \in \{1, \ldots, n\}$. Note that $a_er_{g_i} = r_{g_i}a_e = r_{g_i}$ for every $i \in \{1, \ldots, n\}$. From this we get $a_er = ra_e = r$. In particular, $r \in rR_e \cap R_er$. ■

We end this section by proving the last three corollaries from the introduction.

Proof of Corollary B.1. Suppose that $r \in R$ satisfies $R_gr = \{0\}$ for some $g \in G$. Then, by Lemma 3.11 and the strong grading, we have $r \in R_er = R_{g^{-1}}R_gr = \{0\}$. The first part of the claim now follows from Theorem B. If G is torsion-free, then every finite support group is trivial, and hence all central idempotents will be contained in R_e. ■

Remark 3.12. Mihovski [15] considered the unital case of Corollary B.1 using a different method. His argument is particularly well suited for G-crossed products. However, in the full generality of unital strongly G-graded rings, certain steps in the proof of [15, Lem. 1] do not appear to apply directly without further hypotheses.

Proof of Corollary B.2. A crystalline graded ring $A * G$, as defined in [17], carries a natural G-grading. It is clear from the definition that for every $g \in G$, there is a nonzero element in $(A * G)_g$ that is neither a left nor a right zero-divisor in $A * G$ (see [17, Cor. 2.7]). The first part of the claim now follows

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from Theorem B. If G is torsion-free, then every finite support group is trivial, and hence all central idempotents will be contained in $(A * G)_e = A$. ■

Proof of Corollary B.3. Suppose that R is right non-degenerately G-graded. Seeking a contradiction, suppose that there is a nonzero element $r \in R_h$, for some $h \in G$, such that $R_g r = \{0\}$ for some $g \in \mathrm{Supp}(R)$. Consider the sets $I := R_{g^{-1}} R_g$ and $J := rR_{h^{-1}}$ that are right ideals of R_e. Clearly, by right non-degeneracy of the grading, J is nonzero. By right non-degeneracy of the grading it also follows that $R_{g^{-1}}$ is nonzero, and after using right non-degeneracy of the grading yet again, we see that I must be nonzero. Note that $IJ = R_{g^{-1}} R_g r R_{h^{-1}} = \{0\}$. But R_e is prime, and in a prime ring a product of two nonzero right ideals is nonzero. This is a contradiction. The proof in the left non-degenerately G-graded case is analogous. The first part of the claim now follows from Theorem B. If G is torsion-free, then every finite support group is trivial, and hence all central idempotents will be contained in R_e. ■

4 Applications

In this final section, we apply our results to various types of graded rings.

4.1 Semigroup-graded rings

Recall that a semigroup S is said to be *right reversible* if $Sa \cap Sb \neq \varnothing$ for all a, b in S. Left reversibility is defined analogously. Ore has shown that any right reversible cancellative semigroup can be embedded in a group (see [8, Thm. 1.23]). The same conclusion can be shown to hold for left reversible cancellative semigroups.

The *support semigroup* of a nonzero element r in an S-graded ring is the subsemigroup of S generated by $\mathrm{Supp}(r)$.

Proposition 4.1. Let S be a cancellative semigroup and let R be an S-graded ring. Furthermore, suppose that at least one of the following assertions holds:

(a) S is commutative.
(b) S is right (or left) reversible, and at least one of the following two assertions holds:
  (i) $R_g r \neq \{0\}$ for all nonzero homogeneous $r \in R$ and all $g \in \mathrm{Supp}(R)$.
  (ii) $rR_g \neq \{0\}$ for all nonzero homogeneous $r \in R$ and all $g \in \mathrm{Supp}(R)$.

If f is a nonzero central idempotent in R, then f has finite support semigroup in S.

Proof. Since S is cancellative, and any commutative semigroup is right (and left) reversible, it follows from [8, Thm. 1.23], in both cases (a) and (b), that S embeds in a group G.
Now, R is already S-graded. For each $g \in G \setminus S$, we set $R_g := \{0\}$. We may now view R as a G-graded ring. Let f be a nonzero central idempotent in R.
In case (a), the group G can in fact be assumed to be abelian. By Theorem A, the support group of f in G is finite. Hence, the support semigroup of f in S must also be finite.
In case (b), we may apply Theorem B to reach the same conclusion. ■

4.2 Leavitt path rings

For the definition of Leavitt path algebras and Leavitt path rings, we refer the reader to [1] and [13, Sec. 4], respectively. The following proposition is an immediate consequence of Corollary A.

Proposition 4.2. Let A be a unital ring, let E be a directed graph, and let $R := L_A(E)$ be the corresponding Leavitt path ring. Equip R with its canonical $\mathbb{Z}$-grading. If f is a central idempotent in R, then $f \in R_0$.

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4.3 Fractional skew monoid rings

For the definition of fractional skew monoid rings, we refer the reader to [2]. The next proposition follows from Theorem A.

Proposition 4.3. Let A be a unital ring, let G be an abelian group, and let S be a submonoid of G such that $G = S^{-1}S$. Furthermore, let $\mathbf{Endr}(A)$ be the monoid of not necessarily unital ring endomorphisms of A, and let $\alpha : S \to \mathbf{Endr}(A)$ be a monoid homomorphism. Consider the corresponding fractional skew monoid ring $R := S^{\mathrm{op}} *_\alpha A *_\alpha S$ with its canonical G-grading. If f is a nonzero central idempotent in R, then f has finite support group. Moreover, if G is torsion-free, then every central idempotent lies in R_e.

4.4 Partial skew group rings

For the definition of partial skew group rings, we refer the reader to [12, Sec. 11]. The following proposition is an immediate consequence of Corollary A.

Proposition 4.4. Let G be a torsion-free abelian group and let $(\{\alpha_g\}_{g\in G}, \{D_g\}_{g\in G})$ be a partial action of G on an s-unital ring A, with D_g s-unital for every $g \in G$. Every central idempotent in the partial skew group ring $A \star_\alpha G$ is contained in A.

The next proposition follows from Corollary B.3, [12, Prop. 11.1], and [12, Prop. 2.9].

Proposition 4.5. Let G be a group and let $(\{\alpha_g\}_{g\in G}, \{D_g\}_{g\in G})$ be a partial action of G on an s-unital prime ring A, with D_g s-unital for every $g \in G$. Every nonzero central idempotent in the partial skew group ring $A \star_\alpha G$ has finite support group.

4.5 Algebraic Cuntz–Pimsner rings

For the definition of algebraic Cuntz–Pimsner rings, we refer the reader to [7]. Recall that the Toeplitz ring $\mathcal{T}_{(P,Q,\psi)}$ and the relative Cuntz–Pimsner ring $\mathcal{O}_{(P,Q,\psi)}(J)$ carry natural $\mathbb{Z}$-gradings, and that the image of the coefficient ring is contained in degree zero. The next proposition follows from Corollary A.

Proposition 4.6. Let R be a ring and let (P, Q, ψ) be an R-system. Then every central idempotent in the Toeplitz ring $\mathcal{T}_{(P,Q,\psi)}$ lies in $(\mathcal{T}_{(P,Q,\psi)})_0$. If, moreover, (P, Q, ψ) satisfies Condition (FS) and J is a ψ-compatible ideal of R, then every central idempotent in the relative Cuntz–Pimsner ring $\mathcal{O}_{(P,Q,\psi)}(J)$ lies in $(\mathcal{O}_{(P,Q,\psi)}(J))_0$.

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